Generalized Self-Similarity

Carlos A. Cabrelli Ursula M. Molter

Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Ciudad Universitaria, Pabellón I
1428 Capital Federal
ARGENTINA*

Abstract

We prove the existence of $\mathcal{L}^p$ functions satisfying a kind of self-similarity condition. This is achieved solving a functional equation by means of the construction of a contractive operator on an appropriate functional space. The solution, a fixed point of the operator, can be obtained by an iterative process, making this model very suitable to use in applications such as fractal image and signal compression.
On the other hand, this "generalized self-similarity equation" includes matrix refinement equations of the type $f(x) = \sum c_k f(Ax - k)$ which are central in the construction of wavelets and multiwavelets. The results of this paper will therefore yield conditions for the existence of $\mathcal{L}^p$-refinable functions in a very general setting.

Keywords: Self-Similarity, Functional Equation, Dilation Equation, Refinement Equation, Wavelets, Fixed Points, Fractals, Inverse Problem for Fractals.

*Both authors are partially supported by grants: UBACyT EX048 and a special grant of the Fundación Antorchas

1 Introduction

Self-Similar objects are those that can be constructed out of smaller copies of itself. When we deal with sets, this concept can be formulated using the notion of Iterated Function Schemes (IFS) ([24], [4]): If (X, d) is a metric space and $\Phi = \{w_1, \ldots, w_n\}$ $(w_i : X \to X \ , \ \{w_i\}_{i=1,\ldots,N})$ is a set of maps, then $\mathcal{A} \ \subset \ X$ is self-similar with respect to Φ if $\mathcal{A} = \cup w_i(\mathcal{A})$. It can be shown, that if X is complete, and the maps are contractive, then there exists a unique compact self-similar set with respect to Φ.

This concept can be extended in different ways to different kind of objects: self-similar measures can also be defined using IFS (see [24], [4]) and recently have been studied by Strichartz using Fourier and Wavelet analysis ([26], [28]).

Aiming to recover self-similarity parameters of physical signals, Hwang and Mallat study the self-similarity of the wavelet transform ([25]).

One way to extend the notion of self-similarity to functions, is to require that the graph of the function should be a self-similar set.
If the function is defined on a self-similar set, then we could require that the function share the self-similarity of the domain, i.e.: if $X = \cup_{i=1}^{n} w_i(X)$, then

$$f(w_i(x)) = f(x), \quad i = 1, \ldots, n. \tag{1.1}$$

For this definition we require the w_i to be disjoint (i.e. $w_i(X) \cap w_j(X) = \emptyset, \ i \neq j$).

From IFS-theory it can be shown that if f is a continuous function satisfying the self-similarity condition (1.1), f has to be constant.

In order to consider more general solutions, we relax the condition of self-similarity (1.1), introducing a set of functions $\varphi_1, \ldots, \varphi_n$ and requiring that f satisfy

$$\varphi_i(f(w_i^{-1}(x))) = f(x) \quad x \in w_i(X) \quad i = 1, \ldots, n. \tag{1.2}$$

Finally, to allow overlapping maps in the IFS, we introduce a function $\mathcal{O}$ that combines the values of $\varphi_i \circ f \circ w_i^{-1}(x)$ for $i = 1, \ldots, n$ for the same x.

In this paper we will study the existence of self-similar functions in different contexts and relax even more the self-similarity condition (1.2) allowing *space-dependent* φ_i's and $\mathcal{O}$.

The problem of finding a function $\mathbf{u}$ that satisfies a self-similarity equation of the type:

$$\mathbf{u}(x) = \mathcal{O}\,(x, (\mathbf{u} \circ g_1)(x), \ldots, (\mathbf{u} \circ g_r)(x)), \tag{1.3}$$

has been studied by Bajraktarevic in 1957 ([2]). In the same year, a similar equation was considered by de Rahm ([13]), and conditions for continuous solutions were found.

In [24] Hutchinson, extending the concept of self-similarity to parametric curves, considered a particular case of this equation.

More recently, related functional equations were studied in fractal interpolation, in order to show the existence and construction of continuous fractal functions ([3], [6], [15], [16], [17], [21], [22]) .

Cabrelli et al. in [9] constructed an operator of the type (1.3) introducing a novelty to it: they added a set of grey-level functions φ_i, such that the resulting fixed point of their operator would no longer be strictly self-similar, but φ-self-similar. They worked in a particular setting, in which the functions φ_i had to satisfy very restrictive conditions to guarantee convergence

In this paper we broaden the class of functions and look at different functional spaces and are able to remove most of the previous restrictive conditions making this model much more versatile and therefore more suitable for applications.

We study the more general equation

$$\mathbf{u}(x) = \mathcal{O}(x, \varphi_1(x, (\mathbf{u} \circ g_1)(x)), \ldots, \varphi_r(x, (\mathbf{u} \circ g_r)(x))\,)\ , \tag{1.4}$$

that encloses most of the cases mentioned before and generalizes the concept of self-similar function (1.1). We find conditions on the components in order to assure the existence of solutions.

We construct an operator on a suitable function space and the solution of our equation will be a fixed point of this operator. This not only yields a solution of the equation, but also shows that this solution can be computationally efficiently calculated: we obtain it by iterating the operator.

This functional equation and the easy computation of its solution makes it suitable for many applications. For example it models two situations which are of general interest: using fractal compression in image or signal analysis and the construction of wavelets and multiwavelets.

In the first case, in signal processing, in particular in image representation, a well known problem is the design of an adaptive code for a given target. This has been studied in particular using fractals and self-similar models (see [1], [5], [7], [9], [12], [14], [20]). Some of the advantages of this approach are the compression rates achieved, and the complexity of the images that can be represented. Generally the strategy consists in finding an operator T, whose fixed point is the given target. In [12], it was shown that the previously introduced model ([9]) had the property of being "dense", meaning that for any function and any ε one can construct an operator whose fixed point is closer than ε to the function. However, due to the restrictions on the grey-level maps φ_i, this result was not enough for practical implementations. The functional equation considered in this paper, represents a generalization of the concept of self-similar function extending the applicability of the model to a wider class of images and allowing more flexibility in the choice of the parameters. This should in turn lead to a better compression rate. For work in this direction we refer the reader to [8].

In the second case, in the application to the construction of wavelets and multiwavelets, one wants to find solutions to a *refinement equation* of the type

$$\phi(x) = \sum_{k \in Z^d} c_k \, \phi(Ax - k), \tag{1.5}$$

in order to then construct a wavelet decomposition of $\boldsymbol{\mathcal{L}}^{\mathbf{2}}(\boldsymbol{\mathcal{R}}^d)$. Suitably defining φ_i, g_i and $\mathcal{O}$ in (1.4) will yield (1.5). In the particular case that all the c_k are equal, one yields the equation studied by Gröchenig and Madych in [19] and Strichartz in [27]. Currently there is a growing interest in "multiwavelets", which can be constructed using refinement equations in which the coefficients are matrices and the solutions vector-valued functions ([18], [23]). These matrix refinement equations are particular cases of our functional equation, and solutions to these equations, using generalized self-similar functions are studied in a joined work with C. Heil ([10], [11]). In particular the existence of solutions to this equation in a suitable setting, lead to the construction of the first known example of non-separable orthogonal multiwavelets in $\boldsymbol{\mathcal{R}}^2$ ([11]).

We will analyze two different situations: in section 2 we study the case of bounded solutions with the uniform metric. In section 3 we study $\boldsymbol{\mathcal{L}}^{\mathbf{p}}$

solutions for $1 \leq p < +\infty$. In both cases we give sufficient conditions for the existence of solutions.

2 B(X,E)-case

Let (X, d) be a compact metric space and (E, ℓ) a metric space where E is a closed subset of $\mathcal{R}^m$ (in particular (E could be $\mathcal{R}^m$) and ℓ a distance in E induced by some norm of $\mathcal{R}^m$. Let us also consider a point $t_0 \in E$ that will remain fixed throughout the whole section.

We consider the functional space

$$\boldsymbol{\mathcal{B}}(X, E) = \{\mathbf{u} : X \longrightarrow E,\ \mathbf{u} \text{ bounded}\},$$

with

$$D(\mathbf{u}, \mathbf{v}) = \sup_{x \in X} \ \ell\,(\mathbf{u}(x), \mathbf{v}(x)), \quad \forall \mathbf{u}, \mathbf{v}, \in \ \boldsymbol{\mathcal{B}}(X, E). \tag{2.1}$$

It is well-known that $(\boldsymbol{\mathcal{B}}(X, E), D)$ is a complete metric space.

Let us now define the functions $\mathcal{O}$, $w_i, \varphi_i, i = 1, \ldots, r$ in order to construct an operator $\boldsymbol{\mathcal{T}}$ on $\boldsymbol{\mathcal{B}}(X, E)$.

Let $\mathcal{O} : X \times E^r \longrightarrow E$ be non-expansive for each $x \in X$, i.e. :

$$\ell\,(\mathcal{O}\,(x, \overrightarrow{k^1}), \mathcal{O}\,(x, \overrightarrow{k^2})) \leq \sup_{1 \leq i \leq r} \ \ell\,(k_i^1, k_i^2) \ \ \forall \overrightarrow{k^1}, \overrightarrow{k^2} \ \in E^r. \tag{2.2}$$

Let $w_i : X \to X,\ i = 1, \ldots, r$ be r injective maps, which are not necessarily contractive, and let $\varphi_i : X \times E \to E,\ i = 1, \ldots r$ be r functions that for each $x \in X$ satisfy the Lipschitz condition:

$$\ell\,(\varphi_i(x, k_1), \varphi_i(x, k_2)) \leq c_i\,\ell\,(k_1, k_2), \quad \forall k_1, k_2 \in E, \quad i = 1, \ldots, r \tag{2.3}$$

where $c_i \geq 0$ does not depend on x.

In order to be able to define an operator on $\boldsymbol{\mathcal{B}}(X, E)$, we need some stability conditions. We define a function f to be *stable*, if $f(A)$ is bounded, whenever

A is a bounded set. Hence we shall assume that $\mathcal{O}$ and $\varphi_i, i = 1, \ldots, r$ are stable.

Now we define an operator $\boldsymbol{\mathcal{T}}$ on $\boldsymbol{\mathcal{B}}(X, E)$ in the following way:

$$(\boldsymbol{\mathcal{T}}\mathbf{u})(x) = \mathcal{O}\,(x, \varphi_1(x, \tilde{\mathbf{u}}_1(x)), \ldots, \varphi_r(x, \tilde{\mathbf{u}}_r(x)))\ ; \tag{2.4}$$

$$\text{where} \quad \tilde{\mathbf{u}}_i(x) = \begin{cases} \mathbf{u}(w_i^{-1}(x)) & \text{if } x \in \operatorname{Img}(w_i) \\ t_0 & \text{otherwise} \end{cases} \qquad 1 \le i \le r. \tag{2.5}$$

We shall use $\mathcal{O}\,(x, \overrightarrow{\varphi_i(x, \tilde{\mathbf{u}}_i(x))})$ for the right hand side of (2.4). We can prove the following

Theorem 2.1 *With the above notation, if* $c = \max_{1 \le i \le r} c_i > 0$ *is the Lipschitz constant for the* φ_i*'s, then*

$$\boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{B}}(X, E) \longrightarrow \boldsymbol{\mathcal{B}}(X, E), \qquad \textit{and}$$

$$D(\boldsymbol{\mathcal{T}}\mathbf{u}, \boldsymbol{\mathcal{T}}\mathbf{v}) \le c\, D(\mathbf{u}, \mathbf{v}).$$

In particular, if $c < 1$, $\boldsymbol{\mathcal{T}}$ *is contractive and therefore there exists a unique* $\mathbf{u}^*$ *in* $\boldsymbol{\mathcal{B}}(X, E)$ *such that* $\boldsymbol{\mathcal{T}}\mathbf{u}^* = \mathbf{u}^*$.

Proof : If $\mathbf{u} \in \boldsymbol{\mathcal{B}}(X, E)$ then it is easy to verify that $\boldsymbol{\mathcal{T}}\mathbf{u} \in \boldsymbol{\mathcal{B}}(X, E)$. Now if $\mathbf{u}, \mathbf{v} \in \boldsymbol{\mathcal{B}}(X, E)$ then

$$\begin{aligned}
\ell\,((\boldsymbol{\mathcal{T}}\mathbf{u})(x), (\boldsymbol{\mathcal{T}}\mathbf{v})(x)) &= \ell\,(\,\mathcal{O}\,(x, \overrightarrow{\varphi_i(x, \tilde{\mathbf{u}}_i(x))}), \mathcal{O}\,(x, \overrightarrow{\varphi_i(x, \tilde{\mathbf{v}}_i(x))})\,) \\
&\le \sup_{1 \le i \le r}\ \ell\,(\,\varphi_i(x, \tilde{\mathbf{u}}_i(x)), \varphi_i(x, \tilde{\mathbf{v}}_i(x))\,) \\
&\le \sup_{1 \le i \le r}\ c\ \ell\,(\,\tilde{\mathbf{u}}_i(x), \tilde{\mathbf{v}}_i(x)\,) \\
&\le c\ \sup_{y \in X} \ell\,(\,\mathbf{u}(y), \mathbf{v}(y)\,) \\
&= c\ D(\mathbf{u}, \mathbf{v}).
\end{aligned}$$

Therefore

$$D(\boldsymbol{\mathcal{T}}\mathbf{u}, \boldsymbol{\mathcal{T}}\mathbf{v}) \le c\ D(\mathbf{u}, \mathbf{v}).$$

$\blacksquare$

We then have the following

Corollary 2.2 *If $c < 1$, the functional equation*

$$\mathbf{u} = \mathcal{O}(x, \varphi_1(x, \tilde{\mathbf{u}}_1(x)), \dots, \varphi_r(x, \tilde{\mathbf{u}}_r(x))) \tag{2.6}$$

where the $\tilde{\mathbf{u}}_i$ are as in (2.5), has a unique solution in $\boldsymbol{\mathcal{B}}(X, E)$.

Proof :

The fixed point of the operator $\boldsymbol{\mathcal{T}}$ is the solution of the equation. ■

Note that (2.6) is a generalization of the original functional equation given in (1.3).

In what follows, we will study the operator (2.4) in the $\boldsymbol{\mathcal{L}}^{\mathbf{p}}$spaces.

3 $\boldsymbol{\mathcal{L}}^{\mathbf{p}}$-case

Let now $X \subseteq \boldsymbol{\mathcal{R}}^n$ compact, with μ the n-dimensional Lebesgue measure and let $E = \boldsymbol{\mathcal{R}}^m$ with some norm $\|.\|$. (Note: E could be chosen to be any Banach space.) We consider the functions $\mathbf{u} : X \to E$ such that the real-valued function $\|\mathbf{u}(.)\|$ is Lebesgue-measurable, and, as usual, functions that are equal almost everywhere are identified.

If $1 \leq p < +\infty$, let

$$\boldsymbol{\mathcal{L}}^{\mathbf{p}}(X, E) = \{\mathbf{u} : X \to E : \int_X \|\mathbf{u}(x)\|^p d\mu(x) < +\infty\}$$

with $\|\mathbf{u}\|_p = \left(\int_X \|\mathbf{u}(x)\|^p d\mu(x)\right)^{1/p}$; and

$$\boldsymbol{\mathcal{L}}^{\infty}(X, E) = \{\mathbf{u} : X \to E : \ \|\mathbf{u}(.)\| \quad \text{essentially bounded}\}$$

with $\|\mathbf{u}\|_\infty = \text{ess.sup.}\|\mathbf{u}(.)\|$.

It is well known, that $\boldsymbol{\mathcal{L}}^{\mathbf{p}}(X, E),\ 1 \leq p \leq +\infty$ is a Banach space.

Let as before $\mathcal{O} : X \times E^r \to E^r$ be non-expansive, i.e.

$$\|\mathcal{O}(x, \vec{k^1}) - \mathcal{O}(x, \vec{k^2})\| \leq \left(\sum_{i=1}^{r} \|k_i^1 - k_i^2\|^p\right)^{1/p}. \tag{3.1}$$

For measurable $\mathbf{u} : X \to E$ we define as before the operator (2.4),

$$(\mathcal{T}\mathbf{u})(x) = \mathcal{O}\,(x, \varphi_1(x, \tilde{\mathbf{u}}_1(x)), \ldots , \varphi_r(x, \tilde{\mathbf{u}}_r(x)),$$

where the $w_i's$ and $\varphi_i's$ are as in the previous section, with the following additional conditions:

1. The maps $\{w_i\}$ satisfy a Lipschitz condition, i.e. there exist $s_i > 0$, such that $d(w_i(x), w_i(y)) \leq s_i\, d(x, y)$ where d is the Euclidean distance in $\mathcal{R}^n$.
2. The functions $\varphi_i,\ i = 1, \ldots , r$ and $\mathcal{O}$ are Borel measurable.

These additional conditions are required in order to guarantee the measurability of $\mathcal{T}\mathbf{u}$.

We have the following

Proposition 3.1 *Let $\mathcal{T}$ be defined as above, then $\mathcal{T}\mathbf{u} : X \to E$ is measurable for each measurable function $\mathbf{u} : X \to E$ and also if $\mathbf{u}, \mathbf{v}$ are measurable and $\mathbf{u} = \mathbf{v}$ a.e. then $\mathcal{T}\mathbf{u} = \mathcal{T}\mathbf{v}$ a.e.*

Proof: The measurability of $\mathcal{T}\mathbf{u}$ for measurable $\mathbf{u}$ is a consequence of the stability and the Borel-measurability of $\mathcal{O}$ and the φ_i's and the fact that the w_i's are Lipschitz. Now if $Z = \{x\,:\, \mathbf{u}(x) \neq \mathbf{v}(x)\,\}$, then $\{x\,:\, \mathcal{T}\mathbf{u}(x) \neq \mathcal{T}\mathbf{v}(x)\} \subset \cup_{i=1}^{r} w_i(Z)$. The Lipschitz condition of the w_i's implies that $\mu(w_i(Z)) = 0$ if $\mu(Z) = 0$ and therefore the result follows. ■

Now we consider first the space $\mathcal{L}^\infty$ defined before. The case $\mathcal{L}^{\mathbf{p}}\ \ 1 \leq p < +\infty$ will be treated later.

Theorem 3.2 *Let $\mathcal{T}$ be the operator of proposition 3.1. Then, $\mathcal{T} : \mathcal{L}^\infty \to \mathcal{L}^\infty$ and*

$$\|\mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v}\|_\infty \leq c\|\mathbf{u} - \mathbf{v}\|_\infty, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{L}^\infty.$$

Proof: If $\mathbf{u} \in \mathcal{L}^\infty$ then let $Z \subset X,\ \mu(Z) = 0$ and $\mathbf{u}$ bounded in $X - Z$. If we define $\mathbf{v} : X \to E$ by $\mathbf{v} = \mathbf{u}\,\mathcal{X}_{X-Z}$, then $\mathbf{v} = \mathbf{u}$ a.e. and $\mathbf{v}$ is bounded. Then $\mathcal{T}\mathbf{v}$ is bounded and using the preceding proposition, $\mathcal{T}\mathbf{u} = \mathcal{T}\mathbf{v}$ a.e. and therefore $\mathcal{T}\mathbf{u}\ \in\ \mathcal{L}^\infty$.

From the proof of Theorem 2.1 we see that for $\mathbf{u}$ and $\mathbf{v} \in \boldsymbol{\mathcal{L}}^\infty$ we have

$$\|(\boldsymbol{\mathcal{T}}\mathbf{u})(x) - (\boldsymbol{\mathcal{T}}\mathbf{v})(x)\| \le c\|\mathbf{u}-\mathbf{v}\|_\infty \quad \text{a.e.} \quad \text{on } X \ ,$$

which implies that

$$\|\boldsymbol{\mathcal{T}}\mathbf{u} - \boldsymbol{\mathcal{T}}\mathbf{v}\|_\infty \le c\,\|\mathbf{u}-\mathbf{v}\|_\infty. \qquad \blacksquare$$

We will now analyze the case $\boldsymbol{\mathcal{L}}^{\mathbf{p}}$ $1 \le p < \infty$. We have the following

Theorem 3.3 *Let $\boldsymbol{\mathcal{T}}$ be the operator of proposition 3.1. Then, if $\mathbf{u}, \mathbf{v} \in \boldsymbol{\mathcal{L}}^{\mathbf{p}}(X, E)$, then $(\boldsymbol{\mathcal{T}}\mathbf{u} - \boldsymbol{\mathcal{T}}\mathbf{v}) \in \boldsymbol{\mathcal{L}}^{\mathbf{p}}(X, E)$ and*

$$\|\boldsymbol{\mathcal{T}}\mathbf{u} - \boldsymbol{\mathcal{T}}\mathbf{v}\|_p \le \left(\sum_{i=1}^{r} s_i^n c_i^p \right)^{1/p} \|\mathbf{u}-\mathbf{v}\|_p \ ,$$

where s_i and c_i are the Lipschitz constants of w_i and φ_i respectively, and n is the dimension of X.
Furthermore the finiteness of $\mu(X)$ yields

$$\boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{L}}^{\mathbf{p}}(X, E) \to \boldsymbol{\mathcal{L}}^{\mathbf{p}}(X, E).$$

Proof : If $\mathbf{u}, \mathbf{v} \in \boldsymbol{\mathcal{L}}^{\mathbf{p}}$, then by Proposition 3.1, $\boldsymbol{\mathcal{T}}\mathbf{u} - \boldsymbol{\mathcal{T}}\mathbf{v}$ is measurable and

$$\begin{aligned}
\|\boldsymbol{\mathcal{T}}\mathbf{u} - \boldsymbol{\mathcal{T}}\mathbf{v}\|_p^p &= \int_X \|(\boldsymbol{\mathcal{T}}\mathbf{u})(x) - (\boldsymbol{\mathcal{T}}\mathbf{v})(x)\|^p \, d\mu(x) \\
&= \int_X \|\mathcal{O}(x, \overrightarrow{\varphi_i(x, \tilde{\mathbf{u}}_i(x))}\,) - \mathcal{O}(x, \overrightarrow{\varphi_i(x, \tilde{\mathbf{v}}_i(x))}\,)\,\|^p d\mu(x) \\
&\le \int_X \sum_{i=1}^{r} \|\, \varphi_i(x, \tilde{\mathbf{u}}_i(x)) - \varphi_i(x, \tilde{\mathbf{v}}_i(x))\,\|^p \, d\mu(x) \qquad (\text{ by } 3.1 \text{ }) \\
&\le \sum_{i=1}^{r} c_i^p \int_X \|\, \tilde{\mathbf{u}}_i(x) - \tilde{\mathbf{v}}_i(x)\,\|^p \, d\mu(x) \qquad (\text{ by } 2.3 \text{ }) \\
&\le \sum_{i=1}^{r} c_i^p \int_{w_i(X)} \|\, \mathbf{u}(w_i^{-1}(x)) - \mathbf{v}(w_i^{-1}(x))\,\|^p \, d\mu(x) \\
&\le \sum_{i=1}^{r} s_i^n \, c_i^p \int_X \|\, \mathbf{u}(t) - \mathbf{v}(t)\,\|^p \, d\mu(t) \qquad (\text{by the Lipschitz property of } w_i) \\
&= \sum_{i=1}^{r} s_i^n \, c_i^p \, \|\mathbf{u}-\mathbf{v}\|_p^p.
\end{aligned}$$

From this inequality we see that if $\mathbf{u}, \mathbf{v} \in \boldsymbol{\mathcal{L}}^{\mathbf{p}}$, then

$$\|\boldsymbol{\mathcal{T}}\mathbf{v}\|_p \leq \|\boldsymbol{\mathcal{T}}\mathbf{v} - \boldsymbol{\mathcal{T}}\mathbf{u}\|_p + \|\boldsymbol{\mathcal{T}}\mathbf{u}\|_p \leq \left(\sum_{i=1}^{r} s_i^n \, c_i^p\right)^{1/p} \|\mathbf{u} - \mathbf{v}\|_p + \|\boldsymbol{\mathcal{T}}\mathbf{u}\|_p \ ;$$

what says that if there exists a function $\mathbf{u} \in \boldsymbol{\mathcal{L}}^{\mathbf{p}}$ such that $\boldsymbol{\mathcal{T}}\mathbf{u} \in \boldsymbol{\mathcal{L}}^{\mathbf{p}}$ then $\boldsymbol{\mathcal{T}}$ sends $\boldsymbol{\mathcal{L}}^{\mathbf{p}}$ into $\boldsymbol{\mathcal{L}}^{\mathbf{p}}$, $1 \leq p < +\infty$. Now, since $\mu(X) < +\infty$ then $\boldsymbol{\mathcal{L}}^{\infty} \subset \boldsymbol{\mathcal{L}}^{\mathbf{p}}, 1 \leq p < \infty$ and since, by Theorem 3.2 $\boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{L}}^{\infty} \to \boldsymbol{\mathcal{L}}^{\infty}$, we get the desired result. ■

Corollary 3.4 *If, with the above notation,* $(\sum_{i=1}^{n} s_i^n \, c_i^p)^{1/p} < 1$ *for some* p, $1 \leq p < \infty$, *then* $\boldsymbol{\mathcal{T}}$ *is a contraction map on* $\boldsymbol{\mathcal{L}}^{\mathbf{p}}$ *and the functional equation given by 2.6:*

$$\mathbf{u} = \mathcal{O}(x, \varphi_1(x, \tilde{\mathbf{u}}_1(x)), \ldots , \varphi_r(x, \tilde{\mathbf{u}}_r(x))\,),$$

has a unique solution in $\boldsymbol{\mathcal{L}}^{\mathbf{p}}$.

If the $w_i : X \to X$ are differentiable and $Dw_i(x)$ denotes the differential matrix of w_i at the point x, the proof of the last Theorem shows that we can improve the Lipschitz property of the operator $\boldsymbol{\mathcal{T}}$, replacing s_i^n by $\ell_i = \sup_{x \in X} |\det Dw_i(x)| \leq |s_i|^n$. We then have the following Theorem.

Theorem 3.5 *Let* $\boldsymbol{\mathcal{T}}$ *be as defined by 3.1. Then, if* $\mathbf{u}, \mathbf{v} \in \boldsymbol{\mathcal{L}}^{\mathbf{p}}(X, E)$, *then*

$$\boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{L}}^{\mathbf{p}}(X, E) \to \boldsymbol{\mathcal{L}}^{\mathbf{p}}(X, E),$$

and

$$\|\boldsymbol{\mathcal{T}}\mathbf{u} - \boldsymbol{\mathcal{T}}\mathbf{v}\|_p \leq \left(\sum_{i=1}^{r} \ell_i c_i^p\right)^{1/p} \|\mathbf{u} - \mathbf{v}\|_p \ ,$$

where $\ell_i = \sup_{x \in X} |\det Dw_i(x)|$ *and* c_i *are the Lipschitz constants of* φ_i .

Note that the solution to the functional equation 2.6 presented here can be obtained as the limit of the iteration of the operator $\boldsymbol{\mathcal{T}}$ at any starting function.

Remark In [11] we show that using the same techniques than in this paper, Theorem 3.5 can in some cases be slightly improved weakening the conditions on the φ.

4 Acknowledgments

We wish to acknowledge support from the Faculty of Mathematics and the Department of Pure Mathematics of the University of Waterloo. We would like to thank Chris Heil for many discussions on the subject.

References

[1] S. Abenda and G. Turchetti. Inverse problem of fractal sets on the real line via the moment method. *Nuovo Cimento*, 104B:213–227, 1989.

[2] M. Bajraktarevic. Sur une équation fonctionnelle. *Glasnik Mat.-Fiz. I Astr.*, 12(3):201–205, 1957.

[3] M. F. Barnsley. Fractal functions and interpolation. *Constructive Approximation*, 2:303–329, 1986.

[4] M. F. Barnsley. *Fractals Everywhere*. Academic Press, Inc., San Diego, CA, 1988.

[5] M. F. Barnsley, V. Ervin, D. Hardin, and J. Lancaster. Solution of an inverse problem for fractals and other sets. *Proceedings of the National Academy of Sciences*, 83:1975–1977, 1985.

[6] M. F. Barnsley and A. N. Harrington. The calculus of fractal interpolation functions. *Journal of Approximation Theory*, 57:14–34, 1989.

[7] M. F. Barnsley and A. D. Sloan. A better way to compress images. *BYTE Magazine*, January issue:215–223, 1988.

[8] C. A. Cabrelli, M. C. Falsetti, and U. M. Molter. Block-coding. A functional Approach. Preprint, 1998.

[9] C. A. Cabrelli, B. Forte, U. M. Molter, and E. R. Vrscay. Iterated fuzzy set systems: A new approach to the inverse problem for fractals and other sets. *Journal of Mathematical Analysis and Applications*, 171:79–100, 1992.

[10] C. A. Cabrelli, C. Heil, and U. M. Molter. Generalized self-similarity applied to matrix dilation equations. (Extended Abstract). *Z.Angew.Math.Mech.*, 76:493–494, 1996.

[11] C. A. Cabrelli, C. Heil, and U. M. Molter. Self-Similarity and Multiwavelets in Higher Dimensions. Preprint, 1998.

[12] C. A. Cabrelli and U. M. Molter. Density of fuzzy attractors: A step towards the solution of the inverse problem for fractals and other sets. In J.S. Byrnes, editor, *Proceedings of the NATO ASI on Probabilistic and Stochastic Methods in Analysis with Applications*, pages 163–173, July 1991.

[13] G. de Rahm. Sur quelques courbes définies par des équations fonctionnelles. *Rend. Sem. Mat. dell'Univ. e del Politecnico di Torino*, 16:101–113, 1957.

[14] A. Deliu, J. Geronimo, and R. Schonkwiler. Inverse problem for fractals. Preprint of the School of Mathematics, Georgia Institute of Technology, 1994.

[15] S. Dubuc. Functional equations connected with peculiar curves. In *Iteration Theory and its Functional Equations*, volume 1163 of *Lecture Notes in Mathematics*, pages 33–44. Springer Verlag, 1985.

[16] S. Dubuc. Interpolation through an iterative scheme. *Journal of Mathematical Analysis and Applications*, 114:185–204, 1986.

[17] S. Dubuc and A. Elqortobi. Approximations of fractal sets. *Journal of Computational and Applied Mathematics*, 29:79–89, 1990.

[18] J. S. Geronimo, D. P. Hardin, and P. R. Massopust. Fractal functions and wavelet expansions based on several scaling functions. *Journal of Approximation Theory*, 78:373–401, 1994.

[19] K. Gröchenig and W. Madych. Multiresolution analysis, Haar bases and selfsimilar tilings of R^n. *IEEE, Transactions on Information Theory*, 38:556–568, 1992.

[20] C. Handy and G. Mantica. Inverse problems in fractal construction: Moment method solution. *Physica D*, 43:17–36, 1990.

[21] M. Hata. On the functional equation ... *Journal Math. Kyoto Univ.*, 25(2):357–364, 1985.

[22] M. Hata. Fractals in mathematics. *Patterns and Waves - Qualitative Analysis of Nonlinear Differential Equations*, pages 259–278, 1986.

[23] C. Heil and D. Colella. Matrix refinement equations: Existence and uniqueness. Preprint, 1994.

[24] J. Hutchinson. Fractals and self-similarity. *Indiana University Journal of Mathematics*, 30:713–747, 1981.

[25] W.-L. Hwang and S. Mallat. Characterization of self-similar multifractals with wavelet maxima. *Applied and Computational Harmonic Analysis*, 1(4):316–328, 1994.

[26] R. Strichartz. Wavelet expansions of fractal measures. *Journal of Geometric Analysis*, 1(3):269–289, 1991.

[27] R. Strichartz. Wavelets and Self-Affine tilings. To appear, 1993.

[28] R. Strichartz. Self-similarity in Harmonic Analysis. *The Journal of Fourier Analysis and Applications*, 1(1):1–37, 1994.