HOW TO CONSTRUCT WAVELET FRAMES ON IRREGULAR GRIDS AND ARBITRARY DILATIONS IN $\mathbb{R}^d$

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Abstract. In this article, we present a method for constructing wavelet frames of $L^2(\mathbb{R}^d)$ of the type $\{|\det A_j|^{1/2}\psi(A_jx - x_{j,k}) : j \in J, k \in K\}$ on irregular lattices of the form $X = \{x_{j,k} \in \mathbb{R}^d : j \in J, k \in K\}$, and with an arbitrary countable family of invertible $d \times d$ matrices $\{A_j \in GL_d(\mathbb{R}) : j \in J\}$. Possible applications include image and video compression, speech coding, image and digital data transmission, image analysis, estimations and detection, and seismology.

1. Introduction

In this article we present a general method for constructing well-localized wavelet frames $\{|\det A_j|^{1/2}\psi(A_jx - x_{j,k}) : j \in J, k \in K\}$ of $L^2(\mathbb{R}^d)$ on arbitrary grids $X = \{x_{j,k} \in \mathbb{R}^d : j \in J, k \in K\}$, and with arbitrary dilation matrices $\{A_j\}_{j\in J}$. The construction presented here is a special case of a more general method for constructing time-frequency frame atoms in several variables discussed in [ACM03]. Although there has been considerable interest in trying to obtain wavelet frame decompositions of the space $L^2(\mathbb{R}^d)$, on irregular grids and with unstructured dilation matrices (see [Bal97], [BCHL03], [Chr96], [Chr97], [CH97], [CDH99], [FZ95], [Fei87], [FG89], [FW01], [Grö91], [Grö93],[HK03], [OS92], [RS95], [SZ00], [SZ01], [SZ02], [SZ03], [SZ03]), most of the methods that have been developed are small perturbations of wavelet frames on a regular grid and with a fixed dilation matrix. In contrast, our approach presented in [ACM03] is not a perturbation method and is very general, allowing quite general constructions. The setting includes as particular cases, wavelet frames on irregular lattices and with a set of dilations or transformations that do not have a group structure. For this paper, we will be mainly concerned with an even more particular case consisting of wavelet frames on irregular lattices and with an arbitrary but fixed expansive matrix A (A is said to be expansive if $|\lambda| > 1$ for every eigenvalue λ of A). The case of regular lattices can also be obtained by our system, producing a substantial part of the systems recently characterized by the fundamental work of Guido Weiss and his group [HDW02, HDW03, Lab02] on the decomposition of $L^2(\mathbb{R}^d)$. The wavelet frames obtained by Chui, He, Stöeckler and Sun [CHS], [CHSS03], [CS00] are also

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included in our setting. Wavelet sets and wavelet frame sets studied in [BMM99], [BL99], [BL01], [BS03], [DLS97], [DLS98], [HL00], [Ola03], [OS03] can also be produced by our methods. Furthermore we can obtain wavelet sets with translations on irregular grids.

The method we present relies on combining ideas from four related, but different subjects: 1) Sampling theory; 2) Frame theory; 3) Wavelet theory; and 4) Geometry of $\mathbb{R}^d$. The approach can be considered in the spirit of the classic construction in 1 dimension of smooth regular tight frames done by Daubechies, Grossmann and Meyer in [DGM86]. (See also [HW89] for an expository treatment.) We will say that a set $X = \{x_k \in \mathbb{R}^d : k \in K\}$ is *separated* if

$$\inf_{k,s \in K, k \neq s} |x_k - x_s| > 0.$$

Throughout the paper J and K will denote countable index sets. One of the main ingredients in sampling theory is the notion of *lower Beurling density* $D^-(X)$ [Beu66] of a separated set $X = \{x_k \in \mathbb{R}^d : k \in K\}$, which is defined as:

$$D^-(X) = \lim_{r \to \infty} \frac{\nu^-(r)}{(2r)^d}$$

where $\nu^-(r) := \min_{y \in \mathbb{R}^d} \# \left(X \cap (y + [-r, r]^d) \right)$. $\#(Z)$ denotes the cardinal of the set Z.

The *upper Beurling Density* $D^+(X)$ is defined in a similar way:

$$D^+(X) = \lim_{r \to \infty} \frac{\nu^+(r)}{(2r)^d}$$

where $\nu^+(r) := \max_{y \in \mathbb{R}} \# \left(X \cap (y + [-r, r]^d) \right)$. If $D^-(X) = D^+(X) = D(X)$, then X is said to have *uniform Beurling density* $D(X)$.

Remark. Since X is separated, the limits in the definitions of $D^+(X)$ and $D^-(X)$ exist (see [BW99]).

Beurling [Beu66] introduced also the following notion of density : The *gap* ρ of the set $X = \{x_k : k \in K\}$ is defined as

$$\rho = \rho(X) = \inf \left\{ r > 0 : \bigcup_{k \in K} B_r(x_k) = \mathbb{R}^d \right\}.$$

Equivalently, the gap ρ can be defined as

$$\rho = \rho(X) = \sup_{x \in \mathbb{R}^d} \inf_{x_k \in X} |x - x_k|.$$

A family $\{Q_j : j \in J\}$ is a *covering* of $\mathbb{R}^d$ if $\mathbb{R}^d \setminus \bigcup_j Q_j$ has measure zero. A covering $\{Q_j : j \in J\}$ has *finite index* if every $x \in \mathbb{R}^d$ is at most in i sets of the covering for some fixed positive integer i. The minimum i with this property is called the *covering index*. We will denote by $e_x(w)$ the exponential of frequency x at w, that is $e_x(w) = e^{-2\pi i x w}$.

Let us now state a general theorem on wavelet frames:

Theorem 1.1 (Wavelets)**.** *Let A be an expansive matrix and $V \subset \mathbb{R}^d$ be any measurable bounded set containing 0 in its interior and such that its boundary ∂V has measure zero. Set $Q = \overline{A^T V \setminus V}$, and choose any function $h \in C^r(\mathbb{R}^d), r > 0, h \neq 0$ on Q such that $\operatorname{Supp} h \in Q_\varepsilon$ where $Q_\varepsilon := \{x \in \mathbb{R}^d : dist(x, Q) \leq \varepsilon\}$. If the set $X = \{x_k \in \mathbb{R}^d : k \in K\}$ is separated and such that $\rho(X) < \frac{1}{4\delta}$ where $\delta = Diameter(Q)$, then the following collection is a wavelet frame for $L^2(\mathbb{R}^d)$*

$$\{|\det A|^{j/2}\psi(A^j x - x_k) : k \in K, j \in \mathbb{Z})\}, \tag{1.1}$$

where ψ is the inverse Fourier transform of h.

Remarks.

(1) The result of the theorem remains valid even if the matrix A is not expansive. For example let Q be any closed subset of $\mathbb{R}^d$, and A any invertible matrix. If $\mathbb{R}^d = \cup_j A^j Q_\varepsilon$ with finite covering index, then (1.1) is a frame for $L^2(\mathbb{R}^d)$.
(2) Instead of taking the powers A^j of a single matrix A we can choose a set of invertible matrices $\{A_j \in GL_d(\mathbb{R}) : j \in J\}$ without a particular group structure. In particular the index j can be a multi-index. For example, the set $J = \mathbb{Z} \times \{1, \cdots, N\}$, and the matrices $A_{(i,j)} = D^i R^j$ where R is a rotation and D a dilation matrix, will be used to construct directional wavelets.
(3) The wavelet can be constructed to have polynomial decay of any order by choosing r sufficiently large.
(4) The sets of translations $X_j = \{A^{-j}x_k : k \in K\}$ for each resolution level are not nested. However, the theorem can be easily modified to produce nested sets of translations $X_{j+1} \subset X_j$ for all $j \in \mathbb{Z}$ (c.f. [ACM03]).
(5) For the one dimensional case, ρ can be replaced by the Beurling density $D^-(X)$ which is a weaker condition and allows for arbitrary gaps between sampling points.
(6) If we choose h to be the characteristic function of the set Q_ε, then we obtain a wavelet frame set, and our construction (1.1) gives wavelet sets with translations on irregular grids.

Although the set $\{|\det A|^{j/2}\psi(A^j x - x_k) : j \in J, k \in K\}$ in Theorem 1.1 is a wavelet frame for $L^2(\mathbb{R}^d)$, it is not in general true that for a fixed j the set $\{\psi_{j,x_k}(x) = |\det A|^{j/2}\psi(A^j x - x_k) : k \in K\}$ is a frame. Thus, it appears at first, that the reconstruction of a function $f \in L^2(\mathbb{R}^d)$ from the wavelet coefficients $\{< f, \psi_{j,x_k} >: j \in J, k \in K\}$ cannot be obtained in a stable way by first reconstructing at each level j and then obtaining f by summing over all levels j. But in fact it is always possible to reconstruct each f_j in a stable way and then obtain f by summing up over all levels j, as is described in [ACM03].

2. Examples of wavelet frames on irregular lattices and with arbitrary set of dilation matrices and other transformations

2.1. **Examples of Wavelet frames in $\mathbb{R}^d$.**

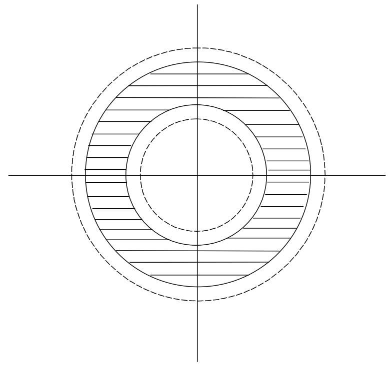

FIGURE 1. Radial wavelet frames that are well localized in space: Region between continuous lines correspond to region Q and region between dashed lines corresponds to Q_ε.

(1) *Isotropic, well-localized wavelets*: Let V be the ball of radius 1/2 centered at the origin. Let $A = 2I$, then $Q = \overline{AV \setminus V} = \{x \in \mathbb{R}^2 : 1/2 \le \|x\| \le 1\}$. Let $\varepsilon = 1/4$, $h(\xi_1, \xi_2) = n\beta_{n-1}\left((\xi_1^2 + \xi_2^2 - 1/4)n\right)$, where β_n is the B-spline of degree n, i.e., the $\beta_n = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ is the n-fold convolution of the characteristic function on $[0,1]$. Let $X = \{x_k \in \mathbb{R}^2 : k \in K\}$ be such that its maximal gap satisfies $\rho(X) < \frac{1}{5}$. Then by Theorem 1.1 the set (1.1) is a wavelet frame for $L^2(\mathbb{R}^2)$ (see Figure 1). By construction, the wavelet ψ is isotropic and has polynomial decay of degree n in space, i.e, $|\psi(x)| \le C(1+|x|)^{-n}$. A theorem in [ACM03] more general than Theorem 1.1 shows that the same wavelet ψ generates a frame for $L^2(\mathbb{R}^2)$ if we replace the dilation matrix $A = 2I$ by any dilation matrix $A = bI$ where $1 < b \le 5/4$. Non-separable radial MRA frame wavelets can be found in [PGKKH1], [PGKKH2]. However, these radial MRA frame wavelets have slow spatial decay, i.e., $|\psi(x)| \le C(1+|x|)^{-1}$.

(2) *Directional wavelet frames*: To construct directional wavelet frames we use a modified version of Theorem 1.1 as suggested by remark (2). Let Q_1 be a region defined by $Q_1 = \{(x, y) \in \mathbb{R}^2 : x = r\cos(\theta), y = r\sin(\theta), 1/2 \le r \le 1, |\theta| \le \frac{\pi}{2N}\}$ as in Figure 2, and define $Q = (-Q_1) \cup Q_1$. Let $B = 2I$, and R be the matrix of a rotation by an angle π/N, where N is any positive integer, $\varepsilon = 0.1$, and $J = \{(j_1, j_2) : j_1 \in \mathbb{Z}, j_2 = 0, \cdots, N-1\}$. Then $\mathbb{R}^2 = \cup_{(j_1,j_2)\in J} B^{j_1} R^{j_2} Q$. Let $X_j = \{x_{j,k} \in \mathbb{R}^2 : k \in K\}$ be a set of points such that $\rho(X_j) < \frac{5}{22}$ for each $j = (j_1, j_2) \in J$. Then the function ψ, such that $\hat{\psi} = \chi_{Q_\varepsilon}$, generates a wavelet frame for $L^2(\mathbb{R}^2)$ of the form $\{\psi_{j,k} = 2^{j_1}\psi_j(2^{j_1}R^{j_2} \cdot - x_{j,k}) : j \in J, k \in \mathbb{Z}\}$. The index j_1 codes for the resolution of the wavelet, while the index j_2 codes for N possible directions. Thus the wavelet frame coefficients encode time scale as well as directional information. The wavelet constructed in this example does not have good space localization, because it is the inverse Fourier transform of a characteristic function. However, we can replace the characteristic function χ_{Q_ε} by a smoother function $h \in C^r$ where $r > 0$ is any positive real number we choose, $h > 0$ on Q and $\operatorname{Supp} h \in Q_\varepsilon$. With this modification we

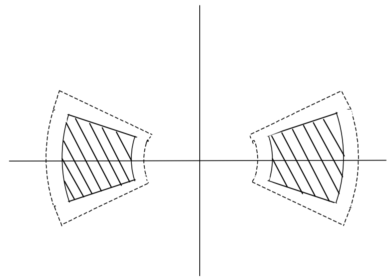

FIGURE 2. Well-localized directional wavelet: Q is the interior of the two regions delineated by continuous lines, and Q_ε is the interior of the two regions delimited by the dashed lines.

obtain a directional wavelet ψ with spatial localization satisfying $|\psi(x)| \leq C(1+|x|)^{-r}$. Moreover the wavelet can be constructed to be real and symmetric with respect to the origin, by choosing a function h who is real and symmetric with respect to the origin. Very nice constructions of smooth directional wavelet frames on regular grids were obtained before in [AHNV01, ADH$^+$03].

(3) *Spiral wavelet: [ACM03]* In this example we will define a dilation A such that its powers A^jQ applied to an annulus sector Q covers $\mathbb{R}^2$ by spiral annulus sectors.

Let $a, b > 1$, and Γ the spiral curve defined by

$$\Gamma(t) = (a^t \cos(bt), a^t \sin(bt)) \quad t \in \mathbb{R}.$$

For $\alpha \in \mathbb{R}$ define R_α to be the rotation of angle α : $R_\alpha = \left[\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\right]$. The curve Γ satisfies:

$$\Gamma(t+\alpha) = a^\alpha R_{b\alpha}\Gamma(t).$$

Note that for positive α the matrix $A = a^\alpha R_{b\alpha}$ is expansive.

Now we are ready to define the covering elements. Set $b = 2\pi$ and $\alpha = \frac{1}{m}$, for some integer $m \geq 2$ so that $A^m = aI_d$. Define the spiral annulus sector $Q = \{x \in \mathbb{R}^2 : x = \lambda\Gamma(\beta), 1 \leq \lambda \leq a, 0 \leq \beta \leq \alpha\}$ (see Figure 3). So Q is compact and $\{A^jQ : j \in \mathbb{Z}\}$ is a disjoint covering of $\mathbb{R}^d \setminus \{0\}$.

Choose $\varepsilon > 0$ sufficiently small and h a smooth function that does not vanish in Q and with support in Q_ε. Define $\hat{\psi} = h$. Select a set $X = \{x_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}^2$ such that $\rho(X) < \frac{1}{2\mathrm{diam}(Q)}$.

The set $\{a^{j/m}\psi(a^{j/m}R_{-2\pi j/m}x - x_k) : k \in \mathbb{Z}, j \in \mathbb{Z}\}$ is a wavelet frame of $L^2(\mathbb{R}^2)$ generated by a single wavelet ψ that is band-limited, with good decay and directional in frequency.

Remarks.

(1) Obviously, all the constructions above can be generalized to $\mathbb{R}^d$ for any dimension $d > 2$.
(2) Some of the wavelet frames may be associated with MRAs. For example, the so called Shannon wavelet frame constructed above, is associated with

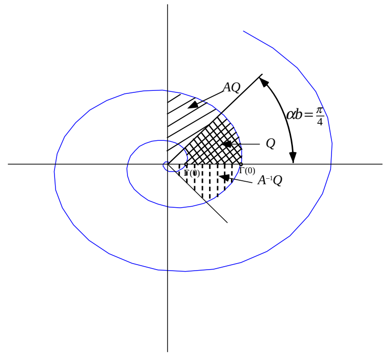

FIGURE 3. Spiral wavelet frames.

the Shannon MRA $V_j = \{f \in L^2(\mathbb{R}) : \operatorname{Supp} \hat{f} \subset [-2^{-j-1}, 2^{-j-1}]\}$, $j \in \mathbb{Z}$. In general however, the precise relation needs further investigation.

3. General Irregular Systems

From the last section we can see that there are three common ingredients in all the examples that we considered.

(1) A covering of the space $\mathbb{R}^d$, whose covering elements are obtained by the action of a countable family of matrices A_j on a basic covering element Q of $\mathbb{R}^d$.
(2) A local frame of non-harmonic exponentials controlled by the Beurling density (in dimension $d = 1$), or the Beurling gap (in dimensions $d \geq 2$) in each covering element.
(3) A smoothing gluing element: the function h with support in Q_ε.

The combination of these three elements produces a wavelet frame of $L^2(\mathbb{R}^d)$ with good decay.

A simple observation with far reaching consequences is the fact that these ideas can be further developed to obtain quite more general constructions of frames of $L^2(\mathbb{R}^d)$. These general constructions include as particular examples irregular wavelet frames and irregular Gabor frames. One nice property of these more general systems of functions is that the frame-elements can still be chosen having good decay in time, or space.

In what follows we will briefly explain the general idea of how this construction works. We will state some of the more relevant results for this exposition and refer the reader to [ACM03] for proofs and the more abstract theory.

In order to understand the heart of the problem, we first pose the following question: Assume that $G = \{g_j\}_{j \in J}$ is a countable family of functions in $L^2(\mathbb{R}^d)$ and let $X = \{x_{j,k} \in \mathbb{R}^d : j \in J, k \in K\}$ be a double indexed countable set in $\mathbb{R}^d$. Consider the system $\mathcal{G}(G, X) = \{g_j(x - x_{j,k}) : j \in J, k \in K\}$ associated to the pair (G, X). The fundamental question is for which pairs (G, X) the system $\mathcal{G}(G, X)$ is a frame of $L^2(\mathbb{R}^d)$. Guido Weiss and his group obtained a characterization of these systems for the case of tight frames on regular grids. The Wavelet set systems mentioned in the introduction, are also particular cases of the general systems $\mathcal{G}(G,X)$.

The general case seems to be much more difficult and probably requires other proof techniques. Our results will produce a substantial part of these systems and provide methods to specifically construct them.

Our method can be described as follows: We start by choosing a completely arbitrary covering of $\mathbb{R}^d$ by bounded sets. This covering is the first ingredient in the description above. Now we choose for the j^{th} covering element a non-harmonic Fourier frame by choosing first a separated family $X_j = \{x_{j,k} \in \mathbb{R}^d : k \in K\}$. The gap of this family should be small enough, in order that $\{e_{x_{j,k}}\}_{k\in K}$ form a frame for the functions supported in a ball centered at the origin, whose diameter is at least equal to the diameter of the considered covering element (or twice the diameter if we are interested in real valued frames or wavelets). This guarantees that this set of exponentials form a local frame for the functions supported in the covering element.

Finally our third ingredient are functions $h_j, j \in J$ that we choose to be C^r, with support included in the ε expanded j^{th} element of the covering, and bounded away from zero on the element itself. We will also need this family to be uniformly bounded. So, if the covering has finite index, we can conclude that the system $\{h_j e_{x_{jk}}\}_{j\in J, k\in K}$ is a frame of $L^2(\mathbb{R}^d)$ provided that the frame bounds for the exponentials are uniformly bounded below and above.

With the construction above, if $\hat{g}_j = h_j$, then the system $\mathcal{G}(G,X) = \{g_j(x - x_{j,k}) : j \in J, k \in K\}$ is a frame of $L^2(\mathbb{R}^d)$ where $X = \bigcup_j X_j$. So we have the following theorem:

Theorem 3.1 (General-Systems). *Let $\{Q_j : j \in J\}$ be a covering of $\mathbb{R}^d$ by bounded sets with finite covering index. Let $h_j \in L^2(\mathbb{R}^d)$ such that $\operatorname{Supp}(h_j) \subset Q_{\varepsilon_j}$, for some $\varepsilon_j > 0$ and $h_j \in C^r$ with $h_j \neq 0$ in Q_j and $|h_j| < c$ for all j. Let $X_j = \{x_{j,k} : k \in K\}$ be a separated set with gap $\rho(X_j) < \frac{1}{4\delta_j}$ where $\delta_j =$diameter(Q_j) and such that the frame bounds of the frame sequence $\{e_{x_{j,k}}\chi_{Q_j}\}_{k\in K}$ are uniformly bounded in J. If $X = \bigcup_j X_j$ and $G = \{g_j =^\vee h_j : j \in J\}$, then the system $\mathcal{G}(G,X)$ is a frame of $L^2(\mathbb{R}^d)$.*

4. Concluding remarks

The results of Theorem 3.1 can be extended in order to obtain atomic decompositions of the space $L^2(\mathbb{R}^d)$ with smooth well-localized frame atoms. The fact that we concentrated on local Fourier frames is to obtain frames of $L^2(\mathbb{R}^d)$ that are localized on irregular grids, e.g., wavelet frames on irregular grids. The general theory allows the construction of general atomic decompositions by pasting together arbitrary local decompositions associated to a covering of the space as is detailed in [ACM03].

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