REFINABLE SHIFT INVARIANT SPACES IN $\mathbb{R}^d$

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Abstract. Let $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ be a compactly supported function which satisfies a refinement equation of the form

$$\varphi(x) = \sum_{k \in \Lambda} c_k \varphi(Ax - k), \quad c_k \in \mathbb{C},$$

where $\Gamma \subset \mathbb{R}^d$ is a lattice, Λ is a finite subset of Γ, and A is a dilation matrix. We prove, under the hypothesis of linear independence of the Γ-translates of φ, that there exists a correspondence between the vectors of the Jordan basis of a finite submatrix of $L = [c_{Ai-j}]_{i,j \in \Gamma}$ and a finite dimensional subspace $\mathcal{H}$ in the shift invariant space generated by φ. We provide a basis of $\mathcal{H}$ and show that its elements satisfy a property of homogeneity associated to the eigenvalues of L. If the function φ has accuracy κ, this basis can be chosen to contain a basis for all the multivariate polynomials of degree less than κ. These latter functions are associated to eigenvalues that are powers of the eigenvalues of A^{-1}. Further we show that the dimension of $\mathcal{H}$ coincides with the local dimension of φ, and hence, every function in the shift invariant space generated by φ can be written locally as a linear combination of translates of the homogeneous functions.

1. Introduction

Let Γ be a lattice, i.e. Γ is the image of $\mathbb{Z}^d$ under any nonsingular linear transformation and A be a *dilation matrix* associated to Γ, (i.e. $A(\Gamma) \subset \Gamma$ and all eigenvalues of A satisfy $|\lambda| > 1$). We will say that a compactly supported function $\varphi : \mathbb{R}^d \longrightarrow \mathbb{C}$ is *refinable* with respect to A and Γ, if it satisfies the *dilation equation*

$$\varphi(x) = \sum_{k \in \Lambda} c_k \varphi(Ax - k), \quad x \in \mathbb{R}^d, \tag{1.1}$$

for some finite subset $\Lambda \subset \Gamma$, and coefficients $c_k \in \mathbb{C}$.

The *Shift Invariant Space* (SIS) generated by φ is the space

$$\mathcal{S}(\varphi) = \left\{ f : \mathbb{R}^d \longrightarrow \mathbb{C} : f(x) = \sum_{k \in \Gamma} y_k \varphi(x + k), \quad y_k \in \mathbb{C}, k \in \Gamma \right\}.$$

Note that since φ is compactly supported, the right hand side of the previous equation is well defined. Even though the SIS is an infinite dimensional space, the fact that its generator is compactly supported yields "locally" a finite number of

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generators. More precisely, let $E \subset \mathbb{R}^d$ be a fundamental domain for the lattice Γ, and

$$E(\varphi) = \{f/_E : E \longrightarrow \mathbb{C} : f/_E(x) = f(x) \; \forall x \in E, f \in \mathcal{S}(\varphi)\}. \tag{1.2}$$

The space $E(\varphi)$ is finite dimensional. A canonical set of generators is the set

$$\{\varphi(x-k)/_E : E \longrightarrow \mathbb{C} : \text{ with } k \text{ such that } |(\mathrm{Supp}(\varphi)+k) \cap E| > 0\}.$$

The algebraic dimension of the vector space $E(\varphi)$ will be called the *local dimension* of $S(\varphi)$ and a basis of $E(\varphi)$ a local basis for $S(\varphi)$.

If φ satisfies a refinement equation (1.1), one may not explicitly know the function, but from properties of the coefficients of (1.1) one often can deduce (properties) of these finite generators. One question is if it is possible to obtain a different set of generators with specific properties. In particular it is important to know if some (or any) of these generators can be chosen to be polynomials. The *accuracy* of φ is the maximum integer κ, such that all polynomials of degree less or equal to $\kappa - 1$ are contained in $S(\varphi)$. Hence, if φ has accuracy κ, one can choose a local basis containing $\alpha_\kappa = \sum_{s=0}^{\kappa-1} d_s$ linearly independent polynomials, where d_s is the number of linearly independent monomials of degree s.

In the one dimensional case, with dilation 2, the generator φ satisfies $\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x-k)$. In this case $\alpha_\kappa = \kappa$ and the accuracy of φ is related to spectral properties of a finite matrix T.

Precisely, under the hypothesis of linear independence of the integer translates of the generator, φ has accuracy κ, if and only if $\{1, 2^{-1}, ..., 2^{-(\kappa-1))}\}$ are eigenvalues of the $(N+1) \times (N+1)$ matrix T defined by $T = \{c_{2i-j}\}_{i,j=0,...,N}$, (the *scale matrix*) **and** there exist polynomials $p_0, ..., p_{\kappa-1}$ of degree$(p_i) = i$ such that each of the the vectors $v_i = \{p_i(k)\}_{k=0,...,N}$ is a left eigenvector of T corresponding to the eigenvalue 2^{-i} ([Dau88], [CHM98]). Here the fact that the powers of $1/2$ are eigenvalues of T is related to the dilation factor 2 in the dilation equation.

So, if φ has accuracy κ, we know κ linearly independent functions in $E(\varphi)$. This set of functions can then be extended to a local basis of $S(\varphi)$. A natural question is, in which way can this completion be done? If some eigenvalues are associated to nice local basis functions, (i.e. local polynomials) would it be possible to extend this set of nice functions to a basis of $E(\varphi)$ using the remaining eigenvalues? What properties will these new functions have?

Blu and Unser [BU02] in the study of radial basis functions, and later Zhou [Zho02] gave the first clue for the answer to these questions. They showed that associated to an arbitrary eigenvalue λ of the matrix T there is a function in $S(\varphi)$ that satisfies that $h(2x) = \frac{1}{\lambda} h(x)$. In particular, the monomial x^k, satisfies this property, for $\lambda = 2^{-k}$. So the set of functions associated in this way to all the eigenvalues of T is a linearly independent set. Then in the case that the matrix T is diagonalizable it is possible to complete the local basis for $S(\varphi)$.

In [CHM03b] the problem was completely solved. They showed that to each vector from a basis that gives the Jordan form of the matrix T it is possible to associate a function h in $S(\varphi)$ that they called (λ, r)-homogeneous and satisfies that

$$(D_2 - \lambda I)^r h = 0.$$

Here for $D_2 f$ is the dilation operator defined by $D_2 f(x) = f(\frac{x}{2})$, and λ is an eigenvalue of T. In particular, the functions associated to eigenvectors are $(\lambda, 1)$-homogeneous, and correspond to the one's obtained before. These (λ, r)-homogeneous functions in $S(\varphi)$ are linearly independent and provide a local basis. This local basis contain all the monomials x^k within the accuracy. The generator φ can be completely obtained from this local basis.

The goal of this paper is to carry on this study to $\mathbb{R}^d$, with a general dilation matrix and an arbitrary full rank lattice.

When moving to higher dimensions, the situation turns much more complicated. In analogy to the one dimensional case, we will consider functions that satisfy the relation $h(Ax) = \frac{1}{\lambda}h(x)$ (see section 4). Here $h : \mathbb{R}^d \to \mathbb{C}$, A is a $d \times d$ invertible matrix and $\lambda \in \mathbb{C}$. More in general, we will consider functions satisfying that $(D_A - \lambda I)^r h = 0$. To avoid any ambiguity, we will say that functions satisfying this equation are in the class $\mathcal{H}(A, \lambda, r)$, in place to use the word homogeneous, since we will also be dealing with polynomials that are homogeneous in the standard way. (i.e. a polynomial p of degree s is homogeneous, if $p(ax) = a^s p(x), x \in \mathbb{R}^d \, \forall a \in \mathbb{R}$).) Note however, that with this definition, for $d = 2$ the monomial $h(x_1, x_2) = x_1x_2$ will be in $\mathcal{H}(A, \lambda, 1)$ only if A is diagonal and $\lambda = \frac{1}{A_{11}A_{22}}$.

When trying to extend the notion of accuracy from one to higher dimensions, it could be seen that the fact that φ had accuracy κ was not immediately related to spectral properties of a finite submatrix of $L = [c_{Ai-j}]_{i,j\in\Gamma}$. The relation was much more subtle and involved (see [CHM98], [CHM99], [CHM00]).

In spite of this, however, in this paper we are able to obtain an analogous result to the 1-dimensional case. Again we are able to show that a local basis of φ can be obtained using solely functions from $\mathcal{H}(A, \lambda, r)$, where λ is an eigenvalue for a finite submatrix T of L. The result is very pleasing, in the sense that the 1-dimensional results are completely recovered and one obtains a *different* way of writing the functions of $\mathcal{S}(\varphi)$, namely, each function in $\mathcal{S}(\varphi)$ can be written locally as a linear combination of the translates of functions in $\mathcal{H}(A, \lambda, r)$ (c.f. equation (4.10)).

In particular, if φ has accuracy κ, then we will find α_κ linearly independent polynomials that are in the class $\mathcal{H}(A, \lambda, r)$ for some eigenvalue λ of T, and some $r \in \mathbb{N}$.

The difficulty here is to find the appropriate matrix T. Since we are in $\mathbb{R}^d$, the indexes vary along a d-dimensional lattice, so to write L as a matrix, one has to *order* the points. Which order is not important, as long as it is always the same. In the one dimensional case, it was straightforward to look at a submatrix of L that was intimately related to the support of φ. In the higher dimensional setting, it may be a difficult problem to determine the support exactly. This is one of the problems one has to overcome to solve the question raised here.

The paper is organized as follows: In section 2 we briefly review some geometric properties of the support of a refinable function φ related to the dilation A and define the finite matrix T whose spectral properties will be fundamental for our analysis of the class $\mathcal{H}(A, \lambda, r)$. In section 3, we relate the spectral properties of the infinite matrix L to those of T, and in section 4 we define the class $\mathcal{H}(A, \lambda, r)$ and show how one associates one of these functions to each vector of the Jordan basis of T. We further prove, that these functions satisfy a vector refinement equation and that they are a basis for the space $E(\varphi)$ (1.2). Finally, in section 5, if φ has accuracy κ, using results from [CHM03a], we show that the space of all functions in the class $\mathcal{H}(A, \lambda, r)$ in $S(\varphi)$, contains α_k linearly independent polynomials.

2. Attractors, Tiles and Admissible Sets

Let Γ be a lattice and A a dilation matrix associated to Γ. Then A has integer determinant and the group $\Gamma/A(\Gamma)$ has order $|\det(A)|$ (see for example [Woj97]). Set

$$m = |\det(A)|,$$

and let $D = \{d_1, \ldots, d_m\}$ be a set of representatives of the group $\Gamma/A(\Gamma)$ of order m. We call D a *full set of digits*, or *digit set*.

The cosets

$$\Gamma_i = A(\Gamma) - d_i = \{Ak - d_i : k \in \Gamma\}, \quad d_i \in D,$$

form a partition of Γ. Assume that $\gamma_1, \ldots, \gamma_d$ is a set of generators for Γ, that is, $\gamma_1, \ldots, \gamma_d$ are linearly independent vectors in $\mathbb{R}^d$ and

$$\Gamma = \{l_1\gamma_1 + \ldots + l_d\gamma_d : l_i \in \mathbb{Z}\}.$$

We will call the set

$$P = \{x_1\gamma_1 + \ldots + x_d\gamma_d : 0 \leq x_i < 1\}$$

a *fundamental domain* for the group $\mathbb{R}^d/\Gamma$.

2.1. Attractors.

For each $k \in \Gamma$, we define $w_k : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ by

$$w_k(x) = A^{-1}(x + k).$$

Since A is a dilation matrix, A^{-1} is contractive for some appropriate norm in $\mathbb{R}^d$, so each w_k is a contractive mapping on $\mathbb{R}^d$ for that norm.

The space

$$\mathcal{H}(\mathbb{R}^d) = \{K \subset \mathbb{R}^d : K \neq \emptyset \text{ and } K \text{ is compact}\},$$

is a complete metric space under the Hausdorff metric d defined by

$$d(B, C) = \inf\{\varepsilon > 0 : B \subset C_\varepsilon \text{ and } C \subset B_\varepsilon\},$$

where

$$B_\varepsilon = \left\{x \in \mathbb{R}^d : \ \operatorname{dist}(x, B) < \varepsilon\right\}.$$

For each finite subset $H \subset \Gamma$, we define

$$w_H(B) = \bigcup_{k \in H} w_k(B) = A^{-1}(B + H)$$

It can be shown that w_H is a contractive map in $\mathcal{H}(\mathbb{R}^d)$ (using that each w_k is a contractive mapping on $\mathbb{R}^d$). Consequently, by the Contraction Mapping Theorem, there exists a unique nonempty compact set $K_H \subset \mathbb{R}^d$ such that

$$w_H(K_H) = K_H, \quad i.e. \quad K_H = A^{-1}(K_H + H).$$

In fact, we can write

$$K_H = \sum_{j=1}^{\infty} A^{-j}(H) = \left\{\sum_{j=1}^{\infty} A^{-j} h_j : h_j \in H\right\}. \tag{2.1}$$

The set K_H is called the *attractor* of the iterated function system generated by $\{w_k\}_{k \in H}$. [Hut81].

2.2. **Tiles.** Given the digit set $D = \{d_1, \ldots, d_m\}$, we consider the attractor

$$Q = K_D = \sum_{j=1}^{\infty} A^{-j}(D) = \left\{ \sum_{j=1}^{\infty} A^{-j}\varepsilon_j : \varepsilon_j \in D \right\} \tag{2.2}$$

of the iterated system generated by $\{w_d\}_{d\in D}$. We have that, for $\gamma \in \Gamma$

$$K_{D+\gamma} = \sum_{j=1}^{\infty} A^{-j}(D+\gamma) = \sum_{j=1}^{\infty} A^{-j}D + (A-I)^{-1}\gamma = K_D + (A-I)^{-1}\gamma. \tag{2.3}$$

Therefore we can assume without loss of generality that $0 \in D$, and hence, by equation (2.2), we have $0 \in Q$.

The set Q satisfies the following properties (see [Ban91] and [GM92]):

a) $\bigcup_{k\in\Gamma} Q + k = \mathbb{R}^d$.
b) $Q^0 \neq \emptyset, Q = \overline{Q^0}$, *and* $|\partial Q| = 0$.
c) $|Q \cap (Q+k)| = 0$ *for every* $k \in \Gamma - \{0\}$ *if and only if* $|Q| = |P|$, *where* P *is a fundamental domain for* $\mathbb{R}^d/\Gamma$. *In this case* $Q \cap (Q+k) \subset \partial Q$ *for all* $k \in \Gamma - \{0\}$.

A longstanding problem was the question of whether for each dilation matrix A there exists a full set of digits D such that the corresponding attractor Q is a tile. A counterexample was found recently. (See [Pot97] and also [LW99].) We will assume in this paper that Q is a *tile*, or a *fundamental domain for* Γ, that is, the Γ- translates $\{Q+k\}_{k\in\Gamma}$ cover $\mathbb{R}^d$ with overlaps of measure zero (hence $|Q| = |P|$). Then the local dimension of φ, will be the dimension of $Q(\varphi)$. See (1.2).

2.3. **Admissible sets.** Let H be a fixed finite subset of Γ

Definition 2.1. We say that a set $\Omega \subset \Gamma$ is H-*admissible* if

$$A^{-1}(\Omega + H) \cap \Gamma \subset \Omega, \tag{2.4}$$

which is equivalent to say that $w_H(\Omega) \cap \Gamma \subset \Omega$.

Remark. If $H \subset H'$ and Ω is H'-admissible, then Ω is H-admissible.

We immediately have the following Proposition:

Proposition 2.1. *If* Ω_H *is defined as* $\Omega_H = K_H \cap \Gamma$, *then* Ω_H *is an* H*-admissible set.*

Proof. Since $\Omega_H \subset K_H$, we have

$$w_H(\Omega_H) \cap \Gamma \subset w_H(K_H) \cap \Gamma = \Omega_H,$$

which shows the desired property. □

Let $\ell(\Gamma)$ be the space of all sequences defined in Γ, and let L be the infinite matrix associated to the refinement equation (1.1), $L_{ij} = c_{Ai-j}$, whenever $Ai - j \in \Lambda$, and $L_{ij} = 0$ in all other cases.

In this paper we will mainly be interested in Λ-admissible sets. The reason for that is that if $\Omega \subset \Gamma$ is Λ-admissible, then the space $\ell(\Omega) = \{Y \in \ell(\Gamma) : y_k = 0, k \notin \Omega\}$ is right invariant under L.

We will need to “extend” finite vectors to infinite ones with certain prescribed properties, and such that they coincide with the finite one if restricted to a finite subset of the lattice. Therefore, the following Proposition found in [CHM04], will be very useful.

Proposition 2.2 (CHM04). *For each finite $H \subset \Gamma$, there exists a strictly increasing sequence $\{\Omega_n\}_{n\geq 0}$ of H-admissible sets whose union is Γ, such that $\Omega_0 = \Omega_H$ and*

$$w_H(\Omega_{n+1}) \cap \Gamma \subset \Omega_n \tag{2.5}$$

for all $n \geq 0$.

Proof. Let $\|\cdot\|$ be any norm in $\mathbb{R}^d$ such that $\|A^{-1}\| < 1$ and fix $\varepsilon > 0$, such that $H \subset B(\varepsilon)$, where $B(\varepsilon) = \{x \in \mathbb{R}^d : \|x\| \leq \varepsilon\}$, the closed ball with radius ε centered at the origin. Now set

$$\delta_0 = \frac{\varepsilon}{\|A^{-1}\|^{-1} - 1}. \tag{2.6}$$

Chose $\delta > \delta_0$ in such a way that $\Omega_H \subset B(\delta)$ and set $F_0 = B(\delta)$. Since $\delta > \delta_0$, we have $\|A^{-1}\|(\delta + \varepsilon) < \delta$. Hence,

$$w_H(F_0) = A^{-1}(B(\delta) + H) \subset A^{-1}(B(\delta + \varepsilon)) \subset B(\|A^{-1}\|(\delta + \varepsilon)) \subset B(\delta) = F_0.$$

We define recursively $F_{j+1} = w_H(F_j)$ for $j \geq 0$. It is easy to see, by induction, that $F_{j+1} \subset F_j$ for every j. Since F_0 is compact, the Contraction Mapping Theorem tells us that $\bigcap F_j = K_H$. It follows that $F_j \cap \Gamma = \Omega_H$ for every j large enough, and consequently $\{F_j \cap \Gamma\}$ is a finite collection of sets. Let

$$\Omega_H = \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_N = F_0 \cap \Gamma$$

be the distinct elements of this collection and fix $0 \leq n < N$. Since there exists a $j \in \mathbb{N}$ such that

$$\Omega_n = F_j \cap \Gamma \subsetneq F_{j-1} \cap \Gamma = \Omega_{n+1},$$

we have

$$w_H(\Omega_{n+1}) \cap \Gamma \subset w_H(F_{j-1}) \cap \Gamma = F_j \cap \Gamma = \Omega_n. \tag{2.7}$$

So inclusion (2.5) holds for $n = 0, 1, \ldots, N-1$.

Now we set $\delta_N = \delta$ and define recursively, $\delta_{n+1} = \frac{\delta_n}{\|A^{-1}\|} - \varepsilon$ for $n \geq N$. The sequence of numbers $\delta_N < \delta_{N+1} < \cdots$ is increasing. Define $\Omega_n = B(\delta_n) \cap \Gamma$ for $n > N$. If $\Omega_{n+1} = \Omega_n$, we skip that one and continue until $\Omega_{n+k} \neq \Omega_n$. In this way we obtain a strictly increasing sequence of sets $\{\Omega_k\}_{k\geq N}$. Combining with the sets $\Omega_0, \ldots, \Omega_N$ constructed previously, we have a strictly increasing sequence $\{\Omega_n\}_{n\geq 0}$. The inclusion

$$w_H(\Omega_{n+1}) \cap \Gamma \subset \Omega_n \tag{2.8}$$

holds for every $n \in \mathbb{N}_0$, since for $n \geq N$, again there exist a $j \in \mathbb{N}$ such that

$$\Omega_n = B(\delta_j) \cap \Gamma \subsetneq B(\delta_{j+1}) \cap \Gamma = \Omega_{n+1},$$

and then

$$w_H(\Omega_{n+1}) = A^{-1}(\Omega_{n+1} + H) \subset B(\|A^{-1}\|(\delta_{j+1} + \varepsilon)) = B(\delta_j). \tag{2.9}$$

We already showed that $\Omega_0 = \Omega_H$ is H-admissible. Since $\Omega_n \subset \Omega_{n+1}$, it follows from (2.8) that Ω_{n+1} is H-admissible for every $n \in \mathbb{N}_0$, which completes the proof. $\square$

Corollary 2.1. *If $H \subsetneq H' \subset \Gamma$, then there exists $n_0 \geq 1$ and a strictly increasing sequence $\{\Omega_n\}_{n\geq 0}$ of H-admissible sets whose union is Γ, such that*

$$\Omega_0 = \Omega_H, \quad \Omega_{n_0} = \Omega_{H'}, \quad \textit{and} \quad w_H(\Omega_{n+1}) \cap \Gamma \subset \Omega_n \textit{ for all } n \geq 0.$$

Proof. First construct the sequence $\{\Omega'_n\}$ associated to H' using the previous proposition. Note that by Remark 2.3 the sets Ω'_n are also H-admissible.

Using the notation of the previous proof, let j_0 be such that $F_{j_0} \cap \Gamma = \Omega_{H'}$. Now consider the sequence $\{G_j\}_{j \geq j_0}$, where $G_{j_0} = F_{j_0}$ and $G_{j+1} = w_H(G_j)$, for $j \geq j_0$. Since

$$G_{j_0+1} = w_H(G_{j_0}) = A^{-1}(F_{j_0} + H) \subseteq A^{-1}(F_{j_0} + H') = w_{H'}(F_{j_0}) \subseteq F_{j_0} = G_{j_0},$$

then $G_{j+1} \subseteq G_j$ and therefore $\cap_{j \geq j_0} G_j = K_H$ and hence, $\{G_j \cap \Gamma\}$ is again a finite collection of sets of say $n_0 + 1$ elements. Consider now the distinct elements

$$\Omega_H = \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_{n_0} = F_{j_0} \cap \Gamma = \Omega_{H'},$$

and let

$$\Omega_{n_0+k} = \Omega'_k.$$

This new sequence satisfies all the desired properties. □

For a more complete treatment of admissible sets see [CHM04] and also Jia [Jia98].

3. Spectral Properties of L

Let us now return to the refinement equation (1.1), $\varphi(x) = \sum_{k \in \Lambda} c_k \varphi(Ax - k)$. If we consider the infinite column vector

$$\Phi(x) = \{\varphi(x + k)\}_{k \in \Gamma}, \tag{3.1}$$

this equation becomes

$$\Phi(x) = L\Phi(Ax). \tag{3.2}$$

It can be shown (see [CHM00]) that the set K_Λ, which is the particular case taking $H = \Lambda$ in (2.1), satisfies that if φ is a compactly supported solution of the refinement equation (1.1), then $\mathrm{Supp}(\varphi) \subset K_\Lambda$. Also, by Proposition 2.1, the set $\Omega_\Lambda = K_\Lambda \cap \Gamma$ is Λ-admissible. However, it is not necessarily true that $\mathrm{Supp}\, \varphi \subset \cup_{\lambda \in \Omega_\Lambda} Q + \lambda$.

We will therefore consider the bigger set $\Omega' = K_{\Lambda'} \cap \Gamma$, where $\Lambda' = \Lambda - D \supset \Lambda$. In [CHM04] it was shown that the translations of Q using all elements of Ω' cover the support of the compactly solution to (1.1). Moreover, Ω' is Λ' admissible, and hence also Λ-admissible. As noted earlier, the Λ-admissibility of Ω' guarantees that the space $\ell(\Omega') = \{Y \in \ell(\Gamma) : y_k = 0, k \notin \Omega'\}$ is right invariant under L.

Let now $\{\Omega_n\}_{n \geq 0}$ be a sequence of subsets of Γ that satisfies:

- $\Omega_0 = \Omega_\Lambda$
- For $i \geq 0$, $\Omega_i \subsetneq \Omega_{i+1}$, and $\cup_i \Omega_i = \Gamma$
- For $i \geq 0$ Ω_i are Λ-admissible and $w_\Lambda(\Omega_{i+1}) \cap \Gamma \subseteq \Omega_i$.
- $\Omega_{n_0} = \Omega'$
- For $i \geq n_0$, Ω_i are Λ'-admissible and $w_{\Lambda'}(\Omega_{i+1}) \cap \Gamma \subseteq \Omega_i$.

These sets exist by Proposition 2.2 and its Corollary 2.1.

We denote by $\{T_n\}_{n \geq 0}$ the finite submatrices of L

$$T_n = [c_{Ai-j}]_{i,j \in \Omega_n}. \tag{3.3}$$

Since $\Omega_n \subset \Omega_{n+1}$, if the order in Γ is appropriately chosen, actually T_n is a submatrix of T_{n+1}, for each n.

Let $Y = \{y_k\}_{k \in \Gamma} \in \ell(\Gamma)$ be an infinite row vector, and $P_n : \ell(\Gamma) \longrightarrow \mathbb{C}^{1 \times \Omega_n}, n \geq 0$ be the restriction mappings defined by

$$P_n Y = \{y_k\}_{k \in \Omega_n}. \tag{3.4}$$

We consider $L - \lambda I : \ell(\Gamma) \longrightarrow \ell(\Gamma)$ the left-multiplication operator who maps $Y \longrightarrow Y(L - \lambda I)$ (where I is the identity operator acting on $\ell(\Gamma)$). By abuse of notation, I will be any identity operator, no matter on which space it is acting on.

Note. In what follows we will use powers of the matrix $(L - \lambda I)$. Note that these powers are point-wise well defined, since the rows of the matrix L have a finite number of non-zero elements.

The next proposition shows the relation between the spectrum of L and T_n:

Proposition 3.1. *Consider $\lambda \in \mathbb{C}$, $r \in \mathbb{N}$ and $n \geq 0$.*

(1) Let $Y \in \ell(\Gamma)$. We have

$$Y \in \operatorname{Ker}(L - \lambda I)^r \quad \textit{implies} \quad P_n Y \in \operatorname{Ker}(T_n - \lambda I)^r. \tag{3.5}$$

Conversely,

(2) If $v \in \operatorname{Ker}(T_n - \lambda I)^r$ and $\lambda \neq 0$, then we can extend v to an infinite row vector Y_v (i.e. $Y_v \in \ell(\Gamma)$ and $P_n Y_v = v$), so that $Y_v \in \operatorname{Ker}(L - \lambda I)^r$.

(3) If $\lambda \neq 0, Y \neq 0$ and $Y \in \operatorname{Ker}(L - \lambda I)^r$, then $P_n Y \neq 0$. In particular the extension in (2) of v to Y_v is unique.

Proof.

(1) First note that $j \in \Omega_n$ and $Ai - j \in \Lambda$ implies $i \in \Omega_n$. For, in this case, $Ai \in \Omega_n + \Lambda$ and since Ω_n is a Λ-admissible set it follows that $i \in A^{-1}(\Omega_n + \Lambda) \cap \Gamma \subset \Omega_n$. Hence

$$j \in \Omega_n, \ i \notin \Omega_n \implies [L - \lambda I]_{ij} = 0. \tag{3.6}$$

Moreover, we will show by induction on r that,

$$\text{if } j \in \Omega_n \text{ and } i \notin \Omega_n, \text{then } [(L - \lambda I)^r]_{ij} = 0. \tag{3.7}$$

(a) The case $r = 1$ is simply (3.6), since we assume that $c_k = 0$ if $k \notin \Lambda$.

(b) Suppose now that (3.7) holds for some fixed $r \geq 1$. Using (a), for $j \in \Omega_n$ we have

$$\begin{aligned}[(L - \lambda I)^{r+1}]_{ij} &= \sum_{k \in \Gamma} [(L - \lambda I)^r]_{ik} [L - \lambda I]_{kj} \\ &= \sum_{k \in \Omega_n} [(L - \lambda I)^r]_{ik} [L - \lambda I]_{kj}.\end{aligned}$$

Now, if $i \notin \Omega_n$, the inductive hypothesis yields that the last sum is zero.

Therefore, the statement is true for all $r \in \mathbb{N}$.

To prove the first part of the Proposition, let $Y \in \ell(\Gamma)$ and $Y \in \operatorname{Ker}(L - \lambda I)^r$.

Applying the preceding equality, we obtain for each $j \in \Omega_n$

$$\begin{aligned}[(P_n Y)(T_n - \lambda I)^r]_j &= \sum_{i \in \Omega_n} y_i [(L - \lambda I)^r]_{ij} \\ &= \sum_{i \in \Gamma} y_i [(L - \lambda I)^r]_{ij} \\ &= [Y(L - \lambda I)^r]_j = 0.\end{aligned}$$

This completes the proof of (1).

(2) Assume that $v \in \mathbb{C}^{1\times\Omega_n}, \lambda \neq 0$ and $v \in \mathrm{Ker}(T_n - \lambda I)^r$. We want to construct a vector $Y_v \in \ell(\Gamma)$ such that $P_n Y_v = v$ and $Y_v \in \mathrm{Ker}(L - \lambda I)^r$.

We now prove by induction on r that,

$$\text{for } j \in \Omega_{k+1},\ i \notin \Omega_k, k \geq 0,\ \left[(L-\lambda I)^r\right]_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ (-\lambda)^r & \text{for } i = j. \end{cases} \tag{3.8}$$

(a) Case $r = 1$. If i were such that $Ai - j \in \Lambda$, then $i \in A^{-1}(\Omega_{k+1} + \Lambda) \cap \Gamma \subset \Omega_k$. Hence, if $i \notin \Omega_k$ then $Ai - j \notin \Lambda$ and so $L_{ij} = 0$, and therefore $[L - \lambda I]_{ij} = 0$ for $i \neq j$, and $[(L - \lambda I)]_{jj} = -\lambda$.

(b) Assume that (3.8) holds for $r \geq 1$. Then for $j \in \Omega_{k+1}$ and $i \notin \Omega_k, k \geq 0$, we have

$$\begin{aligned} \left[(L-\lambda I)^{r+1}\right]_{ij} &= \sum_{\ell\in\Gamma} \left[(L-\lambda I)^r\right]_{i\ell} [L-\lambda I]_{\ell j} \\ &= \sum_{\ell\notin\Omega_k} \left[(L-\lambda I)^r\right]_{i\ell} [L-\lambda I]_{\ell j}, \end{aligned}$$

since by the inductive hypothesis $\left[(L-\lambda I)^r\right]_{i\ell} = 0$ if $\ell \in \Omega_k \subset \Omega_{k+1}$ and $i \notin \Omega_k$. It follows using the case $r = 1$ that

$$\left[(L-\lambda I)^{r+1}\right]_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ (-\lambda)^{r+1} & \text{for } i = j, \end{cases} \tag{3.9}$$

which proves (3.8), for every $r \in \mathbb{N}$.

Define now $y_j = v_j$ for $j \in \Omega_n$, and define recursively, for $j \notin \Omega_n$,

$$y_j = \frac{-1}{(-\lambda)^r} \sum_{i\in\Omega_k} y_i \left[(L-\lambda I)^r\right]_{ij}, \quad j \in \Omega_{k+1}\backslash\Omega_k, k \geq n. \tag{3.10}$$

The vector $Y_v = \{y_j\}_{j\in\Gamma}$ is an extension of v. To see that $Y_v \in \mathrm{Ker}(L-\lambda I)^r$, since $\left(Y(L-\lambda I)^r\right)_j = \sum_{i\in\Gamma} y_i[(L-\lambda I)^r]_{ij}$, we have:

- If $j \in \Omega_n$, then by (3.6)

$$\begin{aligned} \sum_{i\in\Gamma} y_i \left[(L-\lambda I)^r\right]_{ij} &= \sum_{i\in\Omega_n} y_i \left[(T_n-\lambda I)^r\right]_{ij} \\ &= \sum_{i\in\Omega_n} v_i \left[(T_n-\lambda I)^r\right]_{ij} = 0. \end{aligned} \tag{3.11}$$

- If $j \notin \Omega_n$, then there exist $k \in \mathbb{N}_0, k \geq n$ such that $j \in \Omega_{k+1} \setminus \Omega_k$. Therefore,

$$\begin{aligned} \sum_{i\in\Gamma} y_i \left[(L-\lambda I)^r\right]_{ij} &= \sum_{i\in\Omega_k} y_i \left[(L-\lambda I)^r\right]_{ij} + \sum_{i\notin\Omega_k} y_i \left[(L-\lambda I)^r\right]_{ij} \\ &= \sum_{i\in\Omega_k} y_i \left[(L-\lambda I)^r\right]_{ij} + y_j(-\lambda)^r) \\ &= 0. \text{ (by (3.10))} \end{aligned}$$

(3) For the last part of the Proposition, assume that $\lambda \neq 0, Y \neq 0$, and $Y \in \mathrm{Ker}(L - \lambda I)^r$. To show that $P_n Y \neq 0$, take $k_0 \in \Gamma$ such that $Y_{k_0} \neq 0$. If $k_0 \in \Omega_n$, there is nothing to prove. Otherwise, let

$$t_0 = \min\{k \in \mathbb{N} : k_0 \in \Omega_k\}. \tag{3.12}$$

Since $Y(L-\lambda I)^r = 0$,

$$\begin{aligned}\sum_{i\in\Gamma} y_i[(L-\lambda I)^r]_{ik_0} &= \sum_{i\in\Omega_{t_0-1}} y_i[(L-\lambda I)^r]_{ik_0} + \sum_{i\notin\Omega_{t_0-1}} y_i[(L-\lambda I)^r]_{ik_0} \\ &= \sum_{i\in\Omega_{t_0-1}} y_i[(L-\lambda I)^r]_{ik_0} + y_{k_0}(-\lambda)^r = 0.\end{aligned}$$

So, there exist $k_1 \in \Omega_k, 0 < k < t_0$, such that $Y_{k_1} \neq 0$. If $k_1 \in \Omega_n$, we can stop here. If not, we repeat the procedure until $k_j \in \Omega_n$.

□

Remark.

- Since the previous Proposition is true for **any** set of the sequence Ω_n, in fact the smallest matrix T_0 already has all the spectral information of L.
- The extension of the vectors of $\operatorname{Ker}(T_0 - \lambda I)^r$ to vectors of $\operatorname{Ker}(L-\lambda I)^r$ will produce *intermediate* vectors of $\operatorname{Ker}(T_n - \lambda I)^r$, by the construction of the sets Ω_n produced in Corollary 2.1.

For the special case $\lambda = 0$, under some mild assumptions, we have an additional property. We say that the Γ translates $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are *linearly independent*, if for any sequence $\{\alpha_k\}_{k\in\Gamma}$ in $\ell(\Gamma)$,

$$\sum_{k\in\Gamma} \alpha_k \varphi(\cdot - k) \equiv 0 \quad \text{implies} \quad \alpha_k = 0.$$

Lemma 3.1. *If $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent, then the operator $L : \ell(\Gamma) \longrightarrow \ell(\Gamma), Y \longmapsto YL$ is one to one.*

Proof. Let $YL = 0$. Then

$$Y\Phi(x) = YL\Phi(Ax) = 0. \tag{3.13}$$

Since $\{\varphi(\cdot-k)\}_{k\in\Gamma}$ are linearly independent, $Y\Phi = 0$ implies $Y = 0$, so $\operatorname{Ker}(L) = \{0\}$.
□

4. The class $\mathcal{H}(A, \lambda, r)$

Assume $Y \in \operatorname{Ker}(L-\lambda I)^r$. If we define $h(x) = Y\Phi(x)$, we have

$$\begin{aligned}0 = Y(L-\lambda I)^r\Phi(x) &= Y\left(\sum_{k=0}^{r}\binom{r}{k}(-\lambda)^{r-k}L^k\right)\Phi(x) \\ &= Y\left(\sum_{k=0}^{r}\binom{r}{k}(-\lambda)^{r-k}\Phi(A^{-k}x)\right), \\ &= \sum_{k=0}^{r}\binom{r}{k}(-\lambda)^{k}h(A^{k-r}x).\end{aligned}$$

This leads to the following definition:

Definition 4.1. A function h is in the class $\mathcal{H}(A, \lambda, r)$, if it satisfies

$$\sum_{k=0}^{r}\binom{r}{k}(-\lambda)^{k}h(A^{k-r}x) = 0 \quad \text{for every } x \in \mathbb{R}^d. \tag{4.1}$$

If we define the operator D_A by $D_A(f)(x) = f(A^{-1}x)$, then h is in $\mathcal{H}(A,\lambda,r)$ if and only if

$$(D_A - \lambda I)^r h = 0.$$

A function in $\mathcal{H}(A,\lambda,r)$ will also be said to be of class $\mathcal{H}(A,\lambda,r)$.

Note that if $h \in \mathcal{H}(A,\lambda,r)$, then $h \in \mathcal{H}(A,\lambda,s)$ for every $s \geq r$.

Proposition 4.1. *Let $V \subset \mathbb{R}^d$ be a bounded set such that $0 \in V^0$ and $V \subset AV$. Set $C = AV \setminus V$ and $a \in \mathbb{Z}$. Let h be a function of class $\mathcal{H}(A,\lambda,r)$. Then the values of h in $\mathbb{R}^d \setminus \{0\}$ can be determined from its values in any set of the type:*

$$\tilde{C} = \bigcup_{k=a}^{a+r} A^k C.$$

Furthermore, if $\lambda \neq 1$ then $h(0) = 0$.

Proof. Since $h \in \mathcal{H}(A,\lambda,r)$ we get that

$$h(x) = -\sum_{k=1}^{r} \binom{r}{k} (-\lambda)^k h(A^k x) \quad \text{and} \tag{4.2}$$

$$h(x) = -\sum_{k=1}^{r} \binom{r}{k} (-\lambda)^{-k} h(A^{-k} x). \tag{4.3}$$

On the other side, it has been proved in [ACM04] that the set C satisfies:

a) $\bigcup_{j\in\mathbb{Z}} A^j C = \mathbb{R}^d \backslash \{0\}$

b) *The sets $\{A^j C\}_{j\in\mathbb{Z}}$ are pairwise disjoint.*

So, from (4.2) we deduce that the values of h in $A^{a-1}C$ can be obtained from the values in $\tilde{C}$, and analogously, from (4.3) the values of h in $A^{a+r+1}C$ can be obtained from the values in $\tilde{C}$.

Then we proceed inductively to obtain all the values in $\mathbb{R}^d \setminus \{0\}$. Finally, it is immediate from the definition, that $h(0) = 0$ when $\lambda \neq 1$. □

Proposition 4.2. *Suppose $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent. Let $f_1, \dots, f_l \in \mathcal{S}(\varphi)$, $f_i = Y^i\Phi$, where $Y^i \in \ell(\Gamma)$. Then $f_1, \dots, f_l$ are linearly independent functions if and only if $Y^1, \dots, Y^l$ are linearly independent in $\ell(\Gamma)$.*

Proof. Since

$$\sum_{i=1}^{l} \alpha_i f_i = \sum_{i=1}^{l} \alpha_i (Y^i \Phi) = \left(\sum_{i=1}^{l} \alpha_i Y^i \right) \Phi, \tag{4.4}$$

and the translates of φ along the lattice Γ are linearly independent, we conclude that $\sum_{i=1}^{l} \alpha_i f_i \equiv 0$ if and only if $(\sum_{i=1}^{l} \alpha_i Y^i) = 0$, which leads to the desired result. □

Remark. Let $E : \mathcal{S}(\varphi) \longrightarrow \ell(\Gamma)$ be the function that associates to each element of $\mathcal{S}(\varphi)$, its coordinates in $\{\varphi(x+k)\}$. Proposition 4.2 shows that E is an isomorphism.

Proposition 4.3. *Assume that $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent.*

(1) If $h \in \mathcal{S}(\varphi), h = Y\Phi$ and $h \in \mathcal{H}(A,\lambda,r)$, then $Y \in \mathrm{Ker}(L - \lambda I)^r$ and $P_n Y \in \mathrm{Ker}(T_n - \lambda I)^r$.

Conversely

(2) Assume that $\lambda \neq 0, v \in \operatorname{Ker}(T_n - \lambda I)^r$ and that Y_v is the unique extension of v such that $Y_v \in \operatorname{Ker}(L - \lambda I)^r$ (see Proposition 3.1). Then the function $h = Y_v\Phi$ belongs to $\mathcal{H}(A, \lambda, r)$.

Proof. If $h = Y\Phi$ is of class $\mathcal{H}(A, \lambda, r)$, then we have

$$\begin{aligned} 0 &= \sum_{k=0}^{r} \binom{r}{k} (-\lambda)^k h(A^{k-r}x) \\ &= \sum_{k=0}^{r} \binom{r}{k} (-\lambda)^k Y L^{r-k} \Phi(x) \\ &= Y \left(L - \lambda I\right)^r \Phi(x). \end{aligned}$$

Since the Γ translates of φ are linearly independent, it follows that $Y(L-\lambda I)^r = 0$, and consequently, by Proposition 3.1, $P_n Y \in \operatorname{Ker}(T_n - \lambda I)^r$.

To prove the second part, note that if $v = 0$, the statement is trivially true. If $\lambda \neq 0$ and $v \in \operatorname{Ker}(T_n-\lambda I)^r$ and $v \neq 0$, then, by Proposition 3.1 we can extend v to a vector $Y_v \in \operatorname{Ker}(L - \lambda I)^r$, and so the function $h = Y_v\Phi$ is of class $\mathcal{H}(A, \lambda, r)$. □

4.1. **Jordan decomposition of T_n.** Let $m_n = \#\Omega_n$. Consider the set Δ_n of eigenvalues of T_n and the associated Jordan basis $\mathcal{B}_n = \{v_1, \dots, v_{m_n}\}$ of $\mathbb{C}^{m_n}$. For each $v_i \in \mathcal{B}_n$ we have that $v_i \in \operatorname{Ker}(T_n - \lambda I)^k$ and $v_i \notin \operatorname{Ker}(T_n - \lambda I)^{k-1}$ for some $\lambda \in \Delta_n$, and for some $k \geq 1$. So to each v_i there corresponds a unique pair (λ, k). Note however that to two different v_is in the basis there could correspond the same pair. For each vector of $\mathcal{B}_n$, set $v_i = v_i(\lambda, k)$. If $\lambda \neq 0$, by Proposition 4.3 we can associate to each $v_i(\lambda, k)$ a function $h_{v_i(\lambda,k)}$ in $\mathcal{H}(A, \lambda, k) \cap \mathcal{S}(\varphi)$. Since the vectors $v_i(\lambda, k)$ are linearly independent, its extensions $\{Y_{v_i}\}$ are linearly independent in $\ell(\Gamma)$, so the functions $\{h_{v_i(\lambda,k)}\}_{v_i \in \mathcal{B}_n, \lambda \neq 0}$ are linearly independent.

If $h_1, \dots, h_l$ are of class $\mathcal{H}(A, \lambda, k_i)$, for some $k_i, i = 1, \dots, l$, then a linear combination of them is of class $\mathcal{H}(A, \lambda, k)$, with $k = \max_{1 \leq i \leq l}(k_i)$, for if $h = \sum_{i=0}^{l} \alpha_i h_i(\lambda, k_i)$, then

$$(D_A - \lambda I)^k h = (D_A - \lambda I)^k \sum_{i=0}^{l} \alpha_i h_i = \sum_{i=0}^{l} \alpha_i \left(D_A - \lambda I\right)^k h_i = 0,$$

and consequently, $h \in \mathcal{H}(A, \lambda, k)$.

Let

$$\chi_{T_n}(x) = \prod_{\lambda \in \Delta_n} (x - \lambda)^{r_\lambda} \tag{4.5}$$

be the characteristic polynomial of T_n, and set

$$\mathcal{H}_\lambda(\varphi) = \{h \in \mathcal{S}(\varphi) : h \in \mathcal{H}(A, \lambda, k), \text{ for some } k \geq 1 \}, \quad \lambda \in \Delta_n.$$

Then, if $\lambda \neq 0$, using Proposition 4.3, $\dim(\mathcal{H}_\lambda(\varphi)) = r_\lambda$ and a basis for $\mathcal{H}_\lambda(\varphi)$ are the functions of class $\mathcal{H}(A, \lambda, k)$ corresponding to the vectors $v_i \in \mathcal{B}_n$, such that $v_i = v_i(\lambda, k)$, for some $k \geq 1$. So, if we denote

$$\mathcal{H} = \bigoplus_{\lambda \in \Delta_n, \lambda \neq 0} \mathcal{H}_\lambda(\varphi) \subset \mathcal{S}(\varphi),$$

then, $\dim(\mathcal{H}) = m_n - r_0$, where r_0 is the dimension of the subspace generated by the vectors of the Jordan basis associated to $\lambda = 0$. (Note that $m_n = \sum_{\lambda \in \Delta_n, \lambda \neq 0} r_\lambda$.)

In order to be able to include the case $\lambda = 0$ in our analysis, we need to consider the case in which $\operatorname{Supp}\varphi \subset \cup_{\omega\in\Omega_n} Q + \omega$. This will guarantee, that (except for a possible set of measure zero), $\varphi(x+k) = 0$ if $k \notin \Omega_n$.

4.2. **The case in which Ω_n contains** $\operatorname{Supp}(\varphi)$**.** If we recall the choice of the sequence $\{\Omega_n\}$ at the beginning of section 3, it is clear, that for $n \geq n_0$, we have that $\operatorname{Supp}\varphi \subseteq \cup_{\omega\in\Omega_n} Q+\omega$, and hence, the local dimension of φ is $\dim \operatorname{span}\{\varphi(x+k)\}_{k\in\Omega_n}$.

In that case for $x \in Q^\circ$, $\lambda \neq 0$, $h_{v_i(\lambda,k)}(x) = \sum_{l\in\Omega_n} [v_i(\lambda,k)]_l \varphi(x+l)$ since $\varphi(x+j) = 0$ if $j \notin \Omega_n$.

Moreover for $\lambda = 0$ we have the following Proposition:

Proposition 4.4. *Let $n \geq n_0$, and let r_0 be the power of x in χ_{T_n} (c.f. (4.5)). Consider $v_i = v_i(0,r)$, with $r \leq r_0$. Define $h(x) = \sum_{k\in\Omega_n} [v_i]_k \varphi(x+k)$. Then $h \equiv 0$ a.e. on Q.*

We postpone the proof to remark that with this Proposition, if $n \geq n_0$, and $\mathcal{B}_n$ is (as before) the matrix whose columns are the vectors of the Jordan basis for T_n, then

$$\begin{bmatrix} h_{v_1(\lambda,k)}(x) \\ \vdots \\ h_{v_{m_n}(\lambda,k)}(x) \end{bmatrix} = \mathcal{B}_n P_n \Phi(x) \quad \text{a.e. } x \in Q. \tag{4.6}$$

Hence, since the matrix $\mathcal{B}_n$ is invertible the local dimension of φ coincides with $\dim \operatorname{span}\{h_{v_i(\lambda,k)}(x), x \in Q, v_i \in \mathcal{B}_n\}$, which is equal to the dimension of $\mathcal{H}$.

So the local dimension of φ can be found by finding the Jordan form of any of the finite matrices T_n as long as $n \geq n_0$.

Moreover, any function of the shift-invariant space $\mathcal{S}(\varphi)$ can be written as a linear combination of the lattice translates of the homogeneous functions. Namely, let $f \in \mathcal{S}(\varphi)$, then

$$f(x) = \sum_{\gamma\in\Gamma} \alpha_\gamma \varphi(x+\gamma) \quad x \in \mathbb{R}^d, \alpha_\gamma \in \mathbb{C}. \tag{4.7}$$

If we call $\mathbf{g} : \mathbb{R}^n \longrightarrow \mathbb{C}^{m_n}$ and $\mathbf{h} : \mathbb{R}^n \longrightarrow \mathbb{C}^{m_n}$ the functions

$$\mathbf{g}(x) = P_n \Phi(x) \chi_Q(x) \quad \text{and} \quad \mathbf{h}(x) = \begin{bmatrix} h_{v_1(\lambda,k)}(x) \\ \vdots \\ h_{v_{m_n}(\lambda,k)}(x) \end{bmatrix} \chi_Q(x), \tag{4.8}$$

and for $\gamma \in \Gamma$ we denote by $\overline{\alpha}_\gamma = (\alpha_{i_1+\gamma}, \dots, \alpha_{i_{m_n}+\gamma})$ the vector of length m_n whose indices are in $\Omega_n + \gamma$ (here $\Omega_n = \{i_1, \dots, i_{m_n}\}$), then (4.7) becomes

$$f(x) = \sum_{\gamma\in\Gamma} \overline{\alpha}_\gamma \mathbf{g}(x+\gamma), \tag{4.9}$$

and using (4.6) we obtain

$$f(x) = \sum_{\gamma\in\Gamma} \beta_\gamma \mathbf{h}(x+\gamma) \quad \text{where} \quad \beta_\gamma = \overline{\alpha}_\gamma \mathcal{B}_n^{-1}. \tag{4.10}$$

We will now prove Proposition 4.4. For this, let r_0 be the power of x in χ_{T_n} (c.f. (4.5)). Choose $m \geq n$ large enough such that

$$\Omega_m \supset \Omega_{n_0} - \left(D + AD + \cdots + A^{r_0-1}D\right). \tag{4.11}$$

Define the matrices $[(T_k)_d]_{ij} = c_{Ai-j+d}, i, j \in \Omega_k$, for any $k \in \mathbb{N}$, and $d \in D$. It is shown in [CHM04], that if $x \in Q$, for any $r \geq 1$ there exists $y_r \in Q$, $\gamma_r \in \Gamma$, such that

$$x = A^{-r}(y_r + \gamma_r) \quad \text{with} \quad \gamma_r = d_r + Ad_{r-1} + \cdots + A^{r-1}d_1 \tag{4.12}$$

where $d_i \in D, 1 \leq i \leq r$. Therefore, if $k \geq n_0$ and P_k is as in (3.4)

$$P_k\Phi(x) = (T_k)_{d_1} \ldots (T_k)_{d_r} P_k \Phi(A^r x - \gamma_r) \quad x \in Q. \tag{4.13}$$

For convenience, we will call $\Omega = \Omega_m$.

Lemma 4.1. *With the previous notation, if $r \leq r_0$, then for $k \in \Omega$ and $j \in \Omega_{n_0}$, we have*

$$[T_m^r]_{k(j-\gamma_r)} = [(T_m)_{d_1} \ldots (T_m)_{d_r}]_{kj} \, ,$$

where $\gamma_r \in D + AD + \cdots + A^{r-1}D$.

Remark. Note that the preceding equation does not state that both matrices are equal.

Proof. We will prove the Lemma by induction on r. Let γ_r be as in (4.12).

- The case $r = 1$ is trivial by the definition of $(T_m)_{d_1}$.
- $r - 1 \Longrightarrow r$ Observe first that by the choice of Ω,

  $$[T_m]_{u(j-\gamma_r)} = [(T_m)_{d_r}]_{u(j-A\gamma_{r-1})}\,, \quad u \in \Omega, j \in \Omega_{n_0}.$$

  Now

  $$\begin{aligned} [T_m^r]_{k(j-\gamma_r)} &= \sum_{u\in\Omega} \left[T_m^{r-1}\right]_{ku} [T_m]_{u(j-\gamma_r)} \\ &= \sum_{u\in\Omega} \left[T_m^{r-1}\right]_{ku} [(T_m)_{d_r}]_{u(j-A\gamma_{r-1})} \\ &= \sum_{u\in\Omega_{n_0}-\gamma_{r-1}} \left[T_m^{r-1}\right]_{ku} [(T_m)_{d_r}]_{u(j-A\gamma_{r-1})}\,, \end{aligned} \tag{4.14}$$

  where the last equality follows from the Λ'-admissibility of Ω_{n_0}. But

  $$[(T_m)_{d_r}]_{u(j-A\gamma_{r-1})} = [(T_m)_{d_r}]_{(u+\gamma_{r-1})j}\,, u \in \Omega_{n_0} - \gamma_{r-1}, j \in \Omega_{n_0},$$

  and therefore, using induction and the Λ'-admissibility of Ω_{n_0}, (4.14) becomes

  $$\begin{aligned} [T_m^r]_{k(j-\gamma_r)} &= \sum_{u\in\Omega_{n_0}-\gamma_{r-1}} \left[T_m^{r-1}\right]_{ku} [(T_m)_{d_r}]_{(u+\gamma_{r-1})j} \\ &= \sum_{\ell\in\Omega_{n_0}} \left[T_m^{r-1}\right]_{k(\ell-\gamma_{r-1})} [(T_m)_{d_r}]_{\ell j} \\ &= \sum_{\ell\in\Omega_{n_0}} \left[(T_m)_{d_1} \ldots (T_m)_{d_{r-1}}\right]_{k\ell} [(T_m)_{d_r}]_{\ell j} \\ &= \sum_{\ell\in\Omega} \left[(T_m)_{d_1} \ldots (T_m)_{d_{r-1}}\right]_{k\ell} [(T_m)_{d_r}]_{\ell j} \end{aligned}$$

  which completes the inductive step.

□

We can now prove Proposition 4.4.

Proof. Let $\Omega = \Omega_m$ be as before, and let $x \in Q \setminus (\partial Q \cup \bigcup_{i=1}^r A^{-i}\partial Q + A^{-i}D + \cdots + A^{-1}D)$. Note that with this choice of x, $A^r x - \gamma_r \in Q^\circ$, and therefore, if $u \notin \Omega_{n_0}$, then $\varphi(A^r x - \gamma_r + u) = 0$.

Using this, together with the previous lemma, and equation (4.13) for $k = m$, we have

$$\begin{aligned}
[T_m^r P_m \Phi(A^r x)]_k &= \sum_{j\in\Omega} [T_m^r]_{kj}\, \varphi(A^r x + j) \\
&= \sum_{j\in\Omega} [T_m^r]_{k(j-\gamma_r)}\, \varphi(A^r x - \gamma_r + j) \\
&= \sum_{j\in\Omega} [(T_m)_{d_1} \ldots (T_m)_{d_r}]_{kj}\, \varphi(A^r x - \gamma_r + j) \\
&= \varphi(x + k).
\end{aligned}$$

Therefore, for $x \in Q \setminus (\partial Q \cup \bigcup_{i=1}^r A^{-i}\partial Q + A^{-i}D + \cdots + A^{-1}D)$

$$\begin{aligned}
h(x) &= \sum_{k\in\Omega_n} [v_i]_k \varphi(x + k) \\
&= \sum_{k\in\Omega_n} [v_i]_k \sum_{j\in\Omega} [T_m^r]_{kj}\, \varphi(A^r x + j) \\
&= \sum_{j\in\Omega} \left(\sum_{k\in\Omega_n} [v_i]_k\, [T_m^r]_{kj} \right) \varphi(A^r x + j) = 0
\end{aligned}$$

□

5. Accuracy and homogeneous polynomials

In this section we will relate the previously obtained results, to the "accuracy" of a scaling function. We will use the notation of [CHM98].

Definition 5.1. The *accuracy* of φ is the highest degree κ such that all multivariate polynomials q with degree $(q) < \kappa$ are in $S(\varphi)$.

Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. With the standard multi-index notation we write $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, where $\alpha = (\alpha_1, \ldots, \alpha_d)$ with each α_i a nonnegative integer. Denote with $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The number of multi-indices α of degree s is $d_s = \binom{s+d-1}{d-1}$.

For each integer $s \geq 0$, we define the vector valued function $X_{[s]} : \mathbb{R}^d \longrightarrow \mathbb{C}^{d_s}$ by

$$X_{[s]}(x) = [x^\alpha]_{|\alpha|=s}, \quad x \in \mathbb{R}^d.$$

The ordering of the multi-indices α of degree s is not important as long as the same ordering is used throughout.

We will now look at the behavior of $X_{[s]}(x)$ under the multiplication by an arbitrary $d \times d$ matrix Z with scalar entries $z_{i,j}$. If $|\alpha| = s$, then $(Zx)^\alpha$ is not in general a monomial. Instead, it is a new polynomial of degree s, that is still homogeneous, but possibly involves all terms x^β with $|\beta| = s$.

Let $Z_{[s]} = [z^s_{\alpha,\beta}]_{|\alpha|=s,|\beta|=s}$ be the $d_s \times d_s$ matrix whose scalar entries $z^s_{\alpha,\beta}$ are defined by the equation

$$\sum_{|\beta|=s} z^s_{\alpha,\beta} x^\beta = (Zx)^\alpha = \prod_{i=1}^{d} (z_{i,1}x_1 + \cdots + z_{i,d}x_d)^{\alpha_i}.$$

The matrices $Z_{[s]}$ and their properties have been intensively studied in [CHM98], [CHM99]. In particular, if I_d denotes the identity matrix in $\mathbb{R}^d$, we have that

$$(I_d)_{[s]} = I_{d_s}$$

and if Z and U are two matrices,

$$(ZU)_{[s]} = Z_{[s]}U_{[s]} \quad \text{and hence, if } Z \text{ is invertible} \quad (Z^{-1})_{[s]} = (Z_{[s]})^{-1}.$$

Dilation of $X_{[s]}(x)$ by Z obeys the rule

$$X_{[s]}(Zx) = Z_{[s]}X_{[s]}(x),$$

hence, if A is the dilation matrix corresponding to the refinement equation (1.1),

$$X_{[s]}(A^{-1}x) = {A^{-1}}_{[s]}X_{[s]}(x).$$

If J_s is the Jordan form of ${A^{-1}}_{[s]}$, then there exists an invertible $d_s \times d_s$ matrix Q_s such that $Q_s {A^{-1}}_{[s]} Q_s^{-1} = J_s$. So we have that

$$Q_s X_{[s]}(A^{-1}x) = (Q_s {A^{-1}}_{[s]} Q_s^{-1}) Q_s X_{[s]}(x).$$

Denote by $\widetilde{Q}_s(x) = Q_s X_{[s]}(x)$. Observe that $\widetilde{Q}_s(x) = \left(\widetilde{Q}^1_s(x), \ldots, \widetilde{Q}^{d_s}_s(x)\right)^t$ is a column vector of polynomials of degree s that are homogeneous. By the previous equation, we have that

$$\widetilde{Q}_s(A^{-1}x) = J_s \widetilde{Q}_s(x).$$

Let β be an eigenvalue of ${A^{-1}}_{[s]}$ and B the Jordan block of order ℓ associated to β, i.e.,

$$B = \begin{pmatrix} \beta & 0 & \ldots & 0 & 0 \\ 1 & \beta & \ldots & 0 & 0 \\ . & . & \ldots & . & . \\ 0 & 0 & \ldots & \beta & 0 \\ 0 & 0 & \ldots & 1 & \beta \end{pmatrix} \in \mathbb{C}^{\ell \times \ell}.$$

We write $\widetilde{Q}_B(x)$ for the vector that is the restriction of $\widetilde{Q}_s(x)$ to the coordinates that correspond to the block B, i.e, if $j, j+1, \ldots, j+\ell-1$ are the columns of B in J_s then $\widetilde{Q}_B(x) = \left(\widetilde{Q}^j_s(x), \ldots, \widetilde{Q}^{j+\ell-1}_s(x)\right)^t$. Since

$$\widetilde{Q}_s(A^{-1}x) = J_s \widetilde{Q}_s(x),$$

we have

$$\widetilde{Q}_B(A^{-1}x) = B\widetilde{Q}_B(x). \tag{5.1}$$

This relation will enable us to show how, under the hypothesis of accuracy, we can relate the Jordan form of ${A^{-1}}_{[s]}$ to the one of T_n. This relation also gives a necessary condition for φ to have accuracy κ.

Proposition 5.1. *Assume that φ has accuracy κ and that $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent. Let $s < \kappa$. If β is an eigenvalue of ${A^{-1}}_{[s]}$ and B is a Jordan block of ${A^{-1}}_{[s]}$ associated to β of order ℓ, then T_n has a Jordan block associated to β of order ℓ' with $\ell' \geq \ell$.*

Proof. Consider $\widetilde{Q}_B(x) = (\widetilde{Q}_B^1(x), \dots, \widetilde{Q}_B^\ell(x))$. It follows from (5.1) that

$$\begin{array}{rcl} \widetilde{Q}_B^1(A^{-1}x) & = & \beta\widetilde{Q}_B^1(x) \\ \widetilde{Q}_B^2(A^{-1}x) & = & \widetilde{Q}_B^1(x) + \beta\widetilde{Q}_B^2(x) \\ \vdots & & \vdots \\ \widetilde{Q}_B^\ell(A^{-1}x) & = & \widetilde{Q}_B^{\ell-1}(x) + \beta\widetilde{Q}_B^\ell(x). \end{array} \tag{5.2}$$

Since $\widetilde{Q}_B^i(x) \in \mathcal{S}(\varphi)$ for $1 \le i \le \ell$, we can write

$$\widetilde{Q}_B^i(x) = Y^i\Phi(x),$$

for some infinite column vector Y^i. From (5.2) we have for $2 \le i \le \ell$,

$$Y^i\Phi(A^{-1}x) = Y^{i-1}\Phi(x) + \beta Y^i\Phi(x),$$

which implies

$$Y^iL\Phi(x) - \beta Y^i\Phi(x) = Y^{i-1}\Phi(x).$$

So, the linear independence of $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ yields

$$Y^i(L - \beta I) = Y^{i-1}. \tag{5.3}$$

Since $\widetilde{Q}_B^\ell(x) \in \mathcal{H}(A, \beta, \ell)$, by Proposition 4.3 we have that $v = P_nY^\ell \in \mathrm{Ker}(T_n - \beta I)^\ell$. Consider the vectors $v_1 = P_nY^\ell, v_2 = (P_nY^\ell)(T_n-\beta I), \dots, v_\ell = (P_nY^\ell)(T_n-\beta I)^{\ell-1}$. Let us show that $v_1, \dots, v_\ell$ are linearly independent: Assume that

$$\sum_{i=1}^{\ell} \alpha_i v_i = 0. \tag{5.4}$$

Since

$$\begin{aligned} \left(\sum_{i=1}^{\ell} \alpha_i v_i\right)(T_n - \beta I)^{\ell-1} &= \left(\sum_{i=1}^{l} \alpha_i v(T_n - \beta I)^{i-1}\right)(T_n - \beta I)^{\ell-1} \\ &= (\sum_{i=1}^{\ell} \alpha_i v(T_n - \beta I)^{\ell+i-2} \\ &= \alpha_1 v(T_n - \beta I)^{\ell-1}, \end{aligned}$$

it follows from (5.4) that

$$\alpha_1 v(T_n - \beta I)^{\ell-1} = 0.$$

Since for every $Y \in \ell(\Gamma), r \in \mathbb{N}$ we have that

$$(P_nY)(T_n - \beta I)^r = P_n(Y(L - \beta I)^r),$$

part 3 of Proposition 3.1 tells us that $v(T_n - \beta I)^{\ell-1} \neq 0$. Hence $\alpha_1 = 0$. If we multiply each side of (5.4) by $(T_n - \beta I)^{\ell-2}$ we see that $\alpha_2 = 0$. Analogously $\alpha_3 = \ldots = \alpha_\ell = 0$ and therefore $v_1, \dots, v_\ell$ are linearly independent. This implies that we have a Jordan block of T_n associated to β of order at least ℓ. We can repeat this procedure for every Jordan block $B_1, \dots, B_k$ of ${A^{-1}}_{[s]}$ associated to β of respective orders $l_1 \ge l_2 \ge \ldots \ge l_k$. Let, for $1 \le j \le k$

$$\widetilde{Q}_{B_j}^{l_j}(x) = Y_j^{l_j}\Phi(x).$$

All we have to prove now is that

$$\begin{array}{cccc} P_nY_1^{l_1}, & (P_nY_1^{l_1})(T_n-\beta I), & \dots, & (P_nY_1^{l_1})(T_n-\beta I)^{l_1-1}, \\ & & \vdots & \\ P_nY_k^{l_k}, & (P_nY_k^{l_k})(T_n-\beta I), & \dots, & (P_nY_k^{l_k})(T_n-\beta I)^{l_k-1} \end{array}$$

are linearly independent. Let

$$\begin{array}{cccc} \alpha_1^1P_nY_1^{l_1}+ & \alpha_1^2(P_nY_1^{l_1})(T_n-\beta I)+ & \dots+ & \alpha_1^{l_1}(P_nY_1^{l_1})(T_n-\beta I)^{l_1-1}+ \\ & & \vdots & \\ \alpha_k^1P_nY_k^{l_k}+ & \alpha_k^2(P_nY_k^{l_k})(T_n-\beta I)+ & \dots+ & \alpha_k^{l_k}(P_nY_k^{l_k})(T_n-\beta I)^{l_k-1}=0 \end{array} \tag{5.5}$$

Let $B_1,\dots,B_t$ the Jordan blocks of order l_1. If we multiply each side of the previous equation by $(T_n-\beta I)^{l_1-1}$, we obtain

$$\sum_{i=1}^{t}\alpha_i^1(P_nY_i^{l_1})(T_n-\beta I)^{l_1-1}=0, \text{ i.e.}$$

$$P_n\left(\sum_{i=1}^{t}\alpha_i^1Y_i^{l_1}(L-\beta I)^{l_1-1}\right)=0.$$

Since $\sum_{i=1}^{t}\alpha_i^1Y_i^{l_1}(L-\beta I)^{l_1-1}\in \operatorname{Ker}(L-\beta I)$, part 3 of Proposition 3.1 implies that

$$\sum_{i=1}^{t}\alpha_i^1Y_i^{l_1}(L-\beta I)^{l_1-1}=0.$$

So, since by (5.3) and Proposition 4.2, $Y_1^{l_1}(L-\beta I)^{l_1-1},\dots,Y_t^{l_1}(L-\beta I)^{l_1-1}$ are linearly independent, it follows that $\alpha_1^1=\dots=\alpha_t^1=0$. Repeating a similar argument for every $l_j, 2\le j\le k$ we can see that every scalar of (5.5) is equal to zero. This completes the proof. □

Let us now recall (5.1), and notice that

$$\widetilde{Q}_B(A^{-1}x)-\beta\widetilde{Q}_B(x)=(B-\beta I)\widetilde{Q}_B(x).$$

Equivalently, if we recall the definition of D_A of the previous section, $D_A(f)(x)=f(A^{-1}x)$, we have

$$(D_A-\beta I)\widetilde{Q}_B(x)=(B-\beta I)\widetilde{Q}_B(x),$$

where the product on the left side is understood coordinatewise. Moreover, for $k\in\mathbb{N}$,

$$\begin{aligned}(B-\beta I)^k\widetilde{Q}_B(x)&=\sum_{i=0}^{k}\binom{k}{i}(-\beta)^{k-i}B^i\widetilde{Q}_B(x)\\ &=\sum_{i=0}^{k}\binom{k}{i}(-\beta)^{k-i}D_A^i\widetilde{Q}_B(x)\\ &=(D_A-\beta I)^k\widetilde{Q}_B(x).\end{aligned}$$

In particular, since $(B-\beta I)$ is nilpotent of order ℓ, we have

$$(D_A-\beta I)^\ell\widetilde{Q}_B(x)=(B-\beta I)^\ell\widetilde{Q}_B(x)=0.$$

Hence, all entries of $\widetilde{Q}_B(x)$ belong to $\mathcal{H}(A,\beta,\ell)$. We can repeat this argument for every Jordan block associated to β and every eigenvalue β of ${A^{-1}}_{[s]}$. It follows that each component of $\widetilde{Q}_s(x)$ belongs to $\mathcal{H}(A,\lambda,r)$ for some eigenvalue λ of ${A^{-1}}_{[s]}$, and some $r \in \mathbb{N}$. Since Q_s is an invertible matrix and the monomials x^α with $|\alpha| = s$ are linearly independent. It follows that $\widetilde{Q}_s^1(x), \dots, \widetilde{Q}_s^{d_s}(x)$ are linearly independent and all homogeneous polynomials $q(x) = q(x_1, \dots, x_n)$ with $deg(q) = s$, are a linear combination of $\widetilde{Q}_s^1(x), \dots, \widetilde{Q}_s^{d_s}(x)$.

We can now state the next theorem:

Theorem 5.1. *Assume that φ has accuracy κ and that $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent. If q is a homogeneous polynomial in $\mathbb{R}^d$ with $deg(q) < \kappa$, then $q \in \mathcal{H} = \bigoplus_{\lambda\in\Delta_n} \mathcal{H}_\lambda(\varphi)$, where Δ_n is the set of eigenvalues of T_n.*

Proof. Let $s < \kappa$, and let $\widetilde{Q}_s$ and $\widetilde{Q}_B$ be as before. Since φ has accuracy κ, and $s < \kappa$, all components of $\widetilde{Q}_B$ (in fact all components of $\widetilde{Q}_s$) are in $S(\varphi)$, and satisfy

$$\widetilde{Q}_B(A^{-1}x) = \begin{pmatrix} \beta & 0 & \dots & 0 & 0 \\ 1 & \beta & \dots & 0 & 0 \\ . & . & \dots & . & . \\ 0 & 0 & \dots & \beta & 0 \\ 0 & 0 & \dots & 1 & \beta \end{pmatrix} \widetilde{Q}_B(x). \tag{5.6}$$

If we denote by $\widetilde{Q}_B^1(x)$ the first coordinate of $\widetilde{Q}_B(x)$ we see that $\widetilde{Q}_B^1(x)$ is actually of class $\mathcal{H}(A,\beta,1)$. Hence, by Proposition 4.3, $\widetilde{Q}_B^1(x) = Y\Phi$, where $P_nY \in \mathrm{Ker}(T_n - \beta I)$. This means that β is also an eigenvalue of T_n and the theorem follows. $\square$

The following corollary imposes conditions on the eigenvalues of T_n, under the hypothesis of accuracy.

Corollary 5.1. *Assume that φ has accuracy κ and that $\{\varphi(\cdot - k)\}_{k\in\Gamma}$ are linearly independent. Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of A (counted with multiplicity). If $\eta = (\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_d})$, then $[\eta^\alpha]_{|\alpha|=s}$ are eigenvalues of T_n, for $s = 0, 1, \dots, \kappa - 1$.*

Proof. Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of A. By [CHM98], $[\lambda^\alpha]_{|\alpha|=s}$ are the eigenvalues of $A_{[s]}$. Also, recall that since A is invertible, $A_{[s]}$ is also invertible and $(A_{[s]})^{-1} = {A^{-1}}_{[s]}$. So the eigenvalues of ${A^{-1}}_{[s]}$ are $[\eta^\alpha]_{|\alpha|=s}$. We have already proved that if φ has accuracy κ and $s < \kappa$, then every eigenvalue of ${A^{-1}}_{[s]}$ is also an eigenvalue of T_n. So the result follows. $\square$

6. 1-dimensional examples

We conclude the paper by exhibiting two examples of (λ, k)-homogeneous functions, associated to scaling functions in dimension 1. The higher dimensional examples can be obtained in the same way.

6.1. **Daubechies $\mathbf{D_4}$.** Daubechies wavelets, are those refinable functions of N coefficients, that are orthogonal and provide the highest order of accuracy possible. (Note that the splines do not form an orthonormal base).

d_4 is the refinable function that satisfies the refinement equation of 4 coefficients:

$$d_4(x) = \tfrac{1+\sqrt{3}}{4}(d_4(2x) + \tfrac{3+\sqrt{3}}{4}d_4(2x-1) + \tfrac{3-\sqrt{3}}{4}d_4(2x-2) + \tfrac{1-\sqrt{3}}{4}d_4(2x-3). \tag{6.1}$$

d_4 has accuracy 2 (it reproduces the constant and the linear functions).

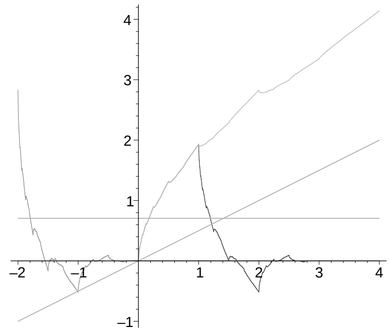

FIGURE 1. Daubechies D4 with the homogeneous functions.

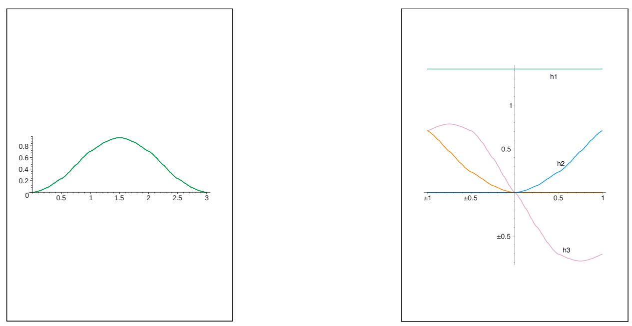

FIGURE 2. Scaling function for coefficients $1/3, 2/3, 2/3, 1/3$ with Homogeneous functions - h_1, h_2, h_3. h_3 is a 2-homogeneous function

In this case the matrix T has eigenvalues 1, $\frac{1}{2}$ and $c_0 = \frac{1+\sqrt{3}}{4}$. So a basis for $\text{span}\{d_4(x), d_4(x-1), d_4(x-2)\}_{x\in[0,1]}$ is also given by $\text{span}\{1, x, h_{c_0}(x)\}_{x\in[0,1]}$ where h_{c_0} is the homogeneous function associated to c_0.

6.2. $(\lambda, 2)$**-Homogeneous function.** In the previous example, we obtained a local basis of $\text{span}\{f(x), f(x-1), f(x-2)\}_{x\in[0,1]}$ just by using 1-homogeneous functions. The following example is to illustrate, that even in the simple case of only 4 coefficients, it may be necessary to use homogeneous functions of order bigger than 1. Consider the function:

$$f(x) = \frac{1}{3}f(2x) + \frac{2}{3}f(2x-1) + \frac{2}{3}f(2x-2) + \frac{1}{3}f(2x-3). \tag{6.2}$$

It can be shown that f has accuracy 1, and the eigenvalues of T are $\{1, \frac{1}{3}\}$. So in this case, $\text{span}\{f(x), f(x-1), f(x-2)\}_{x\in[0,1]} = \text{span}\{1, h_{\{1/3,1\}}(x), h_{\{1/3,2\}}(x)\}_{x\in[0,1]}$, where $h_{\{1/3,1\}}$ is a 1-homogeneous function corresponding to the eigenvalue $1/3$, and $h_{\{1/3,2\}}$ is a 2-homogeneous function corresponding to the eigenvalue $1/3$ (see Figure 2).

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