ON THE HAUSDOFF h-MEASURE OF CANTOR SETS

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We estimate the Hausdorff measure and dimension of Cantor sets in terms of a sequence given by the lengths of the bounded complementary intervals. The results provide the relation between the decay rate of this sequence and the dimension of the associated Cantor set.

It is well known that not every Cantor set on the line is an s-set for some $0 \leq s \leq 1$. However, if the sequence associated to the Cantor set C is non-increasing, we show that C is an h-set for some continuous, concave dimension function h. We construct the function h from the sequence associated to the set C.

1. Introduction

A Cantor set is a compact, perfect, totally disconnected subset of the real line. In this article we will consider only Cantor sets of Lebesgue measure zero. The complement of a Cantor set is a countable union of disjoint open intervals. We will use the term *gap* for any bounded convex component of the complement of a Cantor set.

Every Cantor set is completely determined by its gaps. Since the gaps are disjoint, the sum of their lengths equals the diameter of the Cantor set.

There is a natural way to associate to each summable sequence of positive numbers a unique Cantor set having gaps with lengths corresponding to the terms of the sequence. In this correspondence the order of the sequence is important. Different rearrangements could correspond to different Cantor sets. On the other hand if two sequences correspond to the same Cantor set, one is clearly a rearrangement of the other.

In the first part of this paper we will concentrate on finding the Hausdorff measure of a Cantor set in terms of the decay of the sequence of the lengths of the gaps. In particular we will show that the Hausdorff dimension depends totally on this behavior.

We establish an equivalence relation between sequences and show that Cantor sets in the same equivalence class have the same dimension.

Since the Cantor set depends on the order of the sequence, one expects that the dimension of the resulting set also depends on the order. This is true, and moreover, the arrangement of the sequence in monotone non-increasing order yields the Cantor set with the largest dimension out of all Cantor sets with the same set of gap lengths (see also [**BT54**]).

Let $0 \leq s \leq 1$. An s-set is a set on the line of Hausdorff dimension s, and whose Hausdorff s-measure is finite and positive. Let h be a non-decreasing, right-continuous function taking the value zero at the origin. The Hausdorff h-measure H^h is defined in the same way as the Hausdorff s-measure but replaces the function x^s by $h(x)$, (see [**Rog98**], [**Hau19**]), i.e.

(1)
$$\mathcal{H}^h(A) = \lim_{\delta \to 0} \inf \Big\{ \sum h(\mathrm{diam}(E_i)) \,:\, E_i \text{ open}, \ \cup E_i \supset A, \mathrm{diam}(E_i) \leq \delta \Big\}.$$

A set $A \subset \mathbb{R}$ is an h-set, if $0 < H^h(A) < +\infty$.

Given $0 \leq s \leq 1$, it is not difficult to construct a Cantor set that is an s-set. It is also known that not every Cantor set of dimension s is an s-set. So should a set of dimension s but Hausdorff measure zero or infinity be considered s-dimensional?

Hausdorff proposed the Hausdorff h-measure to further investigate non s-sets. In this paper we prove that every Cantor set C on the line associated to a sequence of non-increasing positive real numbers is an h-set for some continuous concave function h. We explicitly construct h in terms of the sequence that defines the Cantor set. In other words, for *every* sequence, the set with the largest Hausdorff dimension is also an h-set for some appropriate function.

The study of Cantor Sets through the decay of the complementary intervals was initiated by Borel in 1948 [**Bor49**] and continued by Besicovitch and Taylor in their seminal paper [**BT54**]. The present paper explores this subject further and extends some of their results.

On the other hand, Tricot in [**Tri81**] and Falconer in [**Fal97**] obtain results associating properties of the gaps of a Cantor Set with its box dimension. (see also [**Tri95**]). In [**CMPS03**] the particular case of the sequence x^p was thoroughly analyzed.

Throughout the paper, we will use the notation $\dim(A)$ for the Hausdorff dimension of a set A, since it is the only concept of dimension that we are considering. The Hausdorff s-measure of a set A will be denoted by $\mathcal{H}^s(A)$.

2. Cantor sets associated to a sequence

In what follows we will assign to each summable sequence of positive numbers a unique Cantor set with gaps whose lengths correspond to the terms of this sequence. Let $a = \{a_k\}$ be a sequence of real numbers with $a_n > 0$ for all $n = 1, 2, \dots$ and $\sum a_n = S_a < \infty$. Let I be an interval of length $|I| = S_a$. We remove from I an interval of length a_1, then we remove from the left remaining interval an interval of length a_2 and from the right an interval of length a_3. Iterating this procedure, it is easy to see that we end up with a Cantor set which we will call C_a. Note that since $\sum a_k = |I|$, there is only one choice for the location of each interval to be removed in the construction. Therefore this construction defines the Cantor set unequivocally.

The gap of C_a associated with the term a_k will be denoted g_{a_k}. If g and g' are gaps, we will say that $g < g'$ if all $x \in g, y \in g'$ satisfy that $x < y$. Given a sequence a and its associated Cantor set C_a, we define a *cut* of C_a to be a partition of $\mathbb{N} = L \cup R$ such that

$$g_{a_\ell} < g_{a_r} \qquad \text{for all } \ell \in L, r \in R.$$

We will allow L or R to be empty. The following Lemma is an immediate consequence of the definitions.

Lemma 1. *Every point in C_a defines a cut and conversely, every cut of C_a defines a unique point of C_a.*

Let C_a and C_b be Cantor sets associated to sequences a and b respectively. As a result of the definition of C_a and C_b it is clear that if for some $n, m \in \mathbb{N}$

$$g_{a_n} < g_{a_m}, \qquad \text{then} \qquad g_{b_n} < g_{b_m}.$$

This implies that if (L, R) defines a cut of C_a, it also defines a cut of C_b.

Also note, if $x \in C_a$ is defined by a cut (L, R), then $x = \sum_{n \in L} |g_{a_n}|$.

3. Equivalences of Cantor sets

The previous considerations allow us to define a natural map π_{ab} from C_a into C_b, that assigns to the point $x \in C_a$ the point $y \in C_b$ defined by the same cut associated to x, i.e. if $L_a(x) = \{n \in \mathbb{N} : g_{a_n} \subset [0, x]\}$, then

$$y = \pi_{ab}(x) = \sum_{n \in L_a(x)} |g_{b_n}|.$$

Observe that y can be written also as

$$y = \sum_{n \in L_b(y)} |g_{b_n}| \quad \text{with } L_b(y) = \{n \in \mathbb{N} : g_{b_n} \subset [0, y]\}.$$

The map $\pi_{ab} : C_a \to C_b$ is one-to-one and onto. It can be extended linearly to a one-to-one map from $[0, S_a]$ into $[0, S_b]$, by mapping the gap g_{a_n} linearly into the gap g_{b_n}.

Note that π is an increasing function, since given $x, y \in C_a$ with $x < y$, we have

$$\begin{aligned} \pi_{ab}(y) - \pi_{ab}(x) &= \sum_{n \in L_a(y)} b_n - \sum_{n \in L_a(x)} b_n \\ &= \sum_{n \in (L_a(y) \setminus L_a(x))} b_n = \sum_{\{n : g_{a_n} \subset [x, y]\}} b_n > 0. \end{aligned}$$

This shows that π_{ab} is increasing on C_a. This implies that π_{ab} is increasing on $[0, S_a]$. Since $\pi_{ab} : [0, S_a] \to [0, S_b]$ is onto, it must be continuous and consequently $\pi_{ab}^{-1} : [0, S_b] \to [0, S_a]$ is also continuous.

We have therefore proved the following proposition.

Proposition 1. *If C_a and C_b are the Cantor sets associated to arbitrary sequences a and b, then the map $\pi_{ab} : [0, S_a] \to [0, S_b]$ is increasing, one to one, onto and bi-continuous. Furthermore $\pi_{ab}(C_a) = C_b$.*

Definition 1. Given two summable sequences a and b of positive terms, we will say that a *is of lower order than* b,

$$a \prec b \quad \text{if there exists } k > 0 \text{ such that } \quad \frac{a_n}{b_n} < k, \qquad \forall\, n \in \mathbb{N}.$$

If $a \prec b$ and $b \prec a$ we will say that a and b are of *the same order* and we will write $a \sim b$. Note that

$$a \sim b \quad \Longleftrightarrow \quad k_1 < \frac{a_n}{b_n} < k_2, \qquad \forall\, n \in \mathbb{N},$$

for some constants $k_1, k_2 > 0$.

We will need the following result, that appeared in [**CMPS03**].

Proposition 2 (CMPS03). *Let $a = \{a_k\}_{k\in \mathbb{N}}$ be defined by $a_k = \left(\frac{1}{k}\right)^p$, $p > 1$. Then $\dim(C_a) = \frac{1}{p}$, and moreover, C_a is a $\frac{1}{p}$-set, precisely,*

$$\frac{1}{8}\left(\frac{2^p}{2^p-2}\right)^{\frac{1}{p}} \le \mathcal{H}^{\frac{1}{p}}(C_a) \le \left(\frac{1}{p-1}\right)^{\frac{1}{p}}.$$

Theorem 1. *Let C_a and C_b be Cantor sets associated to the sequences a and b. Then we have*

1) *if $a \prec b$ then $\dim(C_a) \le \dim(C_b)$, in particular, if $a \sim b$ then $\dim(C_a) = \dim(C_b)$,*
2) *there exist sequences $a = \{a_n\}$ and $b = \{b_n\}$ such that $\liminf \frac{a_n}{b_n} = 0$, and*

$$\dim(C_a) = \dim(C_b).$$

Proof. For part (1), if $a \prec b$, we will show that the map π_{ba} defined above is Lipschitz. For given $x < y$, $x, y \in C_b$, then

$$\pi_{ba}(y) - \pi_{ba}(x) = \sum_{\{n: g_{b_n} \subset [x,y]\}} a_n \le k \sum_{\{n: g_{b_n} \subset [x,y]\}} b_n = k(y-x). \tag{2}$$

Then we have $\dim(C_a) = \dim(\pi_{ba}(C_b))$, and by an elementary property of Hausdorff dimension we obtain $\dim(\pi_{ab}(C_b)) \le \dim(C_b)$ proving (1).

For part (2), consider a sequence $a = \{a_n\}$ such that for some fixed $p > 1$

$$\lim_{n\to\infty} \frac{a_n}{n^{-p}} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n^{-q}}{a_n} = 0, \quad \text{for all } q > p.$$

Then the maps $\pi_1 : C_{\frac{1}{n^p}} \to C_a$ and $\pi_2 : C_a \to C_{\frac{1}{n^q}}$ are Lipschitz using a similar argument as in the first part. This implies $q-1 \leq \dim(C_a) \leq p-1$, for all $q > p$. Then $\dim(C_a) = \frac{1}{p}$. □

4. Computation of Hausdorff dimensions

In this section we will define some indices associated with a summable sequence. These numbers can be considered as a measure of the decay rate of the sequence. We will then compare their values with the dimension of the associated Cantor set.

We will denote by λ_p the sequence $\lambda_p(n) = 1/n^p$. Let us define now for a sequence $a = \{a_n\}$,

$$\begin{aligned} \beta(a) &= \inf\{s : 0 < s, a \prec \lambda_{1/s}\} \\ \gamma(a) &= \sup\{s : 0 < s, \lambda_{1/s} \prec a\}, \\ \delta(a) &= \inf\{s : 0 < s \leq 1, \sum_n a_n^s < \infty\}, \end{aligned}$$

Note that out of these three constants, only δ is invariant under rearrangements, whereas β and γ are not. Therefore, since we know that for $a_n = \frac{1}{n^p}$, rearrangements can indeed change the dimension (see [**CMPS03**]), we have to discard the intuition that $\delta(a) = \dim(C_a)$.

An historical survey of various indices associated with the decay of gaps (when a_n decreases) and the box dimension is given in Tricot, [**Tri81**], together with more complete results. In particular he shows that

$$\gamma(a) = \underline{\lim} \frac{-\log(n)}{\log(a_n)} \quad \text{and} \quad \beta(a) = \overline{\lim} \frac{-\log(n)}{\log(a_n)},$$

and if $a = \{a_n\}$ is monotonic decreasing, then $\delta(a) = \beta(a)$.

Proposition 3. *If a is a summable sequence of positive terms then*

1) $\gamma(a) \leq \dim(C_a) \leq \beta(a)$.
2) $\gamma(a) \leq \delta(a) \leq \beta(a)$.

Proof. Part (1) is a consequence of Theorem 1 and the definition of $\gamma(a)$ and $\beta(a)$.

For part (2), choose $s > 0$ such that $a \prec \lambda_{1/s}$ then for every n,

$$a_n \leq \frac{c}{n^{1/s}} \quad \text{for some} \quad c > 0, \quad \text{then} \quad a_n^{s+\epsilon} \leq \frac{c'}{n^{(s+\epsilon)/s}}$$

which implies that $s + \epsilon \geq \delta(a)$ for all $\epsilon > 0$ and then $\delta(a) \leq \beta(a)$. Furthermore, for each $\epsilon \geq 0$ we have for every n:

$$\frac{c}{n^{\frac{1}{\gamma(a)-\epsilon}}} \leq a_n \text{ for some} \quad c \quad \text{and so} \quad \sum_n a_n^{\gamma(a)-\epsilon} = +\infty$$

which implies that $\delta(a) \geq \gamma(a) - \epsilon$. Since ϵ is arbitrary, we conclude that $\delta(a) \geq \gamma(a)$. $\square$

A consequence of the preceding proposition and the result by Tricot is that if a is a monotone non-increasing summable sequence of positive terms and $\widetilde{a}$ is any rearrangement of a then $\beta(a) = \delta(a) = \delta(\widetilde{a}) \leq \beta(\widetilde{a})$.

Another immediate consequence of the definition of $\gamma(a)$ and $\beta(a)$ is the following property:

Property. Let a be any summable sequence of positive terms. If $0 < b < \beta(a)$ then $\limsup_{n\to\infty} n^{1/b} a_n = +\infty$, and if $\gamma(a) \leq b$ then $\liminf_{n\to\infty} n^{1/b} a_n = 0$.

This property tells us that if we take a rearrangement $\widetilde{a}$ of a monotone non-increasing sequence a such that $\beta(a) \neq \beta(\widetilde{a})$, (hence $\beta(a) < \beta(\widetilde{a})$), then $\limsup_{n\to\infty} n^{1/\beta(a)} \widetilde{a}_n = +\infty$. Therefore, if $\widetilde{a}$ is a rearrangement of a monotonic non-increasing sequence a, then $\dim(C_{\widetilde{a}}) \leq \beta(a) (\lneq \beta(\widetilde{a}))$.

4.1. Monotone non-increasing sequences. For a non-increasing sequence a, we already know that $\delta(a) = \beta(a)$. In addition, by Proposition 3, we know that $\gamma(a) \leq \dim(C_a) \leq \beta(a)$. Therefore, if $\lim\left(\frac{\log(a_n)}{\log n}\right) = \ell$, then we have $\dim(C_a) = -\frac{1}{\ell}$.

This result extends the result of Falconer [**Fal97**] (pg.55). Moreover, Falconer shows that if that limit does not exist then the upper and lower box-dimensions disagree.

In this case however, we still want to determine the dimension of C_a. For this, we introduce two new constants associated to the sequence a. Let us call $r_n = \sum_{j\geq n} a_j$. Using an argument analogous to the one used in [**CMPS03**], one can see that the s-Hausdorff measure of C_a is bounded by

$$H^s(C_a) \leq c \varliminf n \left(\frac{r_n}{n}\right)^s.$$

We therefore define two constants, associated to the sequence a:

$$\tau(a) = \inf\{s > 0 : \varliminf n \left(\frac{r_n}{n}\right)^s < +\infty\},$$
$$\alpha(a) = \varliminf \alpha_n \quad \text{where} \quad n\left(\frac{r_n}{n}\right)^{\alpha_n} = 1.$$

Note. The constant α associated to a monotone sequence a was introduced in [**BT54**]. In fact they show that $\dim(C_{\tilde{a}}) \leq \alpha(a)$, where $\tilde{a}$ is any rearrangement of a.

It is interesting to remark that $\varlimsup \alpha_n$ was introduced already in 1948 by Emil Borel with the name of *logarithmic density.*

From results in the seminal paper by Besicovitch and Taylor [**BT54**], one can conclude that for a monotonic non-increasing sequence $\dim(C_a) = \alpha(a)$

(see [**CHM03**]), and that for each t and β such that $0 \leq t \leq \beta$, there is a monotone non-increasing sequence $a = \{a_n\}$, such that $\beta(a) = \beta$, and $\alpha(a) = t$. In our next proposition however, we show the surprising result that if $\gamma(a)$ is *strictly smaller* than $\beta(a)$, then $\alpha(a)$ has a smaller than expected bound.

Proposition 4. *With notation as above, for every non-increasing sequence a,*

$$\alpha(a) = \tau(a) \qquad \text{and} \qquad \alpha(a) \leq \frac{\gamma(a)}{1 - \beta(a) + \gamma(a)}.$$

Proof. $\big(\alpha(a) \leq \tau(a)\big)$
Let $s > 0$ be such that $\underline{\lim}\, n \left(\frac{r_n}{n}\right)^s < +\infty$, then

$$n \left(\frac{r_n}{n}\right)^s = n \left(\frac{r_n}{n}\right)^{\alpha_n} \left(\frac{r_n}{n}\right)^{s-\alpha_n} = \left(\frac{r_n}{n}\right)^{s-\alpha_n}$$

So $\underline{\lim} \left(\frac{r_n}{n}\right)^{s-\alpha_n} < +\infty$. Since for each fixed $k > 0$, $\lim \left(\frac{r_n}{n}\right)^{-1/k} = +\infty$, there must exists a subsequence α_{n_k} such that $\alpha_{n_k} < s + 1/k$, for all k. We have $\alpha(a) = \underline{\lim}_n \alpha_n \leq \underline{\lim}\, \alpha_{n_k} \leq s$, and therefore $\alpha(a) \leq \tau(a)$.

For the converse, $\tau(a) \leq \alpha(a)$, assume now that $\alpha(a) < \tau(a)$, and consider s, such that $\alpha(a) < s < \tau(a)$. Let $\{a_{n_k}\}$ be such that $\lim_k a_{n_k} = \alpha(a)$ and $a_{n_k} < s$ for all k. Then

$$+\infty = \underline{\lim}_n\, n \left(\frac{r_n}{n}\right)^s = \underline{\lim}_k\, n_k \left(\frac{r_{n_k}}{n_k}\right)^s = \underline{\lim}_k \left(\frac{r_{n_k}}{n_k}\right)^{s-\alpha_{n_k}} = 0$$

(since $s - \alpha_{n_k} > c > 0$ for some c and for all k). This contradiction shows that $\alpha(a) = \tau(a)$.

For the other inequality, note that if $\gamma(a) = \beta(a)$, then $\frac{\gamma(a)}{1-\beta(a)+\gamma(a)} = \gamma(a)$ and there is nothing to prove. However, if $\gamma(a) < \beta(a)$, then $\frac{\gamma(a)}{1-(\beta(a)-\gamma(a))} < \beta(a)$.

To show that $\alpha(a)$ satisfies the desired inequality, we prove that for each $\varepsilon > 0$, $\alpha(a) \leq \frac{\gamma(a)+\varepsilon}{1-(\beta(a)-\gamma(a)-\varepsilon)}$. For this, we will show, that for each $\varepsilon > 0$, there is a subsequence $\{a_{n_k}\}_k$ of $\{a_n\}_n$ for which r_{n_k} is at most $O\left(\frac{1}{n_k}^{\frac{1-\beta(a)}{(\gamma(a)+\varepsilon)}}\right)$.

Fix $\beta(a) - \gamma(a) \geq \varepsilon > 0$. Let us call $\gamma_\varepsilon = \gamma(a) + \varepsilon$. From the definition of $\gamma(a)$, we immediately see that there is a subsequence n_k so that $a_{n_k} \leq \frac{1}{n_k}^{1/\gamma_\varepsilon}$. This is the subsequence that we desire.

Since a_n is monotone, we can estimate r_{n_k} from above. Fix n_k. Define a new sequence $\{b_n\}_n$ in the following way:

$$b_j = \begin{cases} a_j & \text{for } j \leq n_k, \\ \left(\frac{1}{n_k}\right)^{1/\gamma_\varepsilon} & \text{for } n_k \leq j < \lceil n_k^{\beta(a)/\gamma_\varepsilon} \rceil, \\ \left(\frac{1}{j}\right)^{1/\beta(a)} & \text{for all larger } j, \end{cases}$$

where $\lceil x \rceil$ stands for the smallest integer that is larger or equal than x. So we have that $a_j \leq b_j$ for all j, and therefore $\sum_{j \geq n_k} a_j \leq \sum_{j \geq n_k} b_j$.

We can estimate that

$$\sum_{j=n_k}^{\lceil n_k^{\beta/\gamma_\varepsilon} \rceil} b_j = \frac{\lceil n_k^{\beta(a)/\gamma_\varepsilon} \rceil - n_k}{n_k^{1/\gamma_\varepsilon}} \sim n_k^{(\beta(a)-1)/\gamma_\varepsilon} \qquad \text{for } k \text{ large enough,}$$

and, using an integral comparison, we see that

$$\sum_{j \geq \lceil n_k^{\beta(a)/\gamma_\varepsilon} \rceil} b_j = C \left(n_k^{\beta(a)/\gamma_\varepsilon} \right)^{(\beta(a)-1)/\beta(a)}.$$

Since both of these terms are $O(n_k^{(\beta(a)-1)/\gamma_\varepsilon})$, we have that

$$\alpha(a) \leq \frac{\gamma(a) + \varepsilon}{1 - (\beta(a) - \gamma(a) - \varepsilon)} \quad \text{for every } \varepsilon.$$

□

In [**Tri95**] it is proved that $\beta(a) = \overline{\lim} - \frac{\log n}{\log a_n} = \overline{\lim}\, \alpha_n$. Proposition 4 shows this is false - in general - for the $\underline{\lim}$. Moreover, we know that there are no sequences a, with $\gamma(a) < \beta(a)$ and

$$\frac{\gamma(a)}{1 - \beta(a) + \gamma(a)} < \dim(C_a) \leq \beta(a).$$

So the question now is about the existence of a sequence a, such that

$$\gamma(a) \leq \dim(C_a) \leq \frac{\gamma(a)}{1 - \beta(a) + \gamma(a)}.$$

The next proposition answers this question completely and emphasizes the asymmetry between the $\overline{\lim}$ and the $\underline{\lim}$.

Proposition 5. *Let $0 < \gamma \leq \beta \leq 1$ be given, and let $S = \{a = \{a_n\}$: monotonic non-increasing such that $\gamma(a) = \gamma$ and $\beta(a) = \beta\}$, then for any number t, $\gamma \leq t \leq \frac{\gamma}{1-\beta+\gamma}$, there is a sequence $a \in S$ such that $\dim(C_a) = t$.*

Proof. Let $0 \leq s \leq 1$, and define

$$f(s) = \frac{\gamma(1-s\beta)}{1-\beta+\gamma(1-s)}.$$

For each s we will construct a sequence $a(s) \in S$, such that $\dim(C_{a(s)}) = f(s)$. Since f is decreasing, $f(0) = \frac{\gamma}{1-\beta+\gamma}$ and $f(1) = \gamma$, for any $t \in [\gamma, \frac{\gamma}{1-\beta+\gamma}]$ there is an s_t so that $\dim(C_{a(s_t)}) = t$.

To construct such sequence, let $R = \frac{1-\gamma s}{1-\beta s}\frac{\beta}{\gamma}$ and define $p_n = 2^{R^n}, n = 0, 1, 2, \ldots$. We now define the sequence $a(s) = \{a_n\}$ as follows, $a_0 = a_1 = 1$ and

$$a_j = (p_n)^{-\left(\frac{1-s\gamma}{\gamma}\right)} j^{-s} \quad \text{ when } p_n \leq j < p_{n+1}.$$

Notice that $a_{p_n} = p_n^{-\frac{1}{\gamma}}$ and

$$a_{(p_{n+1}-1)} = p_n^{-\left(\frac{1-s\gamma}{\gamma}\right)} \left(p_n^R - 1\right)^{-s} \sim p_{n+1}^{-\frac{1}{\beta}}.$$

Furthermore, $n^{-1/\gamma} \leq a_n \leq n^{-1/\beta}$. Hence $\gamma(a(s)) = \gamma$ and $\beta(a(s)) = \beta$, so $a \in S$. In addition $a(s)$ verifies,

$$\alpha(a(s)) = \frac{\gamma(1-s)}{(1-\beta)+\gamma(1-s)} = f(s).$$

To show this, we estimate r_{p_n}. We see that

$$r_{p_n} = \sum_{p_n \leq j < p_{n+1}} a_j + \sum_{j \geq p_{n+1}} a_j \sim C p_n^{-\frac{1-s\gamma}{1-s\beta}\frac{1-\beta}{\gamma}}, \tag{3}$$

so that

$$\alpha(a(s)) \leq \frac{\gamma(1-s\beta)}{(1-\beta)+\gamma(1-s)}.$$

To see the other inequality observe that for $i \in \mathbb{N}$, $p_n < i < p_{n+1}$ $\alpha_i = \frac{\ln(1/i)}{\ln(r_i/i)} \geq \frac{\ln(1/p_n)}{\ln(r_{p_n}/p_n) = \alpha_{p_n}}$. This estimate is obtained by noting that if τ is such that $i = p_n^\tau$, $(1 < \tau < R)$, then

$$r_i \approx p_n^{-\frac{1-s\gamma}{\gamma}} \left(p_n^{R(1-s)} - p_n^{\tau(1-s)}\right) + p_n^{R^2(1-s)-\frac{R}{\gamma}},$$

and since $1 < \tau < R$, by 3 asymptotically we have that $r_i/r_{p_n} \to 1$. Thus, for large enough values, we know that $1 < \frac{\ln(r_i)}{\ln(r_{p_n})} < \tau$ which is equivalent to the desired inequality, and therefore $\dim(C_{a(s)}) = f(s)$ as desired. $\square$

We summarize in the next theorem the main results of this section.

Theorem 2. *Let $a = \{a_n > 0\}$ be a summable sequence. Then we have*

1) $0 \le \gamma(a) \le \dim(C_a) \le \alpha(a) \le \dfrac{\gamma(a)}{1-\beta(a)+\gamma(a)} \le \beta(a)$.
   In particular when the sequence a is non-increasing we have $\dim(C_a) = \alpha(a)$.
2) *Given numbers α, β and γ such that $0 \le \gamma \le \alpha \le \frac{\gamma}{1-\beta+\gamma} \le \beta \le 1$, then there exist a summable sequence a (that can be chosen to be non-increasing) such that $\gamma(a) = \gamma, \quad \alpha(a) = \alpha \quad and \quad \beta(a) = \beta$.*

Given a non increasing sequence a it could happen that the $\alpha(a)$-Hausdorff measure of the associate Cantor set C_a is zero or infinite. In the next section we will see that we still can say something in this case.

5. Dimension function

To analyze this situation it will be useful to refine the notion of dimension in the spirit of Hausdorff's original work. Throughout this section we fix a monotonic non-increasing sequence $a = \{a_k\}$ of positive terms such that $\sum a_k = 1$.

We associate to a another non-increasing sequence:

$$b = \{b_n\} \quad b_n = \frac{r_n}{n}, \quad \text{where } r_n \text{ is as before} \quad r_n = \sum_{j=n}^{\infty} a_j. \tag{4}$$

Define the following function h with respect to a decreasing function $f : [1, +\infty) \to \mathbb{R}$ such that $f(k) = b_k$, e.g.

$$f(x) = b_k(k+1-x) + b_{k+1}(x-k), \qquad x \in [k, k+1)$$

then let

$$h(t) = \begin{cases} \frac{1}{f^{-1}(t)} & t \in (0, b_1] \\ h(0) = 0. \end{cases} \tag{5}$$

Then h is a non-decreasing, concave function and $h(b_k) = \frac{1}{k}$.

This function will be useful for determining the dimension of the Cantor set C_a. We will need some auxiliary results and (more!) notation.

Let W denote the set of binary words of finite length:

$$W = \{e\} \bigcup \{w_1 ... w_r : w_i \in \{0,1\},\ r \in \mathbf{N}\},$$

where $\{e\}$ denotes the empty word. If $w, w' \in W$ then ww' will denote the concatenation of w and w', and the length of word w will be denoted by $|w|$. Set $|e| = 0$ and let W^* denote the set of words of positive length. Given w, either an infinite binary word or a finite binary word of length at least k, we will denote by $w(k)$ the truncation $w_1 ... w_k$.

It is convenient to use the elements of W to describe the intervals of our Cantor set C_a. Let I_e denote the initial interval. ($I_e = I_0^0$). If $w \in W$, $|w| = k$ and I_w is an interval of step k in the construction, then we denote by I_{w0} and I_{w1} the left and right intervals obtained by removing the open interval from I_w.

In this way if I_w is an interval of step $|w|$, $I_w = I^{|w|}_{\sum_{j=1}^{|w|} w_j 2^{k-j}}$, then for any w', $I_{ww'}$ is an interval of step $|ww'|$ which is *related* to I_w.

It is worthwhile to note at this stage that in the case of a monotonic non-increasing sequence, the lengths of I_w also form a non-increasing sequence.

For the sequence b_n of (4) we will now denote by b_w the element of the sequence corresponding to b_ℓ, with $\ell = 2^k + \sum_{j=1}^{k} w_j 2^{k-j}$ and $k = |w|$.

In particular note that

$$\text{if} \quad b_w = b_{2^k+l} \quad \text{then} \quad b_{ww'} = b_{2^{k'}(2^k+l)+s}, \tag{6}$$

where $l = \sum_{j=1}^{k} w_j 2^{k-j}$ with $k = |w|$ and $s = \sum_{j=1}^{k'} w'_j 2^{k'-j}$ with $k' = |w'|$.

Lemma 2. *With the above notation, for every $k \geq 1$, and $w, \widetilde{w}$ of length k, and any w',*

$$\frac{1}{2}\,\frac{h(b_{ww'})}{h(b_w)} \;\leq\; \frac{h(b_{\widetilde{w}w'})}{h(b_{\widetilde{w}})} \;\leq\; 2\,\frac{h(b_{ww'})}{h(b_w)}.$$

In particular, for any w' we have $h(b_{ww'}) \;\leq\; 4\; h(b_w)$.

Proof. Recall that $h(b_\ell) = \frac{1}{\ell}$ and let $k' = |w'|$. If we define

$$l = \sum_{j=0}^{k} w_j 2^{k-j}, \quad r = \sum_{j=0}^{k} \widetilde{w_j} 2^{k-j} \quad \text{and} \quad s = \sum_{j=0}^{k'} w'_j 2^{k'-j},$$

then by 6

$$\frac{h(b_{ww'})}{h(b_w)} \;=\; \frac{2^k + l}{2^{k'}(2^k+l)+s} \quad \text{and} \quad \frac{h(b_{\widetilde{w}w'})}{h(b_{\widetilde{w}})} \;=\; \frac{2^k + r}{2^{k'}(2^k+r)+s}.$$

Now noting that $\frac{1}{2} \;\leq\; \frac{2^k+r}{2^k+l} \;\leq\; 2$, we obtain the desired result.

For the second inequality just note that h is non-decreasing and therefore the right-hand side is less or equal than 2 for *any* w'. □

These bounds of the ratios of $h(b_k)$ will be useful for defining a measure on C_a. Since the construction of this Cantor set relies on the size of the gaps, it will be useful to define a measure depending on the size of the gaps.

Proposition 6. *There exists a probability measure μ_h supported on C_a, such that for every $k \geq 1$, $0 \leq \ell \leq 2^k - 1$,*

$$\frac{1}{4}\, h(b_{2^k+\ell}) \;\leq\; \mu_h(I^k_\ell) \;\leq\; 2\; h(b_{2^k+\ell}). \tag{7}$$

Proof. For $m \geq 1$ consider the probability measure μ_m, supported on the intervals I^m_ℓ of level m, such that

$$\mu_m(I^m_t) \;=\; \frac{h(b_{2^m+t})}{\sum_{j=0}^{2^m-1} h(b_{2^m+j})}.$$

Note that, if $k \le m$, and $w = w_1 \dots w_k$ is such that $\sum_{j=0}^{k} w_j 2^{k-j} = t$,

$$\mu_m(I_t^k) \;=\; \mu_m(I_w) \;=\; \sum_{|w'|=m-k} \mu_m(I_{ww'}),$$

and hence

$$\left(\sum_{j=0}^{2^m-1} h(b_{2^m+j})\right) \mu_m(I_t^k) \;=\; \sum_{|w'|=m-k} h(b_{ww'}). \tag{8}$$

But by the bounds found in Lemma 2,

$$h(b_{ww'}) \;\le\; 2\, h(b_w)\, \frac{h(b_{\widetilde{w}w'})}{h(b_{\widetilde{w}})}, \quad \forall \quad \widetilde{w} \quad \text{such that} \quad |\widetilde{w}| = |w| = k.$$

Hence, recalling the definition of w, we obtain (from 8), that for all $\widetilde{w}$ such that $|\widetilde{w}| = k$,

$$\left[h(b_{\widetilde{w}}) \sum_{j=0}^{2^m-1} h(b_{2^m+j})\right] \mu_m(I_t^k) \;\le\; 2\, h(b_w) \sum_{|w'|=m-k} h(b_{\widetilde{w}w'}),$$

and therefore

$$\begin{aligned}\left[\sum_{|\widetilde{w}|=k} h(b_{\widetilde{w}}) \sum_{j=0}^{2^m-1} h(b_{2^m+j})\right] \mu_m(I_t^k) &\le 2\, h(b_w) \left(\sum_{|\widetilde{w}|=k} \sum_{|w'|=m-k} h(b_{\widetilde{w}w'})\right) \\ &= 2\, h(b_w) \sum_{j=0}^{2^m-1} h(b_{2^m+j}),\end{aligned}$$

which yields $\mu_m(I_\ell^k) \;\le\; 2 \frac{h(b_{2^k+\ell})}{\sum_{j=0}^{2^k-1} h(b_{2^k+j})}, \quad k \le m$. But noting that $\frac{1}{2} \;\le\; \sum_{j=0}^{2^m-1} h(b_{2^m+j}) \;\le 1$ and using the other inequality of the Lemma, we finally obtain that for every $1 \le k \le m$, $0 \le \ell \le 2^k - 1$,

$$\frac{1}{2}\, h(b_{2^k+\ell}) \;\le\; \mu_m(I_\ell^k) \;\le\; 4\; h(b_{2^k+\ell}).$$

Let now μ_h be the weak*-limit of μ_m, then (see for example [**Mat95**]) for every $1 \le k$, $0 \le \ell \le 2^k - 1$,

$$\frac{1}{2}\, h(b_{2^k+\ell}) \;\le\; \mu_h(I_\ell^k) \;\le\; 4\, h(b_{2^k+\ell}).$$

□

We are now ready to prove our main result; recall that an h-set was defined in the Introduction.

Theorem 3. *Let $a = \{a_k\}$ be a non-increasing sequence of positive terms such that $\sum a_k = 1$ and C_a the associated Cantor set. Then C_a is an h-set. Moreover*

$$\frac{1}{32} \leq H^h(C_a) \leq 1$$

where H^h is the Hausdorff measure associated to h, and h is the dimension function defined in (5).

Proof. For the upper bound, let $\delta > 0$ and let n_0 be such that $n \geq n_0$, $r_n = \sum_{j \geq n} a_j < \delta$. Then the intervals $E_1, \dots, E_n$ that are the remaining intervals after the gaps associated to $a_1, \dots, a_{n-1}$ are removed, are a δ-covering of C_a, and since h is concave, we have

$$\sum_{i=1}^{n} h(|E_i|) \leq nh\left(\frac{|E_1| + \cdots + |E_n|}{n}\right) = nh\left(\frac{r_n}{n}\right) = 1,$$

and therefore $H^h(C_a) \leq 1$.

For the lower bound, the idea is to try to use the measure μ_h, and apply a generalized version of the *Mass transfer principle.* For this, let U be any open set, and let $\operatorname{diam}(U) = \rho < 1$. Let $k \geq 1$ and $0 \leq \ell \leq 2^k - 2$ be such that $b_{2^k+\ell+1} \leq \rho < b_{2^k+\ell}$ (the case that $b_{2^{k+1}} \leq \rho < b_{2^{k+1}-1}$ will be considered separately). Then, because the length of the intervals I_l^k is a non-increasing sequence

$$\rho < b_{2^k} = \frac{|I_0^k| + \cdots + |I_{2^k-1}^k|}{2^k} < |I_0^k|.$$

Then U can intersect at most 2 consecutive intervals of step $k-1$. Hence

$$\begin{aligned} \mu_h(U) &\leq (\mu_h(I_t^{k-1}) + \mu_h(I_t^{k-1})) \quad \forall\, 0 \leq t \leq 2^k - 2, \\ &\leq 4\, h(b_{2^{k-1}+t}) + 4h(b_{2^{k-1}+t}) \quad \text{by the Proposition} \\ &\leq 8\, h(b_{2^{k-1}}) \leq 32\, h(b_{2^k+\ell+1}) \quad \text{by Lemma 2.} \end{aligned}$$

Therefore since h is non-decreasing, $\mu_h(U) \leq 32\, h(b_{2^k+\ell+1}) \leq 32\, h(\operatorname{diam}(U))$.

Assume now ρ satisfies $b_{2^{k+1}} \leq \rho < b_{2^{k+1}-1}$. Since $\rho < b_{2^k}$, we still have

$$\mu_h(U) \leq 8\, h(b_{2^{k-1}}) = 8\, \frac{1}{2^{k-1}} = 32\, \frac{1}{2^{k+1}} = 32\, h(b_{2^{k+1}}),$$

and so again,$\mu_h(U) \leq 32\, h(\operatorname{diam}(U))$.

Therefore, if $\{U_k\}$ is a δ-covering of C_a, we have

$$\sum_k h(\operatorname{diam}(U_k)) \geq \frac{1}{32} \sum_k \mu_h(U_k) \geq \frac{1}{32}\, \mu_h(C_a).$$

Since this is true for every δ-covering, we obtain:

$$H_\delta^h(C_a) \geq \frac{1}{32}\, \mu_h(C_a), \quad \text{and therefore} \quad H^h(C_a) \geq \frac{1}{32}.$$

$\square$

Among all the dimension functions, one can also establish a certain equivalence relation, namely $h \equiv g$ if there exist constants c_1 and c_2 such that

$$c_1 \leq \varliminf_{x \to 0+} \frac{h(x)}{g(x)} \leq \varlimsup_{x \to 0+} \frac{h(x)}{g(x)} \leq c_2.$$

The following result relates the function h to $\alpha(a)$.

Proposition 7. *If $a \sim \frac{1}{n}^{1/s}$ then $h \equiv x^s$.*

Proof. Since $a \sim \frac{1}{n}^{1/s}$, $\gamma(a) = \beta(a) = s$, and hence there exist $c > 0$ and $d > 0$ such that

$$cn^{-1/s} \leq a_n \leq dn^{-1/s} \quad \text{and therefore} \quad Cn^{-1/s} \leq \frac{r_n}{n} \leq Dn^{-1/s}.$$

Hence

$$0 < c_1 \leq \varliminf_{x \to 0+} \frac{h(x)}{x^s} \leq \varlimsup_{x \to 0+} \frac{h(x)}{x^s} \leq c_2 < +\infty.$$

□

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