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A Complete Gabor System of Zero Beurling Density

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Abstract

It has been shown that there exist complete Gabor systems whose set of time-frequency shifts have an rbitrary small, but positive, Beurling density. A natural question is whether or not the positivity of the Beurling density is necessary for completeness. In this article we give a negative answer to this question. We exhibit a specific example of a complete Gabor System whose set of time-frequency shifts has zero Beurling density.

Key words and phrases: Gabor systems, Schauder basis, Beurling density, frames.

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1 Introduction

A Gabor system generated by a function $f \in L^2(\mathbb{R}^d)$ and a discrete set $\Lambda \subset \mathbb{R}^{2d}$ is defined as $G(f,\Lambda) = \{e^{2\pi ibx}f(x-a) : (a,b) \in \Lambda\}$. If $Q_h(x) = \prod_{i=1}^{d}[x_i - h/2, x_i + h/2)$ for $h > 0$ and $x \in \mathbb{R}^d$ then the upper and lower Beurling densities of a countable set $X \subset \mathbb{R}^d$ are respectively defined as

$$D^+(X) = \limsup_{h\to+\infty} \frac{\sup_{x\in\mathbb{R}^d} \#\,[X \cap Q_h(x)]}{h^d}$$

$$\text{and} \qquad D^-(X) = \liminf_{h\to+\infty} \frac{\inf_{x\in\mathbb{R}^d} \#\,[X \cap Q_h(x)]}{h^d}.$$

If $D^{-}(X) = D^{+}(X)$, then X is said to have uniform Beurling density $D(X) = D^{+}(X)$.

There are several results relating properties of a Gabor system $G(f, \Lambda)$ to the Beurling density of the set Λ of time-frequency shifts. In the next lines we review some of them.

Assume that $d = 1$ and $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$. In this case $D(\Lambda) = 1/ab$. Baggett proved in [2] that if the generated Gabor system is complete in $L^2(\mathbb{R})$ then $D(\Lambda) \geq 1$ (i.e., $ab \leq 1$). This result also follows from Rieffel's work on C^*-algebras [7] . Ramanathan and Steger showed in [6] that it remains true for an arbitrary lattice Λ in $\mathbb{R}^{2d}$ and the associated Gabor system $G(f, \Lambda) \subset L^2(\mathbb{R}^d)$. For the non-lattice case (i.e., Λ is an arbitrary discrete set in $\mathbb{R}^{2d}$), it has been proven that to be a frame for $L^2(\mathbb{R}^d)$, the lower and the upper Beurling densities of a Gabor system must satisfy $1 \leq D^{-}(\Lambda) \leq D^{+}(\Lambda) < \infty$. Moreover, if the Gabor system satisfies the stronger hypothesis of being a Riesz basis, then $D(\Lambda) = 1$; for a complete reference see [6] and [3]. However, as was shown in [4], if the Gabor system is only a Schauder basis for $L^2(\mathbb{R}^d)$, then the upper Beurling density must be $D^{+}(\Lambda) \leq 1$. Although in this case the problem of whether or not $D(\Lambda) = 1$ for an arbitrary generating function is still open, the following sufficient conditions were obtained by Deng and Heil [4]: if the generating function belongs to the modulation space $M^{1,1}(\mathbb{R}^d)$ or if the Gabor system possesses a lower frame bound, then $D(\Lambda) = 1$.

These results do not answer however the question about the existence (for arbitrary discrete Λ) of a complete Gabor system whose set of time-frequency shifts has Beurling density strictly less than one. This was affirmed by J. J. Benedetto, C. Heil, and D. F. Walnut in [1]. As shown in their paper, for any $M \in \mathbb{N}$ there is a complete Gabor system of $L^2(\mathbb{R})$ whose index set Λ satisfies $D(\Lambda) = 2/M$. Another surprising example is due to Y. Wang [8]. For an arbitrary $\varepsilon > 0$, Wang constructs a complete Gabor system $G(f, \Lambda) \subset L^2(\mathbb{R})$ such that $D(\Lambda) < \varepsilon$. In this example Λ satisfies the additional property of being a subset of a lattice Δ with density $D(\Delta) > 1$, which is interesting considering the above mentioned Ramanathan and Steger result.

In view of these results one can still question whether or not there exists a Gabor system that is complete and its set of time-frequency shifts has zero Beurling density. In this note we give an affirmative answer to this question. We exhibit a specific example constructed on the same bases as those proposed in [1].

2 Preliminaries

For a, $b \in \mathbb{R}^d$ let M_b and T_a denote the modulation and translation operators on $L^2(\mathbb{R}^d)$ defined by

$$M_b f(t) = e^{2\pi i b t} f(t) \quad \text{and} \quad T_a f(t) = f(t-a).$$

With this notation, for a function $f \in L^2(\mathbb{R}^d)$ and a discrete set $\Lambda \subset \mathbb{R}^{2d}$, the generated Gabor system is the set $G(f,\Lambda) = \{M_b T_a f\}_{(a,b)\in\Lambda}$.

As in the Gabor systems given in [1] the next theorem due to H. J. Landau [5] is essential for our construction.

For any closed interval I we denote by S_n the set $S_n = \bigcup_{j=-n}^{n-1}(I+j)$ and by $C(S_n)$ the space of continuous functions with domain S_n equipped with the L^∞ norm.

Theorem 1 (Landau [5]) *Given any $\varepsilon > 0$ there exists a symmetric real sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ (i.e., $\lambda_k = \lambda_{-k}$) such that for all $0 < \delta < 1$ and for all closed intervals I of length less than $1-\delta$:*

(1) $|\lambda_k - k| < \varepsilon \quad \forall k \in \mathbb{Z}$,

(2) $\{e^{2\pi i \lambda_k t}\}_{k\in\mathbb{Z}}$ *is complete in* $C(S_n) \quad \forall n \in \mathbb{N}$.

Since S_n is a closed set of finite measure for every n, we have that $C(S_n) \subset L^2(S_n)$. Furthermore, $\|f\|_2 \leq |S_n|^{1/2}\,\|f\|_\infty$ for each $f \in C(S_n)$. Combining this with the fact that $C(S_n)$ is dense in $L^2(S_n)$ we can replace $C(S_n)$ by $L^2(S_n)$ in the last theorem.

We will also need the following simple lemma, which is straightforward:

Lemma 1 *Let $\{\varphi_n\}_{n\in\mathbb{Z}}$ be complete in $L^2(E)$ with E a measurable subset of $\mathbb{R}$ and $g \in L^\infty(E)$ such that $g \neq 0$ a.e. in E. Then $\{g\varphi_n\}_{n\in\mathbb{Z}}$ is complete in $L^2(E)$.*

3 Construction of the Gabor system

In what follows we construct a Gabor system of $L^2(\mathbb{R})$ that is complete and whose set of time-frequency shifts has zero Beurling density.

Let ε be a fixed real number such that $0 < \varepsilon < 1/4$ and take $\{\lambda_m\}_{m\in\mathbb{Z}}$ as in *Theorem 1* for this ε. Fix also any $0 < \delta < 1/4$ and let I be the interval $I = [0, 1-\delta]$. We denote with $\mathcal{X}_E$ the characteristic function of a set E.

The Gabor system we propose here will be generated by the function $h(t) = 2^{-t}\mathcal{X}_G(t)$ with $G = \bigcup_{n=0}^{\infty}(I+n)$; i.e.,

$$h(t) = \begin{cases} 2^{-t} & \text{if } t \in I+n \\ 0 & \text{if } t \notin \bigcup_{n=0}^{\infty}(I+n) \end{cases}$$

Clearly, $h \in L^2(\mathbb{R})$.

Set $A = A_1 \cup (A_1 + 1/2)$ with $A_1 = \{2^n : n \in \mathbb{N}\} \cup \{-2^n : n \in \mathbb{N}\}$ and set $B = \{\lambda_m : m \in \mathbb{Z}\}$.

Our set of time-frequency shifts $\Lambda \subset \mathbb{R}^2$ is then defined by $\Lambda = A \times B$.

We show now that $D(\Lambda) = 0$. An immediate consequence of the fact that $|\lambda_m - m| < \varepsilon$ for all integers m is that in each cube of sides of length 2^h there are at most $4(h+1)(2^h+1)$ points of Λ. Using this we have:

$$\begin{aligned} 0 \le D^-(\Lambda) \le D^+(\Lambda) &= \limsup_{h\to\infty} \frac{\sup_{x\in\mathbb{R}^2} \#\left[\Lambda \cap Q_h(x)\right]}{h^2} \\ &\le \limsup_{h\to\infty} \frac{4(h+1)(2^h+1)}{(2^h)^2} = 0. \end{aligned}$$

Thus $D(\Lambda) = 0$.

Finally we must prove that $G(h,\Lambda) = \{M_b T_a h : (a,b) \in \Lambda\}$ is complete in $L^2(\mathbb{R})$ (i.e., $\operatorname{span}(G(h,\Lambda))$ is dense in $L^2(\mathbb{R})$).

We denote with S_n the set $S_n = \bigcup_{k=-n}^{n-1}(I+k)$ and with T_n the set $T_n = S_n + 1/2 = \bigcup_{k=-n}^{n-1}(I+k+1/2)$. With this notation, $G-n$ is the disjoint union of S_n and $G+n$, also $G-n+1/2$ is the disjoint union of T_n and $G+n+1/2$.

Let H_k be the set $H_k = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(f) \subset [-2^k, 2^k + \frac{1}{2} - \delta]\}$. Notice that $\bigcup_{k=1}^{\infty} H_k$ is dense in $L^2(\mathbb{R})$. Hence it is enough to show that for each $\xi > 0$, $k \in \mathbb{N}$ and each $f \in H_k$ there exists a function $g \in \operatorname{span}(G(h,\Lambda))$ such that $\|f-g\| < \xi$.

First choose some $k \in \mathbb{N}$ and observe that for any $m \in \mathbb{Z}$ we can compute

$$\begin{aligned} &M_{\lambda_m} T_{-2^k} h(t) - 2^{-(2^{k+1})} M_{\lambda_m} T_{2^k} h(t) = \\ &M_{\lambda_m}\left(2^{-(t+2^k)}\mathcal{X}_G(t+2^k) - 2^{-(t-2^k+2^{k+1})}\mathcal{X}_G(t-2^k)\right) = \\ &M_{\lambda_m} 2^{-(t+2^k)}\left(\mathcal{X}_{G-2^k}(t) - \mathcal{X}_{G+2^k}(t)\right) = M_{\lambda_m} 2^{-(t+2^k)} \mathcal{X}_{S_{2^k}}(t). \end{aligned}$$

In a similar way we obtain

$$M_{\lambda_m} T_{-2^k+1/2} h(t) - 2^{-(2^{k+1})} M_{\lambda_m} T_{2^k+1/2} h(t) = M_{\lambda_m} 2^{-(t+2^k)} \mathcal{X}_{T_{2^k}}(t).$$

In other words, if we call

$$\psi_1(t) = 2^{-(t+2^k)} \mathcal{X}_{S_{2^k}}(t) \quad \text{and} \quad \psi_2(t) = 2^{-(t+2^k)} \mathcal{X}_{T_{2^k}}(t),$$

we have that

$$M_{\lambda_m} \psi_i \in \operatorname{span}(G(h, \Lambda)), \qquad \forall m \in \mathbb{Z}, \quad i = 1, 2. \tag{1}$$

From *Theorem 1* we know that the sequence $\{M_{\lambda_m} \mathcal{X}_{S_{2^k}}\}_{m \in \mathbb{Z}}$ is complete in the closed subspace $\{f \in L^2(\mathbb{R}) : \operatorname{supp}(f) \subset S_{2^k}\}$ of $L^2(\mathbb{R})$. Lemma 1 applied to this sequence and to the function $\varphi(t) = 2^{-(t+2^k)} \mathcal{X}_{[-2^k, 2^k + \frac{1}{2} - \delta]}(t) \in L^\infty(\mathbb{R})$ yields that the sequence $\{M_{\lambda_m} \psi_1\}_{m \in \mathbb{Z}}$ is also complete in that subspace because for each m, $M_{\lambda_m} \psi_1 = M_{\lambda_m} \mathcal{X}_{S_{2^k}} \varphi$. By a similar argument the sequence $\{M_{\lambda_m} \psi_2\}_{m \in \mathbb{Z}}$ is complete in the subspace $\{f \in L^2(\mathbb{R}) : \operatorname{supp}(f) \subset T_{2^k}\}$.

Choose now $\xi > 0$ and a function $f \in H_k$. Note that its support is the union of the sets S_{2^k} and T_{2^k}, hence we can write $f = \mathcal{X}_{S_{2^k}} f + \mathcal{X}_{T_{2^k} \setminus S_{2^k}} f$. By the completeness of the sequences above it follows that there exist functions $g_1 \in \operatorname{span}(\{M_{\lambda_m} \psi_1\}_{m \in \mathbb{Z}})$ and $g_2 \in \operatorname{span}(\{M_{\lambda_m} \psi_2\}_{m \in \mathbb{Z}})$ such that

$$\left\| \mathcal{X}_{S_{2^k}} f - g_1 \right\|_{L^2(\mathbb{R})} < \frac{\xi}{2} \quad \text{and} \quad \left\| \mathcal{X}_{T_{2^k} \setminus S_{2^k}} f - g_2 \right\|_{L^2(\mathbb{R})} < \frac{\xi}{2}. \tag{2}$$

Therefore, taking $g = g_1 + g_2$, by (1) and (2) we can conclude that $g \in \operatorname{span}(G(h, \Lambda))$ and that

$$\|f - g\| = \left\| \mathcal{X}_{S_{2^k}} f + \mathcal{X}_{T_{2^k} \setminus S_{2^k}} f - g_1 - g_2 \right\| < \xi,$$

which completes the proof.

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