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LOCAL BASES FOR REFINABLE SPACES

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ABSTRACT. We provide a new representation of a refinable shift invariant space with a compactly supported generator, in terms of functions with a special property of homogeneity. In particular these functions include all the homogeneous polynomials that are reproducible by the generator, what links this representation to the accuracy of the space. We completely characterize the class of homogeneous functions in the space and show that they can reproduce the generator. As a result we conclude that the homogeneous functions can be constructed from the vectors associated to the spectrum of the scale matrix (a finite square matrix with entries from the mask of the generator). Furthermore, we prove that the kernel of the transition operator has the same dimension than the kernel of this finite matrix. This relation provides an easy test for the linear independence of the integer translates of the generator. This could be potentially useful in applications to approximation theory, wavelet theory and sampling.

1. INTRODUCTION

A function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ is called *refinable* if it satisfies the equation:

$$\varphi(x) = \sum_{k=0}^{N} c_k \varphi(2x - k), \tag{1.1}$$

for some complex scalars $c_0, ..., c_N$. The scalars c_k are the *mask* of the refinable function. We consider the case in which φ is compactly supported. Define the *Shift Invariant Space* (SIS) generated by φ as:

$$\mathcal{S}(\varphi) = \{f : \mathbb{R} \longrightarrow \mathbb{C} : f(x) = \sum_{k \in \mathbb{Z}} y_k \varphi(x + k), y_k \in \mathbb{C}\}.$$

A refinable SIS is a SIS with a refinable generator. Refinable SIS and refinable generators have been studied extensively, since they are very important in Approximation Theory and Wavelet Theory.

Many properties of φ can be obtained imposing conditions on the mask. One fundamental question is when the space $S(\varphi)$ contains polynomials and of which degree. The *accuracy* of φ is the maximum integer n such that all the polynomials of degree less or equal than $n-1$ are contained in $S(\varphi)$.

The accuracy is related to the approximation order of $S(\varphi)$, ([Jia95], [dB90] and references therein), and with the zero moments and the smoothness of the associated wavelet when φ generates a Multiresolution Analysis [Mey92]. There are many well known equivalent conditions for accuracy. The one that interests us here is the following, [Dau92], [CHM98]:

Proposition 1. *Let φ be a compactly supported function satisfying* (1.1). *Then the following statements are equivalent*

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(1) *The function* φ *has accuracy* n.
(2) *The numbers* $\{1, 2^{-1}, ..., 2^{-(n-1)}\}$ *are eigenvalues of the* $(N+1) \times (N+1)$ *matrix* T *defined by* $T = \{c_{2i-j}\}_{i,j=0,\dots,N}$*, (the* scale matrix*)* **and** *there exist polynomials* $p_0, ..., p_{n-1}$ *with degree*$(p_i) = i$ *such that the vector* $v_i = \{p_i(k)\}_{k=0,\dots,N}$ *is a left eigenvector of* T *corresponding to the eigenvalue* 2^{-i}, $i = 0, ..., n-1$.

Here, and always throughout the paper, we assume $c_t = 0$, if $t \neq 0, \dots, N$. One interesting property is that if φ has accuracy n, then for $s = 0, 1, ..., n-1$ it is true that $x^s = \sum_{k \in \mathbb{Z}} p_s(k)\varphi(x-k)$,where p_s is the polynomial that provides the eigenvector for the eigenvalue 2^{-s}. So, if the polynomial x^s is in $S(\varphi)$, then the left eigenvector of T, corresponding to the eigenvalue 2^{-s}, provides the coefficients needed to write x^s as a linear combination of the translates of φ.

A *local basis* for $S(\varphi)$ is a set of functions in $S(\varphi)$ whose restriction to the $[0,1]$-interval form a basis for the space of all the functions in $S(\varphi)$, restricted to the $[0,1]$-interval.

In the case that φ is the B-spline of order m (so $N = m$), then all polynomials of degree less or equal than $m-1$ are in $S(\varphi)$. Moreover, the set $\{1, x, x^2, \dots, x^{m-1}\}$ is a local basis for $S(\varphi)$, and the spectrum of T consists exactly of the numbers $\{1, 2^{-1}, \dots, 2^{-(m-1)}\}$.

Now, if φ is not a B-spline then T could have some eigenvalue λ different from a power of $1/2$. If powers of $1/2$ are associated to homogeneous polynomials, which functions in $S(\varphi)$ are associated to an arbitrary eigenvalue λ? Will the functions, associated to all the eigenvalues, provide also local bases of $S(\varphi)$, or equivalently, will these functions reproduce the generator φ? If φ has accuracy $p < N$, then no polynomials of degree bigger or equal than p will be in $S(\varphi)$. Will the extra eigenvalues of T tell us something about the order of approximation of $S(\varphi)$?

In the case that λ is a simple eigenvalue, Blu and Unser [BU02] and later Zhou [Zho02] showed that λ is associated to what they call a 2-scale λ-homogeneous function, that is a function in the SIS that satisfies the homogeneity relation $h(x) = \lambda h(2x)$.

However, to obtain a complete representation of the space it is necessary to consider the whole spectrum of T. This motivates the study of the spectral properties of T for a *general* refinable φ.

This is achieved in this paper: we are able to completely characterize the SIS in terms of functions associated to the spectrum of T. We prove that these functions provide a local basis of $S(\varphi)$. The advantage of this local basis is that it contains all possible homogeneous polynomials in the space, and those functions in the basis which are not polynomials, still preserve some kind of homogeneity. Furthermore this basis can be easily obtained from the spectrum of the finite matrix T. We are also able to proof that this matrix is necessarily invertible if the translates of φ are linearly independent.

Definition 1. Let $\lambda \in \mathbb{C}, \lambda \neq 0$ and $r \geq 1$ an integer. A function h is (λ, r)-*homogeneous* if it satisfies the following equation:

$$\sum_{k=0}^{r} \binom{r}{k} (-\lambda)^{r-k} h(2^{-k}x) = 0 \text{ a.e..} \tag{1.2}$$

For $r = 1$ these functions are called *two-scale homogeneous* and they satisfy $h(x) = \frac{1}{\lambda} h(\frac{x}{2})$.

If $\mathcal{H} \subset S(\varphi)$ is the span of all the (λ, r)-homogeneous functions in $S(\varphi)$, for any $\lambda \in \mathbb{C}$ and any positive integer r, we show that under the hypothesis of linear independence of the translates of φ, $\dim(\mathcal{H}) = N + 1$, and that there is a basis of $\mathcal{H}$, corresponding to the spectrum of T. More precisely, given a basis $\mathcal{B} = \{v_0, \dots, v_N\}$ of $\mathbb{C}^{N+1}$ that yields the Jordan form of T we associate to each vector $v \in \mathcal{B}$ a unique (λ, r)-homogeneous function in $S(\varphi)$, where λ and r, satisfy $v(T - \lambda I)^r = 0$.

The first N of these functions are a local basis of functions in $S(\varphi)$ restricted to $[0,1]$. This allows to reconstruct the generator φ from the homogeneous functions, and gives a new representation for the functions in $S(\varphi)$.

As a corollary of these results we obtain that if the integer translates of φ are linearly independent, then the subdivision operator S_c (see (2.2)) is one to one.

Furthermore, we show that to each non-zero vector in the kernel of T, there corresponds a non-trivial linear combination of the integer translates of φ yielding the zero function.

This paper is organized as follows: We first introduce in Section 2 some notation and some necessary tools. In Section 3 we show that each non-zero left eigenvector of T can be extended to an infinite left-eigensequence of S_c, and every vector in a basis for the Jordan form of T can be extended to a sequence that satisfies the same relation but for S_c. Using the theory of difference equations, we then show that the dimension of the Kernel of T is the same than the one of S_c and finally, we introduce the (λ, r)-homogeneous functions that completely characterize the shift invariant space.

2. Notation

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a function supported in $[0, N]$ satisfying (1.1).

We will often use an infinite column vector associated to φ, namely

$$(\phi(x))^t = [\ldots,\ \varphi(x-1),\ \varphi(x),\ \varphi(x+1),\ \ldots\ ]. \tag{2.1}$$

Let $\ell(\mathbb{Z})$ be the space of all the sequences defined in $\mathbb{Z}$. We say that the integer translates of φ are *globally linearly independent*, or *linearly independent* if $\sum_{k\in\mathbb{Z}} \alpha_k \varphi(\cdot - k) \equiv 0 \quad \Longrightarrow \quad \alpha_k = 0\ \forall k$, for any sequence $\alpha \in \ell(\mathbb{Z})$.

The *subdivision operator* associated to the mask c_k is the operator

$$S_c : \ell(\mathbb{Z}) \to \ell(\mathbb{Z}) \quad \text{defined by} \quad S_c(\alpha)_j = \sum_{i\in\mathbb{Z}} \alpha_i c_{2i-j}. \tag{2.2}$$

Note. The subdivision operator is sometimes defined in a different but equivalent way as $\tilde{S}_c(\alpha)_j = \sum_{i\in\mathbb{Z}} \alpha_i c_{j-2i}$. If $h : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ is the operator $h(\alpha)_k = \alpha_{-k}$, then $h\tilde{S}_c h = S_c$ and therefore S_c and $\tilde{S}_c$ share most of the properties. For a nice account of properties of the subdivision operator see [BJ02]

If $L = L_\varphi$ is the double infinite matrix $L = [c_{2i-j}]_{i,j\in\mathbb{Z}}$, then the refinement equation can be written as $\phi(x) = L\phi(2x)$.

Using the matrix L, the subdivision operator (2.2) can be recast as: $S_c\alpha = \alpha L, \quad \alpha \in \ell(\mathbb{Z})$, where α on the right hand side of the equation is thought as an infinite row vector. Note that the scaling matrix T defined in Proposition 1 is a finite submatrix of L. We will consider in our analysis the matrices M, T_0, T_1, that are submatrices of T and are defined as: $M = [c_{2i-j}]_{i,j=1,\ldots,N-1}, \quad T_0 = [c_{2i-j}]_{i,j=0,\ldots,N-1}, \quad T_1 = [c_{2i-j}]_{i,j=1,\ldots,N}$. That is,

$$T = \begin{bmatrix} c_0 & 0 & \ldots\ldots\ldots & & & 0 \\ c_2 & c_1 & c_0 & 0 & \ldots & 0 \\ \vdots & c_3 & c_2 & c_1 & \ldots & \vdots \\ 0 & \vdots & \vdots & \vdots & \vdots & c_{N-2} \\ 0 & \ldots\ldots\ldots & & & 0 & c_N \end{bmatrix} = \left[\begin{array}{c|c|c} c_0 & 0 & 0 \\ \hline \vdots & M & \vdots \\ \hline 0 & 0 & c_N \end{array}\right]. \tag{2.3}$$

Note that c_0 and c_N must be non-zero, since $\mathrm{Supp}(\varphi) = [0, N]$.

Now, if $Y \in \ell(\mathbb{Z})$, define Y^0 and Y^M as the restriction of Y to the indexes $\{0, \ldots, N\}$, and $\{1, \ldots, N-1\}$, respectively, i.e., $Y^0 = (Y_0, \ldots, Y_N)$,
$Y^M = (Y_1, \ldots, Y_{N-1})$.

Note. Throughout this paper, $(L-\lambda I)$ is considered as an operator on $\ell(\mathbb{Z})$, defined by left-multiplication, (i.e. $Y \longmapsto Y(L-\lambda I)$, where Y is a double infinite row vector). I is the identity operator acting on $\ell(\mathbb{Z})$. By an abuse of notation, we will use the notation I for all identity operators, without distinguishing the space they are acting on. Note also that properties of the matrix L translate directly into properties of the subdivision operator S_c.

3. The point spectrum of L

The following proposition, will show, how the spectral properties of L are related to those of T. The case $r = 1$ has been studied earlier by [CHM00], [JRZ98], [Zho01, Zho02].

Proposition 2. *Let $\lambda \in \mathbb{C}$.*

(1) *Let $Y \in \ell(\mathbb{Z})$ and $r \in \mathbb{N}$, $r \geq 1$.*

$$\text{If} \quad Y \in \operatorname{Ker}(L-\lambda I)^r \quad \text{then} \quad Y^0 \in \operatorname{Ker}(T-\lambda I)^r. \tag{3.1}$$

Moreover, if $\lambda \neq 0$, $Y \neq 0$ and $Y \in \operatorname{Ker}(L-\lambda I)^r$, then $Y^0 \neq 0$.

(2) *If $v \in \operatorname{Ker}(T-\lambda I)^r$ and $\lambda \neq 0$, then there exists an extension $Y_v \in \ell(\mathbb{Z})$ of v, (i.e. $Y_v^0 = v$) such that $Y_v \in \operatorname{Ker}(L-\lambda I)^r$.*

Proof. The matrix L can be decomposed in blocks as

$$L = \left[\begin{array}{c|c|c} R & 0 & 0 \\ \hline P & T & Q \\ \hline 0 & 0 & S \end{array}\right], \tag{3.2}$$

where we decompose $\mathbb{Z}$ as

$$\mathbb{Z} = A^- \cup \quad A^0 \cup \quad A^+, \tag{3.3}$$

with $A^- = \mathbb{Z} \cap (-\infty, -1]$, $A^0 = \mathbb{Z} \cap [0, N]$ and $A^+ = \mathbb{Z} \cap [N+1, +\infty)$, and

$$R = L|_{A^- \times A^-} \quad P = L|_{A^0 \times A^-} \quad T = L|_{A^0 \times A^0} \quad Q = L|_{A^0 \times A^+} \quad S = L|_{A^+ \times A^+}.$$

This block form of the matrix, is closed under multiplication. So if $r \geq 1, r \in \mathbb{N}$

$$L^r = \left[\begin{array}{c|c|c} R^r & 0 & 0 \\ \hline P_r & T^r & Q_r \\ \hline 0 & 0 & S^r \end{array}\right],$$

where

$$P_r = \sum_{k=0}^{r-1} T^k P R^{r-k-1} \quad \text{and} \quad Q_r = \sum_{k=0}^{r-1} T^k Q S^{r-k-1}. \tag{3.4}$$

Since

$$(L-\lambda I) = \left[\begin{array}{c|c|c} R-\lambda I & 0 & 0 \\ \hline P & T-\lambda I & Q \\ \hline 0 & 0 & S-\lambda I \end{array}\right],$$

$$(L-\lambda I)^r = \left[\begin{array}{c|c|c} (R-\lambda I)^r & 0 & 0 \\ \hline P_r^\lambda & (T-\lambda I)^r & Q_r^\lambda \\ \hline 0 & 0 & (S-\lambda I)^r \end{array}\right],$$

where P_r^λ and Q_r^λ are as in (3.4) with the obvious changes.

Note that the matrix S is upper triangular, with diagonal $(0, 0, 0, \dots)$, and hence $(S - \lambda I)^r$ is upper triangular, with diagonal $((-\lambda)^r, (-\lambda)^r, (-\lambda^r), \dots)$.

Analogously, we observe that R is lower triangular with zeroes in the main diagonal, so $(R - \lambda I)^r$ is lower triangular with diagonal $((-\lambda)^r, (-\lambda)^r, (-\lambda)^r, \dots)$.

If $Y = (Y^-, Y^0, Y^+)$, then

$$\begin{aligned} Y(L-\lambda I)^r &= \\ &= (Y^-(R-\lambda I)^r + Y^0 P_r^\lambda, Y^0(T-\lambda I)^r, Y^0 Q_r^\lambda + Y^+(S-\lambda I)^r). \end{aligned} \tag{3.5}$$

So if $Y \in \operatorname{Ker}(L-\lambda I)^r$, then $Y^0 \in \operatorname{Ker}(T-\lambda I)^r$.

We now want to show that if $Y \in \operatorname{Ker}(L-\lambda I)^r$, $\lambda \neq 0$, $Y \neq 0$, then $Y^0 \neq 0$.

For this, let $k_0 \in \mathbb{Z}$ be such that $Y_{k_0} \neq 0$. If $0 \leq k_0 \leq N$, we are done. Assume that $k_0 > N$. Then, since $Y(L-\lambda I)^r = 0$, in particular, the k_0 element of this product is 0. But since $\lambda \neq 0$, $(S-\lambda I)^r$ is upper triangular with $(-\lambda)^r$ in the diagonal, therefore the only nonzero elements of column k_0 of $(L-\lambda I)^r$ are between 0 and k_0. Hence there has to be a $k_1, 0 \leq k_1 < k_0$ such that $Y_{k_1} \neq 0$. Again, if $0 \leq k_1 \leq N$ we are done, otherwise we repeat the argument until k_j is in the desired interval.

If $k_0 < 0$, the argument works in the same way, reversing the role of $(S-\lambda I)^r$ and $(R-\lambda I)^r$.

For the proof of part 2, assume that $v \in \mathbb{C}^{N+1}$, $v \in \operatorname{Ker}(T-\lambda I)^r$. We want to find an infinite vector $Y \in \ell(\mathbb{Z})$, such that $Y^0 = v$ and $Y \in \operatorname{Ker}(L-\lambda I)^r$. From equation (3.5) we know that if $Y \in \ell(\mathbb{Z})$,

$$\begin{aligned} [Y(L-\lambda I)^r]^+ &= Y^0 Q_r^\lambda + Y^+(S-\lambda I)^r, \quad \text{and} \\ [Y(L-\lambda I)^r]^- &= Y^0 P_r^\lambda + Y^-(R-\lambda I)^r. \end{aligned}$$

Therefore, if $Y \in \operatorname{Ker}(L-\lambda I)^r$, and $Y^0 = v$, then Y^+ and Y^- have to satisfy

$$Y^+(S-\lambda I)^r = -vQ_r^\lambda \quad \text{and} \quad Y^-(R-\lambda I)^r = -vP_r^\lambda. \tag{3.6}$$

Using again that $(S-\lambda I)^r$ and $(R-\lambda I)^r$, are triangular, if $\lambda \neq 0$, there are unique solutions for Y^+ and Y^- and they can be obtained recursively.

This completes the proof. □

As a consequence of the preceding Proposition, looking at the structure of the eigenvectors of the operator L, we have the following Corollary.

Corollary 1. *The operator L does not have eigenvectors in ℓ^p, $1 \leq p \leq +\infty$.*

The last proposition tells us that the elements of the spectrum of T are intimately related to those of L. But by the special form of T (see equation (2.3)), we can actually use the $(N-1) \times (N-1)$ matrix M to obtain the spectrum of T, as the following proposition shows:

Proposition 3. *Let $\lambda \neq 0 \in \mathbb{C}$.*

1. *Let $v^0 = (v_0, \dots, v_N) \in \mathbb{C}^{N+1}$ and $r \in \mathbb{N}$, $r \geq 1$, then*
$$\textit{if} \quad v^0 \in \operatorname{Ker}(T-\lambda I)^r \quad \textit{then} \quad v^M = (v_1, \dots, v_{N-1}) \in \operatorname{Ker}(M-\lambda I)^r.$$
Moreover, if $\lambda \neq c_0$, $\lambda \neq c_N$, $v^0 \in \operatorname{Ker}(T-\lambda I)^r$, and $v^0 \neq 0$, then $v^M \neq 0$.
2. *if $v^M = (v_1, \dots, v_{N-1}) \in \operatorname{Ker}(M-\lambda I)^r$ and $\lambda \neq c_0$ and $\lambda \neq c_N$, then there exists an extension $v^0 \in \mathbb{C}^{N+1}$ of v, such that $v^0 \in \operatorname{Ker}(T-\lambda I)^r$.*

Proof. The proof is immediate by noting the special block-form of T given in equation (2.3). □

4. The kernel of L

The case $\lambda = 0$ could not be handled with the methods of Proposition 2, since the matrices R and S in (3.2) have zeros in the main diagonal. Instead, we need some results from the theory of difference equations which we present below [Hen62].

4.1. **Difference Equations.** Consider the linear difference equation with constant coefficients of order r

$$u_0 y_n + u_1 y_{n+1} + \cdots + u_r y_{n+r} = 0 \quad y = \{y_n\}_{n \in \mathbb{Z}}, \tag{4.1}$$

where $u_k \in \mathbb{C}, u_0 \neq 0, u_r \neq 0$ with characteristic polynomial $P(x) = \sum_{k=0}^r u_k x^k$.

A *solution* to the equation (4.1) is a sequence Y in $\ell(\mathbb{Z})$, that satisfies (4.1) **for all** $k \in \mathbb{Z}$. A vector $y = (y_0, \dots, y_m)$ with $m \geq r+1$ is a *finite solution* of (4.1), if it satisfies (4.1) for $n = 0$ to $n = m-r-1$.

The *space of solutions* $S \subset \ell(\mathbb{Z})$, has dimension r, and a basis of this space (the fundamental basis) can be written in the following way:

Let $h \geq 1$ be an integer, $d_1, \dots, d_h$ arbitrary non-zero complex numbers with $d_i \neq d_j$ if $i \neq j$. Let $r_1, \dots, r_h$ be positive integers. To each pair (d_i, r_i), $i = 1, \dots, h$ we will associate a sequence $a_i = \{a_{ik}\}_{k \in \mathbb{Z}}$ defined as follows: Set $r = r_1 + \cdots + r_h$ and $r_0 = 0$.Let $0 \leq i \leq r-1$ and $s = s(i), j = j(i)$ be the unique integers that satisfy

$$r_0 + \cdots + r_{s-1} \leq i < r_1 + \cdots + r_s, \quad j(i) = i - \sum_{k=0}^{s(i)-1} r_k.$$

Define

$$a_{ik} = \begin{cases} \frac{|k|! \mathrm{sg}(k)}{(|k|-j(i))!} d_{s(i)}^k & \text{for } |k| \geq j(i) \\ 0 & |k| < j(i). \end{cases} \quad i = 0, \dots, r-1, \quad k \in \mathbb{Z}. \tag{4.2}$$

So, if P is the characteristic polynomial associated to equation (4.1), consider the pairs $\{(d_i, r_i) :$ where d_i is a root of P and r_i its multiplicity$\}$. The sequences $\{a_{ik}\}_{k \in \mathbb{Z}}$, $i = 0, \dots, r-1$ form a basis of S, the subspace of $\ell(\mathbb{Z})$ of the solutions to (4.1).

It is also known from the theory of difference equations, that every solutions is determined unequivocally by any r consecutive elements of it. Hence, if y is a solution such that r consecutive elements are 0, then y is the zero solution.

We will now associate to the pairs $\{(d_i, r_i) : i = 1, \dots, h\}$ the $r \times r$ matrix $A = [a_{ij}]_{i,j=0,\dots,r-1}$. Then (cf. Henrici [Hen62], pg. 214)

$$\det(A) = \prod_{1 \leq l < s \leq h} (d_l - d_s)^{r_l + r_s} \prod_{i=1}^{h} (r_i - 1)!!, \tag{4.3}$$

where $0!! = 1$ and $k!! = k!(k-1)! \dots 1!$.

Since $d_i \neq d_j$ for $i \neq j$, $\det(A) \neq 0$ and A is invertible.

Let us now consider a system of k linear difference equations with constant coefficients of order r.

$$u_{i0} y_n + \cdots + u_{ir} y_{n+r} = 0, \quad i = 1, \dots k, \quad n \in \mathbb{Z}, \tag{4.4}$$

and let P_i be the characteristic polynomial of equation i, $P_i(x) = \sum_{j=0}^{r} u_{ij} x^j$. Define

$$\mathcal{D} = \cup_{i=1}^{k} \{d : P_i(d) = 0\} = \{d_1, \dots, d_s\}, \tag{4.5}$$

and for each $d \in \mathcal{D}$ define

$$r_d = max\{r_i : r_i \text{ is the multiplicity of } d \text{ in } P_i\}. \tag{4.6}$$

Note that $r_d \geq 1 \; \forall d \in \mathcal{D}$. We then have the pairs $(d_i, r_{d_i}) = (d_i, r_i)$. Define the index of the system to be $t = \sum_{d \in \mathcal{D}} r_d \leq kr$. Let ℓ be the degree of the maximum common divisor p, of $\{P_a, \dots, P_k\}$. Hence, $P_i(x) = p(x) \tilde{P}_i(x)$, with degree $\tilde{P}_i = r - \ell$. (Note that ℓ could be 0). With the above notation, we have the following proposition (probably known):

Proposition 4. *The space S_k of solutions to the system* (4.4) *has dimension ℓ, where ℓ is the degree of the maximum common divisor of the characteristic polynomials.*

Proof.
Let p be the maximum common divisor of $P_1, \dots, P_k$, and let $\ell = \deg(p)$. It is clear, that $\dim(S_k) \geq \ell$.

For the other inequality, consider the $t \times t$ matrix $A = [a_{ij}]_{i,j=0,\dots,t-1}$, with a_{ij} defined in (4.2) for the pairs $\{(d_i, r_i)\}$ defined above and t being the index of the system (4.4). Since $d_i \neq d_j$ for $i \neq j$, $\det(A) \neq 0$ by (4.3) and A is invertible.

Assume now that $y \in S_k$, then y is a solution to all k difference equations, hence there exist $\alpha^1, \dots, \alpha^k$ vectors of length r, such that

$$A_i \alpha^i = [y_0, \dots, y_{t-1}]^t \quad 1 \leq i \leq k, \tag{4.7}$$

where A^i is an $t \times r$ matrix whose columns are a fundamental system for equation i. Note that A^i is a sub-matrix of A, whose columns correspond to some columns $\{i_1, \ldots, i_r\}$ of A.

Let now $\tilde{\alpha}^i$ be vectors of length t, such that $\tilde{\alpha}^i_h = 0$ whenever $h \notin \{i_1, \ldots, i_r\}$ and $\tilde{\alpha}^i_{i_s} = \alpha_s$, $s = 1, \ldots, r$. Then we have for $i, j = 1, \ldots, k$

$$A\tilde{\alpha}^i = A_i \alpha^i = [y_0, \ldots, y_{t-1}]^t = A_j \alpha^j = A\tilde{\alpha}^j, \tag{4.8}$$

and hence $A(\tilde{\alpha}^i - \tilde{\alpha}^j) = 0$, for all $i \neq j$ and therefore, by the invertibility of A, $\tilde{\alpha}^i = \tilde{\alpha}^j$, for all $i \neq j$. Therefore the only non-zero elements of α_i can be those corresponding to the columns associated to the roots of p. Hence y is a linear combination of ℓ columns, and therefore $\dim(S_k) \leq \ell$. □

By noting that for the previous proof, we only used the first t coordinates of the infinite sequences, we have the following immediate Corollary.

Corollary 2. *If z is a vector of length t that satisfies* (4.4)*, then it can be extended to a sequence $y_z = \{y_j\}_{j \in \mathbb{Z}}$ solution of* (4.4) *and such that $y_j = z_j$, $j = 1, \ldots, t$.*

4.2. **The** $\mathrm{Ker}(L)$**.** We can now return to our double infinite matrix L and look at the special case $\lambda = 0$. As it turns out, the kernel of L is characterized by the vectors in the kernel of M. Since c_0 and c_N are non-zero, the matrices T and M have kernels of the same dimension.

Moreover, we have the following Proposition:

Proposition 5. *Consider the polynomials p_e and p_o of degree $q = \frac{N-1}{2}$ (we assume N to be odd)*

$$p_e(x) = c_0 + c_2 x + \cdots + c_{2q} x^q, \quad p_o(x) = c_1 + c_3 x + \cdots + c_{2q+1} x^q. \tag{4.9}$$

Then

$$\dim(\mathrm{Ker}(L)) = \dim(\mathrm{Ker}(M)) = \mathrm{degree}(p), \tag{4.10}$$

where p is the maximum common divisor of the polynomials p_e and p_o. In particular, if $\dim(\mathrm{Ker}(M)) >$ 0*, p_e and p_o have a common root.*

Furthermore

1. *For every $Y \in \mathrm{Ker}(L)$, $Y \neq 0$, we have $Y^M \neq 0$ and $Y^M \in \mathrm{Ker}(M)$.*
2. *Conversely, for each $v \in \mathrm{Ker}(M), v \neq 0$, we have $Y_v \neq 0$ and $Y_v \in \mathrm{Ker}(L)$.*

Proof. Let us observe first, that $Y \in \ell(\mathbb{Z})$ is in the Kernel of L, if and only if Y satisfies the system of difference equations:

$$\begin{cases} c_0 v_n + c_2 v_{n+1} + \cdots + c_{2q} v_{n+q} & = 0 \\ c_1 v_n + c_3 v_{n+1} + \cdots + c_{2q+1} v_{n+q} & = 0. \end{cases} \tag{4.11}$$

Therefore, by Proposition 4, $\mathrm{Ker}(L)$ is the subspace generated by the fundamental solutions associated to the roots of p, the maximum common divisor of p_o and p_e. This shows that $\dim(\mathrm{Ker}(L)) = \mathrm{degree}(p)$.

On the other side, if $Y \in \mathrm{Ker}(L)$, since $(YL)^M = Y^M M$ we conclude that $Y^M \in \mathrm{Ker}(M)$, and if Y^M is the zero vector, then the solution Y of (4.11) has $N-1$ consecutive zeros, so $Y = 0$. Hence, if $Y \neq 0$, then $Y^M \neq 0$, which proves (1).

To see that if $v = (v_1, \ldots, v_{N-1})$ satisfies $vM = 0$, then v can be extended, just note that the sequence $v_1, \ldots, v_{N-1}$ must satisfy the difference equations system of order $q = \frac{N-1}{2}$ (we assumed N to be odd) given by (4.11). Since the index t of the system 4.11 satisfies $t \leq 2q = N-1$, and v is a non-trivial common solution of length $N-1$, by Corollary 4, this solution can be extended in such a way that the extension satisfies both difference equations. This proves (2).

From (1) and (2) it is immediate that $\dim(\mathrm{Ker}(L)) = \dim(\mathrm{Ker}(M))$. □

4.3. **Invertibility of L.** Propositions 2 and 3, relate the spectral properties of the matrix M to the ones of the operator L. The next proposition shows a necessary condition for the independence of the integer translates of the function φ, in terms of the matrix M.

Proposition 6. *With the above notation, consider the following properties*

(1) $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ *are globally linearly independent,*
(2) *The operator* $L : \ell(\mathbb{Z}) \longrightarrow \ell(\mathbb{Z}), Y \longmapsto YL$ *is one-to-one,*
(3) *The matrix* M *is invertible.*

Then (2 $\iff$ 3) and (1 $\implies$ 2).

Proof. **(1 $\implies$ 2)**
Assume $YL = 0$. Define $F(x) = Y\phi(x)$. Then we have

$$F(x) = Y\phi(x) = YL\phi(2x) = 0.$$

Now, $Y\phi(x) = 0 \implies Y = 0$, therefore $\mathrm{Ker}(L) = \{0\}$.
(2 $\iff$ 3)
Proposition 4.11. □

Note: We do not know if either (2) or (3) implies (1).

5. Homogeneous functions

Assume now that $Y \in \mathrm{Ker}(L - \lambda I)^r$, and define the function $h \in S(\varphi)$ as $h(x) = Y\phi(x)$. So, h satisfies:

$$0 = Y(L-\lambda I)^r\phi(x) = Y\left(\sum_{k=0}^{r}\binom{r}{k}(-\lambda)^k L^{r-k}\right)\phi(x)$$
$$= Y\left(\sum_{k=0}^{r}\binom{r}{k}(-\lambda)^k\phi(\frac{x}{2^{r-k}})\right) = \sum_{k=0}^{r}\binom{r}{k}(-\lambda)^k h(\frac{x}{2^{r-k}}).$$

So, if D_a is the operator defined by $(D_a f)(x) = f(ax)$, we have that h satisfies

$$(D_{1/2} - \lambda I)^r h = 0. \tag{5.1}$$

We will say that a function h is $(2, \lambda, r)$ homogeneous, if h satisfies (5.1) (r is the order of homogeneity, and λ is the degree), and we will denote by $\mathcal{H}(2, \lambda, r)$, the space of all $(2, \lambda, r)$ homogeneous functions.

Remark (1). Note that if

$$h \in \mathcal{H}(2,\lambda,r) \quad \text{then} \quad h \in \mathcal{H}(2,\lambda,s) \quad \text{for every} \quad s \geq r.$$

Therefore the "order of homogeneity" will be defined by $\min\{s : h \in \mathcal{H}(2,\lambda,s)\}$.

Remark (2). If h is homogeneous (of any order) and $\lambda \neq 1$, then $h(0) = 0$. The values of any homogeneous function of order r in $(0, +\infty)$, are completely determined by its values on any interval of the type $\left[\frac{1}{2^{k+r}}, \frac{1}{2^k}\right)$, $k \in \mathbb{Z}$. (Analogously, the values on $(-\infty, 0)$, are obtained from the values in any interval of the type $\left(-\frac{1}{2^k}, -\frac{1}{2^{k+r}}\right]$). To see this, note that after some algebraic manipulation, equation (1.2) is equivalent to

$$h(x) = -\sum_{j=1}^{r}\binom{r}{j}(-\lambda)^{-j}h(2^{-j}x) = -\sum_{j=1}^{r}\binom{r}{j}(-\lambda)^{j}h(2^{j}x). \tag{5.2}$$

So, if $x \in \left[\frac{1}{2^{k+r+1}}, \frac{1}{2^{k+r}}\right)$, then for $j = 1, \ldots, r$, $\;2^j x \in \left[\frac{1}{2^{k+r-j+1}}, \frac{1}{2^{k+r-j}}\right) \subset \left[\frac{1}{2^{k+r}}, \frac{1}{2^k}\right)$, and using (5.2), we determine $h(x)$. Iterating this procedure, we see that all values in the interval $\left(0, \frac{1}{2^{k+r}}\right)$ can be determined.

On the other hand, for $x \in \left[\frac{1}{2^k}, +\infty\right)$, we use (5.2), and observe that if $x \in \left[\frac{1}{2^k}, \frac{1}{2^{k-1}}\right)$, for $j = 1, \ldots, r$, $2^{-j}x \in \left[\frac{1}{2^{k+j}}, \frac{1}{2^{k+j-1}}\right) \subset \left[\frac{1}{2^{k+r}}, \frac{1}{2^k}\right)$.

Remark (3). In the case of order of homogeneity 1, (e.g. $r = 1$), h is a 2-scale homogeneous function as described in [Zho02].

Proposition 7. *Assume $\{\varphi(\cdot - k)\}$ are linearly independent. Let ϕ be as in (2.1). If $g_1, \ldots, g_n \in \mathcal{S}(\varphi)$, $g_i = Y^i\phi$, then $\{g_1, \ldots, g_n\}$ are linearly independent functions if and only if $\{Y^1, \ldots, Y^n\}$ are linearly independent in $\ell(\mathbb{Z})$.*

Proof. We observe that

$$\sum_{i=1}^{n} \alpha_i g_i = \sum_{i=1}^{n} \alpha_i \left(Y^i \phi\right) = \left(\sum_{i=1}^{n} \alpha_i Y^i\right) \phi.$$

This equation, together with the linear independence of the translates of φ, tells us that $\sum_i \alpha_i g_i \equiv 0$ if and only if $\left(\sum_i \alpha_i Y^i\right) = 0$, which proves the desired result. □

Theorem 1. *Assume that $\{\varphi(\cdot - k)\}$ are linearly independent. If $h = Y\phi$, ($h \in \mathcal{S}(\varphi)$), and $h \in \mathcal{H}(2, \lambda, r)$ then $v_h = Y^0 \in \operatorname{Ker}(T - \lambda I)^r$. Reciprocally, if $v \in \operatorname{Ker}(T - \lambda I)^r$, then the function $h = Y_v\phi$ is in $\mathcal{H}(2, \lambda, r)$. (Here Y_v is the unique extension of v to a vector in $\operatorname{Ker}(L - \lambda I)^r$ by Prop. 2.)*

Proof. For the first claim, note that

$$0 = \sum_{k=0}^{r} \binom{r}{k} (-\lambda)^{r-k} h(2^{-k}x) = \sum_{k=0}^{r} \binom{r}{k} (-\lambda)^{r-k} Y L^k \phi(x) = Y\left(L - \lambda I\right)^r \phi(x).$$

Then, $Y\left(L - \lambda I\right)^r = 0$ and by Proposition 2, $v_h \in \operatorname{Ker}(T - \lambda I)^r$.

For the converse first observe that if $v = 0$ the result is trivial. Assume $v \neq 0$ and $v \in \operatorname{Ker}(T - \lambda I)^r$, then by Proposition 6 $\lambda \neq 0$. Hence (by Prop. 2) there is a unique extension $Y_v \in \operatorname{Ker}(L - \lambda I)^r$, so $h = Y_v\phi$ is in $\mathcal{H}(2, \lambda, r)$. □

5.1. **Jordan decomposition of T.** Now, let Λ be the set of eigenvalues of T, and let us consider a basis $\mathcal{B} = \{v_0, \ldots, v_N\}$ of $\mathbb{C}^{N+1}$ that gives the Jordan form of T.

Remark. Note that we can choose both $v_0 = (1, 0, \ldots, 0)$ **and** $v_N = (0, \ldots, 0, 1)$ to be in the basis $\mathcal{B}$, corresponding to the eigenvalues c_0 and c_N respectively.

If $v_i \in \mathcal{B}$, $(0 \leq i \leq N)$, then $v_i \in \operatorname{Ker}(T - \lambda I)^k$ and $v_i \notin \operatorname{Ker}(T - \lambda I)^{k-1}$, for some $\lambda \in \Lambda$, and $k \geq 1$. So to each $v_i \in \mathcal{B}$, we can associate a unique pair (λ, k). Let us denote such $v_i = v(\lambda, k)$. (Note that by the previous observation, $v_0 = v(c_0, 1)$ and $v_N = v(c_N, 1)$).

After Theorem 1, we can associate to each $v(\lambda, k)$ a function $h_{v(\lambda,k)}$ in $\mathcal{H}(2, \lambda, k) \cap \mathcal{S}(\varphi)$. Furthermore, the functions $\{h_{v(\lambda,k)}\}_{v \in \mathcal{B}}$, are linearly independent.

For this, observe that since the vectors in $\mathcal{B}$ are linearly independent, its extensions $\{Y_v\}$ are linearly independent in $\ell(\mathbb{Z})$, and therefore the functions $\{h_{v(\lambda,k)}\}_{v \in \mathcal{B}}$ are linearly independent.

One can see that if a finite number of functions are homogeneous for the same λ, then a linear combination of them is also homogeneous for the same λ. More precisely,

$$\sum_{i=0}^{n} \alpha_i h_i(\lambda, k_i) = h(\lambda, k), \quad \text{where} \quad k = \max_i(k_i),$$

for

$$\left(D - \lambda I\right)^k h = \left(D - \lambda I\right)^k \sum_{i=0}^{n} \alpha_i h_i = \sum_{i=0}^{n} \alpha_i \left(D - \lambda I\right)^k h_i = 0.$$

If p_T, the characteristic polynomial of T, is factorized as: $p_T(x) = \prod_{\lambda \in \Lambda} (x - \lambda)^{r_\lambda}$,and we denote

$$\mathcal{H}_\lambda(\varphi) = \{h \in \mathcal{S}(\varphi) : h \in \mathcal{H}(2, \lambda, k), \text{ for some } k \geq 1 \}, \quad \lambda \in \Lambda,$$

then $\dim(\mathcal{H}_\lambda) = r_\lambda$ and a basis of $\mathcal{H}_\lambda$ is the set of (λ, k)-homogeneous functions associated to the vectors $v \in \mathcal{B}$, such that $v = v(\lambda, k)$, for some $k \geq 1$.

Now consider

$$\mathcal{H} = \bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda \subset \mathcal{S}(\varphi).$$

We have, $\dim(\mathcal{H}) = N + 1$.

The previous observations are synthesized in the following theorem.

Theorem 2. *With the notation above we have: The correspondence*

$$v(\lambda, k) \in \mathcal{B} \longmapsto h_{v(\lambda,k)} \in \mathcal{H},$$

extends linearly to a linear isomorphism between $\tau : C^{N+1} \longrightarrow \mathcal{H}$. On the other side, the extension

$$v(\lambda, k) \longmapsto Y_{v(\lambda,k)},$$

defined earlier, can also be extended to an isomorphism $C^{N+1} \longrightarrow W \subset \ell(\mathbb{Z})$. We therefore have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^{N+1} & \longleftrightarrow & \mathcal{H} \\ \updownarrow & \nearrow & \\ W & & \end{array}$$

Here W is the span of $\{Y_v : v \in \mathcal{B}\}$.

5.2. **Self-similarity of homogeneous functions.** Let $\mathcal{B} = \{v_0, \dots, v_N\}$ be as before, a Jordan basis for T, and let B be the $(N+1) \times (N+1)$ matrix that has the vectors v_i as rows. So BTB^{-1} is in Jordan form. Let

$$\phi^0(x) = [\varphi(x), \varphi(x+1), \dots, \varphi(x+N)]^t \quad \text{and} \quad h(x) = [h_0(x), h_1(x), \dots, h_N(x)]^t,$$

where h_i is the homogeneous function associated to the vector v_i. We have, by the refinability of φ, and the fact that the support is in $[0, N]$,

$$\text{if} \quad x \in [-\frac{1}{2}, \frac{1}{2}] \qquad \phi(x) = T\phi(2x). \tag{5.3}$$

Since $h(x) = B\phi^0(x)$ we have that $B^{-1}h(x) = \phi^0(x)$. From the refinement equation (5.3) we get that

$$h(x) = BTB^{-1}h(2x) \quad x \in [-1/2, 1/2],$$

where the matrix BTB^{-1} is in Jordan form. So h satisfies a scaling equation with the scaling matrix being the Jordan form of T.

6. Examples for $N = 3$

6.1. **B-spline.** The simplest case of refinable functions are the *B-splines*. They are the (normalized) convolutions of the characteristic function of $[0, 1]$ with itself. In particular the B-spline of order 3 is the refinable function that satisfies the refinement equation of 4 coefficients:

$$b(x) = \frac{1}{4}b(2x) + \frac{3}{4}b(2x-1) + \frac{3}{4}b(2x-2) + \frac{1}{4}b(2x-3). \tag{6.1}$$

The B-splines are those functions, for which the accuracy is maximum and so coincides with the dimension of the matrix T_0, so in this case, the eigenvalues of T_0 are 1 (for the constant functions), $\frac{1}{2}$ (for the linear functions), and $\frac{1}{4}$ (for the quadratic functions).

6.2. **Daubechies D_4.** Daubechies wavelets, are those refinable functions of N coefficients, that are orthogonal and provide the highest order of accuracy possible. (Note that the splines do not form an orthonormal basis).

d_4 is the refinable function that satisfies the refinement equation of 4 coefficients:

$$d_4(x) = \frac{1+\sqrt{3}}{4}d_4(2x) + \frac{3+\sqrt{3}}{4}d_4(2x-1) + \frac{3-\sqrt{3}}{4}d_(2x-2) + \frac{1-\sqrt{3}}{4}d_4(2x-3). \tag{6.2}$$

d_4 has accuracy 2 (it reproduces the constant and the linear functions). In this case the matrix T_0 has eigenvalues 1, $\frac{1}{2}$ and $c_0 = \frac{1+\sqrt{3}}{4}$. So a basis for $\operatorname{span}\{d_4(x), d_4(x-1), d_4(x-2)\}_{x\in[0,1]}$ is also given by $\operatorname{span}\{1, x, h_{c_0}(x)\}_{x\in[0,1]}$ where h_{c_0} is the homogeneous function associated to c_0.

6.3. $(\lambda,1)$**-Homogeneous functions are not enough.** In the two previous examples, we could always obtain a basis of $\operatorname{span}\{f(x), f(x-1), f(x-2)\}_{x\in[0,1]}$ just by using 1-homogeneous functions. The following example is to illustrate, that even in the simple case of only 4 coefficients, it may be necessary to use homogeneous functions of order bigger than 1. Consider the function:

$$f(x) = \frac{1}{3}f(2x) + \frac{2}{3}f(2x-1) + \frac{2}{3}f(2x-2) + \frac{1}{3}f(2x-3). \tag{6.3}$$

It can be shown that f has accuracy 1, and the eigenvalues of T are $\{1, \frac{1}{3}\}$. So in this case, $\operatorname{span}\{f(x), f(x-1), f(x-2)\}_{x\in[0,1]} = \operatorname{span}\{1, h_{\{1/3,1\}}(x), h_{\{1/3,2\}}(x)\}_{x\in[0,1]}$, where $h_{\{1/3,1\}}$ is a 1-homogeneous function corresponding to the eigenvalue 1/3, and $h_{\{1/3,2\}}$ is a 2-homogeneous function corresponding to the eigenvalue 1/3.

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