Sums of Cantor sets

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Abstract

We find conditions on the ratios of dissection of a Cantor set so that the group it generates under addition has positive Lebesgue measure. In particular we answer affirmatively a special case of a conjecture posed by J. Palis.

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1 Introduction

Cantor sets are sets that can be constructed in a similar fashion to the classical middle third Cantor set but rather than using the ratio 1/3 at each step, we allow the removed intervals to be variable in length and not necessarily centered.

Cantor sets appear in many different settings - and very often one is interested in knowing if the sum (or difference) of two Cantor sets has positive Lebesgue measure. For example, in dynamical systems, in connection with the study of homoclinic tangencies Palis asked if the difference of two Cantor sets is either of Lebesgue measure zero, or contains an interval ([10], pg. 151). This is known to be false in full generality (see for example [12], [1]), however it is of interest to know for which Cantor sets it is true. Sums of Cantor sets arise in connection with the study of intersections of translates of Cantor sets [7], and their topological structure has been studied in [8]. In harmonic analysis, sums of Cantor sets have been studied by Brown and Moran ([4], [3]) in connection with understanding the algebraic structure of the space of measures.

In this paper we concentrate on the question of whether there exists an integer n such that the Lebesgue measure of $(n)C = \underbrace{C + \cdots + C}_{n}$ is positive, as is well known for the middle third Cantor set with $n = 2$ (c.f. [13], [2]). We prove that for certain Cantor sets which admit a construction with ratios of dissection bounded away from 0, such an integer n can always be found. This is true, for example, if the removed interval in the Cantor set is always centred. The case in which the removed interval is not centred is more complicated and surprisingly very anti-intuitive because of the loss of symmetry, however, we are still able to find sufficient conditions on the ratios of dissection for the result to be true for a wide class of these sets. We show that Cantor sets with ratios not bounded away from zero may generate groups of Lebesgue measure zero. We analyze the Hausdorff dimension of Cantor sets and prove that having positive Hausdorff dimension is a necessary but not sufficient condition for the sum of Cantor sets to have positive Lebesgue measure. One consequence of our results is to yield an affirmative answer to Palis' conjecture for the case of central Cantor sets with constant ratios of dissection.

It is worth noticing here, that the group generated by any measurable set E under addition has positive Lebesgue measure, if and only if the group generated by E under addition *mod 1* has positive Lebesgue measure. Hence the results that are proved for one case should be understood as holding for the other. Our criteria to choose one or the other has been to improve clarity in the exposition and simplicity in the proofs.

2 Cantor sets of positive Lebesgue measure sum

Definition: A *Cantor set* is a compact, totally disconnected, perfect subset of the real line. Cantor sets can be constructed in a similar fashion to the classical middle third Cantor set. We begin with a compact interval on the real line and remove from it a finite number of open intervals, leaving a finite union of closed intervals within the initial interval, called the intervals of step one. The ratios between the lengths of each of these intervals and the initial one are referred to as the ratios of dissection at step one. A similar operation is then executed in each interval of step one producing the intervals of step two and the ratios of dissection of step two. In order to avoid isolated points, we require that any two of the removed intervals have no common endpoints. This construction defines a decreasing sequence of closed sets, the intersection of which is a Cantor set. Note that this is a very general definition of a Cantor set, and includes all *dynamically defined* Cantor sets (see [10]).

Every Cantor set C constructed removing several open intervals from each interval at every step, contains a Cantor set C' in which only one interval is removed from each interval at each step. To see this, simply observe that if $J_1, \ldots, J_r$ are the open intervals removed from I at some step, then C' can be constructed by removing the smallest open interval containing $\cup J_i$. In order to prove that the measure of $(n)C$ is positive, it certainly suffices to show that the measure of $(n)C'$ is positive. We will therefore restrict our attention to Cantor sets of the latter type.

In addition, we will only consider Cantor sets constructed in the interval $[0,1]$. It is clear that these assumptions do not represent any loss of generality and are only imposed to simplify the proofs.

Notation

Let T^* be the set of binary words of finite length, i.e:

$$T^* = \{ \ \alpha_1 \ldots \alpha_s, : \alpha_i \in \{0,1\}, \ s \in \mathcal{N} \ \} \ .$$

Let e be the empty word, and $T = T^* \cup \{e\}$. If $\alpha = \alpha_1 \ldots \alpha_s$, $\beta = \beta_1 \ldots \beta_r \in T$ then $\alpha\beta \ \in \ T$ where $\alpha\beta \equiv \alpha_1 \ldots \alpha_s \beta_1 \ldots \beta_r$ is the concatenation of α and β.

For $\alpha \in T$ we denote by $|\alpha|$ the length of α and we define $|e| = 0$.

Let us call $I = I_e = [0,1]$, I_0 and I_1 the left and right closed intervals remaining after the first open interval is removed from I, and let $F_1 = I_0 \cup I_1$. We call I_0, I_1 intervals of step 1.

In general $I_{\alpha 0}$ and $I_{\alpha 1}$, for $\alpha \in T$ and $|\alpha| = k$, will be the left and right intervals of step $k+1$ obtained after an open interval is removed from I_α, where I_α is one of the 2^k closed intervals of step k. If $F_k = \cup\{I_\alpha, \alpha \in T, |\alpha| = k\}$, then the collection $\{I_\alpha\}_{\alpha \in T}$ univocally defines the set $C = \cap_{k \in \mathcal{N}} F_k$.

Let us also denote by $\xi_{\alpha 0} = \dfrac{|I_{\alpha 0}|}{|I_\alpha|}$ and $\xi_{\alpha 1} = \dfrac{|I_{\alpha 1}|}{|I_\alpha|}$ for every $\alpha \ \in \ T$. The numbers $\{\xi_\alpha\}_{\alpha \in T^*}$ are the rates of dissection of the Cantor set and $\{\xi_\alpha\}_{|\alpha| = k}$ the rates of dissection of step k.

Note that these rates satisfy that $\xi_\alpha \ \in \ (0,1)$ for every $\alpha \in T^*$ and $\xi_{\alpha 0} + \xi_{\alpha 1} < 1$ for every $\sigma \in T$. Conversely, any family $\mathcal{A} = \{\xi_\alpha\}_{\alpha \in T^*}$ satisfying $\xi_\alpha \in (0,1)$ for every $\alpha \in T^*$ and $\xi_{\alpha 0} + \xi_{\alpha 1} < 1$ for every $\alpha \in T$ defines univocally the Cantor set $C_{\mathcal{A}}$. It is interesting to note that constructions with different ratios of dissection can produce the same Cantor set. A surprising example of this is the fact that the classical middle third Cantor set can be constructed with ratios of dissection $\{\xi_\alpha\}$ satisfying $\inf \xi_\alpha = 0$.

Central Cantor sets are Cantor sets where $\xi_{\alpha 0} = \xi_{\alpha 1}$ for every $\alpha \in T^*$. A particular case of central Cantor sets, which we will denote by C_a, is when the ratios of dissection are constant, i.e. $\xi_\alpha = a$ for every $\alpha \in T^*$.

We will denote by $\dim_H(C)$ the Hausdorff dimension of C, and by $\tau(C)$ the thickness of C (for the definition see [10]). Newhouse has shown that if $\tau(C_1)\tau(C_2) > 1$ then $C_1 + C_2 = [0, 2]$ ([9], [10] pg. 63), however it is not clear how this condition can be generalized to answer the question of when $C_1 + \cdots + C_n$ contains an interval (or even has positive Lebesgue measure).

In this section we will instead find explicit conditions on the ratios of dissection which will enable us to conclude that $(n)C \equiv \{c_1 + \cdots + c_n : c_i \in C\}$ has positive Lebesgue measure for a wide class of Cantor sets.

We will begin with a simple Lemma.

Lemma 2.1 *Let $C = \cap F_k$ be a Cantor set. Then*

$$(n)C = \bigcap (n)F_k,$$

and, in particular, if $(n)F_k \supseteq I$ for some interval I, for every k, then $(n)C \supseteq I$.

The proof is straightforward.

First we consider the case of a central Cantor set C_a of constant ratio of dissection a. Since $\tau(C_a) = \frac{a}{1-2a}$, Newhouse's result implies that $C_a + C_a = [0, 2]$ if $a > \frac{1}{3}$. Using a different approach, we are able to show that $\tau(C_a) \geq 1$ is necessary and sufficient for this result to hold true, and we generalize it to n-fold sums.

Proposition 2.2 *Let C_a be the central Cantor set of ratio of dissection $a > 0$. Then for $\dfrac{1}{n+1} \leq a < \dfrac{1}{n}$ we have*

i) $(q)C_a = [0, q]$ *for $q \geq n$, and*

ii) $\dim_H((q)C_a) = \dfrac{-\ln(q+1)}{\ln a} < 1$ *for $q = 1, \ldots, n-1$.*

Proof:

i) Observe that

$$F_1 = [0, a] \cup [1-a, 1] \quad \text{and hence,} \quad F_1 + F_1 = [0, 2a] \bigcup [1-a, 1+a] \bigcup [2(1-a), 2].$$

In general

$$(q)F_1 = \bigcup_{j=0}^{q} I_j^1 = \bigcup_{j=0}^{q} [j(1-a), (q-j)a + j]. \tag{2.1}$$

Now if $\dfrac{1}{q+1} \leq a$ then $I_j^1 \cap I_{j+1}^1 \neq \emptyset, j = 0, \ldots, q-1$ and therefore $\bigcup_{j=0}^{q} I_j^1 = [0, q]$.

Since $\dfrac{1}{n+1} \leq a$ we have that if $q \geq n$, $(q)F_1 = [0, q]$.

Now assume that $(q)F_k = [0, q]$. Since $F_k = \bigcup_{|\sigma|=k} I_\sigma$ where I_σ are intervals of step k, then

$$(q)F_k = (q) \bigcup_{|\sigma|=k} I_\sigma = \cup\{I_{\sigma_1} + \cdots + I_{\sigma_q} : |\sigma_i| = k, i = 1, \ldots, q\}. \tag{2.2}$$

We are now going to show, that each member of this union can be covered by sums of q intervals of step $k+1$. For this we first note that

$$I_{\sigma_1} + \cdots + I_{\sigma_q} = (I_0^k + \tau_{\sigma_1}) + \cdots + (I_0^k + \tau_{\sigma_q}) = (q)I_0^k + \sum \tau_{\sigma_i},$$

where I_0^k is the leftmost interval at step k, i.e. $I_0^k = I_\alpha$ with α being the word of length k having k zeros, and the τ_{σ_k} are the corresponding translations. It is straightforward to see that if $q \geq n$

$$(q)I_0^k = (q)(I_0^{k+1} \cup \ I_1^{k+1}),$$

where I_0^{k+1} and I_1^{k+1} are the two intervals of step $k+1$ corresponding to the interval I_0^k. But

$$\begin{aligned} (q)(I_0^{k+1} \cup \ I_1^{k+1}) + \sum_{i=1}^{q} \tau_{\sigma_i} &= \bigcup_{\ell=0}^{q} \left((\ell)I_0^{k+1} + (q-\ell)I_1^{k+1} \right) + \sum_{i=1}^{q} \tau_{\sigma_i} \\ &= \bigcup_{\ell=0}^{q} \left(\sum_{i=1}^{\ell} (I_0^{k+1} + \tau_{\sigma_i}) + \sum_{i=\ell+1}^{q} (I_1^{k+1} + \tau_{\sigma_i} \right) \quad . \end{aligned}$$

Now observing that $I_s^{k+1} + \tau_{\sigma_i}$ for $s = 0, 1$ is a sub-interval of step $k+1$ of I_{σ_i} $(i = 1, \ldots, q)$, we obtain $(q)F_k \subseteq (q)F_{k+1}$ and hence the equality holds and $(q)F_{k+1} = [0, q]$.

ii) If $q < n$ we observe that $(q)C_a$ is a Cantor set in $[0, q]$ where q intervals are removed at each step (the union in (2.2) is disjoint), and the ratio of dissection is a. Hence (see for example Falconer [6])

$$\dim_H((q)C_a) = \frac{-\ln(q+1)}{\ln(a)}. \qquad \blacksquare$$

Corollary 2.3 *Let C_a be the central Cantor set of ratio of dissection $a > 0$. Then either $(n)C_a$ has Lebesgue measure zero or it contains an interval, in which case $(n)C = [0, n]$, and the latter happens if and only if $\frac{1}{n-2} > \tau(C_a) \geq \frac{1}{n-1}$.*

Remark: This Corollary completely answers (affirmatively) Palis' conjecture for $C_a + C_a$. It also shows that in order for the sum to be an interval (or even to have positive Lebesgue measure) it is necessary (as well as sufficient) for the thickness to be greater than or equal to 1.

We now want to extend the previous result to central Cantor sets with variable ratios of dissection. We will show that if the ratios of dissection are bounded away from zero, then the Cantor set will generate a group of positive Lebesgue measure.

We will first need the following Lemma:

Lemma 2.4 *For each $i = 1, \ldots, n$ consider an arbitrary interval $I_i = [a_i, b_i]$ and two subintervals I_i^1, I_i^2, such that $a_i \in I_i^1 \subset I_i$ and $b_i \in I_i^2 \subset I_i$. Let $a = \min\limits_{\substack{k=1,2 \\ i=1,\ldots,n}} \left(\frac{|I_i^k|}{|I_i|} \right)$. If $a \geq \dfrac{\max_i |I_i|}{(n+1) \min_i |I_i|}$, then*

$$\sum_{i=1}^{n} (I_i^1 \cup I_i^2) = \sum_{i=1}^{n} I_i.$$

Proof: The proof is straightforward writing down the expressions for the left and right endpoints of the intervals. ∎

Theorem 2.5 *Let C be the central Cantor set where the ratio of dissection at the k^{th} step is ξ_k. If $\inf \xi_k \geq \dfrac{1}{n+1}$, then $(n)C = [0, n]$. In particular, the Lebesgue measure of $(n)C$ is positive.*

Proof: We will prove that $(n)F_k = (n)[0,1]$, for each k. We are going to use induction on the steps used in the construction of C.

In the first step we deduce from the previous lemma that

$$(n)[0,1] = \sum_{i=1}^{n} (I_1^1 \cup I_2^1) = \sum_{i=1}^{n} F_1 = (n) F_1.$$

(In general, I_s^k, for $s = 1, \ldots, 2^k$, will denote the intervals of step k.)

Now assume that $(n)F_k = (n)[0,1]$, that is

$$(n)[0,1] = \bigcup_j (I_{j_1}^k + \cdots + I_{j_n}^k).$$

To complete the proof of the theorem, it is enough to show that each member of this union can be covered by sums of n intervals of step $k+1$. But again, by the previous lemma,

$$I_{j_1}^k + \cdots + I_{j_n}^k = \sum_{i=1}^{n} (I_{j_i,1}^{k+1} \cup I_{j_i,2}^{k+1})$$

where $I_{j_i,1}^{k+1}$ and $I_{j_i,2}^{k+1}$ are the two intervals of step $k+1$ corresponding to the interval $I_{j_i}^k$. ∎

It is easy to see that if $\sup \xi_k < \dfrac{1}{n+1}$ then $(n)C$ does not contain an interval. However, it can still have positive measure ([1]).

Note that in the proof of Theorem 2.5 we could apply Lemma 2.4 since the intervals at a given step were the same. It should be clear to the reader that we can, in fact, obtain a slightly more general version of the last theorem:

Theorem 2.6 *Let $C_{\mathcal{A}}$ be the Cantor set of rates $\mathcal{A} = \{\xi_\sigma\}_{\sigma \in T}$. Suppose there is an $N \geq 1$ such that $\dfrac{1}{N} < \dfrac{|I_\alpha|}{|I_\beta|} < N$, for every $|\alpha| = |\beta| = k$, and for all $k \in \mathcal{N}$. If $\inf \xi_\sigma \geq \dfrac{N}{n+1}$, then $(n)C_{\mathcal{A}} = [0, n]$.*

Let us now consider the more general case in which not necessarily $\xi_{\sigma 0} = \xi_{\sigma 1}$. It is worth pointing out that this translates into a loss of symmetry in the construction of the Cantor sets which makes the problem significantly harder. Note for example that in this case the ratio between two intervals at step k can be unbounded (above and below!).

Consider first the following case: for $0 < a, b$ and $a + b < 1$ denote by C_{ab} the Cantor set with rates $\xi_{\sigma 0} = a$ and $\xi_{\sigma 1} = b$ for all $\sigma \in T^*$. Newhouse's result ([9]) implies for this case that

$$\frac{1-(a+b)}{\min(a,b)} < 1 \quad \Longrightarrow C_{ab} + C_{ab} = [0,2]\ . \tag{2.3}$$

E. Dubuc [5] proved the following result, which is weaker (for $n = 2$) than (2.3) but works for $n > 2$.

Theorem 2.7 *([5]) Let $a, b \in \mathcal{R}$, $a, b > 0$ be such that $a + b < 1$ and let C_{ab} denote the Cantor set that can be constructed using the rates a and b at each step.*

$$\textit{If} \quad \frac{1-(a+b)}{ab} \leq n-1 \qquad \textit{then} \quad (n)C_{ab} = [0, n].$$

In particular the Lebesgue measure of $(n)C_{ab}$ is positive.

In what follows, we consider the more general case of non-central Cantor sets in which the rates are allowed to vary from step to step. Let $\overline{a} = \{a_k\}, \overline{b} = \{b_k\}$ be sequences of positive real numbers, satisfying $a_k + b_k < 1$ for every k. Consider the Cantor set $C_{\overline{a}\overline{b}}$ with rates $\{\xi_\sigma\}_{\sigma \in T^*}$ where $\xi_{\sigma 0} = a_k$, and $\xi_{\sigma 1} = b_k$, for $|\sigma| = k - 1$. We will show that by imposing certain restrictions on these sequences one can prove that the group generated by the set has positive measure.

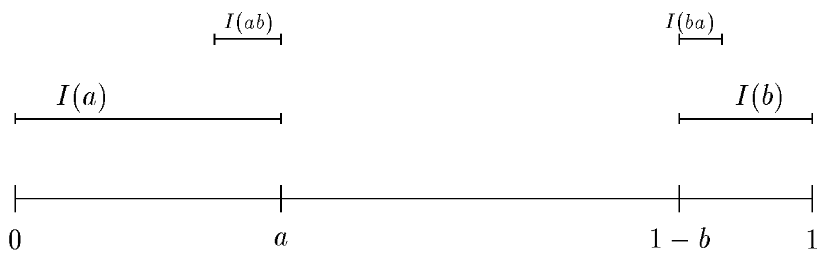
We will use the following notation: Let I be an interval and $0 < a, b$ and $a + b < 1$. We denote

$I(a)$ the subinterval of I of *rate* a respect to I, that has the same *left* endpoint than I,

$I(b)$ the subinterval of I of *rate* b respect to I, that has the same *right* endpoint than I,

$I(ab)$ the subinterval of $I(a)$ of *rate* b respect to $I(a)$, that has the same *right* endpoint than $I(a)$,

$I(ba)$ the subinterval of $I(b)$ of *rate* a respect to $I(b)$, that has the same *left* endpoint than $I(b)$.



Lemma 2.8 *Let $I_1, \ldots, I_n$ be n intervals of lengths $r_1, \ldots, r_n$ respectively, and a, b, real numbers satisfying $0 < a, b$ and $a + b < 1$, . Let $\{i_1, \ldots, i_n\}$ be a permutation of $\{1, 2, \ldots, n\}$ and assume that the following n conditions hold*

$$1 - (a+b) \leq b\frac{(r_{i_1} + \cdots + r_{i_{\ell-1}})}{r_{i_\ell}} + a\frac{(r_{i_{\ell+1}} + \cdots + r_{i_n})}{r_{i_\ell}} \tag{2.4}$$

where $\ell = 1, \ldots, n$ and $r_{i_0} = r_{i_{n+1}} = 0$.

$$\textit{Then} \qquad \sum_{j=1}^{n} I_j = \bigcup_{h=0}^{n} J_h$$

$$\begin{aligned} \textit{where} \quad J_0 &= \sum_{j=1}^{n} I_j(a) \\ \textit{and} \quad J_\ell &= \sum_{j=1}^{\ell} I_{i_j}(b) + \sum_{j=\ell+1}^{n} I_{i_j}(a) \ , \quad \ell = 1, \ldots, n-1 \ , \\ \textit{and} \quad J_n &= \sum_{j=1}^{n} I_j(b). \end{aligned}$$

Proof: It is sufficient to prove the lemma for the case that the permutation is the identity (i.e. $(i_1, \ldots, i_n) = (1, \ldots, n)$, just by renumbering the intervals. We can also assume that the left endpoint of the intervals $\{I_i\}$ is the origin, since otherwise the result follows by translating the intervals.

Let us call $H = \sum_{i=1}^{n} I_i$.

H has the same left endpoint as J_0 and the same right endpoint as J_n. We are now going to see that condition ℓ in (2.4) is precisely the condition needed so that the left endpoint of J_ℓ is smaller than, or equal to the right endpoint of $J_{\ell-1}$ ($\ell = 1, \ldots, n$). We show this for $\ell = 2$ to $n-1$, the cases $\ell = 1$ and $\ell = n$ are similar. The result then follows.

We have:

$$I_i = [0, r_i], \quad I_i(a) = [0, ar_i], \quad I_i(b) = [(1-b)r_i, r_i] \ ,$$

and therefore, if $2 \leq \ell \leq n$

$$\begin{aligned} J_\ell &= \sum_{j=1}^{\ell} I_i(b) + \sum_{j=\ell+1}^{n} I_i(a) \\ &= \left[(1-b)\sum_{j=1}^{\ell} r_j \ , \ \sum_{j=1}^{\ell} r_j + a \sum_{j=\ell+1}^{n} r_j \right] \end{aligned}$$

Now, if we call $J_\ell = [c_\ell, d_\ell]$ we see, that condition ℓ in (2.4) implies $c_\ell < d_{\ell-1}$. ■

We can now prove the following theorem:

Theorem 2.9 *Let $\overline{a} = \{a_k\}$ and $\overline{b} = \{b_k\}$ be sequences of positive numbers with $a_k + b_k < 1$, and assume that there exists a constant $N \geq 1$ such that the sequences $\overline{a}$ and $\overline{b}$ verify*

$$\frac{1}{N} \leq \frac{b_k}{a_k} \leq N \quad \forall\ k \in \boldsymbol{\mathcal{N}}. \tag{2.5}$$

If n is such that

$$1 - (a_k + b_k) \leq \frac{\min(a_k, b_k)}{N}(n-1) \quad \forall k\ \in \boldsymbol{\mathcal{N}} \tag{2.6}$$

then

$$(n)C_{\overline{a}\overline{b}} = [0, n].$$

Proof: Let F_k be the union of the Cantor intervals of step k, with $F_0 = [0,1]$, i.e. $C_{\overline{a}\overline{b}} = \cap F_k$. By Lemma 2.1 $(n)C_{\overline{a}\overline{b}} = \cap_k((n)F_k)$.

To prove that $(n)F_k = [0, n]$ for every k, we will first prove by induction that $(n)F_k$ can be written as a finite union of intervals, each of which is the sum of n intervals of step $k+1$ where the ratio between any two of these intervals is bounded. More precisely we will prove inductively that:

"If the rates satisfy (2.5) and (2.6), then for every non-negative integer $k, (n)F_k$ is a finite union of intervals, $(n)F_k = \cup_{s=1}^{m_k} J_s$, with each J_s being the sum of n intervals, $J_s = I_1 + I_2 + \cdots + I_n$, where the sets I_j are all Cantor intervals of step $k+1$ and satisfy $\frac{1}{N} \leq \frac{|I_i|}{|I_j|} \leq N$ for all i, j."

For $k = 0$, we apply Lemma 2.8 to $I_1 = I_2 = \cdots = I_n = F_0, a_1, b_1$ and any permutation. Since

$$1 - (a_1 + b_1) \leq \frac{\min(a_1, b_1)}{N}(n-1) \qquad \text{and}$$

$$\frac{\min(a_1, b_1)}{N}(n-1) \leq \min(a_1, b_1)(n-1) \leq b_1(\ell - 1) + a_1(n - \ell)\ ,\ \ell = 1, \ldots, n,$$

the conditions of the Lemma are satisfied. In addition, the intervals $I_i(a_1)$ and $I_i(b_1)$ are Cantor intervals of step 1 and satisfy $\frac{|I_i(a_1)|}{|I_j(a_1)|} = 1 = \frac{|I_i(b_1)|}{|I_j(b_1)|}$ for every (i,j), and since $|I_i(a_1)| = a_1$ and $|I_i(b_1)| = b_1$, by (2.5) we obtain the boundedness condition for the intervals, and therefore the induction step for $k = 0$.

Assume now that the proposition is true for $k-1$, i.e. $(n)F_{k-1} = \cup_{s=1}^{m_{k-1}} J_s$ with J_s the sum of n intervals of step k. Fix $s, (1 \leq s \leq m_{k-1})$, and assume that $J_s = I_1 + \cdots + I_n$, with I_i a Cantor interval of step k and $\frac{1}{N} \leq \frac{|I_i|}{|I_j|} \leq N$ for every i, j. We will show that we can apply Lemma 2.8 to $I_1, \ldots, I_n, a_{k+1}, b_{k+1}$, and any permutation $(i_1, \ldots, i_n)$.

To see this, we first observe that by hypothesis

$$1 - (a_{k+1} + b_{k+1}) \leq \frac{\min(a_{k+1}, b_{k+1})}{N}(n-1).$$

Also, if $|I_i| = r_i$, the boundedness condition on the intervals I_i implies

$$\begin{aligned} \frac{\min(a_{k+1}, b_{k+1})}{N}(n-1) &\leq \min(a_{k+1}, b_{k+1}) \left(\sum_{j \neq \ell} \frac{r_{i_j}}{r_{i_\ell}} \right) \leq \\ & b_{k+1} \frac{(r_{i_1} + \cdots + r_{i_{\ell-1}})}{r_{i_\ell}} + a_{k+1} \frac{(r_{i_{\ell+1}} + \cdots + r_{i_n})}{r_{i_\ell}}. \end{aligned}$$

Therefore the n conditions of the Lemma are satisfied, and hence $J_s = U_0 \cup \cdots \cup U_n$ where

$$\begin{aligned} U_0 &= \sum I_i(a_{k+1}) \ , \\ U_\ell &= \sum_{h=1}^{\ell} I_{i_h}(b_{k+1}) + \sum_{h=\ell+1}^{n} I_{i_h}(a_{k+1}) \ , \quad \ell = 1, \ldots, n-1 \ , \\ U_n &= \sum I_i(b_{k+1}) \ . \end{aligned}$$

Note that all the intervals in the sums are Cantor intervals of step $k+1$.

Next we establish boundedness condition. For the intervals in U_0 and U_n, we see that

$$\frac{|I_i(a_{k+1})|}{|I_j(a_{k+1})|} = \frac{|I_i|}{|I_j|} = \frac{|I_i(b_{k+1})|}{|I_j(b_{k+1})|} \ ,$$

and therefore the conditions are fulfilled by the inductive hypothesis.
In order to check the condition for the sets U_ℓ for $1 \leq \ell < n$, the appropriate choice of the permutation is crucial. We choose the permutation with the following criteria: Let $(i_1, \ldots, i_n)$ be such that $|I_{i_1}| = r_{i_1} \leq \cdots \leq |I_{i_n}| = r_{i_n}$.

$$\begin{aligned} &\text{If} \quad \frac{b_{k+1}}{a_{k+1}} > 1 \quad \text{we select the permutation } (i_1, \ldots, i_n), \\ &\text{if} \quad \frac{b_{k+1}}{a_{k+1}} < 1 \quad \text{we select the permutation } (i_n, \ldots, i_1), \end{aligned}$$

otherwise we choose either one of these two permutations.

We will consider the case $1 \leq \dfrac{b_{k+1}}{a_{k+1}}$; the other case is similar. The intervals in the sum are:

$$I_{i_1}(b_{k+1}), \ldots, I_{i_\ell}(b_{k+1}), I_{i_{\ell+1}}(a_{k+1}), \ldots, I_{i_n}(a_{k+1}).$$

Observe that if $h, t \leq \ell$ or $h, t > \ell$ the intervals I_{i_h}, I_{i_t} are scaled by the same numbers, and therefore the rate between them does not change.

If $1 \leq t \leq \ell < h \leq n$ we have

$$\begin{aligned} 1 &\leq \frac{b_{k+1}}{a_{k+1}} \leq N \quad \text{and} \\ \frac{1}{N} &\leq \frac{|I_{i_t}|}{|I_{i_h}|} \leq 1 \quad (\text{ by the choice of the permutation}) \ , \end{aligned}$$

and combining these two inequalities we get

$$\frac{1}{N} \leq \frac{|I_{i_t}(b_{k+1})|}{|I_{i_h}(a_{k+1})|} \leq N$$

which shows that the condition is fulfilled.

We have therefore proved that each J_s can be written as a finite union of sums of n intervals of step $k+1$ with the necessary boundedness condition, and thus $(n)F_{k-1} = \cup J_s$, has the same property. Since the inductive hypothesis implies $(n)F_{k-1} \subset (n)F_k$ and we always have the reverse inclusion, it follows that $(n)F_k = (n)F_{k-1}$ completing the induction proof. Because $(n)F_0 = [0, n]$, the proof is complete. ■

We have the following corollary:

Corollary 2.10 *Let $\overline{a} = \{a_k\}$, $\overline{b} = \{b_k\}$ be sequences of positive real numbers, with $a_k + b_k < 1$ for every $k \in \mathcal{N}$ and let $a = \inf\{a_k\}$ and $b = \inf\{b_k\}$. If $a, b > 0, a \leq b$ and $n \in \mathcal{N}$ satisfies*

$$\tau(C_{\overline{ab}}) = \frac{a}{1-(a+b)} \geq \frac{b}{a(n-1)} \tag{2.7}$$

then $(n)C_{\overline{ab}} = [0, n]$. In particular the Lebesgue measure of $(n)C_{\overline{ab}}$ is positive.

Proof: This is a particular case of Theorem 2.9 where $N = \dfrac{b}{a}$. ■

Remark: For the case $n = 2$ condition (2.7) is weaker than $\tau(C_{\overline{ab}})\tau(C_{\overline{ab}}) > 1$ when $a \neq b$, however, it does give results when $\tau(C_{\overline{ab}}) \leq 1$.

In some cases we are able to improve Theorem 2.9 generalizing Theorem 2.7 to the case where the rates are allowed to vary at each step.

Lemma 2.11 *Let $I_1, \ldots, I_n$ be n intervals of lengths $r_1, \ldots, r_n$ respectively, and a_i, b_i, $i = 1, 2$ real numbers satisfying $0 < a_i, b_i$, $a_i + b_i < 1$, $i = 1, 2$. Let $\{i_1, \ldots, i_n\}$ be a permutation of $\{1, 2, \ldots, n\}$ and assume that the following n conditions hold*

$$1 - (a_1 + b_1) \leq b_1 a_2 \frac{(r_{i_1} + \cdots + r_{i_{\ell-1}})}{r_{i_\ell}} + a_1 b_2 \frac{(r_{i_{\ell+1}} + \cdots + r_{i_n})}{r_{i_\ell}} \tag{2.8}$$

where $\ell = 1, \ldots, n$ and $r_{i_0} = r_{i_{n+1}} = 0$.

$$\textit{Then} \qquad \sum_{j=1}^{n} I_j = \bigcup_{h=0}^{n} J_h$$

$$\begin{aligned} \textit{where}\quad J_0 &= \sum_{j=1}^{n} I_j(a_1)\ , \\ J_\ell &= \sum_{j=1}^{\ell} I_{i_j}(b_1a_2) + \sum_{j=\ell+1}^{n} I_{i_j}(a_1b_2)\ , \quad \ell = 1,\ldots,n-1\ , \\ \textit{and}\quad J_n &= \sum_{j=1}^{n} I_j(b_1). \end{aligned}$$

Proof: The proof is essentially the same as for Lemma 2.8. ■

We can then prove the following theorem:

Theorem 2.12 *Let $\overline{a} = \{a_k\}$ and $\overline{b} = \{b_k\}$ be sequences of positive numbers with $a_k + b_k < 1$, and assume that there exists a constant $N \geq 1$ such that the sequences $\overline{a}$ and $\overline{b}$ verify*

$$\frac{1}{N} \leq \frac{b_k a_{k+1}}{a_k b_{k+1}} \leq N \quad \forall\ k \in \mathcal{N}. \tag{2.9}$$

If n is such that

$$1 - (a_k + b_k) \leq \frac{\min(b_k a_{k+1}, a_k b_{k+1})}{N}(n-1) \quad \forall k\ \in \mathcal{N} \tag{2.10}$$

then

$$(n)C_{\overline{a}\overline{b}} = [0, n].$$

Proof: The proof works as in Theorem 2.9, except that we now have to prove the following statement by induction:

"If the rates satisfy (2.9) and (2.10), then for every non-negative integer k, $(n)F_k$ is a finite union of intervals, $(n)F_k = \cup_{s=1}^{m_k} J_s$, with each J_s being the sum of n intervals, $J_s = I_1 + I_2 + \cdots + I_n$, where the sets I_j are all Cantor intervals of **the same step** $k_s \geq k+1$ and satisfy $\frac{1}{N} \leq \frac{|I_i|}{|I_j|} \leq N$ for all i, j."
■

This Theorem allows us to find examples of Cantor sets $C_{\mathcal{A}}$ with rates $\mathcal{A} = \{\xi_\sigma\}_{\sigma \in T}$ satisfying $\inf(\xi_\sigma) = 0$, and yet there still exists n such that $(n)C_{\mathcal{A}} = [0, n]$.

Example: Let $\{b_k\}$ be the sequence $b_k = \dfrac{1}{k+3}$ and define the sequence $\{a_k\}$ by $a_k = 1 - \dfrac{3}{k+3}$. Then the conditions of the Theorem are fulfilled with $N = 2$ and $n = 21$. Note that in this case $\tau(C_{\mathcal{A}}) = \frac{1}{2}$.

It is still an unanswered question whether any Cantor set whose rates $\{\xi_\sigma\}_{\sigma \in T}$ are bounded away from zero generates a group of positive Lebesgue measure. We conjecture that this statement is true.

3 Sums of Cantor sets with unbounded ratios of dissection

In the case of unbounded ratios of dissection, both positive Lebesgue measure and Lebesgue measure zero (even with Hausdorff dimension one) are possible for the group generated by Cantor sets.

Let us begin with some remarks about the Hausdorff dimension of central Cantor sets.

Proposition 3.1 *Let C be a central Cantor set with ratios of dissection ξ_k. Then*

$$\dim_H(C) = \underline{\dim}_B(C) = \underline{\lim} \frac{N \ln 2}{|\ln \xi_1 \cdots \xi_N|}.$$

(Here $\underline{\dim}_B$ denotes the lower box dimension of C, see for example Falconer [6] p.41.)

Proof: First note that C can be covered by 2^N intervals of length $\xi_1 \cdots \xi_N$.
Thus

$$\dim_H(C) \le \underline{\dim}_B(C) \le \underline{\lim} \frac{N \ln 2}{|\ln \xi_1 \cdots \xi_N|} \equiv t.$$

Fix $\varepsilon > 0$ and choose N such that for all $n \ge N$

$$\frac{\ln 2}{|\ln(\xi_1 \cdots \xi_n)^{\frac{1}{n}}|} > t - \varepsilon,$$

or equivalently $(\xi_1 \cdots \xi_n)^{(t-\varepsilon)} > 2^{-n}$. Let U be any set with diameter $\delta = \delta(U)$ satisfying

$$\xi_1 \cdots \xi_{n+1} \le \delta(U) < \xi_1 \cdots \xi_n \ ,$$

for $n \ge N$, and let μ denote the associated Cantor measure. The set U intersects at most one interval in the Cantor set construction at level n and thus

$$\mu(U) \le 2^{-n} \le 2\,(\xi_1 \cdots \xi_{n+1})^{t-\varepsilon}.$$

By Falconer [6], p.55

$$t \le \dim_H(C). \qquad \blacksquare$$

Using this proposition we are now able to see that the group generated by a central Cantor set has positive Hausdorff dimension (a necessary condition to have positive Lebesgue measure) if and only if the Cantor set itself has positive Hausdorff dimension.

Proposition 3.2 *Let A be any measurable set. If $\underline{\dim}_B(A) = 0$ then $\dim_H(Gp(A)) = 0$.*

Proof: Observe that if A is covered by $N(\delta)$ cubes of side δ, then $\underbrace{A \times \cdots \times A}_{n}$ is covered by the $N(\delta)^n$ cubes formed by their products. Thus

$$\dim_H(A \times \cdots \times A) \le \underline{\dim}_B(A \times \cdots \times A) \le n\ \underline{\dim}_B(A) = 0.$$

The map $f : \mathcal{R}^n \to \mathcal{R}$ given by $f(x_1, \ldots, x_n) = \sum_1^n \varepsilon_i x_i$, where $\varepsilon_i = \pm 1$, is Lipschitz, thus

$$\dim_H(\varepsilon_1 A + \cdots + \varepsilon_n A) = 0.$$

By countable stability, $\dim_H(\mathrm{Gp}(A)) = 0$. ■

Corollary 3.3 *If C is a central Cantor set, then the following are equivalent*

i) $\dim_H(C) = 0$,

ii) $\dim_H(Gp(C)) = 0$,

iii) $\underline{lim}(\xi_1 \ldots \xi_n)^{1/n} = 0$.

Proof: For $i) \Leftrightarrow ii)$ observe that $\dim_H(C) = \underline{\dim}_B(C)$ and then use Proposition 3.2.
For $i) \Leftrightarrow iii)$ use Proposition 3.1. ■

Corollary 3.4 *If C is a central Cantor set satisfying any of the conditions of Corollary 3.3 then $m(Gp(C)) = 0$, where m is the Lebesgue measure.*

Cantor sets can also be constructed by choosing the ratios of dissection randomly. Salem [11] has shown that if μ is the uniform Cantor measure supported on a random central Cantor set C with ratios of dissection ξ_k satisfying

$$\underline{\lim}(\xi_1 \ldots \xi_n)^{\frac{1}{n}} > 0 \ ,$$

then $\widehat{\mu} \in \boldsymbol{\ell}^{\mathrm{p}}$ for some $p < \infty$ a.s. It follows that the k'th convolution power of μ, μ^k, belongs to $\boldsymbol{\mathcal{L}}^{\mathbf{2}}$ for sufficiently large k, and as $\mu^k \neq 0$ and μ^k is supported on $(k)C$, the Lebesgue measure of $(k)C$ is positive. Consequently the group generated by a random central Cantor set is $[0,1]$ a.s. if and only if $\dim_H(C) > 0$.

These almost sure results do not extend to the non-random case. To see this, we first show that the group generated by a Cantor set has measure zero for some particular sequences $\{\xi_k\}$.

Proposition 3.5 *Suppose C is a central Cantor set with ratios of dissection ξ_k, with $\xi_k^{-1} \in \mathcal{N}$. If $\inf \xi_k = 0$ then the measure of the group generated by C, under addition modulo 1, is zero.*

Proof: Choose an increasing sequence m_n with $\xi_{m_n}^{-1} > n^2 2^n$. If $c \in C$ then $c = \sum_{k=1}^{\infty} r_k \xi_1 \ldots \xi_k$ where $r_k = 0$ or $(\xi_k^{-1} - 1)$, so that

$$\begin{aligned}(n)C &= \underbrace{C + \cdots + C}_{n} \\ &= \left\{ \sum_{k=1}^{\infty} (r_k^{(1)} + \cdots + r_k^{(n)}) \xi_1 \ldots \xi_k \ : \ r_k^{(i)} = 0,\ \xi_k^{-1} - 1 \text{ for } i = 1, \ldots, n \right\}.\end{aligned}$$

Note that

$$\sum_{j=k+1}^{\infty} n(\xi_j^{-1}-1)\xi_1\ldots\xi_j = n\xi_1\ldots\xi_k,$$

thus if $x \in (n)C$ has expansion $x = \sum_{k=1}^{\infty} c_k\xi_1\ldots\xi_k$ with $c_k \in \{0,1,\ldots,\xi_k^{-1}-1\}$, then

$$c_k \in \Big\{(r^{(1)}+\cdots+r^{(n)}+m) \bmod \xi_k^{-1} : r^{(i)} = 0 \text{ or } \xi_k^{-1}-1 \text{ for } i=1,\ldots,n \ . \\ \text{and } m \in \{0,1,\ldots,n\}\Big\}$$

In particular there are at most $(n+1)2^n$ choices from $\{0,1,\ldots,\xi_{m_n}^{-1}-1\}$ for c_{m_n}, and thus

$$m((n)C) \le n2^n\xi_{m_n} < \frac{1}{n}.$$

Because C is symmetric, $\mathrm{Gp}(C) = \bigcup_{n=1}^{\infty}(n)C$ and as $(n)C \supseteq (n-1)C$, it follows that

$$m(\mathrm{Gp}(C)) = \lim_n m((n)C) = 0. \qquad \blacksquare$$

In fact, positive Hausdorff dimension of C is not even enough to ensure that $\dim_H(\mathrm{Gp}(C)) = 1$ as we see below.

Proposition 3.6 *Let C be a central Cantor set with ratios of dissection ξ_k, with $\xi_k^{-1} \in \mathcal{N}$ and $\inf\xi_k = 0$. It is possible to choose ξ_k so that $\dim_H(C) > 0$ and $\dim_H(Gp(C)) < 1$.*

Proof: The proof of Proposition 3.5 shows that $(n)C$ can be covered by $(n+1)2^n(\xi_1\cdots\xi_{k-1})^{-1}$ intervals of width $\xi_1\cdots\xi_k$. Thus

$$\begin{aligned}\underline{\dim}_B((n)C) &\le \underline{lim}_k \frac{\ln(n+1)2^n(\xi_1\cdots\xi_{k-1})^{-1}}{|\ln(\xi_1\cdots\xi_k)|} \\ &= \underline{lim}_k \frac{\ln(n+1)2^n + |\ln(\xi_1\cdots\xi_{k-1})|}{|\ln\xi_k| + |\ln(\xi_1\cdots\xi_{k-1})|}\end{aligned}$$

Choose any subsequence k_m and suppose that the ratios ξ_k are given by $\xi_k = \frac{1}{3}$ if $k \ne k_m$ and $\xi_{k_m} = \prod_1^{k_m-1}\xi_i$. Then, for any fixed n

$$\underline{\dim}_B((n)C) \le \underline{lim}_m \frac{\ln(n+1)2^n + |\ln(\xi_1\cdots\xi_{k_m-1})|}{2|\ln(\xi_1\cdots\xi_{k_m-1})|} = \frac{1}{2}\ .$$

By countable stability

$$\dim_H(\mathrm{Gp}(C)) = \sup_n \dim_H(n)C \le \sup \underline{\dim}_B((n)C) \le \frac{1}{2}.$$

If the subsequence $\{k_m\}$ tends to infinity sufficiently fast, we can still arrange for

$$0 < \dim_H(C) = \underline{\lim}\frac{N\ln 2}{|\ln\xi_1\cdots\xi_N|}.$$

For this we merely need $\inf\,(\xi_1\cdots\xi_N)^{1/N} < \infty$. Set $k_1 = 1$ and inductively define k_m so that $\left(\prod_1^{k_m-1}\xi_i\right)^{-1} \le (3.5)^{k_m-1}$. Then, as

$$\begin{array}{rcll} \left(\prod_1^{k_m}\xi_i\right)^{-1} & \le & (3.5)^{(k_m-1)2} \le 16^{k_m} & \text{and} \\ \left(\prod_1^{k_m+\ell}\xi_i\right)^{-1/(k_m+\ell)} & \le & 3\left(\prod_1^{k_m}\xi_i\right)^{-1/k_m} & \text{for}\quad k_m+\ell < k_{m+1}, \end{array}$$

we have the desired result. ∎

We finish this section by observing that even $\dim_H(\mathrm{Gp}(C)) = 1$ is not sufficient for $m(\mathrm{Gp}(C)) > 0$.

Example: Suppose C is a central Cantor set with ratios of dissection $\xi_k \le \frac{1}{3}$. Then $\dim_H(C+C) = 1$ if $\underline{\lim}(\xi_1\ldots\xi_N)^{\frac{1}{N}} = \frac{1}{3}$. The proof of this fact is similar to the proof of Proposition 3.1, and so we omit it here. Choosing the ξ_k as in the preceding proposition, we can arrange for $m(\mathrm{Gp}(C)) = 0$.

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