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The Sizes of Rearrangements of Cantor Sets

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Abstract. A linear Cantor set C with zero Lebesgue measure is associated with the countable collection of the bounded complementary open intervals. A *rearrangment* of C has the same lengths of its complementary intervals, but with different locations. We study the Hausdorff and packing h-measures and dimensional properties of the set of all rearrangments of some given C for general dimension functions h. For each set of complementary lengths, we construct a Cantor set rearrangement which has the maximal Hausdorff and the minimal packing h-premeasure, up to a constant. We also show that if the packing measure of this Cantor set is positive, then there is a rearrangement which has infinite packing measure.

1 Introduction

Given E, a compact subset of the real line contained in the interval I, its complement $I \setminus E$ is the union of a countable collection of open intervals, say

$$I \setminus E = \bigcup_j A_j.$$

Clearly the intervals A_j determine E but, surprisingly, some geometric information is obtainable from knowing only the lengths (for example, the pre-packing (upper-box) dimension, see [6]) and not the positioning of the A_j's.

In this paper we are interested in singular sets, so we assume that the Lebsegue measure of E is zero. Furthermore, for simplicity, we assume that the endpoints of I are contained in E so that $|I| = |E|$ (where by $|S|$ we mean the diameter of $S \subset \mathbb{R}$). These two assumptions imply that $\sum_n a_n = |I|$, where $a_n = |A_n|$.

For a given positive, summable and non-increasing sequence $a = (a_n)$ there are many possible linear closed sets E such that the complementary intervals have lengths given by the terms of the sequence. Such a rearrangement E will be said to *belong to the sequence* (a_j) or $E \in \mathscr{C}_a(I)$ (or shortly, $\mathscr{C}_a$). Our main interest lies in the properties of the collection $\mathscr{C}_a$ for a fixed sequence a, particularly in the dimensional behaviour as we range over $\mathscr{C}_a$.

These sets were first studied by Borel [1] and Besicovitch and Taylor [2]. In their seminal paper, Besicovitch and Taylor studied the s-Hausdorff dimension and measures of these cut-out sets. In particular, they proved that

$$\{\dim_H(E) : E \in \mathscr{C}_a\} \text{ is a closed interval} \tag{1.1}$$

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and constructed a Cantor set $C_a \in \mathscr{C}_a$, as described below, with maximal Hausdorff dimension and measure. Cabrelli et al. [4] and Garcia et al. [9] continued this study and, among other things, constructed a concave dimension function h so that C_a is an h-set (that is, $0 < \mathcal{H}^h(C_a) \leq \mathcal{P}^h(C_a) < \infty$). Xiong and Wu [19] showed that $\mathscr{C}_a$ is a compact metric space under the Hausdorff distance ρ and studied density-type properties in $(\mathscr{C}_a(I), \rho)$. Lapidus and co-workers (see [10,13] and the references therein) studied these sets under the name "fractal strings" and were especially interested in inverse spectral problems and a surprising relationship with the Riemann zeta function and the Riemann Hypothesis.

We prove a generalization of (1.1) for arbitrary dimension functions h for both Hausdorff and packing measures. In contrast to the Besicovitch and Taylor result for Hausdorff measure and despite the fact that the (pre)packing dimension of the Cantor set C_a is maximal over all $E \in \mathscr{C}_a$, we show that C_a has the minimal packing h-premeasure of the sets in $\mathscr{C}_a$ (up to a constant). Furthermore, if the packing h-measure of C_a is positive (such as if $h(x) = x^s$ when C_a is an s-set), then there is some rearrangement $E \in \mathscr{C}_a$ with infinite packing h-measure. In fact, $\{\mathcal{P}^h(E) : E \in \mathscr{C}_a\}$ is either equal to $\{0\}$ or is equal to $[0, \infty]$. Finally, we also generalize a density result from [19] to arbitrary dimension functions.

2 Notation

2.1 The Sets C_a and D_a

There are two sets belonging to a given sequence $a = (a_n)$ to which we will often refer.

One is built using a Cantor construction and will be denoted by C_a. We begin with a closed interval I of length $\sum a_n$ and remove from it an open interval with length a_1. This leaves two closed intervals, I_1^1 and I_2^1, called the *intervals of step one*. If we have constructed $\{I_j^k\}_{1 \leq j \leq 2^k}$, the intervals of step k, we remove from each interval I_j^k an open interval of length $a_{2^k + j - 1}$, obtaining two closed intervals of step $k+1$, namely I_{2j-1}^{k+1} and I_{2j}^{k+1}. We define

$$C_a := \bigcap_{k \geq 1} \bigcup_{1 \leq j \leq 2^k} I_j^k.$$

This process uniquely determines the set C_a. For instance, the position of the first interval to be removed (of length a_1) is uniquely determined by the property that the length of the remaining interval on the left is $a_2 + a_4 + a_5 + a_8 + \cdots$. The classical middle-third Cantor set is the set C_a associated with the sequence $a = (a_n)$, where $a_j = 3^{-n}$ if $2^{n-1} \leq j < 2^n$.

The set C_a is compact, perfect and totally disconnected. The average length of a step k interval is $r_{2^k}/2^k$, where $r_n = \sum_{i \geq n} a_i$. Since the sequence (a_n) is decreasing, any interval of step $k-1$ has length at least the average length at step k, and this, in turn, is at least the length of any interval of step $k+1$.

The other important set in the class $\mathscr{C}_a(I)$ is a countable set that will be denoted by D_a. If $I = [\alpha, \beta]$, where $\beta = \alpha + \sum_{j \geq 1} a_j$, and $x_n = \sum_{j \leq n} a_j$, then

$$D_a := \{\alpha\} \cup \{\alpha + x_n : n \geq 1\} \cup \{\beta\}.$$

2.2 Dimension Functions

We will say that $h\colon (0,\infty) \to \mathbb{R}$ is a *dimension function* if h is increasing, continuous, doubling, *i.e.,* $h(2x) \le c\,h(x)$, and satisfies $\lim_{x\to 0} h(x) = 0$. The class of dimension functions will be denoted $\mathcal{D}$.

Given two dimension functions $g, h,$, we say $g \prec h$ if $\lim_{t\to 0} h(t)/g(t) = 0$ and $g \sim h$ (and say g is comparable to h) if there are positive constants c_1, c_2 such that $c_1 h(t) \le g(t) \le c_2 h(t)$ for t small. We will write $g \preceq h$ if either $g \prec h$ or $g \sim h$.

2.3 Hausdorff and Packing h-Measures

For any dimension function h, the *Hausdorff h-measure* $\mathcal{H}^h$ can be defined in a similar fashion to the familiar Hausdorff measure (see [15]). Given E, a subset of $\mathbb{R}$, we denote by $|E|$ its diameter. A *δ-covering of E* is a countable family of subsets with diameters at most δ, whose union contains E. Define

$$\mathcal{H}^h_\delta(E) = \inf\Big\{ \sum_{i\ge 1} h(|E_i|) : (E_i) \text{ is a } \delta\text{-covering of } E \Big\},$$

$$\mathcal{H}^h(E) = \lim_{\delta\to 0} \mathcal{H}^h_\delta(E).$$

The h-packing measure and premeasure can be defined similarly (see [17]). A *δ-packing of a set E* is a disjoint family of open intervals, centred at points in E, and with diameters at most δ. Define

$$\mathcal{P}^h_\delta(E) = \sup\Big\{ \sum_{i\ge 1} h(|E_i|) : (E_i) \text{ is a } \delta\text{-packing of } E \Big\}.$$

The *h-packing premeasure* $\mathcal{P}^h_0$ is given by

$$\mathcal{P}^h_0(E) = \lim_{\delta\to 0} \mathcal{P}^h_\delta(E).$$

As $\mathcal{P}^h_0$ is not a measure, we also define the *h-packing measure* of E, $\mathcal{P}^h(E)$, as

$$\mathcal{P}^h(E) = \inf\Big\{ \sum_i \mathcal{P}^h_0(E_i) : E = \bigcup_{i=1}^{\infty} E_i \Big\}.$$

Clearly, $\mathcal{P}^h(E) \le \mathcal{P}^h_0(E)$ for any set E and since h is doubling, $\mathcal{H}^h(E) \le \mathcal{P}^h(E)$ ([16]).

In the special case when $h_s(x) = x^s$, $\mathcal{H}^{h_s}$ is the usual s-dimensional Hausdorff measure and similarly for the s-packing (pre)measure.

For a given set E put

$$N(E,\varepsilon) = \min\{k : E \subset \bigcup_{i=1}^{k} B(x_i,\varepsilon)\},$$

$$P(E,\varepsilon) = \max\{k : \exists \text{ disjoint } (B(x_i,\varepsilon))_{i=1}^{k} \text{ with } x_i \in E\}.$$

Elementary geometric reasoning shows that for any set E

$$N(E,2\varepsilon) \le P(E,\varepsilon) \le N(E,\varepsilon/2). \tag{2.1}$$

Furthermore, it is obvious that

$$\mathcal{H}^h(E) \le \liminf_r N(E,r)h(r) \quad \text{and} \quad \mathcal{P}_0^h(E) \ge \limsup_r P(E,r)h(r).$$

Also, if $f \preceq h$, then for any set E there is a constant c such that $\mathcal{H}^h(E) \le c\mathcal{H}^f(E)$, and similarly for packing (pre)measures.

The upper box dimension of E is given by

$$\limsup_{r\to 0} \frac{\log(N(E,r))}{-\log r} = \limsup_{r\to 0} \frac{\log(P(E,r))}{-\log r}$$

and is known to coincide with the pre-packing dimension of E, *i.e.*, the index given by the formula $\inf\{s : \mathcal{P}_0^{h_s}(E) = 0\}$ ([17]).

3 Hausdorff Measures of Rearrangements

In [2], Besicovitch and Taylor gave bounds for the Hausdorff s-measures of Cantor sets C_a in terms of the asymptotic rate of decay of the tail sums,

$$r_n = \sum_{i\ge n} a_i,$$

of the sequence. In [9], those estimates were extended to h-Hausdorff and packing premeasures.

Theorem 3.1 ([9]) *Suppose $h \in \mathcal{D}$. Then*

(i) $1/4 \liminf_{n\to\infty} nh(r_n/n) \le \mathcal{H}^h(C_a) \le 4 \liminf_{n\to\infty} nh(r_n/n)$,
(ii) $1/8 \limsup_{n\to\infty} nh(r_n/n) \le \mathcal{P}_0^h(C_a) \le 8 \limsup_{n\to\infty} nh(r_n/n)$.

A set E is called an *s-set* if $0 < \mathcal{H}^s(E) \le \mathcal{P}^s(E) < \infty$. Although not all Cantor sets C_a are s-sets, Cabrelli et al. [4] proved that for any non-increasing sequence (a_n) there is a concave function $h_a \in \mathcal{D}$ such that $h_a(r_n/n) \sim 1/n$. Thus C_a is an h_a-set. Any function with the property $h(r_n/n) \sim 1/n$ is called an *associated dimension function* and all associated dimension functions for a given sequence a are comparable. The set C_a has Hausdorff and packing h-premeasure finite and positive if and only if h is an associated dimension function [3].

Given $E \subseteq \mathbb{R}$ and $\varepsilon > 0$, let $E(\varepsilon) = \{x \in \mathbb{R} : |x-y| < \varepsilon \text{ for some } y \in E\}$. Falconer [6, 3.17] observed that if $E, E' \in \mathscr{C}_a$, then $\mathcal{L}(E(\varepsilon)) = \mathcal{L}(E'(\varepsilon))$, where $\mathcal{L}$ denotes the Lebesgue measure. Observe that any union of ε-balls with centres in E is contained in $E(\varepsilon)$ and any union of 2ε-balls covers $E(\varepsilon)$ if the union of the ε-balls with the same centres covers E. Thus we have

$$P(E,r)2r \le \mathcal{L}(E(r)) \le N(E,r)4r. \tag{3.1}$$

Combining (2.1) and (3.1) gives the following useful geometric fact.

Lemma 3.2 *For any $E \in \mathscr{C}_a$ and $\varepsilon > 0$,*

$$P(C_a, \varepsilon) \le 2N(E, \varepsilon) \le 2P(E, \varepsilon/2) \le 4N(C_a, \varepsilon/2).$$

Besicovitch and Taylor [2] showed that C_a has maximal $\mathcal{H}^s$ measure in $\mathscr{C}_a$. Our first result extends this (up to a constant) for arbitrary h. We remark that if h is assumed to be concave, the same arguments as given in [2] show that $\mathcal{H}^h(E) \le \liminf_{n\to\infty} nh(r_n/n)$ for any $E \in \mathscr{C}_a$.

Proposition 3.3 *If $h \in \mathcal{D}$ and $E \in \mathscr{C}_a$, then $\mathcal{H}^h(E) \le c\mathcal{H}^h(C_a)$, where c depends only on the doubling constant of h.*

Proof Since h is a doubling function, the lemma above together with the definitions of $\mathcal{H}^h$ and $N(E, r)$ imply

$$\mathcal{H}^h(E) \le \liminf_{r\to 0} N(E, r)h(r) \le c \liminf_{r\to 0} N(C_a, r)h(r).$$

Temporarily fix $r > 0$ and choose n such that

$$\frac{r_{2^{n-1}}}{2^{n-1}} \ge r \ge \frac{r_{2^n}}{2^n}.$$

Since the length of any Cantor interval at step $n+1$ is at most the average of the lengths of the step n intervals, the 2^{n+1} intervals centred at the right end points of the Cantor intervals of step $n+1$ and radii $r_{2^n}/2^n$ cover C_a. Thus $N(C_a, r) \le 2^{n+1}$ and hence

$$N(C_a, r)h(r) \le 2^{n+1} h\Big(\frac{r_{2^{n-1}}}{2^{n-1}}\Big) \le 4 \cdot 2^{n-1} h\Big(\frac{r_{2^{n-1}}}{2^{n-1}}\Big).$$

Therefore, Theorem 3.1 implies

$$\mathcal{H}^h(C_a) \ge \frac{1}{4} \liminf_{n\to\infty} 2^n h\Big(\frac{r_{2^n}}{2^n}\Big) \ge \frac{1}{16} \liminf_{r\to 0} N(C_a, r)h(r) \ge \frac{1}{16c}\mathcal{H}^h(E). \qquad \blacksquare$$

Remark 3.4 If C_a corresponds to a middle-τ Cantor set, then $\mathcal{H}^s(C_a) = 1 = \liminf_n n(r_n/n)^s$, where $s = -\log 2/\log(\tau)$. Thus the comment immediately before the proposition shows we may take $c = 1$ in the proposition and C_a has the maximal $\mathcal{H}^s$ measure amongst $E \in \mathscr{C}_a$ in this case. For the general case, it is unknown what the minimal constant c is and which set $E \in \mathscr{C}_a$ (if any) has the maximum Hausdorff measure.

Besicovitch and Taylor [2] also show that if $s < \dim_H C_a$, then for any $\gamma \ge 0$ there is a rearrangement E such that $\mathcal{H}^s(E) = \gamma$. We extend this result to dimension functions and also prove that, in addition, E can be chosen to be perfect.

Theorem 3.5 *Let I be an interval with $|I| = \sum a_i$. If $h \prec h_a$ and $\gamma \ge 0$, then there is a perfect set $E \in \mathscr{C}_a(I)$ such that $\mathcal{H}^h(E) = \gamma$.*

Proof As shown in [3], the assumption $h \prec h_a$ implies that $\mathcal{H}^h(C_a) = \infty$, thus by [12] there exists a closed subset $E \subset C_a$ with $\mathcal{H}^h(E) = \gamma$. The set E might not be perfect or belong to the sequence (a_n), so we will modify it in order to obtain the desired properties.

Since both E and C_a are closed, there are collections of open intervals A_j and (α_j, β_j) such that

$$I \setminus C_a = \bigcup_{i\geq 1} A_i \qquad I \setminus E = \bigcup_{j\geq 1} (\alpha_j, \beta_j).$$

Fix $j \geq 1$ and define $\Lambda_j = \{i : (\alpha_j, \beta_j) \supset A_i\}$. Of course, $\sum_{i\in\Lambda_j} |A_i| = \sum_{i\in\Lambda_j} a_i = \beta_j - \alpha_j$. Since C_a is perfect, Λ_j is either a singleton or infinite. In the first case the length of the gap (α_j, β_j) is a term of the sequence (a_n).

If, instead, Λ_j is infinite, consider the terms $\{a_i : i \in \Lambda_j\}$ in decreasing order and call this subsequence $a^{(j)}$. For each fixed j, we will decompose the subsequence $a^{(j)}$ into countably many subsubsequences $a^{(j,k)}$ for $k = 1, 2, \ldots$.

First, fix a sequence d_n such that $h(d_n) \leq n^{-2}$. We start by defining $a^{(j,1)}$ and begin by putting $a_1^{(j,1)} = a_1^{(j)}$. Assume $a_i^{(j,1)}$ are defined for $i = 1, 2, \ldots, m-1$ and $a_{m-1}^{(j,1)} = a_{N'}^{(j)}$. Pick the first integer $N > N'$ satisfying $a_N^{(j)} \leq d_m - d_{m+1}$ and define $a_m^{(j,1)} = a_{N+m}^{(j)}$. (We do not just take $a_N^{(j)}$ in order to have enough terms to build $a^{(j,k)}$ for $k \geq 1$.)

Now inductively assume $a^{(j,k)}$ have been defined for $k = 1, 2, \ldots, m-1$. We let $a_1^{(j,m)}$ be the first term of $a^{(j)}$ that was not picked in $a^{(j,k)}$ for $k < m$. If also the terms $a_i^{(j,m)}$ are defined for $i = 1, 2, \ldots, l-1$, pick N to be the first integer satisfying:

(i) $a_N^{(j)}$ is not an element of one of the sequences $a^{(j,k)}$, $k = 1, \ldots, m-1$, that have already been defined;

(ii) $N > N'$, where N' is defined by $a_{l-1}^{(j,m)} = a_{N'}^{(j)}$;

(iii) $a_N^{(j)} \leq d_l - d_{l+1}$.

Then put $a_l^{(j,m)} = a_k^{(j)}$, where $k \geq N + l$ is the minimal index not already chosen. Note that the union of $a^{(j,k)}$ for $k = 1, 2, \ldots$ is $a^{(j)}$ and by (iii),

$$\liminf_{n\to\infty} nh\Big(\sum_{i\geq n} a_i^{(j,k)}/n\Big) = 0 \quad \text{for all } j, k.$$

Inside each interval $[\alpha_j, \beta_j]$ consider the subintervals $I_m^{(j)}$ with length equal to $\sum_i a_i^{(j,m)}$. Construct within each such subinterval the Cantor set $C^{(j,m)}$ associated with the sequence $a^{(j,m)}$. By Theorem 3.1 we have $\mathcal{H}^h(C^{(j,k)}) = 0$ for any pair (j, k). The set $E \cup (\bigcup_{j,m} C^{(j,m)})$ is perfect, belongs to $\mathscr{C}_a(I)$ and has the same Hausdorff h-measure as E. ■

A direct consequence of this theorem is the following extension of Theorem 2 in [19]. Let ρ be the Hausdorff metric defined for compact subsets of the real line by

$$\rho(A, B) = \max\{\sup_{y\in B} \inf_{x\in A} d(x, y),\ \sup_{x\in B} \inf_{y\in A} d(x, y)\}.$$

In [19] it was shown that $(\mathscr{C}_a(I), \rho)$ is compact.

Corollary 3.6 *Let (a_n) be a decreasing, positive and summable sequence and let I be an interval with $|I| = \sum a_k$. If $h \prec h_a$ and $\gamma \geq 0$, the set $\Gamma = \{E \in \mathscr{C}_a(I) : \mathcal{H}^h(E) = \gamma\}$ is dense in $\mathscr{C}_a(I)$ with the Hausdorff metric.*

Proof Fix $E \in \mathscr{C}_a(I)$ and $n \in \mathbb{N}$. As usual, assume $I \setminus E = \bigcup_j A_j$ where the lengths of A_j are decreasing. There is a permutation σ of $\{1, \ldots, n\}$ (determined by n and E) such that $A_{\sigma(1)}, \ldots, A_{\sigma(n)}$ are placed from left to right, meaning that if $x_i \in A_{\sigma(i)}$, then $x_1 < x_2 < \cdots < x_n$. We define a subfamily of $\mathscr{C}_a(I)$ by

$$\mathscr{C}_a^n(I) = \{F \in \mathscr{C}_a(I) : \text{ if } x_i \in A^F_{\sigma(i)}, \text{ then } x_1 < x_2 < \cdots < x_n\},$$

where $\{A^F_i\}$ are the intervals (in order of decreasing lengths) whose union is the complement of F.

In [19] the authors proved that $\operatorname{diam}(\mathscr{C}_a^n(I)) \leq 3r_{n+1}$, thus it is enough to prove that $\Gamma \cap \mathscr{C}_a^n(I) \neq \varnothing$.

Put $I = [\alpha, \beta]$, $\tilde{I} = [\alpha + \sum_{k=0}^n a_k, \beta]$, and $\tilde{a}_k = a_{n+k}$. By Theorem 3.5, there is a set $\tilde{E} \in \mathscr{C}_{\tilde{a}}(\tilde{I})$ with $\mathcal{H}^h(\tilde{E}) = \gamma$. The set $F = \{\alpha + \sum_{k=0}^j a_k : 0 \leq j \leq n\} \cup \tilde{E}$ belongs to $\Gamma \cap \mathscr{C}_a^n(I)$. ∎

4 Packing Measures and Packing Premeasures of Rearrangements

In contrast to the case for Hausdorff measure, it was shown in [6] that the prepacking dimension is the same for any set $E \in \mathscr{C}_a$. Furthermore, as we show next, the packing premeasure of the Cantor set is (up to a constant) the *least* premeasure of any set with the same gap lengths. This result is dual to Proposition 3.3.

Proposition 4.1 *There is a constant c such that if $E \in \mathscr{C}_a$, then $\mathcal{P}_0^h(C_a) \leq c\mathcal{P}_0^h(E)$.*

Proof Similar arguments to Proposition 3.3 show that

$$\mathcal{P}_0^h(C_a) \leq c \limsup_{r \to 0} P(C_a, r) h(r).$$

But $P(C_a, r) \leq 2P(E, r/2)$ for any $E \in \mathscr{C}_a$ and for any set E, $\limsup_{r\to 0} P(E, r)h(r)$ is a lower bound for $\mathcal{P}_0^h(E)$. Combine these observations. ∎

As is the case with Hausdorff measures, the sharp value of c and the exact set from $\mathscr{C}_a$ which minimizes $\mathcal{P}_0^h$ is unknown, even in the case of the middle-third Cantor set C_a. It is known that $4^s = \mathcal{P}^s(C_a) \neq \limsup n(r_n/n)^s = 1$, where $s = \log 2/\log 3$ (see [7, 8]).

Corollary 4.2 *If $\mathcal{P}_0^h(C_a) > 0$, then $\mathcal{P}_0^h(D_a) = \infty$. In particular, $\mathcal{P}_0^{h_a}(D_a) = \infty$.*

Proof Since D_a is countable, $\mathcal{P}^h(D_a) = 0$ (for any h). By virtue of the previous proposition, for this particular h we have $\mathcal{P}_0^h(D_a) > 0$. It was proved in [18] that if $\mathcal{P}_0^h(D_a) < \infty$, then $\mathcal{P}_0^h(D_a) \leq c\mathcal{P}^h(D_a)$ for a suitable constant c. But this is not the case. ∎

From here on we will be more restrictive with the dimension functions and require, in addition, that they are subadditive, *i.e.,* there is a constant C such that $h(x+y) \leq C(h(x)+h(y))$ for all x, y. Peetre showed that any function equivalent to a concave function is subadditive [14]. Since every sequence admits an associated dimension function that is concave [4], any function h which makes C_a an h-set will be subadditive.

Lemma 4.3 *If $h \in \mathcal{D}$ is subadditive, then $\mathcal{P}_0^h(E) \leq 2\mathcal{P}_0^h(D_a)$ for all $E \in \mathscr{C}_a$.*

Proof Without loss of generality $0 \in I$ and $E = I \setminus \bigcup_{j\geq 1} A_j$ where A_j are open intervals with decreasing lengths, $|A_j| = a_j$. Consider any δ-packing of E, say $\{B_j\}$. For each j, let $\Delta_j = \{i : A_i \cap B_j \text{ is not empty}\}$.

Let B_i' denote the interval centered at $x_i = \sum_{n\leq i} a_n$ (where $x_0 = 0$) and diameter equal to $\min(a_{i+1}, \delta)$. The balls $\{B_i'\}$, $i = 0, 1, 2, \ldots$ form a δ-packing of D_a, thus $\sum h(|B_j'|) \leq \mathcal{P}_\delta^h(D_a)$. By subadditivity,

$$\sum h(|B_j|) \leq \sum_j \sum_{i\in\Delta_j} h(|A_i \cap B_j|) \leq \sum_j \sum_{i\in\Delta_j} h(\min(a_i, \delta))$$
$$\leq 2\sum_{i\geq 1} h(|B_{i-1}'|) \leq 2\mathcal{P}_\delta^h(D_a),$$

where the penultimate inequality holds because each i belongs to Δ_j for at most two choices of j. Since $\{B_j\}$ was an arbitrary δ-packing of E, the result follows. ∎

It is known that for Cantor sets C_a the packing dimension coincides with the pre-packing dimension [3]. Since the pre-packing dimension of all sets in $\mathscr{C}_a$ coincide and the pre-packing dimension is an upper bound for the packing dimension of a set, it follows that $\dim_p C_a \geq \dim_p E$ for any $E \in \mathscr{C}_a$. Despite this, we have the following theorem.

Theorem 4.4 *If $h \in \mathcal{D}$, is subadditive the following statements are equivalent.*

(i) *There exists a set $E \in \mathscr{C}_a$ with $\mathcal{P}_0^h(E) > 0$.*
(ii) $\mathcal{P}_0^h(D_a) = \infty$.
(iii) $\sum h(a_i) = \infty$.
(iv) *There exists a perfect set $E \in \mathscr{C}_a$ with $\mathcal{P}^h(E) = \infty$.*

Proof (iv) $\Rightarrow$ (i). is trivial as $\mathcal{P}_0^h(E) \geq \mathcal{P}^h(E)$.

(i) $\Rightarrow$ (ii). By Lemma 4.3, $\mathcal{P}_0^h(D_a) > 0$ and this forces $\mathcal{P}_0^h(D_a) = \infty$ as in Corollary 4.2.

(ii) $\Rightarrow$ (iii). Since $\mathcal{P}_0^h(D_a) = \infty$, given $\delta > 0$ and M, there is a δ-packing of D_a, say $\{B_i\}$, such that $\sum h(|B_i|) \geq M$. Put $\Delta_j = \{i : (x_i, x_{i+1}) \cap B_j \neq \varnothing\}$. Since a gap of D_a can intersect at most two of these intervals B_j, we have

$$\sum_j h(|B_j|) \leq \sum_j h\Big(\sum_{i\in\Delta_j} (x_i, x_{i+1})\Big) \leq \sum_j \sum_{i\in\Delta_j} h(x_i, x_{i+1}) \leq 2\sum_i h(a_i)$$

and therefore the series $\sum h(a_j)$ is divergent.

(iii) $\Rightarrow$ (iv). Take the interval $I_0 = [0, \sum a_i]$. Choose N_0 such that

$$\sum_{1 \le i \le N_0 - 1} h(a_i) \ge 1$$

and remove from I_0 a total of $N_0 - 1$ open intervals with lengths $a_1, \dots, a_{N_0-1}$, respectively, where we remove these intervals in order from left to right. This produces N_0 closed intervals, denoted by I_j^1 for $j = 1, \dots, N_0$, which we will call the intervals of step one.

Put $N_0^1 = N_0$ and for $1 \le j \le N_0$, choose N_j^1 such that

$$\sum_{N_{j-1}^1 \le i \le N_j^1 - 1} h(a_i) \ge 2.$$

From each I_j^1 we remove $N_j^1 - N_{j-1}^1 - 1$ open intervals with lengths a_i for $i = N_{j-1}^1, \dots, N_j^1 - 1$, again removing them in order from left to right. This produces a total of $S_1 := N_{N_0}^1 - N_0$ closed intervals of step 2 that will be labeled $(I_j^2)_{1 \le j \le S_1}$.

We proceed inductively and assume we have constructed S_{k-1} intervals of step k, $I_1^k, \dots, I_{S_{k-1}}^k$. Put $N_0^k = N_{S_{k-1}}^{k-1}$ and for $j = 1, \dots, S_{k-1}$ pick N_j^k such that

$$\sum_{i = N_{j-1}^k}^{N_j^k - 1} h(a_i) \ge 2^k.$$

From I_j^k remove, from left to right, $N_j^k - N_{j-1}^k - 1$ intervals of lengths a_i for $i = N_{j-1}^k, \dots, N_j^k - 1$ obtaining $S_k := N_{S_k}^k - N_0^k$ closed intervals of step $k+1$, denoted $(I_j^{k+1})_{1 \le j \le S_k}$.

Put $E = \bigcap_{k \ge 1} \bigcup_{1 \le j \le S_k} I_j^{k+1} \in \mathscr{C}_a$. As with the construction of C_a, the fact that $|I| = \sum a_j$ ensures that this construction uniquely determines E. Clearly, $E \in \mathscr{C}_a$ and is perfect.

We claim that $\mathcal{P}^h(E) = \infty$. To see this, suppose that $E \subset \bigcup_i E_i$ with E_i closed. By Baire's Theorem there is (at least) one E_i with non-empty interior and therefore one of the sets E_i contains an interval from some step in the construction. It follows that in order to prove $\mathcal{P}^h(E) = \infty$, it is enough to prove that $\mathcal{P}_0^h(E \cap I_j^k) = \infty$ for any interval I_j^k.

Fix such an interval I_j^k. It will be enough to show that for any $\delta > 0$ and M there is a δ-packing $\{B_i\}$ of $E \cap I_j^k$ with $\sum h(|B_i|) \ge M$. Pick K such that $a_j < \delta$ if $j \ge K$, $2^K \ge M$ and $K \ge k$. Inside I_j^k take an interval of step K, say $I_{j'}^K$. Denote by $A_i = (\alpha_i, \beta_i)$ the gap with length a_i. For $i = N_{j'-1}^K, \dots, N_{j'}^k - 1$ the gaps A_i are inside the interval $I_{j'}^K$. Now take the δ-packing $B_i = (\alpha_i - a_i/2, \alpha_i + a_i/2)$ for $N_{j'-1}^K \le i < N_{j'}^K$. These sets satisfy

$$\sum_{i = N_{j-1}^K}^{N_j^K - 1} h(|B_i|) = \sum_{i = N_{j-1}^K}^{N_j^K - 1} h(a_i) \ge 2^K \ge M. \qquad \blacksquare$$

Since the associated dimension function h_a is subadditive and $\mathcal{P}_0^{h_a}(C_a) > 0$, we immediately obtain the following corollary.

Corollary 4.5 *There exists $E \in \mathcal{C}_a$ such that $\mathcal{P}^{h_a}(E) = \infty$.*

For example, if C_a is the classical middle-third Cantor set, then there exists $E \in \mathcal{C}_a$ such that $\mathcal{P}^s(E) = \infty$ for $s = \log 2/\log 3$.

One can even find functions $f \succ h_a$ for which this is true.

Example 4.6 Take $\{a_n\} = \{n^{-1/p}\}$ for $p < 1$; the associated dimension function is $h_a(x) = x^p$. If we put $f(x) = x^p/|\log x|$, then $f/h \to 0$ as $x \to 0$, f is concave, and $\sum f(a_n) = \infty$. Hence $\mathcal{P}_0^f(D_a) = \infty$ and $\mathcal{P}^f(E) = \infty$ for some $E \in \mathcal{C}_a$.

However, since all sets with the same gap lengths have the same pre-packing dimension there is a severe restriction on the functions f with the property above.

Proposition 4.7 *Suppose $\mathcal{P}_0^f(E) > 0$ for some $E \in \mathcal{C}_a$. Then* $\liminf \frac{\log f}{\log h_a} \le 1$.

Proof Our proof is a modification of Lemma 3.7 in [5].

If the conclusion is not true, then for some $s > 1$ and suitably small x we have $f(x) \le h_a^s(x)$.

Assume $\mathcal{P}_0^f(E) \ge \varepsilon > 0$. For each $\delta > 0$ there are disjoint balls $\{B_i\}$, with diameter at most δ and centred in E, such that $\sum f(|B_i|) \ge \varepsilon$. For each k, let n_k denote the number of balls B_i with $r_{2^{k+1}}/2^{k+1} \le |B_i| < r_{2^k}/2^k$. In terms of this notation we have

$$\varepsilon \le \sum_i f(|B_i|) \le \sum_i h_a^s(|B_i|) \le \sum_k n_k h_a^s(r_{2^k}/2^k) \le \sum_k n_k 2^{-ks}$$

and $P(E, r_{2^{k+1}}/2^{k+1}) \ge n_k$.

Fix $t \in (1, s)$. The previous inequality implies that $n_k \ge \varepsilon 2^{kt}(1-2^{t-s})$ for infinitely many k. For such k,

$$\begin{aligned}\limsup_k \varepsilon 2^{kt}(1-2^{t-s})2^{-(k+1)} &\le \limsup_k n_k 2^{-(k+1)} \\ &\le \limsup_k P\Big(E, \frac{r_{2^{k+1}}}{2^{k+1}})\Big) h_a\Big(\frac{r_{2^{k+1}}}{2^{k+1}}\Big) \\ &\le \mathcal{P}_0^{h_a}(C_a) < \infty.\end{aligned}$$

But since $t > 1$, the left-hand side of this inequality is ∞, and this is a contradiction. ∎

We finish with analogues of Theorem 3.5 and Corollary 3.6 for packing measure.

Theorem 4.8 *Suppose $h \preceq h_a$ and $\gamma > 0$. There is a perfect set $E \in \mathscr{C}_a$ with $\mathcal{P}^h(E) = \gamma$.*

Proof Corollary 4.5 implies that there is a perfect set $E \in \mathscr{C}_a$ with $\mathcal{P}^h(E) = \infty$. Analogous reasoning to that used in the proof of Theorem 3.5 shows that it will be enough to establish that for any fixed γ there is a closed subset of E of h-packing measure γ. In [11], Joyce and Preiss proved that if a set has infinite h-packing measure (for any $h \in \mathcal{D}$), then the set contains a compact subset with finite h-packing measure. With a simple modification of their proof, in particular Lemma 6, we obtain a set of finite packing measure greater than γ. Then, using standard properties of regular, continuous measures, we get the desired closed set. ■

Corollary 4.9 *Let a be a decreasing, positive, and summable sequence and let I be an interval with $|I| = \sum a_k$. If $h \preceq h_a$, the set $\{E \in \mathscr{C}_a(I) : \mathcal{P}^h(E) = \gamma\}$ is dense in $(\mathscr{C}_a(I), \rho)$.*

Proof The proof is analogous to the one of Corollary 3.6. ■

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