DETERMINING SETS OF SHIFT INVARIANT SPACES

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Abstract. The problem of determining an appropriate signal or image model from experimental data is addressed. Specifically, given a finite set of signals or images belonging to a fixed but unknown shift invariant space, the problem is whether the known signals at hand are sufficient for determining the unknown shift invariant space to which they belong. This problem gives rise to the concept of determining sets for shift invariant spaces, and we obtain necessary and sufficient conditions needed for determining an unknown shift invariant space $V(\Phi)$ from a finite subset $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ of $V(\Phi)$.

1. introduction

In many signal and image processing applications, images and signals are assumed to belong to some shift invariant space of the form:

$$V(\Phi) := \{f = \sum_{i=1}^{n} \sum_{j\in\mathbb{Z}^d} \alpha_i(j)\phi_i(\cdot + j) : \ \alpha_i \in l^2, \ i = 1, \ldots, n\} \tag{1.1}$$

where $\Phi = [\phi_1, \phi_2 \ldots \phi_n]^t$ is a column vector consisting of functions in $L^2(\mathbb{R}^d)$ called *a generator* for the space $V = V(\Phi)$ (see e.g., [2]). For example, if $n = 1, d = 1$ and $\phi(x) = \operatorname{sinc}(x)$, then the underlying space is the space of band limited functions (often used in communications). However, the assumed model (often the band limitedness assumption) is seldom derived from experimental data.

Thus, given a class of signals belonging to a certain fixed - but unknown - shift invariant space V, the problem is whether it is possible to determine the space V from a set of m experimental data $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$, where f_i are observed functions (signals) belonging to $V(\Phi)$. If a finite set $\mathcal{F}$ is sufficient to determine $V(\Phi)$, we will call it a *determining set for* $V(\Phi)$. The goal is to see if we can perform operations on the observations $\mathcal{F} = \{f_1, f_2, \ldots, f_m\}$ to deduce whether they are sufficient to determine the unknown shift invariant space $V(\Phi)$, and if so, use them to find some generator Ψ for $V(\Phi)$, i.e., find Ψ such that $V(\Psi) = V(\Phi)$. If the observations are not sufficient to determine $V(\Phi)$, then we need to obtain more observations until a determining set is found.

In this paper we give necessary and sufficient conditions for a finite subset $\mathcal{F}$ of the space $V(\Phi)$ to be a determining set for $V(\Phi)$. In the case $\mathcal{F}$ is a determining set, we shall exhibit an orthonormal generator for $V(\Phi)$ that is written in terms of the elements of $\mathcal{F}$ and thereby reconstruct the whole space.

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The paper is organized as follows. In section 2, we will recall some known properties of shift invariant spaces, Riesz basis and the Gramian matrix. In section 3, we will introduce the notion of determining set and will give a necessary and sufficient condition on $\mathcal{F}$ to determine $V(\Phi)$. Finally, in section 4, we will prove our results.

2. Notation and preliminaries

Throughout this paper, we assume that the unknown space V can be generated by some generator $\Phi = (\phi_1, \dots, \phi_n)^t$ such that $\{\phi_i(x-k) : i = 1, \dots, n,\ k \in \mathbb{Z}^d\}$ forms a Riesz basis for V. This Riesz basis assumption can be restated in the Fourier domain using the Gramian matrix of Φ. Specifically, the Gramian G_Θ of a vector function $\Theta = [\theta_1, \cdots, \theta_n]^t$ is defined by

$$G_\Theta(\omega) = \sum_{k\in\mathbb{Z}^d} \widehat{\Theta}(\omega+k)\widehat{\Theta}^*(\omega+k) \tag{2.1}$$

where $\widehat{\Theta}(\omega) := \int_{\mathbb{R}^d} \Theta(x)e^{-2\pi i\omega x}dx$, and $\widehat{\Theta}^*$ is the adjoint of $\widehat{\Theta}$. With this definition, it is well-known that Φ induces a Riesz basis for V if and only if there exists two positive constants $A > 0$ and $B > 0$ such that

$$AI \leq G_\Phi(\omega) \leq BI, \quad a.e.\ \omega, \tag{2.2}$$

where I is the $n \times n$ identity matrix (see e.g., [1, 3, 4]). The set $B = \{\phi_i(x-k) : i = 1, \dots, n,\ k \in \mathbb{Z}^d\}$ forms an orthonormal basis if and only if $A = B = 1$ in (2.2). Throughout the paper we assume that $\Phi = [\phi_1, \cdots, \phi_n]^t$ satisfies (2.2).

For a set $A \subseteq \mathbb{R}^d$, the complementary set in $\mathbb{R}^d$ will be denoted by A^c.

We use $\mathcal{F}$ to indicate a set of functions and we use F to denote the vector valued function whose components are the elements of $\mathcal{F}$ in some fixed order.

3. Main results

Our main goal is to find necessary and sufficient conditions on subsets $\mathcal{F} = \{f_1, \cdots, f_m\}$ of $V(\Phi)$ such that any $g \in V$ can be recovered from $\mathcal{F}$. A set $\mathcal{F}$ with such a property will be called a *determining set* for $V(\Phi)$. Specifically,

Definition 1. The set $\mathcal{F} = \{f_1, f_2, \dots, f_m\} \subset V(\Phi)$ is said to be a determining set for $V(\Phi)$, if any $g \in V(\Phi)$ can be written as

$$\widehat{g} = \widehat{\alpha}_1\widehat{f}_1 + \widehat{\alpha}_2\widehat{f}_2 + \dots + \widehat{\alpha}_m\widehat{f}_m \tag{3.1}$$

where $\widehat{\alpha}_1, \dots, \widehat{\alpha}_m$ are 1-periodic measurable functions. In addition, if $\mathcal{F}$ is a determining set of $V(\Phi)$ we will say that $V(\Phi)$ is determined by $\mathcal{F}$.

Remarks

(i) The vector $F = [f_1, \dots, f_m]^t$ need not be a generator for V. In fact, series of the form $\sum_{i=1}^m \sum_k c_i f_i(x-k)$ need not even be convergent for all $c_i \in l^2$.

(ii) An equivalent definition of a determining set can be obtained (e.g., see [3, Theorem 1.7]): a set $\mathcal{F}$ is a determining set for $V(\Phi)$ if and only if $V(\Phi) \subset \text{closure}_{L_2}\big(\text{span}\{f_i(x-k) :\ f_i \in \mathcal{F}\}\big)$.

It is not surprising that if the generators of V are vector-functions of size n, then the cardinality m of a determining set $\mathcal{F}$ must be larger or equal to n. This result is stated in the following proposition:

Proposition 1. *Let V be a shift invariant space generated by some Riesz basis $\{\phi_i(x-k) : i = 1, \dots, n,\ k \in \mathbb{Z}^d\}$, where $\Phi = [\phi_1, \dots, \phi_n]^t$ is a vector of functions in V. If $\mathcal{F}$ is a determining set for V then card$(\mathcal{F}) \geq n$.*

Because of the proposition above, we will only consider sets $\mathcal{F}$ with cardinality m larger than or equal to the size n of the generators for V. Given such a set $\mathcal{F}$ there are $L = \begin{pmatrix} m \\ n \end{pmatrix}$ subsets $\mathcal{F}_\ell \subset \mathcal{F}$ of size n. For each such subset $\mathcal{F}_\ell$ of size n, we define the set

$$A_\ell = \{\omega : \det G_{F_\ell}(\omega) \neq 0\}, \qquad 1 \leq \ell \leq L, \tag{3.2}$$

where G_{F_ℓ} is the $n \times n$ Gramian matrix for the vector F_ℓ. We now state our main theorem on determining sets.

Theorem 1. *A set $\mathcal{F} = \{f_1, \cdots, f_m\} \subset V(\Phi)$ is a determining set for $V(\Phi)$ if and only if the set $Z = \bigcap_{\ell=1}^{L} A_\ell^c$ has Lebesgue measure zero.*

Moreover, if $\mathcal{F}$ is a determining set for $V(\Phi)$, then the vector function

$$\widehat{\Psi}(\omega) := G_{F_1}^{-\frac{1}{2}}(\omega)\widehat{F_1}(\omega)\chi_{B_1}(\omega) + \cdots + G_{F_L}^{-\frac{1}{2}}(\omega)\widehat{F_L}(\omega)\chi_{B_L}(\omega) \tag{3.3}$$

where $B_1 := A_1, B_\ell := A_\ell - \bigcup_{j=1}^{\ell-1} A_j,\ \ell = 2, \dots, L$, generates an orthonormal basis $\{\psi_i(x-k) : i = 1, \dots, n,\ k \in \mathbb{Z}^d\}$ of $V(\Phi)$.

Remarks

(i) Theorem 1 provides a method for checking whether and when a set of functions generates a fixed (yet unknown) shift-invariant space generated by some unknown Φ of known size n. Since other than the value n the only requirement is that the set of functions belong to the same (unknown) shift-invariant space, we can apply the theorem to a set of *observed* functions (the data) if we know that they are all from some shift-invariant space V and either determine the space or conclude that we do not have enough data to do so and need to acquire more data.

(ii) The orthonormal basis constructed in the Theorem is only in L^2 but not in L^1 in general. Further investigation is needed for the construction of better localized bases.

4. Proofs

4.1. Proof of Proposition 1.

Proof. Let $\mathcal{F} = \{f_1, f_2, \dots, f_m\}$, and $F = [f_1, f_2, \dots, f_m]^t$. Since $\mathcal{F}$ is a determining set, there exists $\beta = \beta(\omega)$ an $n \times m$ matrix of 1-periodic functions such that $\widehat{\Phi} = \widehat{\beta}\widehat{F}$. Then

$$\begin{aligned} G_\Phi(\omega) &= \sum_{k\in\mathbb{Z}} \widehat{\Phi}(\omega+k)\widehat{\Phi}^*(\omega+k) \\ &= \sum_{k\in\mathbb{Z}} \widehat{\beta}(\omega+k)\widehat{F}(\omega+k)\widehat{F}^*(\omega+k)\widehat{\beta}^*(\omega+k) \\ &= \sum_{k\in\mathbb{Z}} \widehat{\beta}(\omega)\widehat{F}(\omega+k)\widehat{F}^*(\omega+k)\widehat{\beta}^*(\omega) \\ &= \widehat{\beta}(\omega)G_F(\omega)\widehat{\beta}^*(\omega). \end{aligned}$$

Therefore rank $(G_\Phi) \leq$ rank $(G_F) \leq m$, since the rank of the product of matrices is not greater than the rank of any of its factors, see [5]. In addition, since $\{\phi_1(x-k), \phi_2(x-k)\dots\phi_n(x-k)\}$ form a Riesz basis, the rank $(G_\phi) = n$. $\square$

4.2. **Proof of Theorem 1.** In order to prove Theorem 1, we need to prove the following lemma:

Lemma 1. *If a vector function $T = [t_1, \dots, t_n]^t$ satisfies that $\widehat{T}(\omega) = \widehat{C}_T(\omega)\widehat{\Phi}(\omega)$, a.e. ω for some $n \times n$ measurable matrix function $\widehat{C}_T(\omega)$, with entries in $L^2([0,1]^d)$ and if $G_T(\omega) = I\chi_\Omega(\omega)$ then there exist constants, $B' > 0$ and $B'' > 0$ such that $\|\widehat{C}_T\|_{L^\infty(\Omega)} \leq B'$, and $\|\widehat{C}_T^{-1}\|_{L^\infty(\Omega)} \leq B''$.*

Proof. The Gramian of T is given by $G_T = \widehat{C_T} G_\Phi \widehat{C_T}^*$. Since by (2.2) G_Φ is positive self adjoint a.e. ω, the square root $G_\Phi^{1/2}$ makes sense and we can write G_T as

$$G_T = \widehat{C_T} {G_\Phi}^{\frac{1}{2}} {G_\Phi}^{\frac{1}{2}} \widehat{C_T}^*. \tag{4.1}$$

Define the $n \times n$ matrix function $U(\omega)$ by $U = \widehat{C}_T {G_\Phi}^{\frac{1}{2}}$. Then, from (4.1) and our assumption on G_T, we have that $I\chi_\Omega = UU^*$ and hence $U^* = U^{-1}$ on Ω. Thus we also have $I\chi_\Omega = U^*U$. Therefore ${G_\Phi}^{\frac{1}{2}} \widehat{C_T}^* \widehat{C_T} {G_\Phi}^{\frac{1}{2}} = I\chi_\Omega$ and,

$$\widehat{C_T}^* \widehat{C_T} \chi_\Omega = {G_\Phi}^{-\frac{1}{2}} {G_\Phi}^{-\frac{1}{2}} \chi_\Omega = G_\Phi^{-1}\chi_\Omega.$$

Using (2.2), we conclude that $AI\chi_\Omega \leq \widehat{C_T}^* \widehat{C_T} \leq BI\chi_\Omega$ where I is the identity matrix, and A, B are positive constants independent of ω.

But trace $(\widehat{C_T}^* \widehat{C_T}\chi_\Omega) = \sum_{r=1}^n \lambda_r(\omega)$ where λ_r is an eigenvalue of

$$D(\omega)\chi_\Omega := \widehat{C_T}^*(\omega)\widehat{C_T}(\omega)\chi_\Omega,$$

thus

$$nA \leq \text{trace}\,(D(\omega)\chi_\Omega) = \sum_{r=1}^n \lambda_r(\omega) \leq nB.$$

It follows that

$$\|\widehat{C_T}\|_{L^\infty(\Omega)} \leq \sum_{i,j} |(\widehat{C_T})_{i,j}(\omega)|^2 \leq B' = nB.$$

Similarly, since $\widehat{C_T^{-1}}^* \widehat{C_T^{-1}} \chi_\Omega = G_\Phi \chi_\Omega$, there exists B'' such that $\|\widehat{C}_T\|_{L^\infty(\Omega)} \leq B''$. $\square$

We can now prove Theorem 1.

Proof of Theorem 1. $\Longleftarrow$ Let A_ℓ be as in (3.2), and define $B_1 := A_1, B_\ell := A_\ell - \bigcup_{j=1}^{\ell-1} A_j$, $\ell = 2, \dots, L$. We will proceed in four steps:

Step 1): Since $\mathcal{F}_l \subset V(\Phi)$ and card $\mathcal{F}_l = n$, we can write $\widehat{F_\ell} = \widehat{C_{F_\ell}}\widehat{\Phi}$ for some $n \times n$ square matrix $\widehat{C_{F_\ell}}$ with $L^2([0,1]^d)$ entries, and we have

$$\begin{aligned} G_{F_\ell}(\omega) &= \sum_k (\widehat{C_{F_\ell}}(\omega+k)\widehat{\Phi}(\omega+k))(\widehat{C_{F_\ell}}(\omega+k)\widehat{\Phi}(\omega+k))^* \\ &= \sum_k \widehat{C_{F_\ell}}(\omega+k)\widehat{\Phi}(\omega+k)\widehat{\Phi}^*(\omega+k)\widehat{C_{F_\ell}}^*(\omega+k) \\ &= \widehat{C_{F_\ell}}(\omega)G_\Phi(\omega)\widehat{C_{F_\ell}}^*(\omega), \end{aligned}$$

since $\widehat{C_{F_\ell}}(\omega)$ is 1-periodic.

Moreover, since Φ induces a Riesz basis then G_Φ is positive definite, and there exists $A > 0$ such that $vG_\Phi(\omega)v^* \geq A\|v\|^2 > 0$ for almost all ω and for all $v \neq 0$ in $\mathbb{R}^n$. Hence for any non-zero vector $z \in \mathbb{R}^n$

$$\begin{aligned} zG_{F_\ell(\omega)}z^* &= z(\widehat{C_{F_\ell}}(\omega)G_\Phi(\omega)\widehat{C_{F_\ell}}(\omega)^*)z^* \\ &= (z\widehat{C_{F_\ell}}(\omega))G_\Phi(\omega)(z\widehat{C_{F_\ell}}(\omega))^* \end{aligned}$$

must be positive in B_ℓ if we prove that $z\widehat{C_{F_\ell}}(\omega) \neq 0$ in B_ℓ. But this is true since $\det G_{F_\ell} = \left(\det \widehat{C_{F_\ell}}\right)^2 \det G_\phi$ and $\det G_{F_\ell} \neq 0$ on B_ℓ, so $\widehat{C_{F_\ell}}(\omega)$ is non-singular a.e. $\omega \in B_\ell$. Thus, G_{F_ℓ} is self adjoint and positive definite on B_ℓ.

Step 2) The vector $\widehat{H}_\ell$ defined by $\widehat{H}_\ell := G_{F_\ell}^{-\frac{1}{2}}\widehat{F_\ell}\chi_{B_\ell}$ is well defined and belongs to $(L^2(\mathbb{R}^d))^n$ (we are abusing notation and writing $G_{F_\ell}^{-\frac{1}{2}}$ even though $G_{F_\ell}^{-\frac{1}{2}}$ only makes sense on B_ℓ^c, however there should be no ambiguity about the definition of $\widehat{H}_\ell$ because of the term χ_{B_ℓ}).

Since by step 1) the Gramian $G_{F_\ell}(\omega)$ is self adjoint and positive for a.e. $\omega \in B_\ell$, the inverse square root $G_{F_\ell}^{-\frac{1}{2}}$ makes sense in B_ℓ. Moreover

$$\begin{aligned} G_{H_\ell}(\omega) &= \left(\sum_k (G_{F_\ell}^{-\frac{1}{2}}(\omega+k)\widehat{F_\ell}(\omega+k)(G_{F_\ell}^{-\frac{1}{2}}(\omega+k)\widehat{F_\ell}(\omega+k))^*\right)\chi_{B_\ell}(\omega) \\ &= G_{F_\ell}^{-\frac{1}{2}}(\omega)\left(\sum_k \widehat{F_\ell}(\omega+k)\widehat{F_\ell}(\omega+k)^*\right)(G_{F_\ell}^{-\frac{1}{2}})^*(\omega)\chi_{B_\ell}(\omega) \\ &= G_{F_\ell}^{-\frac{1}{2}}(\omega)G_{F_\ell}(\omega)G_{F_\ell}^{-\frac{1}{2}}(\omega)\chi_{B_\ell}(\omega) \\ &= I\chi_{B_\ell}(\omega). \end{aligned}$$

But $\widehat{H}_\ell(\omega) = G_{F_\ell}^{-\frac{1}{2}}(\omega)\widehat{F_\ell}(\omega)\chi_{B_\ell}(\omega) = G_{F_\ell}^{-\frac{1}{2}}(\omega)\widehat{C}_{F_\ell}(\omega)\widehat{\Phi}(\omega)\chi_{B_\ell}(\omega)$, since $\widehat{F_\ell}(\omega) = \widehat{C}_{F_\ell}(\omega)\widehat{\Phi}(\omega)$ for some $n \times n$ matrix $\widehat{C_{F_\ell}}$ with $L^2([0,1]^d$ entries. Hence we can apply Lemma 1 to get that $\widehat{C_{H_\ell}}(\omega) = G_{F_\ell}^{-\frac{1}{2}}(\omega)\widehat{C_{F_\ell}}(\omega)\chi_{B_\ell}(\omega)$ belongs to $L^\infty([0,1]^d) \subset L^2([0,1]^d)$. Hence $H_\ell \in V(\Phi) \subset (L^2(\mathbb{R}^d))^n$.

Step 3): We now construct an orthonormal basis of V as follows: Define $\Psi = [\psi_1, \dots, \psi_n]^t$ by the the following equation:

$$\widehat{\Psi}(\omega) := G_{F_1}^{-\frac{1}{2}}(\omega)\widehat{F_1}(\omega)\chi_{B_1}(\omega) + \cdots + G_{F_L}^{-\frac{1}{2}}(\omega)\widehat{F_L}(\omega)\chi_{B_L}(\omega) \tag{4.2}$$

where B_ℓ are defined as in the beginning of the proof, and F_ℓ as in the statement of the theorem. By claims 1 and 2, $\widehat{\Psi}$ is well defined and $\psi_i \in V(\Phi), i = 1, \dots, n$.

Step 4): We claim that the set $\{\psi_i(x-k) : i = 1, \dots, n\ k \in \mathbb{Z}^d\}$ forms an orthonormal basis for $V(\Phi)$. To see this, we note that since $\psi_i \in V(\Phi)$, $i = 1, \dots, n$, $\widehat{\Psi} = \widehat{C_\Psi}\widehat{\Phi}$. Moreover given ω, there exists some unique ℓ such that $\omega \in B_\ell$, and $G_\Psi(\omega) = G_{H_\ell}(\omega) = I$. Thus $G_\Psi(\omega) = I$, *a.e.* ω. Therefore by Lemma 1, there exist positive constants B' and B'', such that $\|C_\Psi\|_{L^\infty} \leq B'$, and $\|C_\Psi^{-1}\|_{L^\infty} \leq B''$. It follows that $\{\psi_i(x-k) : i = 1, \dots, n,\ k \in \mathbb{Z}^d\}$ is a Riesz basis for $V(\Phi)$ with Gramian $G_\Psi = I$, hence an orthonormal basis for $V(\Phi)$. In particular $V(\Psi) = V(\Phi)$.

Now, since the translates of $\{\psi_1, \dots, \psi_n\}$ are an orthonormal basis, we have that if $g \in V(\Phi)$, then there exists $\widehat{C} \in (L^2([0,1]^d))^n$ such that $\widehat{g} = \widehat{C}\widehat{\Psi}$. So

$$\widehat{g}(\omega) = \widehat{C}(\omega) \sum_{\ell} (G_{F_\ell}^{-\frac{1}{2}}(\omega)\chi_{B_\ell}(\omega))\widehat{F_\ell}(\omega) = \sum_{\ell} \left(\widehat{C}(\omega)G_{F_\ell}^{-\frac{1}{2}}(\omega)\chi_{B_\ell}(\omega)\right)\widehat{F_\ell}(\omega),$$

and this last equation can be rewritten as: $\widehat{g} = \sum_{i=1}^m \widehat{\alpha}_i \widehat{f}_i$.

This completes the proof of one of the implications of the Theorem. For the converse, we proceed by contradiction. Suppose that the set $Z = \bigcap_{\ell=1}^L A_\ell^c$ has positive Lebesgue measure.

Since each $f_i \in V(\Phi)$, there exists a 1-periodic measurable $m \times n$ matrix $\widehat{A}$ with entries in $L^2([0,1]^d)$ such that $\widehat{F} = \widehat{A}\widehat{\Phi}$. If we denote by $\widehat{A_i}, i = 1, \dots, m$ the i-th row of $\widehat{A}$, so $\widehat{f_i} = \widehat{A}_i\widehat{\Phi}$, then for each choice $\tilde{\imath} = (i_1, \dots, i_n)$, the vector

$$F_{\tilde{\imath}} = \begin{bmatrix} \widehat{f}_{i_1} \\ \widehat{f}_{i_2} \\ \vdots \\ \widehat{f}_{i_n} \end{bmatrix} = \begin{bmatrix} \widehat{A}_{i_1}\widehat{\Phi} \\ \widehat{A}_{i_2}\widehat{\Phi} \\ \vdots \\ \widehat{A}_{i_n}\widehat{\Phi} \end{bmatrix},$$

and if $A_{\tilde{\imath}}$ denotes the $n \times n$ matrix whose rows are $A_{i_1}, \dots, A_{i_n}$, then we have that $G_{F_{\tilde{\imath}}} = A_{\tilde{\imath}} G_\Phi A_{\tilde{\imath}}^*$.

Thus, $\det \widehat{A}_{\tilde{\imath}}^{\,2} = 0$, a.e. on Z, since $\det G_\Phi \neq 0$ and we are assuming that $\det G_{F_{\tilde{\imath}}} = 0$ for all choices of $\tilde{\imath}$. Hence any minor of $\widehat{A}$ of order n is 0 and we conclude that $\operatorname{rank} \widehat{A} < n$ a.e. on Z.

Since F is a determining set, given any $\widehat{C}$, a 1-period vector in $(L^2([0,1]^d)^n$, there exists $\widehat{D}$ a 1-periodic measurable vector of length m such that, $\widehat{C}\widehat{\phi} = \widehat{D}\widehat{f} = \widehat{D}(\widehat{A}\widehat{\phi})$

Thus for a.e. ω

$$\widehat{C}(\omega)\widehat{\phi}(\omega) = (\widehat{D}(\omega)\widehat{A}(\omega))\widehat{\phi}(\omega), \quad \text{i.e.} \quad \left(\widehat{C}(\omega) - \widehat{D}(\omega)\widehat{A}(\omega)\right)(\widehat{\Phi}) = 0.$$

Hence,

$$\left(\widehat{C}(\omega) - \widehat{D}(\omega)\widehat{A}(\omega)\right)\widehat{\Phi}(\omega)\widehat{\Phi}^*(\omega)\left(\widehat{C}(\omega) - \widehat{D}(\omega)\widehat{A}(\omega)\right)^* = 0,$$

and therefore

$$\left(\widehat{C}(\omega) - \widehat{D}(\omega)\widehat{A}(\omega)\right)G_\Phi(\omega)\left(\widehat{C}(\omega) - \widehat{D}(\omega)\widehat{A}(\omega)\right)^* = 0.$$

Since G_Φ is positive definite a.e., it follows that

$$\widehat{C}(\omega) - \widehat{D}(\omega)\widehat{A}(\omega) = 0 \quad \text{a.e. } \omega \in \mathbb{R}. \tag{4.3}$$

But since $\operatorname{rank} \widehat{A}(\omega) < n$ in Z, we can always choose $\widehat{C}(\omega) \in (\operatorname{range}\widehat{A})^\perp$ with the property that $\|\widehat{C}(\omega)\| = 1$ and $\widehat{C}(\omega) = \widehat{C}(\omega + k)$ for all $k \in \mathbb{Z}$ so that $\widehat{C} \in (L(0,1)^d)^{(n)}$. Finally, by construction of $\widehat{C}(\omega)$, no $\widehat{D}$ can be found such that (4.3) is satisfied giving a contradiction. $\square$

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