Classifying Cantor sets by their multifractal spectrum

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Abstract. The multifractal spectrum of a measure is the dimension of the sets

$$\mathcal{E}(\mu,\alpha) := \left\{ x \in supp(\mu) : \lim_{r\to 0^+} \frac{\log \mu(B_r(x))}{\log r^\alpha} = 1 \right\}.$$

In this work we focus on Cantor measures μ and the Cantor sets which support them. We consider the more general sets, $\mathcal{E}(\mu,h)$, defined by replacing r^α by $h(r)$, and the sets, $\mathcal{D}(\mu,x)$, formed by functions h for which a given element x belongs to $\mathcal{E}(\mu,h)$. We propose a classification for the Cantor sets in terms of the sets $\mathcal{D}(\mu,x)$ that is finer than the classification by dimension, but not as fine as the classification given by the equivalence of associated dimension functions. We characterize this latter classification in terms of particular subsets of $\mathcal{D}(\mu,x)$.

We also give estimates of dimensions of the sets $\mathcal{E}(\mu,h)$, extending the earlier multifractal analysis carried out for central Cantor sets and measures.

Keywords: Cantor sets, Multifractal Analysis, Hausdorff measures, Packing measures.

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1. Introduction

In the study of sets of Lebesgue measure zero the Hausdorff (or packing) s-measure, where s is the Hausdorff (resp., packing) dimension of the set, plays a crucial role. However, not all sets of dimension s have finite and positive s-measure, consequently, nearly a hundred years ago, Hausdorff in [11] introduced the more refined notion of Hausdorff h-measures, where the function t^s is replaced in the definition of the Hausdorff measure by $h(t)$. More recently, h-packing measures (defined similarly) were introduced by Tricot in [20] and both classes of measures have been extensively studied.

An important tool in understanding the nature of singular measures (and their support sets) is the concept of the local dimension of the measure, that is, the limiting behaviour of $\log \mu(B_r(x))/\log r$ as $r \to 0$, and its multifractal spectrum, meaning the dimension of the sets

$$\mathcal{E}(\mu, \alpha) := \left\{ x \in supp(\mu) : \lim_{r \to 0^+} \frac{\log \mu(B_r(x))}{\log r^\alpha} = 1 \right\}. \tag{1}$$

Multifractal spectra have been used in many applications in hydrodynamic turbulence, finance, genomics, computer network traffic. For many classes of measures, including (quasi) self-similar measures, measures on cookie cutters and p-Cantor measures on central Cantor sets, the dimensions of the sets $\mathcal{E}(\mu, \alpha)$ have been calculated and the multifractal formalism investigated (see [7, 10, 14] and the references therein). In this paper we investigate h-local dimensions, where we replace r^α by $h(r)$, and the analogous sets $\mathcal{E}(\mu, h)$. Unlike the set $\bigcup_{\alpha>0} \mathcal{E}(\mu, \alpha)$, whose complement can be of full dimension (cf. [1, 15]), the set $\bigcup_h \mathcal{E}(\mu, h)$ is a decomposition of the support of μ since the function $h(r) = \mu(B_r(x))$ has the property that $x \in \mathcal{E}(\mu, h)$.

We are interested in analyzing Cantor-like sets with gaps given by a sequence of real numbers. Specifically, for a given positive and summable sequence $a = \{a_n\}_{n=1}^\infty$ we associate a Cantor set C_a constructed in a similar fashion to the classical middle-third Cantor set. We begin by considering a closed interval I with length $\sum a_n$. We remove from I an open interval with length a_1, obtaining two closed intervals I_1^1 and I_2^1, the intervals of step one. If $I_1^k, \ldots, I_{2^k}^k$, the intervals of step k, have been constructed, we remove from I_j^k an open interval of length a_{2^k+j-1}, obtaining two closed intervals I_{2j-1}^{k+1} and I_{2j}^{k+1}. The Cantor set C_a is the compact, perfect set defined by

$$C_a := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{2^k} I_j^k.$$

Note that the condition $|I| = \sum a_n$ implies that the Lebesgue measure of the set C_a is zero. Moreover, the construction uniquely determines the location of the gaps at each step. For instance, the location of the first gap is determined by the rule that the interval to the left of the gap has length $a_2 + a_4 + a_5 + a_8 + \ldots$. The classical middle-third Cantor set, for example, is the set C_a with $a_n = 1/3^k$ for $n = 2^{k-1}, ..., 2^k - 1$. Such Cantor sets can be thought of as generalized Moran sets (see [6]) and need not be central or (quasi) self-similar. Indeed, note that all compact, perfect, totally disconnected, measure zero subsets of $\mathbb{R}$ can be obtained in this fashion.

In contrast to the situation for arbitrary compact sets (see [3]), it was shown in [4, 9] that for Cantor sets with gaps given by a decreasing sequence of positive real numbers, it is always possible to find a function h such that both the Hausdorff and packing h-measures of the Cantor set are positive and finite. In [9] such an associated dimension function h was shown to be unique up to comparability of functions (4), thus we may speak about 'the' associated dimension function to a Cantor set. Cabrelli et al in [5] introduced a classification of Cantor sets C_a by the associated dimension functions. They proved that this classification is finer than the one given by Hausdorff and packing dimensions and characterized it in terms of the sequences of gaps.

In this paper our interest is in the study of the sets $\mathcal{D}(\mu, x)$ consisting of the dimension functions h for which a given x belongs to $\mathcal{E}(\mu, h)$. We prove that the classification of Cantor sets introduced in [5] can be characterized in terms of suitable subsets of these sets (see Theorem 4.2). Motivated by this, we define a second classification in terms of the asymptotic equivalence of the logarithms of the associated dimension functions. This new classification is again finer than that given by Hausdorff and packing dimensions, but it is not as restrictive as the [5] classification. We characterize this new classification in terms of the sequences of gaps, as was done for the classification of [5], and also in terms of equality of the sets $\mathcal{D}(\mu_p, \boldsymbol{w})$ where μ_p is the p-Cantor measure and $\boldsymbol{w}$ is the unique element of the Cantor set whose 'address' is given by the infinite binary word w. Furthermore, we prove in Theorem 4.6 that if $\mathcal{D}(\mu_p^a, \boldsymbol{w}_a) = \mathcal{D}(\mu_p^b, \boldsymbol{w}_b)$ for one choice of $p \in (0,1)$ and one infinite binary word w, then these sets are equal for all p and all words w. A similar statement is deduced for the classification of [5] (Theorem 4.2).

Generalizing the (usual) multifractal analysis, we also study the Hausdorff and packing measures of the sets $\mathcal{E}(\mu, h)$ when μ is a p-Cantor measure on a Cantor set C_a associated with a decreasing sequence of gaps (see Theorem 5.1). An important point in the proof is the fact that the symbolic multifractal spectrum (see, for example, [15]) agrees with the classical one (Corollary 3.8).

2. Notation and definitions

2.1. h-Hausdorff and packing measures

A function $h : [0, \infty) \to [0, \infty)$ is said to be a *dimension function* if $h(0) = 0$ and h is continuous, non-decreasing and doubling, i.e., there is a constant τ such that $h(2t) \leq \tau h(t)$ for all t. The set of all dimension functions will be denoted by $\mathcal{D}$.

We recall here the definition of h-Hausdorff and packing measures and dimensions (for a detailed explanation see [7, 19]). Given A, a subset of $\mathbb{R}$, and $\delta > 0$, a collection $(A_i)_{i=1}^{\infty}$ is called a δ-covering of A if $A \subset \bigcup A_i$ and $|A_i| \leq \delta$, where $|A_i|$ denotes its diameter. Given $h \in \mathcal{D}$, the *h-Hausdorff measure* is defined as

$$\mathcal{H}^h(A) = \lim_{\delta \to 0^+} \left(\inf \left\{ \sum h(|A_i|) : \ (A_i) \text{ is a } \delta\text{-covering of } A \right\} \right).$$

When the dimension function is of the form $h_s(t) = t^s$, $\mathcal{H}^{h_s}$ is the usual s-Hausdorff measure, $\mathcal{H}^s$. It is very well known that given a set A there is a real number t such that $\mathcal{H}^s(A) = 0$ if $s > t$ and $\mathcal{H}^s(A) = \infty$ if $s < t$. We call t the Hausdorff dimension of A and write $t = \dim_H(A)$.

A δ-packing of the set A is a disjoint family of open balls centered at points of A, with diameters smaller than δ. The *h-packing pre-measure* is defined as

$$\mathcal{P}_0^h(A) = \lim_{\delta \to 0^+} \left(\sup \left\{ \sum h(|B_i|) : (B_i) \text{ is a } \delta\text{-packing of } A \right\} \right).$$

$\mathcal{P}_0^h$ is not a measure since is not σ-additive and hence the *h-packing measure* is given by

$$\mathcal{P}^h(A) = \inf\left\{\sum \mathcal{P}_0^h(A_i) : A = \bigcup A_i\right\}.$$

Analogously to the case of Hausdorff dimension, the packing dimension, $\dim_P(A)$, is the unique number satisfying $\mathcal{P}^s(A) = 0$ when $s > \dim_P(A)$ and $\mathcal{P}^s(A) = \infty$ when $s < \dim_P(A)$.

The doubling assumption, although not essential to the definitions, ensures that $\mathcal{H}^h(A) \leq \mathcal{P}^h(A)$ for any A. Notice that the function $h(r) = \mu(B(x,r))$ (for fixed x) is doubling if μ is a doubling measure and continuous if μ is a continuous measure.

2.2. Cantor Sets

As remarked in the introduction, each positive and summable sequence, $a = \{a_n\}_{n=1}^\infty$, is uniquely associated with a Cantor set, C_a. The sets W and W_∞ of finite (resp. infinite) binary words on the letters $0, 1$ are useful for describing the intervals in the construction (and the points in the set). The interval we begin with will be labelled by the empty word (i.e. $I_e^a := I$. Then for $w \in W$ $I_w := I_j^k$ if $k = |w|$ and $j = 1 + \sum_{i=1}^k w_i 2^{k-i}$ (using $|w|$ to denote its lenght).

The elements in the Cantor set can also be labelled by the (infinite) binary words. We denote by $w|k$ the finite word formed by the first k letters of w. Observe that $x \in C_a$ if and only if there is a unique $w \in W_\infty$ such that $x \in I_{w|k}^a$ for all k. We will follow this convention:

Notation. Given $w \in W_\infty$ we will denote by $\boldsymbol{w}_a$ the unique point in C_a such that $\boldsymbol{w}_a \in I_{w|k}^a$ for all k.

Finally, we will denote by $I_k^a(x)$ the unique interval of step k containing the Cantor set element x. Notice that $I_k^a(\boldsymbol{w_a}) = I_{w|k}^a$.

For the remainder of the paper $a = \{a_k\}$ will be a decreasing, positive and summable sequence.

We will consider the following three sequences associated to the sequence a:

$$r_n^a = \sum_{k \geq n} a_k, \quad R_n^a = \frac{r_n^a}{n} \quad \text{and} \quad s_n^a = R_{2^n}^a. \tag{2}$$

The superscript $'a'$ in our notation may be suppressed if the sequence a is understood.

The number r_{2^n} is the sum of the lengths of the intervals remaining at step n and s_n is the average length of an interval of step n. The decreasing assumption ensures that any Cantor interval of step n has length at least s_{n+1} and at most s_{n-1}.

The study of the dimension of Cantor sets C_a in terms of their gaps was initiated by Besicovitch and Taylor in [2]. From their work and [21] one can deduce the following formulas:

$$\dim_H C_a = \frac{\log 2}{\limsup\limits_{n\to\infty} \frac{1}{n}|\log s_n^a|} \text{ and } \dim_P C_a = \frac{\log 2}{\liminf\limits_{n\to\infty} \frac{1}{n}|\log s_n^a|}. \tag{3}$$

Extending this work, Garcia et al in [9] obtained the following estimates on the h-Hausdorff and packing pre-measures of the Cantor sets C_a.

Theorem 2.1. *[9] For every $h \in \mathcal{D}$ and Cantor set C_a*

(i) $\frac{1}{4}\liminf\limits_{n\to\infty} nh(R_n^a) \leq \mathcal{H}^h(C_a) \leq 4 \liminf\limits_{n\to\infty} nh(R_n^a)$.

(ii) $\frac{1}{8}\limsup_{n\to\infty} nh(R_n^a) \le \mathcal{P}_0^h(C_a) \le 8\limsup_{n\to\infty} nh(R_n^a)$.

This motivates the following definition adopted first in [5].

Definition 2.2. We say that a dimension function h is *associated to the sequence* a (or the Cantor set C_a) if and only if there are positive constants c_1, c_2 such that:

$$\frac{c_1}{n} \le h(R_n^a) \le \frac{c_2}{n} \text{ for all } n.$$

Of course, a consequence of Theorem 2.1 is that C_a is an h-set, meaning $0 < \mathcal{H}^h(C_a) \le \mathcal{P}_0^h(C_a) < \infty$, if and only if h is associated to the sequence a.

Observe that two dimension functions, g, h, are associated to the sequence a if and only if g is *comparable* to h, that is, there are positive constants A, B such that

$$Ag(x) \le h(x) \le Bg(x) \quad \forall x > 0. \tag{4}$$

2.3. p-Cantor measures

By a measure we will mean a finite, regular, Borel measure. We will focus on a (natural) class of measures supported on the Cantor set C_a. Given $0 < p < 1$, the unique probability measure, μ_p^a, satisfying $\mu_p^a(I_{w0}) = p\mu_p^a(I_w)$ and $\mu_p^a(I_{w1}) = (1-p)\mu_p^a(I_w)$ will be called the *p-Cantor measure*. Equivalently, if for a given $w \in W$ we define

$$n_0(w) = \sharp\{i : w_i = 0\} \quad \text{and} \quad n_1(w) = \sharp\{i : w_i = 1\},$$

then the p-Cantor measure is the probability measure satisfying $\mu_p^a(I_w) = p^{n_0(w)}(1-p)^{n_1(w)}$ for all $w \in W$.

The multifractal analysis of p-Cantor measures was studied for the special cases of Cantor sets that are self-similar (see [6, 7, 13]), quasi self-similar [17] and central [10]. One can easily verify that our Cantor sets need not have any of these properties.‡

3. Preliminary Results

The sets, $\mathcal{E}(\mu, \alpha)$, can be viewed as a partial decomposition of the support of the measure μ into a family of subfractals. The complement of the set $\bigcup_{\alpha>0} \mathcal{E}(\mu, \alpha)$ has aroused interest in recent years (cf. [1, 15]), and when μ is a self similar measure satisfying the strong separation condition and the weights are not the canonical ones (such as a p-Cantor measure, with $p \neq 1/2$, on the classical middle-third Cantor set) the complement has been proven to be a set of full dimension.

Thus it is natural to consider a more general class of sets.

Definition 3.1. Let μ be a measure and $h : [0, \infty) \to [0, \infty)$ a dimension function. We define

$$\mathcal{E}(\mu, h) := \left\{ x \in support(\mu) : \lim_{r\to 0^+} \frac{\log \mu(B_r(x))}{\log h(r)} = 1 \right\}.$$

‡ Arguments similar to [10] show that the Cantor set C_a is quasi self-similar if and only if there exists a constant c such that $1/c \le s_j s_k / s_{j+k} \le c$ for all j, k.

The sets $\mathcal{E}_h(\mu)$ and $\mathcal{E}^h(\mu)$ will be defined similarly, but with lim inf and lim sup replacing the limit respectively.

Note that the functions $h(t) = t^\alpha$ belong to $\mathcal{D}$ and in this case $\mathcal{E}(\mu, h)$ is the set $\mathcal{E}(\mu, \alpha)$.

In the multifractal analysis of self-similar sets the strong separation condition (SSC) is commonly assumed. This was replaced in the multifractal analysis of Cantor measures on central Cantor sets by the assumption that the ratios of the lengths of the gaps to the lengths of the intervals at the same step was bounded away from zero [10]. Below we give equivalent conditions that could be considered a generalization of the SSC for Cantor sets C_a.

Proposition 3.2. *The following conditions are equivalent for a Cantor set C_a:*

(i) $\sup\{s_{k+1}/s_k : k \geq 1\} < 1/2$;

(ii) There is an $\varepsilon > 0$ such that $a_j \geq \varepsilon |I_\omega|$ for all $w \in W$ and $j \leq 2^{|\omega|}$;

(iii) There is an $\varepsilon > 0$ such that $a_j \geq \varepsilon s_k$ for all $j \leq 2^k$.

Proof. $(i) \Rightarrow (ii)$ The hypothesis ensures that there exists $\delta > 0$ such that $2\delta s_k \leq s_k - 2s_{k+1}$ for all $k \geq 1$. Since $\{a_j\}$ decreases,

$$2\delta s_k \leq s_k - 2s_{k+1} = 2^{-k}(a_{2^k} + \ldots + a_{2^{k+1}-1}) \leq a_{2^k}.$$

Now fix ω. For a suitable j with $0 \leq j \leq 2^{|\omega|} - 1$ we have

$$|I_\omega| = |I_{\omega 0}| + |I_{\omega 1}| + a_{2^{|\omega|}+j} \leq 2s_{|\omega|} + a_{2^{|\omega|}} \leq (1 + 1/(2\delta))a_{2^{|\omega|}}.$$

$(ii) \Rightarrow (iii)$ is clear since the length of some interval of step k must be at least the average length, s_k.

$(iii) \Rightarrow (i)$ Assumption (iii) implies $\varepsilon s_{k+1} \leq a_{2^{k+1}} \leq s_k - 2s_{k+1}$, hence $s_k \geq (2 + \varepsilon)s_{k+1}$. □

The next proposition provides one situation in which the conditions of Proposition 3.2 holds. Others can be found in [12].

Proposition 3.3. *Let h be an associated dimension function for C_a and suppose there is an increasing function ϕ satisfying $\phi(n)/n \to 0$ as $n \to \infty$ such that $h(nx) \leq \phi(n)h(x)$. Then the (equivalent) conditions of Proposition 3.2 hold.*

Remark 3.4. When C_a is an s-set, the function $h(t) = t^s$ is an associated dimension function satisfying the hypothesis in this Proposition.

Proof. We have $c_1 2^{-n} \leq h(s_n) \leq c_2 2^{-n}$ since h is an associated dimension function. As $n/\phi^{-1}(n) \to 0$ when $n \to \infty$, we can pick m such that $2^m/\phi^{-1}(c_1 c_2^{-1} 2^m) \leq 1/2$.

As above, the decreasingness of (a_j) implies that $a_{2^j} \geq 2^{-m}s_j - s_{m+j}$, and so

$$a_{2^j} \geq 2^{-m}h^{-1}(c_1 2^{-j}) - h^{-1}(c_2 2^{-(m+j)}).$$

Applying the relationship $h^{-1}(nx) \geq \phi^{-1}(n)h^{-1}(x)$ with $x = c_2 2^{-(m+j)}$ and $n = c_1 c_2^{-1} 2^m$, we deduce that

$$\begin{aligned} a_{2^j} &\geq 2^{-m}h^{-1}(c_1 2^{-j})(1 - 2^m/\phi^{-1}(c_1 c_2^{-1} 2^m)) \\ &\geq 2^{-(m+1)}h^{-1}(c_1 2^{-j}). \end{aligned}$$

Condition (iii) above follows taking $\varepsilon = \phi(c_1 c_2^{-1})2^{-(m+1)}$. □

The (equivalent) conditions in Prop. 3.2 will allow us to show that for many calculations we can replace balls by Cantor intervals. **In the remainder of the paper, it should be understood that these equivalent conditions are assumed to hold,** although this will not be explicitly stated.

We first prove a useful geometric property.

Lemma 3.5. *There exists a positive integer N such that given any positive integer n and $x \in C_a$,*

$$I_n(x) \cap C_a \subseteq B_{|I_n(x)|}(x) \cap C_a \subseteq I_{n-N}(x) \cap C_a.$$

Proof. As $x \in I_n(x)$, the fact that $I_n(x) \subseteq B_{|I_n(x)|}(x)$ is obvious. To prove the other inclusion, with $\varepsilon > 0$ chosen as in Prop. 3.2(*iii*), choose N such that $2^{-N} < \varepsilon/2$. Then

$$r := |I_n(x)| \leq s_{n-1} \leq 2^{1-N} s_{n-N} \leq 2^{1-N} a_j/\varepsilon \text{ for any } j \leq 2^{n-N}.$$

The two gaps in the Cantor set adjacent to the Cantor interval $I_{n-N}(x)$ have lengths belonging to the set $\{a_j : j \leq 2^{n-N}\}$ and hence exceed r. As $x \in I_{n-N}(x)$, it follows that $B_r(x)$ is contained in the union of $I_{n-N}(x)$ and its two adjacent gaps. Thus $B_r(x) \cap C_a \subseteq I_{n-N}(x) \cap C_a$. □

Corollary 3.6. *For any $0 < p < 1$, p-Cantor measures are doubling.*

Proof. Given $r > 0$, choose k such that $s_k \leq r < s_{k-1}$. For any $x \in C_a$, the support of the measure μ_p^a, we have $r \geq |I_{k+1}(x)|$ and $2r < 2s_{k-1} \leq s_{k-2} \leq |I_{k-3}(x)|$. Thus the previous Lemma implies that $B(x, 2r) \cap C_a \subset I_{k-3-N}(x) \cap C_a$ and hence

$$\mu_p^a(B(x, 2r)) \leq \mu_p^a(I_{k-3-N}(x)) \leq \tau \mu_p^a(I_{k+1}(x)) \leq \tau \mu_p^a(B(x, r)),$$

where $\tau = (\min(p, 1-p))^{-(N+4)}$. □

The previous Lemma will enable us to prove that in the definition of $\mathcal{E}(\mu, h)$ we can replace balls by Cantor intervals.

Theorem 3.7. *If μ is a doubling measure supported on a Cantor set C_a, then for all $x = \boldsymbol{w}_a \in C_a$, the limiting behaviours of*

$$\frac{\log \mu(I_n(x))}{\log h(s_n)} = \frac{\log \mu(I_{w|n})}{\log h(s_n)} \text{ and } \frac{\log \mu(B_r(x))}{\log h(r)}$$

as $n \to \infty$ or $r \to 0^+$ are the same.

As an immediate consequence we obtain the following important observation.

Corollary 3.8. *If μ is a doubling measure supported on a Cantor set C_a, then*

$$\mathcal{E}(\mu, h) = \left\{ x = \boldsymbol{w_a} : \lim_{k \to \infty} \frac{\log \mu(I_{w|k})}{\log h(s_k)} = 1 \right\}. \tag{5}$$

A similar statement holds for $\mathcal{E}_h(\mu)$ and $\mathcal{E}^h(\mu)$.

Remark 3.9. In [15], the authors propose a symbolic multifractal analysis for self similar measures. The sets on the right side of the equation (5) could be considered a symbolic multifractal analysis for Cantor sets. In this sense, the Corollary shows that the classic and the symbolic multifractal analysis coincide.

Proof. Since the lengths of the gaps decrease, for any $w \in W$ we have $I_{w1} \supseteq B_r(z) \cap C_a$ where z is the left endpoint of I_{w1} and r is the length of the gap with endpoint z. If $|w| = n$, then $r = a_{2^n+k}$ for some $k \in \{0, 1, ..., 2^n - 1\}$. By assumption Prop. 3.2(ii), $r \geq \varepsilon |I_{w0}|$ and hence $I_{w0} \subseteq B_{r(1+\varepsilon^{-1})}(z)$. The doubling property of μ implies that

$$\mu(I_{w1}) \geq \mu\left(B_r(z)\right) \geq c\mu\left(B_{r(1+\varepsilon^{-1})}(z)\right) \geq c\mu(I_{w0}).$$

We can similarly find a constant c' such that $\mu(I_{\omega 1}) \leq c'\mu(I_{\omega 0})$ for all $w \in W$.

Thus, for any n and $x \in C_a$, the μ-measure of $I_n(x)$ is comparable to the measure of $I_{n+1}(x)$ and hence also to the measure of $I_{n\pm 2N}(x)$ for any fixed N. It follows by repeated application of Lemma 3.5 that $\mu(B_r(x))$ and $\mu(I_n(x))$ are also comparable for any r with $|I_{n+N}(x)| \leq r \leq |I_{n-N}(x)|$.

Because $|I_{n+1}(x)| \geq s_{n+2}$ and $|I_n(x)| \leq s_{n-1}$, this fact implies that there are positive constants c_1, c_2, independent of n and x, such that if $|I_{n+1}(x)| \leq r \leq |I_n(x)|$, then

$$\frac{\log(c_1\mu(I_{n+2}(x)))}{\log h(s_{n+2})} \leq \frac{\log \mu(B_r(x))}{\log h(r)} \leq \frac{\log(c_2\mu(I_{n-1}(x)))}{\log h(s_{n-1})},$$

Similarly, as $|I_{n+1}(x)| \leq s_n \leq |I_{n-1}(x)|$, there are constants c_3, c_4 such that

$$\frac{\log\left(c_3\mu(B_{|I_{n+1}(x)|}(x))\right)}{\log h(|I_{n+1}(x)|)} \leq \frac{\log \mu(I_n(x))}{\log h(s_n)} \leq \frac{\log\left(c_4\mu(B_{|I_{n-1}(x)|}(x))\right)}{\log h(|I_{n-1}(x)|)}.$$

□

Corollary 3.10. $\bigcup \mathcal{E}(\mu_p^a, h) = C_a$ *where the union is taken over all* $h \in \mathcal{D}$.

Proof. If $x = \boldsymbol{w_a} \in \boldsymbol{C}_a$, then the piecewise linear function h defined by the rule $h(s_k) = p^{n_0(w|k)}(1-p)^{n_1(w|k)}$ and $h(0) = 0$ is a dimension function. The Corollary 3.8 implies $x \in \mathcal{E}(\mu_p^a, h)$. □

4. The Classification of Cantor sets.

In this section we will classify Cantor sets by considering certain subsets of the space of dimension functions $\mathcal{D}$, namely

$$\mathcal{D}(\mu, x) := \{h \in \mathcal{D} : x \in \mathcal{E}(\mu, h)\}$$

and

$$\Lambda(\mu, x) := \left\{h \in \mathcal{D} : 0 < \liminf_{r\to 0^+} \frac{\mu(B_r(x))}{h(r)} \leq \limsup_{r\to 0^+} \frac{\mu(B_r(x))}{h(r)} < \infty\right\},$$

where μ is any (or all) of the p-Cantor measures.

We recall the following equivalence relation on the class of Cantor sets C_a introduced in [5].

Definition 4.1. Two Cantor sets, C_a, C_b, are said to be *equivalent*, denoted $C_a \sim C_b$, if their associated dimension functions are equivalent.

In [5] it was shown that this equivalence relation can be described in terms of properties of the tail sequences (the equivalence of (i) and (ii) in the Theorem 4.2 below). We will use this fact to prove that the $\sim$ equivalence relation on Cantor sets can also be characterized in terms of the sets $\Lambda(\mu_p^a, \boldsymbol{w_a})$ where $\boldsymbol{w_a} \in C_a$. To simplify notation we write $\Lambda_p(\boldsymbol{w_a}) := \Lambda(\mu_p^a, \boldsymbol{w_a})$. In view of Theorem 3.7, we have

$$\Lambda_p(\boldsymbol{w_a}) = \left\{ h \in \mathcal{D} : 0 < \liminf_{k\to\infty} \frac{\mu_p^a(I_{w|k})}{h(s_k)} \leq \limsup_{k\to\infty} \frac{\mu_p^a(I_{w|k})}{h(s_k)} < \infty \right\}.$$

Theorem 4.2. *Let a and b be positive, decreasing, summable sequences. The following are equivalent:*

(i) $C_a \sim C_b$.

(ii) There exists an integer N such that for all $k > N$, $s_{k+N}^a \leq s_k^b \leq s_{k-N}^a$.

(iii) $\Lambda_p(\boldsymbol{w_a}) = \Lambda_p(\boldsymbol{w_b})$ *for all* $0 < p < 1$ *and all* $w \in W_\infty$.

(iv) $\Lambda_p(\boldsymbol{w_a}) = \Lambda_p(\boldsymbol{w_b})$ *for some* $0 < p < 1$ *and some* $w \in W_\infty$.

Remark 4.3. Following the ideas in [16], for $\mathbf{p} = (p_1, \ldots, p_m)$ we could have considered the more general sets

$$\Lambda_{\mathbf{p}}^m(\boldsymbol{w_a}) := \left\{ (h_1, \ldots, h_m) \in \mathcal{D}^m : 0 < \liminf_{k\to\infty} \frac{\mu_{p_i}^a(I_{w|k})}{h_i(s_k)} \leq \limsup_{k\to\infty} \frac{\mu_{p_i}^a(I_{w|k})}{h_i(s_k)} < \infty \right\}.$$

However, a consequence of $(iii) \iff (iv)$ is that $\Lambda_{\mathbf{p}}^m(\boldsymbol{w_a}) = \Lambda_{\mathbf{p}}^m(\boldsymbol{w_b})$ if and only if $\Lambda_{p_1}(\boldsymbol{w_a}) = \Lambda_{p_1}(\boldsymbol{w_b})$. So, these sets give exactly the same classification as the one we considered.

Remark 4.4. In [18] the authors proved that two Cantor sets are Lipschitz equivalent (i.e. there is a bi-Lipschitz map between them) if and only if the sequences of gaps are equivalent. It is easy to see that there are sequences satisfying condition *(ii)* that are not equivalent (see [5] Example 4.3), hence, this notion is weaker than Lipschitz equivalence.

Proof. $(i) \iff (ii)$ is given in [5, Thm. 4]. Of course $(iii) \Rightarrow (iv)$ is trivial.

$(iv) \Rightarrow (ii)$ For $w \in W_\infty$ satisfying the hypothesis, put $n_0(w|k) = n_k$ and define $h \in \mathcal{D}$ by $h(s_k^a) = \mu_p^a(I_{w|k}^a) = p^{n_k}(1-p)^{k-n_k}$. Obviously, $h \in \Lambda_p(\boldsymbol{w_a})$ and therefore $h \in \Lambda_p(\boldsymbol{w_b})$. Thus for $\theta = a, b$ there are positive constants, A, B, such that

$$A \leq \frac{\mu_p^\theta(I_{w|k}^\theta)}{h(s_k^\theta)} \leq B \text{ for all } k.$$

Since $\mu_p^a(I_{w|k}^a) = \mu_p^b(I_{w|k}^b)$, we can conclude that there are positive constants C_1 and C_2 such that

$$C_1 h(s_k^a) \leq h(s_k^b) \leq C_2 h(s_k^a).$$

Choose a sequence $j(k)$ such that $s_{j(k)+1}^a \leq s_k^b \leq s_{j(k)}^a$. By monotonicity,

$$h(s_{j(k)+1}^a) \leq h(s_k^b) \leq h(s_{j(k)}^a).$$

Put $c = \min\{p, 1-p\}$. Combining these inequalities we obtain

$$\begin{aligned} C_2 p^{n_k}(1-p)^{k-n_k} &= C_2 h(s_k^a) \geq h(s_k^b) \geq h(s_{j(k)+1}^a) \\ &\geq c p^{n_{j(k)}}(1-p)^{j(k)-n_{j(k)}} \end{aligned}$$

and, similarly,

$$C_1 p^{n_k}(1-p)^{k-n_k} \leq p^{n_{j(k)}}(1-p)^{j(k)-n_{j(k)}}.$$

Together these inequalities yield,

$$c/C_2 \leq p^{n_k-n_{j(k)}}(1-p)^{k-n_k-(j(k)-n_{j(k)}))} \leq 1/C_1. \tag{6}$$

Without loss of generality assume $p \geq 1-p$. Define the integers $m(k) = k - j(k)$ and $d(k) = n_k - n_{j(k)}$.With this notation we can reformulate (6) as

$$c/C_2 \leq p^{m(k)} \left(\frac{1-p}{p}\right)^{m(k)-d(k)} \leq 1/C_1.$$

Since the signs of $m(k) - d(k)$ and $m(k)$ coincide, these two inequalities ensure $m(k)$ is bounded, say $|m(k)| \leq N$. Then $s^a_{k+N+1} \leq s^b_k \leq s^a_{k-N}$.

(ii) $\Rightarrow$ *(iii)* Fix $p \in (0,1)$ and $w \in W_\infty$. By assumption $s^a_{n+N} \leq s^b_n \leq s^a_{n-N}$, thus we have

$$\frac{\mu^a_p(I^a_{w|n})}{h(s^a_{n-N})} \leq \frac{\mu^a_p(I^a_{w|n})}{h(s^b_n)} = \frac{\mu^b_p(I^b_{w|n})}{h(s^b_n)} \leq \frac{\mu^a_p(I^a_{w|n})}{h(s^a_{n+N})}.$$

For any fixed N, the sequences $(\mu^a_p(I_{w|n}))$ and $(\mu^a_p(I_{w|n\pm N}))$ are equivalent, hence as $h \in \Lambda_p(\boldsymbol{w_a})$ we can deduce that

$$\liminf_{n\to\infty} \frac{\mu^a_p(I^a_{w|n})}{h(s^a_{n-N})} > 0 \text{ and } \limsup_{n\to\infty} \frac{\mu^a_p(I^a_{w|n})}{h(s^a_{n+N})} < \infty.$$

Together, these bounds clearly imply $h \in \Lambda_p(\boldsymbol{w_b})$. □

Motivated by this, we propose to introduce a weaker equivalence relation for functions and Cantor sets.

Definition 4.5. We will say two non-negative functions, f, g, are *logarithmically equivalent,* and write $f \diamond g$, if and only if

$$\lim_{x\to 0} \frac{\log f(x)}{\log g(x)} = 1.$$

We will say that two Cantor sets are *logarithmically equivalent*, and write $C_a \diamond C_b$, if their associated dimension functions are logarithmically equivalent.

Since equivalent functions are clearly logarithmically equivalent, this is a well defined equivalence relation on the Cantor sets, C_a.

As with the $\sim$ relation, logarithmic equivalence can be characterized in terms of the (tails of the) sequence itself. It can also be characterized by the sets $\mathcal{D}_p(\boldsymbol{w_a}) := \mathcal{D}(\mu^a_p, \boldsymbol{w_a})$.

The notation $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ will denote the floor and ceiling functions, respectively.

Theorem 4.6. *The following are equivalent:*

(i) $C_a \diamond C_b$.

(ii) *For each* $\alpha > 1$, *there exists an integer* $N = N(\alpha)$ *such that for all* $n \geq N$,

$$s^a_{\lceil n\alpha\rceil} \leq s^b_n \leq s^a_{\lfloor n/\alpha\rfloor}.$$

(iii) $\mathcal{D}_p(\boldsymbol{w_a}) = \mathcal{D}_p(\boldsymbol{w_b})$ *for all* $0 < p < 1$ *and* $w \in W_\infty$.

(iv) $\mathcal{D}_p(\boldsymbol{w_a}) = \mathcal{D}_p(\boldsymbol{w_b})$ *for some* $0 < p < 1$ *and some* $w \in W_\infty$.

Proof. $(i) \Rightarrow (ii)$ We will denote by h_a and h_b the associated dimension functions to the sequences a and b respectively. By assumption $h_a \diamond h_b$, thus given $\varepsilon > 0$ there exists N such that if $n \geq N$,

$$(1-\varepsilon)\log h_b(s_n^a) \geq \log h_a(s_n^a) \geq (1+\varepsilon)\log h_b(s_n^a).$$

Put $\alpha = 1/(1-\varepsilon)$. Since all associated dimension functions corresponding to a given sequence are logarithmically equivalent there is no loss of generality in assuming that for $\theta = a$ or b we have $h_\theta(s_n^\theta) = 2^{-n}$ and that h_θ is strictly increasing. Thus for $n \geq N$,

$$h_b(s_n^a) \geq 2^{-n\alpha} \geq 2^{-\lceil n\alpha \rceil} = h_b\left(s^b_{\lceil n\alpha \rceil}\right)$$

and therefore $s_n^a \geq s^b_{\lceil n\alpha \rceil}$. Since a and b play symmetric roles, this gives (ii).

$(ii) \Rightarrow (i)$ Again, there is no loss of generality in assuming that $h_\theta(s_n^\theta) = 2^{-n}$ for $\theta = a, b$. Fix $\alpha > 1$. By (ii) and monotonicity, for n sufficiently large $h_b(s_n^a) \geq h_b(s^b_{\lceil n\alpha \rceil}) = 2^{-\lceil n\alpha \rceil}$, hence

$$\limsup_{n\to\infty} \frac{\log h_b(s_n^a)}{\log h_a(s_n^a)} \leq \limsup_{n\to\infty} \frac{\lceil n\alpha \rceil}{n} = \alpha.$$

Analogously, we obtain

$$\liminf_{n\to\infty} \frac{\log h_b(s_n^a)}{\log h_a(s_n^a)} \geq \frac{1}{\alpha}.$$

As $\alpha > 1$ was arbitrary and $\log h_\theta(s_n^\theta)/\log h_\theta(s_{n+1}^\theta) \to 1$, this suffices to prove $h_a \diamond h_b$, i.e., $C_a \diamond C_b$.

$(iv) \Rightarrow (ii)$ For w satisfying the hypothesis, put $n_k = n_0(w|k)$. Define $h(s_k^a) = p^{n_k}(1-p)^{k-n_k}$, so that $h \in \mathcal{D}_p(\boldsymbol{w_a})$. The assumption of (iv) ensures that also $h \in \mathcal{D}_p(\boldsymbol{w_b})$ and therefore

$$\lim_{k\to\infty} \frac{\log \mu_p^a(I^a_{w|k})}{\log h(s_k^a)} = \lim_{k\to\infty} \frac{\log \mu_p^b(I^b_{w|k})}{\log h(s_k^b)} = 1.$$

Since $\mu_p^a(I^a_{w|k}) = \mu_p^b(I^b_{w|k})$, we can conclude that $\lim_{k\to\infty} \frac{\log h(s_k^a)}{\log h(s_k^b)} = 1$. Fix $\varepsilon > 0$ and choose K such that if $k \geq K$,

$$(1-\varepsilon)\log h(s_k^a) \geq \log h(s_k^b) \geq (1+\varepsilon)\log h(s_k^a). \tag{7}$$

Choose the sequence $j(k)$ such that $s^a_{j(k)+1} \leq s_k^b < s^a_{j(k)}$ and put $j(k) = k + k\alpha_k$. For $c = \min\{p, 1-p\}$ we have

$$cp^{n_{j(k)}}(1-p)^{j(k)-n_{j(k)}} \leq h(s^a_{j(k)+1}) \leq h(s_k^b) \tag{8}$$

$$\text{and} \qquad h(s_k^b) \qquad \leq h(s^a_{j(k)}) = p^{n_{j(k)}}(1-p)^{j(k)-n_{j(k)}}. \tag{9}$$

Combining (8) with (7) we obtain

$$cp^{n_{j(k)}}(1-p)^{j(k)-n_{j(k)}} \leq \left(p^{n_k}(1-p)^{k-n_k}\right)^{1-\varepsilon}. \tag{10}$$

Assume $p \leq 1-p$, say $1-p = p^t$ for suitable $0 < t \leq 1$. (The other case is symmetric.) With this notation (10) becomes

$$0 < c \leq p^{(n_k - n_{j(k)})(1-t) - n_k\varepsilon(1-t) - tk(\alpha_k+\varepsilon)}.$$

If $\alpha_k \leq 0$, then also $n_{j(k)} - n_k \leq 0$ and thus $c \leq p^{-n_k \varepsilon(1-t) - tk(\alpha_k+\varepsilon)}$. This can only be true if there is a constant c_0 such that $-n_k\varepsilon(1-t) - tk(\alpha_k+\varepsilon) \leq c_0$. Since $\varepsilon > 0$ was arbitrary, this ensures $\alpha_k \to 0$ as $k \to \infty$. If, instead, $\alpha_k \geq 0$, we can argue similarly from (9).

Fix $\alpha > 1$ and pick $k_0 \geq K$ such that if $k \geq k_0$, then $1 + \alpha_k + 1/k \leq \alpha$. The choice of $j(k)$ and monotonicity implies

$$s_k^b \geq s_{j(k)+1}^a = s_{k(1+\alpha_k+1/k)}^a \geq s_{\lceil k\alpha \rceil}^a.$$

Analogous reasoning gives the other relation.

$(ii) \Rightarrow (iii)$ Fix $p \in (0,1)$ and $w \in W_\infty$, and assume $h \in \mathcal{D}_p(\boldsymbol{w_a})$. We need to prove that $h \in \mathcal{D}_p(\boldsymbol{w_b})$. It is a routine exercise to verify that for any $p \in (0,1)$ and any $\alpha > 0$,

$$\limsup_k \left| \frac{\log \mu_p^a(I_{w|\{k\alpha\}}^a)}{\log \mu_p^a(I_{w|k}^a)} - 1 \right| \leq |\alpha - 1| \tag{11}$$

(where $\{k\alpha\}$ denotes either the floor or the ceiling).

Temporarily fix $\alpha > 1$. The assumption of (ii) and the fact that $\mu_p^b(I_{w|k}^b) = \mu_p^a(I_{w|k}^a)$ implies that for large enough k,

$$\frac{\log \mu_p^a(I_{w|k}^a)}{\log h(s_{\lceil k\alpha \rceil}^a)} \leq \frac{\log \mu_p^a(I_{w|k}^a)}{\log h(s_k^b)} = \frac{\log \mu_p^b(I_{w|k}^b)}{\log h(s_k^b)} \leq \frac{\log \mu_p^a(I_{w|k}^a)}{\log h(s_{\lfloor k/\alpha \rfloor}^a)}.$$

As h belongs to $\mathcal{D}_p(\boldsymbol{w_a})$, these bounds, together with (11), imply that

$$\limsup_k \left| \frac{\log \mu_p^b(I_{w|k}^b)}{\log h(s_k^b)} - 1 \right| \leq |\alpha - 1|.$$

Because this holds for all $\alpha > 1$, we conclude that $h \in \mathcal{D}_p(\boldsymbol{w_b})$. □

One consequence of this characterization is that logarithmically equivalent Cantor sets have the same packing and Hausdorff dimension.

Corollary 4.7. *If $C_a \diamond C_b$, then the packing and Hausdorff dimensions of C_a and C_b coincide.*

Proof. This follows directly from the characterization (ii) above and the formulas given in (3). □

Remark 4.8. One cannot obtain a similar characterization of logarithmic equivalence if, instead of considering all dimension functions, only functions of the form $h(t) = t^s$ are considered. To see this, suppose $a_n = \alpha^{k-1}(1-2\alpha)$ and $b_n = \beta^{k-1}(1-2\beta)$ when $2^{k-1} \leq n \leq 2^k - 1$. If $w \in W_\infty$ is chosen such that $\lim \frac{n_0(w|k)}{k}$ does not exist and $\mathcal{D}'_p(\boldsymbol{w_a}) = \{s \in \mathbb{R} : \boldsymbol{w_a} \in \mathcal{E}(\mu_p^a, s)\}$, then for $p \neq 1/2$, $\mathcal{D}'_p(\boldsymbol{w_a}) = \mathcal{D}'_p(\boldsymbol{w}_b)$ (both are empty). But clearly C_a and C_b are not logarithmically equivalent when $\alpha \neq \beta$ (they have different dimension, in fact).

5. The Size of the Sets $\mathcal{E}(\mu, h)$.

In this section our purpose is to extend the multifractal analysis of Cantor measures on central Cantor sets developed in [10] (see Cor. 5.5 below). Our method of proof was inspired by [10, Section 5] and [7, 11.2].

We will use the following notation: For $p \in (0,1)$ and a real number q, let

$$b_q = \frac{p^q \log p + (1-p)^q \log(1-p)}{p^q + (1-p)^q}, \qquad \theta_q = q - \frac{\log(p^q + (1-p)^q)}{b_q},$$

$$b_{\min} = \min(\log p, \log(1-p)) \qquad \text{and} \qquad b_{\max} = \max(\log p, \log(1-p)).$$

When $p = 1/2$, $b_q = -\log 2$ and $\theta_q = 1$ for all q. Otherwise, as b_q is a convex combination of $\log p$ and $\log(1-p)$ the set $\{b_q : q \in \mathbb{R}\}$ is the interval $(b_{\min}, b_{\max})$.

We will write $\mathcal{E}_h$ and $\mathcal{E}^h$ for $\mathcal{E}_h(\mu)$ and $\mathcal{E}^h(\mu)$ respectively.

Theorem 5.1. *Let μ_p^a be the p-Cantor measure on the Cantor set C_a and let $h \in \mathcal{D}$.*

(i) Suppose there is a real number q such that $\liminf_{k\to\infty} \frac{1}{k} \log h(s_k) = b_q$. Then $\mathcal{H}^{h^{\lambda\theta_q}}(\mathcal{E}_h) \geq 1$ for all $\lambda < 1$.

(ii) Suppose there is a real number q such that $\limsup_{k\to\infty} \frac{1}{k} \log h(s_k) = b_q$. Then $\mathcal{P}^{h^{\lambda\theta_q}}(\mathcal{E}^h) \leq 1$ for all $\lambda > 1$.

In order to prove this Theorem, we will introduce an auxiliary measure. For a given $q \in \mathbb{R}$ we define $\nu = \nu_q$ on Cantor intervals as

$$\nu(I_w) = p^{qn_0(w)}(1-p)^{qn_1(w)} \left(p^q + (1-p)^q\right)^{-k} \tag{12}$$

where $k = |w|$. It is easy to verify that $\nu(I_{w0}) + \nu(I_{w1}) = \nu(I_w)$, hence the measure is well defined and concentrated on the Cantor set. The same reasoning as for p-Cantor measures proves that ν is a doubling measure.

The following Lemma establishes conditions on h and q which implies that ν_q is concentrated in $\mathcal{E}_h$ or $\mathcal{E}^h$.

Lemma 5.2. *Assume $\nu = \nu_q$ is as defined above.*

(i) If $\liminf_{k\to\infty} k^{-1} \log h(s_k) = b_q$, then $\nu_q(\mathcal{E}_h) = 1$.

(ii) If $\limsup_{k\to\infty} k^{-1} \log h(s_k) = b_q$, then $\nu_q(\mathcal{E}^h) = 1$.

Proof. (i) For fixed $\varepsilon > 0$ and integer k, consider the set

$$E_k := \{x \in supp(\mu) : \log \mu(I_k(x)) \geq (1-\varepsilon) \log h(s_k)\}.$$

For any $\delta > 0$, we have the following estimation:

$$\begin{aligned} \nu(E_k) &= \nu\left\{x \in supp(\mu) : \mu(I_k(x)) \geq h(s_k)^{1-\varepsilon}\right\} \\ &\leq \int \mu(I_k(x))^\delta h(s_k)^{\delta(\varepsilon-1)} d\nu(x) = \sum_{|w|=k} h(s_k)^{\delta(\varepsilon-1)} \mu(I_w)^\delta \nu(I_w) \\ &= h(s_k)^{\delta(\varepsilon-1)} \left(p^q + (1-p)^q\right)^{-k} \left(p^{q+\delta} + (1-p)^{q+\delta}\right)^k := \Phi^+(k). \end{aligned}$$

Calculating the Taylor expansion of first order to the function $\log(p^t + (1-p)^t)$, we obtain that

$$\log \Phi^+(k) = k\delta \left((\varepsilon - 1) \frac{\log h(s_k)}{k} + b_q + O(\delta) \right).$$

Since $\liminf_{k\to\infty} k^{-1} \log h(s_k) = b_q$, for any $\varepsilon_0 > 0$ there exists an integer k_0 such that for all $k \geq k_0$ we have $k^{-1} \log h(s_k) \geq b_q - \varepsilon_0$. Thus for all $k \geq k_0$,

$$\log \Phi^+(k) \leq k\delta \left(\varepsilon b_q - \varepsilon_0(\varepsilon - 1) + O(\delta) \right).$$

Since $b_q < 0$, by taking ε_0 and δ suitably small we obtain $\Phi^+(k) \leq \exp\left(\frac{k\delta b_q \varepsilon}{2}\right)$ for all $k \geq k_0$ and this proves that $\nu(E_k)$ is summable. By the Borel Cantelli Lemma we have that for ν-almost every $x \in supp(\mu)$ there is an integer k_1 such that $\log \mu(I_k(x)) < (1-\varepsilon) \log h(s_k)$ for all $k \geq k_1$. Thus

$$\liminf_{k\to\infty} \frac{\log \mu(I_k(x))}{\log h(s_k)} \geq 1 - \varepsilon \qquad \text{for } \nu\text{-almost every } x.$$

Analogously, if we define

$$E^k := \{x \in supp(\mu) : \log \mu(I_k(x)) < (1+\varepsilon) \log h(s_k)\}$$

with an estimation similar to (13) we obtain

$$\nu(E^k) \leq h(s_k)^{\delta(1+\varepsilon)} \left(p^q + (1-p)^q\right)^{-k} \left(p^{q-\delta} + (1-p)^{q-\delta}\right)^k := \Phi_-(k).$$

Also,

$$\log \Phi_-(k) \leq k\delta \left((\varepsilon + 1) \frac{\log h(s_k)}{k} - b_q + O(\delta) \right).$$

But the assumption that $\liminf_{k\to\infty} k^{-1} \log h(s_k) = b_q$ implies that along a suitable subsequence

$$\log \Phi_-(k) \leq k\delta \left(\varepsilon b_q + \varepsilon_0(\varepsilon + 1) + O(\delta) \right) \leq k\delta(\varepsilon b_q/2),$$

for small enough ε_0 and δ. Thus $\nu(E^k)$ is summable in a subsequence and this proves

$$\liminf_{k\to\infty} \frac{\log \mu(I_k(x))}{\log h(s_k)} \leq 1 + \varepsilon \text{ for } \nu\text{-almost every } x.$$

Consequently, $\nu(\mathcal{E}_h) = 1$.

(ii) is similar. □

The following Proposition can be viewed as a mass distribution principle for h-Hausdorff and packing measures.

Proposition 5.3. *Let ν be any measure and let h be a dimension function.*

(i) If $\limsup_{r\to 0^+} \frac{\log \nu(B_r(x))}{\log h(r)} \leq \theta < \infty$ for all $x \in E$, then $\mathcal{P}^{h^{\lambda\theta}}(E) \leq \nu(E)$ for all $\lambda > 1$.

(ii) If $\liminf_{r\to 0^+} \frac{\log \nu(B_r(x))}{\log h(r)} \geq \theta > 0$ for all $x \in E$, then $\mathcal{H}^{h^{\theta\lambda}}(E) \geq \nu(E)$ for all $\lambda < 1$.

Proof. (i) Consider an open set $V \supset E$ such that $\nu(V) \le \nu(E) + \varepsilon$ and define

$$V_k = \left\{ x \in V : h(r)^{\lambda\theta} \le \nu(B_r(x)) \text{ and } B_r(x) \subset V \quad \forall r < 1/k \right\}.$$

Note that $V_k \subset V_{k+1}$ and, by hypothesis, $E \subset \bigcup V_k$.

For a fixed k, let $\left\{ B_j = B_{r_j}(x_j) \right\}$ an arbitrary δ packing of V_k with $\delta < 1/k$. We have that

$$\nu(V) \ge \nu\left(V \cap (\cup B_j)\right) = \sum_{j\ge1} \nu(V \cap B_j) = \sum_{j\ge1} \nu(B_j) \ge \sum_{j\ge1} h(r_j)^{\lambda\theta}.$$

Since the packing was arbitrary, we conclude that $\nu(V) \ge \mathcal{P}^{h^{\lambda\theta}}(V_k)$, and the desired inequality follows by continuity.

(ii) Argue in a similar fashion to [8, Prop. 4.9(a)]. $\square$

Proof of the Theorem 5.1. (i) Let $\nu = \nu_q$ be the auxiliary measure for the specified q. Clearly,

$$\frac{\log \nu(I_k(x))}{\log h(s_k)} = q \frac{\log \mu(I_k(x))}{\log h(s_k)} - \frac{\log(p^q + (1-p)^q)}{\frac{1}{k} \log h(s_k)}, \tag{13}$$

thus if $x \in \mathcal{E}_h$, then

$$\liminf_{k\to\infty} \frac{\log \nu(I_k(x))}{\log h(s_k)} \ge q - \frac{\log(p^q + (1-p)^q)}{b_q} = \theta_q.$$

It is elementary to verify that $\log(p^q + (1-p)^q) \ge q \max\{\log p, \log(1-p)) > -q\,|b_q|$, thus $\theta_q > 0$.

Since ν is a doubling measure, Theorem 3.7 shows that the ν-measure of balls and (suitable) Cantor intervals are compararable. Invoking Lemma 5.2 and Prop. 5.3 completes the proof.

(ii) is similar. $\square$

Corollary 5.4. *If* $\lim_{k\to\infty} k^{-1} \log h(s_k) = b_q$*, then* $\mathcal{H}^{h^{\lambda\theta_q}}(\mathcal{E}(h)) \ge 1$ *for all* $\lambda < 1$ *and* $\mathcal{P}^{h^{\lambda'\theta_q}}(\mathcal{E}(h)) \le 1$ *for all* $\lambda' > 1$.

Proof. By Lemma 5.2 we have $\nu(\mathcal{E}_h \cap \mathcal{E}^h) = 1$. Looking at (13), for $x \in \mathcal{E}_h \cap \mathcal{E}^h$ we have

$$\lim_{k\to\infty} \frac{\log \nu(I_k(x))}{\log h(s_k)} = q - \frac{\log(p^q + (1-p)^q)}{b_q} = \theta_q.$$

Since θ_q is always positive and finite, using Proposition 5.3 we obtain the result. $\square$

Corollary 5.5. *Suppose* $\underline{\tau} = \liminf_{k\to\infty} k^{-1} \left|\log s_k^a\right|$ *and* $\overline{\tau} = \limsup_{k\to\infty} k^{-1} \left|\log s_k^a\right|$.

(i) If $\overline{\tau} < \infty$ *and* $\alpha = |b_q| / \overline{\tau}$*, then* $\dim_H \mathcal{E}_\alpha \ge q\alpha + (\log(p^q + (1-p)^q))/\overline{\tau}$.

(ii) If $\underline{\tau} < \infty$ *and* $\alpha = |b_q| / \underline{\tau}$ *, then* $\dim_P \mathcal{E}^\alpha \le q\alpha + (\log(p^q + (1-p)^q))/\underline{\tau}$.

(iii) Suppose $\underline{\tau} = \overline{\tau} = \tau < \infty$ *and* $\alpha \in (-b_{\max}/\tau, -b_{\min}/\tau)$*. If* q *is chosen such that* $\alpha\tau = |b_q|$*, then*

$$\dim_H \mathcal{E}(\alpha) = \dim_P \mathcal{E}(\alpha) = q\alpha + \frac{\log(p^q + (1-p)^q)}{\tau}.$$

This result was previously obtained for central Cantor sets in [10, Section 5]. We remind the reader that $\underline{\tau} = \log 2/\dim_H C_a$ and $\overline{\tau} = \log 2/\dim_P C_a$ (see (3)).

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