

DENSITY OF THE SET OF GENERATORS OF WAVELET SYSTEMS

CARLOS CABRELLI AND URSULA M. MOLTER

ABSTRACT. Given a function ψ in $\mathcal{L}^2(\mathbb{R}^d)$, the affine (wavelet) system generated by ψ , associated to an invertible matrix a and a lattice Γ , is the collection of functions $\{|\det a|^{j/2}\psi(a^j x - \gamma) : j \in \mathbb{Z}, \gamma \in \Gamma\}$. In this article we prove that the set of functions generating affine systems that are a Riesz basis of $\mathcal{L}^2(\mathbb{R}^d)$ is dense in $\mathcal{L}^2(\mathbb{R}^d)$.

We also prove that a stronger result is true for affine systems that are a frame of $\mathcal{L}^2(\mathbb{R}^d)$. In this case we show that the generators associated to a fixed but arbitrary dilation are a dense set.

Furthermore, we analyze the orthogonal case in which we prove that the set of generators of orthogonal (not necessarily complete) affine systems, that are compactly supported in frequency, are dense in the unit sphere of $\mathcal{L}^2(\mathbb{R}^d)$ with the induced metric. As a byproduct we introduce the p -Grammian of a function and prove a convergence result of this Grammian as a function of the lattice. This result gives insight in the problem of oversampling of affine systems.

1. INTRODUCTION

Let ψ be an $\mathcal{L}^2(\mathbb{R}^d)$ -function, a an invertible $d \times d$ matrix and Γ a lattice in \mathbb{R}^d . The *wavelet system* or *affine system* (ψ, a, Γ) , generated by ψ and associated to a and Γ , is the collection of functions,

$$(\psi, a, \Gamma) = \{D_{a^j} T_\gamma \psi : j \in \mathbb{Z}, \gamma \in \Gamma\}. \quad (1)$$

Here T_y and D_a denote the the unitary operators in $\mathcal{L}^2(\mathbb{R}^d)$ defined by $(T_y \psi)(x) = \psi(x - y)$, $y \in \mathbb{R}^d$ and $(D_a \psi)(x) = |\det a|^{1/2} \psi(ax)$, $a \in GL_d(\mathbb{R})$.

Affine systems have been studied in depth during the last 25 years mainly because of their importance in applications. In addition they proved to be very useful in a variety of theoretical problems. On the other hand they were studied in the context of Hilbert spaces where the translation and dilation operators were replaced by a general group of unitary operators [GLT93, DL98]. Generalizations of wavelet systems in $\mathcal{L}^2(\mathbb{R}^d)$ with translations not necessarily on a lattice and using different dilations were also considered [ACM04], [GLL⁺04].

One of the relevant questions about these systems is whether the collection (ψ, a, Γ) form an orthonormal basis, a Riesz basis or a frame of the space $\mathcal{L}^2(\mathbb{R}^d)$.

The construction of affine systems with specific prescribed properties is a difficult problem. Usually it is accomplished imposing conditions on the generators. This has been one of the core problem from the beginning of wavelet research

Date: May 29, 2006.

1991 Mathematics Subject Classification. Primary:42C40, Secondary: 42C30.

Key words and phrases. Wavelet Set, affine systems, Riesz basis wavelets, Wavelet Frames.

[Dau88, Mey88, Mey92, BF94, CHM04] (and references therein). Still there are many unanswered questions and open problems in the study of affine systems.

One way to get a better understanding of these systems is considering the set of generators of affine systems having some particular structure, and trying to answer global questions about these sets. For example, wavelet systems forming a tight frame of the space have been completely characterized by a set of equations imposing conditions on the Fourier transform of the generators [CCMW02, CS00, Bow03]. Another example is the problem of connectivity of the set of generators which has received considerable attention lately, [Con98, Spe99].

In this article, motivated by a question posed by David Larson [Lar05], we study the problem of density of the set of generators of affine systems. We prove (Theorem 4.1) that the set of generators forming Riesz bases is dense in $\mathcal{L}^2(\mathbb{R}^d)$ when we allow diagonal matrices and arbitrary lattices. This theorem gives an even stronger result, since the generators in the dense set have Fourier transform supported on *wavelet-sets* and as a consequence have orthogonal dilations. The proof is obtained as a combination of the techniques used in the proof of density for wavelet frames (section 3) and the theory of wavelets sets that has been developed recently by different groups of researches [BL01, BMM99, BS02, BS04, DLS97, DLS98, ILP98, SW98, Wan02, Zak96].

The question about density of generators of wavelet frames, is also answered positively. We obtain a very general result, that confirms the flexibility in their construction. If a dilation matrix a is chosen arbitrarily, the set of frame generators associated to the dilation a is dense in $\mathcal{L}^2(\mathbb{R}^d)$ (Theorem 3.2). The main tool here is the use of the general scheme in frame construction that appears in [ACM04].

Finally in section 5 we study the case of orthogonal affine systems. Since generators of orthonormal systems have norm one the question here is whether they are dense in the unit sphere of $\mathcal{L}^2(\mathbb{R}^d)$. It is easy to see that this is not the case when either the dilation or the lattice is fixed (see section 5).

We first show that for a given function of norm one, we can always find a lattice Γ and a function with orthonormal translates in that lattice, that is close to the original function. We then prove that a dilation can be chosen in such a way that the associated affine system is orthonormal. This proves the density of generators of orthonormal wavelet systems (not necessarily complete in $\mathcal{L}^2(\mathbb{R}^d)$) (Theorem 5.2).

The strategy here is the study of the behavior of the Grammian of the generator, as a function of the lattice. We obtain an interesting result on the convergence of the p -Grammian. We prove that the p -Grammian of a function converges pointwise and in $\mathcal{L}^p([0, 1]^d)$ to the constant function 1 when the lattice expands. This result is a statement about oversampling.

However the question whether complete orthonormal affine systems are dense in the sphere remains unanswered.

2. NOTATION

We will denote by μ the Lebesgue measure in \mathbb{R}^d and by $GL_d(\mathbb{R})$ the usual group of invertible matrices in $\mathbb{R}^{d \times d}$. By a *lattice* Γ we mean $\Gamma = c\mathbb{Z}^d$, where $c \in GL_d(\mathbb{R})$.

The Fourier transform of a function $f \in \mathcal{L}^2(\mathbb{R}^d)$ is

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \omega} dx.$$

For a measurable $\Omega \subset \mathbb{R}^d$, K_Ω will denote the functions in $\mathcal{L}^2(\mathbb{R}^d)$ with support in Ω ,

$$K_\Omega = \{f \in \mathcal{L}^2(\mathbb{R}^d) : \text{supp}(f) \subseteq \Omega\}. \quad (2)$$

We will say that a matrix a is *expansive* if $a \in GL_d(\mathbb{R})$ and $|\lambda| > 1$ for all eigenvalues λ of a .

Since affine systems depend on the matrix and the lattice involved we consider the following sets of generators:

$$\mathcal{W}_O(a, \Gamma) := \{\psi \in \mathcal{L}^2(\mathbb{R}^d) : (\psi, a, \Gamma) \text{ is an orthonormal basis (onb) of } \mathcal{L}^2(\mathbb{R}^d)\}$$

and

$$\begin{aligned} \mathcal{W}_O(a) &:= \bigcup_{c \in GL_d(\mathbb{R})} \mathcal{W}_O(a, c\mathbb{Z}^d) \\ \mathcal{W}_O(\Gamma) &:= \bigcup_{a \in GL_d(\mathbb{R})} \mathcal{W}_O(a, \Gamma) \\ \mathcal{W}_O &:= \bigcup_{a, c \in GL_d(\mathbb{R})} \mathcal{W}_O(a, c\mathbb{Z}^d). \end{aligned}$$

Similarly, we use the notations \mathcal{W}_R or \mathcal{W}_F for the corresponding generators of wavelet Riesz basis and frames respectively.

3. DENSITY OF THE SET OF FRAME GENERATORS

In this section we will prove that for a fixed expansive matrix a , the set $\mathcal{W}_F(a)$ is dense in $\mathcal{L}^2(\mathbb{R}^d)$.

In what follows $\text{diam}(S)$ will denote the diameter of a set $S \subset \mathbb{R}^d$ and $B(0, t)$ the ball in \mathbb{R}^d centered at 0 and with radius t .

Let X be a discrete set in \mathbb{R}^d . The *gap* ρ of X is defined as:

$$\rho = \rho(X) = \sup_{x \in \mathbb{R}^d} \inf_{\gamma \in X} |x - \gamma|.$$

The set X is said to be *separated* if $\inf_{\gamma \neq \gamma'} |\gamma - \gamma'| > 0$.

For a separated set X Beurling [Beu66] proved the following result:

Theorem (Beurling). *Let $X \subset \mathbb{R}^d$ be separated, and $\Omega = B(0, r)$. If $r\rho < 1/4$, then $\{e^{-2\pi i \gamma \omega} \chi_\Omega(\omega) : \gamma \in X\}$ is a frame for \mathcal{K}_Ω .*

For a very clear exposition of some of the Beurling density results see [BW99].

We need now the following result which is a particular case of the more general theorems in [ACM04]. We include its proof here for completeness of this presentation.

Theorem 3.1. *Let a be an expansive matrix and h a bounded compactly supported function such that $|h(\omega)| > c > 0$, for almost every $\omega \in \mathcal{U} := \text{supp}(h)$, and $0 \notin \text{supp}(h)$. If $X \subset \mathbb{R}^d$ is a separated set such that the gap $\rho(X) < \frac{1}{4R}$ where R is such that $U \subset B(0, R)$ then*

$$\{D_{a^j} T_\gamma \psi : j \in \mathbb{Z}, \gamma \in X\}$$

is a frame of $\mathcal{L}^2(\mathbb{R}^d)$ where $\hat{\psi} = h$.

Proof. By Beurling's theorem we know that

$$\{e^{-2\pi i\gamma\omega} \chi_{B(0,R)}(\omega) : \gamma \in X\}$$

is a frame of $K_{B(0,R)}$ with some frame bounds $0 < m \leq M < +\infty$. In particular

$$\{e^{-2\pi i\gamma\omega} \chi_{\mathcal{U}}(\omega) : \gamma \in X\}$$

is a frame of $K_{\mathcal{U}}$. Using that D_a is a unitary operator and calling $b = (a^{-1})^t$, we conclude that for each $j \in \mathbb{Z}$

$$\{|\det(b)|^{j/2} e^{-2\pi i b^j \gamma \omega} \chi_{\mathcal{U}}(b^j \omega) : \gamma \in X\}$$

is a frame of $K_{b^{-j}\mathcal{U}}$ with the same bounds m, M .

Further, by Proposition 5.9 in [ACM04] there exist $0 < p \leq P < +\infty$ such that

$$p < \sum_{j \in \mathbb{Z}} |h(b^j \omega)|^2 < P, \text{ a.e. } \omega \in \mathbb{R}^d.$$

Now, since $\text{supp}(h) = \mathcal{U}$, if we set $h_j(\omega) = h(b^j \omega)$ then $\text{supp}(h_j) = b^{-j}\mathcal{U}$.

Given now $f \in \mathcal{L}^2(\mathbb{R}^d)$ and calling $f_j = \overline{h_j} f$ we will see that

$$p\|f\|^2 \leq \sum_j \|f_j\|^2 \leq P\|f\|^2, \quad (3)$$

for

$$p\|f\|^2 = \int p|f|^2 \leq \int \sum_j |h_j|^2 |f|^2 = \sum_j \int |h_j f|^2 = \sum_j \|f_j\|^2,$$

and the upper inequality can be obtained similarly. Using that $f_j \in K_{b^{-j}\mathcal{U}}$ we have

$$m\|f_j\|^2 \leq \sum_{\gamma} |\langle f_j(\omega), |\det(b)|^{j/2} e^{-2\pi i b^j \gamma \omega} \chi_{b^{-j}\mathcal{U}}(\omega) \rangle|^2 \leq M\|f_j\|^2. \quad (4)$$

In addition, since $\text{supp}(f_j) = b^{-j}\mathcal{U}$, we have that

$$\langle f_j(\omega), |\det(b)|^{j/2} e^{-2\pi i b^j \gamma \omega} \chi_{b^{-j}\mathcal{U}}(\omega) \rangle = \langle f(\omega), |\det(b)|^{j/2} e^{-2\pi i b^j \gamma \omega} h_j(\omega) \rangle. \quad (5)$$

Using (5) in (4) and summing in j by (3) we get

$$pm\|f\|^2 \leq \sum_j \sum_{zg} |\langle f(\omega), |\det(b)|^{j/2} e^{-2\pi i b^j \gamma \omega} h_j(\omega) \rangle|^2 \leq PM\|f\|^2.$$

The claim is now a consequence of Plancherel's Theorem. \square

Now we are ready to state and proof the main result of this section.

Theorem 3.2. *The set $\mathcal{W}_F(a)$ is dense in $\mathcal{L}^2(\mathbb{R}^d)$, for every $d \times d$ expansive matrix a .*

Proof. Assume that a function $f \in \mathcal{L}^2(\mathbb{R}^d)$ and a positive number ε are given. We want to approximate f by a wavelet frame function ψ . We will do this by constructing the Fourier transform of ψ .

- Select $g \in \mathcal{L}^2(\mathbb{R}^d)$ such that \hat{g} is a continuous function and $\|\hat{f} - \hat{g}\|_2 < \varepsilon/2$.

- Choose $0 < r < R < \infty$ such that

$$aB(0, r) \subset B(0, R) \quad \text{and} \quad \int_{\mathbb{R}^d \setminus \mathcal{U}} |\hat{g}(\omega)|^2 d\omega < \frac{\varepsilon^2}{8},$$

where $\mathcal{U} = \{\omega : r \leq |\omega| \leq R\}$.

- Choose $\lambda > 0$, such that $4\lambda^2\mu(\mathcal{U}) < \frac{\varepsilon^2}{8}$, and let

$$E_\lambda := \{\omega \in \mathbb{R}^d : |\hat{g}(\omega)| > \lambda\}. \quad (6)$$

- Define a function h in the following way:

$$h(\omega) := \begin{cases} \hat{g}(\omega) & \omega \in \mathcal{U} \cap E_\lambda \\ \lambda & \omega \in \mathcal{U} \setminus E_\lambda \\ 0 & \text{else} \end{cases}. \quad (7)$$

- Choose X to be any separated set with gap $\rho(X) < \frac{1}{4R}$. (Note that X can be chosen to be a lattice.)

We will now see that we are indeed under the hypothesis of Theorem 3.1. Since

$$\mathcal{Q} := aB(0, r) \setminus B(0, r) \subset B(0, R) \setminus B(0, r) = \mathcal{U}$$

we note that by Lemma 5.11 of [ACM04] $\{a^j \mathcal{Q}\}$ is a covering of $\mathbb{R}^d \setminus \{0\}$ and therefore $\{a^j \mathcal{U}\}$ is also a covering of $\mathbb{R}^d \setminus \{0\}$.

Finally, h and X were defined to match the hypothesis of Theorem 3.1. Therefore we can apply that Theorem to conclude that (ψ, a, X) is a frame for $\mathcal{L}^2(\mathbb{R}^d)$, with $\psi = \hat{h}$.

Furthermore,

$$\begin{aligned} \|\hat{g} - h\|_2^2 &= \int_{\mathcal{U}} |\hat{g} - h|^2 + \int_{\mathbb{R}^d \setminus \mathcal{U}} |\hat{g} - h|^2 \\ &< \int_{\mathcal{U} \setminus E_\lambda} |\hat{g} - h|^2 d\omega + \frac{\varepsilon^2}{8} \\ &\leq 4\lambda^2\mu(\mathcal{U} \setminus E_\lambda) + \frac{\varepsilon^2}{8} < \frac{\varepsilon^2}{4} \end{aligned}$$

where the last inequality comes from the choice of λ .

Finally

$$\|f - \psi\|_2 \leq \|\hat{f} - \hat{g}\|_2 + \|\hat{g} - h\|_2 < \varepsilon.$$

□

4. THE RIESZ BASIS CASE

In this section we will prove that the set of generators for Riesz basis is dense when we allow diagonal matrices and arbitrary lattices. Since the requirement of building a Riesz basis is much stronger than to form a frame, the construction of the function requires more subtle techniques than for the previous case.

The idea is again to approximate the given function on the Fourier side with an appropriate function. We will use the construction for the frame wavelets and adapt it to the Riesz basis case. The main difficulty here lies in the fact that the elements of the affine system need to be independent, which forces the construction to be more involved.

It is worth to remark here, that the construction in the previous section was very general and *stable* in the sense that we had freedom to move things a little bit - and still obtain a satisfactory result. In this section, the construction is tight and very adapted to the function we want to approximate.

The method we use, is based on a standard *wavelet-set* construction. Wavelet sets have been studied and developed by many groups (for references see the Introduction). We adapt the construction of [Zak96] and [BS02] to obtain functions supported on a wavelet set, but which are not constant. In this way we obtain a Riesz basis, instead of a wavelet basis.

In what follows we will say that two sets A and B in \mathbb{R}^d are *almost disjoint* if $\mu(A \cap B) = 0$.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^d$ be a set of finite measure. If $\{\lambda_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}^d$ satisfies that $\{e^{2\pi i \lambda_k \omega} \chi_\Omega(\omega) : k \in \mathbb{Z}\}$ is a Riesz basis for \mathcal{K}_Ω with bounds A and B and $h \in \mathcal{L}^2(\mathbb{R}^d)$ satisfies that $0 < p < |h(\omega)| < P < +\infty$ then*

$$\{h(\omega)e^{2\pi i \lambda_k \omega} : k \in \mathbb{Z}\}$$

is a Riesz basis for \mathcal{K}_Ω , with Riesz bounds $pA\mu(\Omega)$ and $PB\mu(\Omega)$.

If $a \in GL_d(\mathbb{R})$ and Ω satisfies that $\cup_{j \in \mathbb{Z}} a^j \Omega = \mathbb{R}^d$ up to a set of zero measure, with the union being almost disjoint, and $\{g_k : k \in \mathbb{Z}\}$ is a Riesz basis for \mathcal{K}_Ω , then

$$\{D_a^j g_k : k, j \in \mathbb{Z}\}$$

is a Riesz basis for $\mathcal{L}^2(\mathbb{R}^d)$ with the same bounds.

Proof. The first assertion is immediate, and the second one follows from the fact that the dilation is a unitary operator in $\mathcal{L}^2(\mathbb{R}^d)$. \square

We are now ready to state the main theorem of this section.

Theorem 4.1. *The set \mathcal{W}_R is dense in $\mathcal{L}^2(\mathbb{R}^d)$. Precisely, if $f \in \mathcal{L}^2(\mathbb{R}^d)$, and $\varepsilon > 0$, there exists a function $\psi \in \mathcal{L}^2(\mathbb{R}^d)$, an expansive matrix $a \in \mathbb{R}^{d \times d}$, and a lattice Γ , such that*

- $\|f - \psi\|_2 < \varepsilon$ and
- (Ψ, a, Γ) is a Riesz basis for $\mathcal{L}^2(\mathbb{R}^d)$.

Before proceeding with the proof let us give the following definition.

Definition 4.1. Let Γ be an arbitrary lattice. A set $E \subset \mathbb{R}^d$ is Γ -congruent to a set $\tilde{E} \subset \mathbb{R}^d$, if there exist partitions $\{E_s : s \in \Gamma\}$ of E and $\{\tilde{E}_s : s \in \Gamma\}$ of \tilde{E} in measurable sets such that for every $s \in \Gamma$, $E_s = \tilde{E}_s + s$.

In the proof of Theorem 4.1 we will use the \mathcal{L}^∞ -norm of \mathbb{R}^d . In this way we will obtain an orthogonal basis of exponentials supported on the cube $B_\infty(0, R/2)$ for some appropriate R . If we multiply the elements in this basis by a function that is bounded above and bounded away from zero, by the previous Lemma, we will have a Riesz basis on $\mathcal{K}_{B_\infty(0, R/2)}$. Therefore, looking at the previous construction in the frame case, we would now need a set \mathcal{U} on which $|\hat{g}|$ is bounded away from zero, that is Γ -congruent to $B_\infty(0, R/2)$, and furthermore, that tiles the plane by dilations by a matrix a . This forces us to be more careful in the choice of r , and will also limit our choices of Γ and a .

4.1. Proof of Theorem 4.1.

Proof. Let as before $g \in \mathcal{L}^2(\mathbb{R}^d)$ be such that \hat{g} is continuous and $\|\hat{f} - \hat{g}\|_2 < \frac{\varepsilon}{2}$. We will approximate \hat{g} with an appropriate function.

- Let $R > 0$ be such that $\int_{\mathbb{R}^d \setminus B_\infty(0, R/2)} |\hat{g}(\omega)|^2 d\omega < \frac{\varepsilon^2}{16}$.
- We now select $r > 0$ small enough such that:

$$r < \frac{R}{3} \tag{8}$$

$$\int_{B_\infty(0, r/2)} |\hat{g}(\omega)|^2 d\omega < \frac{\varepsilon^2}{16} \tag{9}$$

$$\text{and } mr^d < \frac{\varepsilon^2}{16} \quad \text{where } m := \max\{|\hat{g}(\omega)|^2 : \omega \in B_\infty(0, R/2)\}. \tag{10}$$

Let now

$$\Gamma = R\mathbb{Z}^d \quad \text{and} \quad a = \frac{R}{r} I_{d \times d}.$$

We define

$$\begin{aligned} \tilde{T} : \mathbb{R}^d &\longrightarrow \mathbb{R}^d \\ x &\mapsto x + R\xi_j \quad \text{if } x \in j^{\text{th}}\text{-quadrant} \end{aligned}$$

where ξ_j is the vertex of the cube $[-1, 1]^d$ that lies in the j^{th} -quadrant.

Let us call

$$A_0 = B_\infty(0, r/2) \quad \text{and define} \quad A_i := (a^{-1} \circ \tilde{T})^i(A_0), \quad i = 1, 2, \dots \tag{11}$$

It will be convenient to use the notation A_0^j for the intersection of the set A_0 with the j^{th} -quadrant. With this notation, note that

$$\begin{aligned} A_i &= \bigcup_{1 \leq j \leq 2^d} (a^{-1} \circ \tilde{T})^i(A_0^j) \\ &= \bigcup_{1 \leq j \leq 2^d} T_{Ra^{-1}\xi_j + \dots + Ra^{-i}\xi_j}(a^{-i}(A_0^j)). \end{aligned} \tag{12}$$

Here T_y denotes the usual translation by y in $\mathcal{L}^2(\mathbb{R}^d)$. Therefore we have that

$$\sum_{i=1}^{\infty} \mu(A_i) = \mu(A_0) \sum_{i=1}^{\infty} \left(\frac{r}{R}\right)^{di} = r^d \frac{r^d}{R^d - r^d} < r^d, \tag{13}$$

where the last inequality holds by (8).

We define the set

$$\mathcal{U} := a \left(\bigcup_{i=0}^{\infty} A_i \right) \setminus \left(\bigcup_{i=0}^{\infty} \tilde{A}_i \right),$$

and for $\lambda = \frac{\varepsilon}{8(R)^{d/2}}$, the function h by

$$h(\omega) := \begin{cases} \hat{g}(\omega) & x \in \mathcal{U} \cap E_\lambda \\ \lambda & x \in \mathcal{U} \setminus E_\lambda \\ 0 & \text{else} \end{cases}$$

where as before

$$E_\lambda := \{\omega \in \mathbb{R}^d : |\hat{g}(\omega)| > \lambda\}.$$

Our claim is that ψ with $\hat{\psi} = h$ satisfies both conditions of the Theorem.

For the first, let us compute

$$\|f - \psi\|_2 = \|\hat{f} - h\|_2 \leq \|\hat{f} - \hat{g}\|_2 + \|\hat{g} - h\|_2 < \frac{\varepsilon}{2} + \|\hat{g} - h\|_2.$$

To compute $\|\hat{g} - h\|_2$ we note that $\|\hat{g} - h\|_2^2$ can be split into 3 integrals

$$\int_{\mathcal{U} \cap E_\lambda} |\hat{g} - h|^2 + \int_{\mathcal{U} \setminus E_\lambda} |\hat{g} - h|^2 + \int_{\mathbb{R}^d \setminus \mathcal{U}} |\hat{g} - h|^2. \quad (14)$$

By the definition of h , the first term in (14) vanishes.

For the second, note that on $\mathcal{U} \setminus E_\lambda$, $|\hat{g}(\omega)| \leq \lambda$ and therefore

$$\int_{\mathcal{U} \setminus E_\lambda} |\hat{g} - h|^2 \leq 4\lambda^2 \mu(\mathcal{U} \setminus E_\lambda) < \frac{\varepsilon^2}{16}. \quad (15)$$

For the last term, since $h(\omega) = 0$ if $\omega \notin \mathcal{U}$ we have

$$\int_{\mathbb{R}^d \setminus \mathcal{U}} |\hat{g}|^2 < \int_{B_\infty(0, R/2) \setminus \mathcal{U}} |\hat{g}|^2 + \int_{\mathbb{R}^d \setminus B_\infty(0, R/2)} |\hat{g}|^2. \quad (16)$$

The right hand side of (16) can be written,

$$\int_{A_0} |\hat{g}|^2 + \int_{\cup_{i \geq 1} A_i} |\hat{g}|^2 + \int_{\mathbb{R}^d \setminus B_\infty(0, R/2)} |\hat{g}|^2. \quad (17)$$

The first and last term in this sum are each smaller than $\varepsilon^2/16$ by our choice of r and R . For the middle one, we use the computation about the measure of $\cup_{i \geq 1} A_i$ done in (13) and the choice of r in (10) to obtain

$$\int_{\cup_{i \geq 1} A_i} |\hat{g}|^2 d\omega \leq m\mu(\cup_{i \geq 1} A_i) < mr^d < \frac{\varepsilon^2}{16}. \quad (18)$$

Putting all this together, we obtain

$$\|\hat{g} - h\|_2^2 < \frac{\varepsilon^2}{4} \quad \text{and therefore} \quad \|f - \psi\|_2 < \varepsilon. \quad (19)$$

This proves that the function h can be chosen as close to \hat{g} as we wish. It remains to show, that (ψ, a, Γ) is a Riesz basis. For this, we observe that:

- By construction, \mathcal{U} tiles $\mathbb{R}^d \setminus 0$ by dilations by a .
- Furthermore \mathcal{U} tiles \mathbb{R}^d by translations on Γ . For this, we first note that if $x \in A_n$ then

$$\frac{rR}{R-r} \left(1 - \left(\frac{r}{R}\right)^n\right) \leq \|x\|_\infty \leq \frac{rR}{R-r} \left(1 - \left(\frac{r}{R}\right)^n \left(\frac{r+R}{2R}\right)\right). \quad (20)$$

This fact allows us to conclude that:

- (1) $A_i \subset (B_\infty(0, R/2) \setminus A_0)$, $i \geq 1$,
- (2) $A_i \cap A_j = \emptyset$ $i \neq j$

which allows us to rewrite \mathcal{U}

$$\begin{aligned} \mathcal{U} &= a \left(\bigcup_{i=0}^{\infty} A_i \right) \setminus \left(\bigcup_{i=0}^{\infty} A_i \right) \\ &= \left(aA_0 \setminus \left(\bigcup_{i=0}^{\infty} A_i \right) \right) \cup \left(\bigcup_{i=0}^{\infty} \tilde{T}A_i \right). \end{aligned}$$

This shows that \mathcal{U} is Γ -congruent to $B_\infty(0, R/2)$.

On the other hand, since $\lambda < |h(\omega)| < m$ on \mathcal{U} by Lemma 4.1

$$\left\{ h(\omega) \frac{1}{R^{d/2}} e^{2\pi i \frac{k}{R} \omega} : k \in \mathbb{Z}^d \right\}$$

is a Riesz basis for $\mathcal{K}_{\mathcal{U}}$.

Thus we found a Riesz basis for $\mathcal{K}_{\mathcal{U}}$, and \mathcal{U} is a *wavelet set* for the dilation a and translation Γ . Therefore, the system (ψ, a, Γ) is a Riesz basis of $\mathcal{L}^2(\mathbb{R}^d)$. \square

Remark. In the above proof, we constructed a function that is supported on a wavelet set. In this way we in fact prove that a proper subset of \mathcal{W}_R is dense in $\mathcal{L}^2(\mathbb{R}^d)$, since the functions in the dense set are supported on a wavelet set. Further, they generate *quasi-orthogonal affine systems*, since they have orthogonal dilations (i.e. $\langle D_{a^j} T_{\gamma} \psi, D_{a^{j'}} T_{\gamma'} \psi \rangle = 0$ if $j \neq j'$).

5. DENSITY AND ORTHOGONAL WAVELETS

In this section we will consider the problem of the density for orthonormal wavelets. Since orthonormal wavelet functions have norm one, the natural question in this context is whether they are dense on the unit sphere of $\mathcal{L}^2(\mathbb{R}^d)$ with the induced metric (i.e. $\mathcal{S}_d := \{f \in \mathcal{L}^2(\mathbb{R}^d) : \|f\|_2 = 1\}$).

An immediate argument, that we will see later, shows that if we fix either the dilation matrix a or the lattice Γ , then the sets $\mathcal{W}_O(a)$ and $\mathcal{W}_O(\Gamma)$ are not dense in \mathcal{S}_d . If we allow both, the matrix a and the lattice Γ to be arbitrary, it is an open problem if the set \mathcal{W}_O is dense in \mathcal{S}_d .

In this section we will prove that the set of functions

$$\widetilde{\mathcal{W}}_O := \{\psi : \exists \Gamma \text{ and } a \in GL_d(\mathbb{R}), \text{ such that } (\psi, a, \Gamma) \text{ is an orthonormal system}\}$$

is dense in \mathcal{S}_d . Note that in $\widetilde{\mathcal{W}}_O$ we removed the completeness requirement of the system.

In particular we prove an interesting property of the Gramian of an arbitrary function in $\mathcal{L}^p([0, 1]^d)$ which is of independent interest. This property is then used to derive some consequences which in particular imply the above mentioned result.

Throughout this section, b will be a matrix in $GL_d(\mathbb{R})$. Our first Lemma is a known result (see for example [Mal89]). We state it here in the form we need it.

Lemma 5.1. *Let $f \in \mathcal{L}^2(\mathbb{R}^d)$. The following statements are equivalent*

- (1) *The system $\{T_{((b^*)^{-1}k)} f : k \in \mathbb{Z}^d\}$ is orthonormal in $\mathcal{L}^2(\mathbb{R}^d)$.*
- (2) *The system $\{\hat{f}(\omega) e^{-2\pi i \omega \cdot (b^*)^{-1}k} : k \in \mathbb{Z}^d\}$ is orthonormal in $\mathcal{L}^2(\mathbb{R}^d)$.*
- (3) $\sum_{s \in \mathbb{Z}^d} |\det b|^{1/2} |\hat{f}(b(\omega + s))|^2 = 1$ a.e. in $[0, 1]^d$.
- (4) $\sum_{s \in \mathbb{Z}^d} |\hat{f}(\omega + bs)|^2 = |\det b|^{-1}$ a.e. in $b([0, 1]^d)$.

Proof. We will show (1) \Leftrightarrow (3). The rest is trivial.

The system $\{T_{((b^*)^{-1}k)} f : k \in \mathbb{Z}^d\}$ is orthonormal in $\mathcal{L}^2(\mathbb{R}^d)$, if and only if

$$\begin{aligned} \delta_{0,k} &= \int_{\mathbb{R}^d} f(x - (b^*)^{-1}k) \overline{f(x)} dx = \int_{\mathbb{R}^d} e^{-2\pi i \omega \cdot (b^*)^{-1}k} \hat{f}(\omega) \overline{\hat{f}(\omega)} d\omega \\ &= \int_{[0,1]^d} (|\det b| \sum_s |\hat{f}(b(\omega + s))|^2) e^{-2\pi i \omega \cdot k} d\omega. \end{aligned}$$

Hence, it will be orthonormal if and only if (3) is satisfied. \square

Note that the sum in (3) can be rewritten as $\sum_{s \in \mathbb{Z}^d} |T_s D_b \hat{f}(\omega)|^2$, which motivates the following definition.

Definition 5.1. If $g \in \mathcal{L}^2(\mathbb{R}^d)$ and $c \in GL_d(\mathbb{R})$, the Gramian of g with respect to c is the function

$$g_c(\omega) = \left(\sum_{s \in \mathbb{Z}^d} |T_s D_c g(\omega)|^2 \right)^{1/2} = |\det c|^{1/2} \left(\sum_{s \in \mathbb{Z}^d} |g(c(\omega + s))|^2 \right)^{1/2}$$

Note that g_c is a \mathbb{Z}^d -periodic function.

In what follows the set $E(g, b) := \{\omega \in \mathbb{R}^d : g_b(\omega) > 0\}$ will be relevant. It is immediate to verify that, if $g \in \mathcal{L}^2(\mathbb{R}^d)$, then

$$\|g_b\|_{\mathcal{L}^2([0,1]^d)} = \|g\|_{\mathcal{L}^2(\mathbb{R}^d)}. \quad (21)$$

For $g \in \mathcal{L}^2(\mathbb{R}^d)$, we define u_g by

$$u_g(\omega) := \begin{cases} \frac{g(\omega)}{g_b(b^{-1}\omega)} & \text{if } \omega \in bE(g, b) \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

We have the following Lemma, whose proof is immediate.

Lemma 5.2. Let $g \in \mathcal{L}^2(\mathbb{R}^d)$ and $b \in \mathbb{R}^{d \times d}$ an invertible matrix. Then the function u_g defined in (22) satisfies that

$$(u_g)_b(\omega) = |\det b|^{1/2} \left(\sum_{s \in \mathbb{Z}^d} |u_g(b(\omega + s))|^2 \right)^{1/2} = 1 \quad \text{a. e. } \omega \in E(g, b).$$

In particular, if $\mu(E(g, b) \cap [0, 1]^d) = 1$ then the function φ defined by $\hat{\varphi} = u_g$ has orthonormal $(b^*)^{-1}\mathbb{Z}^d$ translates (i.e. $\{T_{(b^*)^{-1}k}\varphi : k \in \mathbb{Z}^d\}$ is orthonormal).

The next Lemma states that for each invertible matrix b the distance in $\mathcal{L}^2(\mathbb{R}^d)$ of two arbitrary functions is bigger than the distance of its Grammians in $\mathcal{L}^2([0, 1]^d)$.

Lemma 5.3. Let $g, h \in \mathcal{L}^2(\mathbb{R}^d)$, $b \in GL_d(\mathbb{R})$ and u_g as in (22). Then we have

$$\begin{aligned} \|g - h\|_{\mathcal{L}^2(\mathbb{R}^d)} &\geq \|g_b - h_b\|_{\mathcal{L}^2([0,1]^d)} \quad \text{and} \\ \|g - u_g\|_{\mathcal{L}^2(\mathbb{R}^d)} &= \|g_b - \chi_{E(g,b) \cap [0,1]^d}\|_{\mathcal{L}^2([0,1]^d)}. \end{aligned}$$

Proof. By (21)

$$\begin{aligned} \|g - h\|_{\mathcal{L}^2(\mathbb{R}^d)}^2 &= \|(g - h)_b\|_{\mathcal{L}^2([0,1]^d)}^2 \\ &= \int_{[0,1]^d} |\det b| \sum_s |g(b(\omega + s)) - h(b(\omega + s))|^2 d\omega \\ &\geq \|g_b - h_b\|_{\mathcal{L}^2([0,1]^d)}^2, \end{aligned}$$

where in the last inequality we used for the $\ell_2(\mathbb{Z})$ norm the inequality $\|x - y\| \geq \|\|x\| - \|y\|\|$.

For the second equation, we compute directly

$$\begin{aligned} \|g - u_g\|_2^2 &= \int_{bE(g,b)} \left| g(\omega) - \frac{g(\omega)}{(|\det b| \sum_s |g(\omega + bs)|^2)^{1/2}} \right|^2 d\omega \\ &= \int_{bE(g,b)} \frac{|g(\omega)|^2}{|\det b| \sum_s |g(\omega + bs)|^2} \left| \left(|\det b| \sum_s |g(\omega + bs)|^2 \right)^{1/2} - 1 \right|^2 d\omega. \end{aligned}$$

If we now periodize and change variables, we obtain the result. \square

Remark. Note that if $\mu(E(g, b) \cap [0, 1]^d) = 1$ then

$$\|g - u_g\|_{\mathcal{L}^2(\mathbb{R}^d)} = \|g_b - \chi_{[0,1]^d}\|_{\mathcal{L}^2([0,1]^d)}.$$

A consequence of Lemma 5.3 is the following: Assume that we want to be able to find a function ψ that is close to a given function $f \in \mathcal{S}_d$ and that has orthonormal translates with respect to a lattice $(b^*)^{-1}\mathbb{Z}^d$. Then we will need that \hat{f}_b is close (in $\mathcal{L}^2([0, 1]^d)$) to the constant function 1. Now, if for a given matrix b , \hat{f}_b is far from $\chi_{[0,1]^d}$, will the choice of a different matrix b improve the error? The next theorem establishes the interesting result that \hat{f}_b in fact converges almost everywhere and also in norm to $\chi_{[0,1]^d}\|f\|_2$ when $\|b\| \rightarrow 0$.

There is a natural interpretation in time domain of this result: when $\|b\|$ becomes small, then $\|(b^*)^{-1}\|$ grows, that is the associated lattice $(b^*)^{-1}\mathbb{Z}^d$ becomes sparser and since our functions are in $\mathcal{L}^2(\mathbb{R}^d)$ they will have some decay at infinity which will imply that the scalar product between two of its translates will be small.

In fact the theorem is more general. The convergence also holds for the p -Grammians of a function $f \in \mathcal{L}^p$ that we denote by $f_{b,p}$ and are defined as $f_{b,p}(\omega) = (|\det b| \sum_{s \in \mathbb{Z}^d} |f(b(\omega + s))|^p)^{1/p}$. Note that for $p = 2$, $f_{b,2}$ coincides with our previous f_b .

The following theorem is a generalization of a result proved in [JWW05] for the \mathcal{L}^1 case in a completely different context.

Theorem 5.1. *For any $f \in \mathcal{L}^p(\mathbb{R}^d)$,*

$$f_{b,p} \longrightarrow \|f\|_p \chi_{[0,1]^d} \quad \text{a.e. and in } \mathcal{L}^p([0, 1]^d) \quad \text{when } \|b\| \rightarrow 0. \quad (23)$$

We postpone the proof of the theorem until the last section.

Let us now see, how we deduce from Lemma 5.3 immediately that when the matrix b is fixed, that is the lattice $\Gamma = (b^*)^{-1}$ is fixed, then the set $\mathcal{W}_O(\Gamma)$ is not dense in the sphere: By Parseval, Lemma 5.3 and using that $\hat{g}_b = 1$ a. e. for every $g \in \mathcal{W}_O(\Gamma)$ we have for $f \in \mathcal{S}_d$ and $g \in \mathcal{W}_O(\Gamma)$:

$$\|f - g\|_{\mathcal{L}^2(\mathbb{R}^d)} \geq \|\hat{f}_b - \hat{g}_b\|_{\mathcal{L}^2([0,1]^d)} = \|\hat{f}_b - \chi_{[0,1]^d}\|_{\mathcal{L}^2([0,1]^d)}$$

The proof is completed choosing a function f in \mathcal{S}_d whose Grammian (with respect to b) is far from $\chi_{[0,1]^d}$.

The following argument from Yang Wang [private communication] shows that if the dilation a is fixed, then $\mathcal{W}_O(a)$ is not dense in \mathcal{S}_d : Assume that $\mathcal{W}_O(a)$ is dense in \mathcal{S}_d . Let $f \in \mathcal{S}_d$ be an arbitrary function such that $\langle f, D_a f \rangle \neq 0$. Choose $\psi_n \in \mathcal{W}_O(a)$ such that $\psi_n \rightarrow f, n \rightarrow \infty$ in \mathcal{S}_d . Then we have:

$$0 = \langle \psi_n, D_a \psi_n \rangle \rightarrow \langle f, D_a f \rangle,$$

which is a contradiction.

Now we are ready to prove a density result for the set of generators of orthonormal (not necessarily complete) wavelet systems:

Theorem 5.2. *The set $\widetilde{\mathcal{W}}_O$ of generators of orthonormal wavelet systems is dense in \mathcal{S}_d in the induced $\mathcal{L}_2(\mathbb{R}^d)$ metric.*

Proof. Let $f \in \mathcal{S}_d$ and $\varepsilon > 0$ be given. Choose r, R, g and h as in Theorem 3.2.

By Theorem 5.1 there exists $b \in GL_d(\mathbb{R})$ with small enough norm, such that

$$\|h_b - \chi_{[0,1]^d}\|_{\mathcal{L}_2([0,1]^d)} < \frac{\varepsilon}{2}$$

and $h_b > 0$ a.e. Note that this is possible since $|h| > 0$ in $\{\omega : r \leq \|\omega\| \leq R\}$. Now using Lemma 5.3

$$\|h - u_h\|_{\mathcal{L}^2(\mathbb{R}^d)} < \frac{\varepsilon}{4}$$

and therefore the inverse Fourier transform of u_h has orthonormal translates in the lattice $(b^*)^{-1}\mathbb{Z}^d$.

By choosing the dilation a to be $a = \frac{R}{r}I_d$, the set

$$\left\{ |\det a|^{j/2} u_h(a^j \omega) e^{-2\pi a^j \omega (b^*)^{-1} k} : k \in \mathbb{Z}^d, j \in \mathbb{Z} \right\}$$

is orthonormal, and consequently, if ψ is such that $\hat{\psi} = u_h$

$$\left\{ |\det a|^{j/2} \psi(a^j x - (b^*)^{-1} k) : k \in \mathbb{Z}^d, j \in \mathbb{Z} \right\}$$

is an orthonormal wavelet system, and so $\psi \in \mathcal{W}_O$. Further we have

$$\|f - \psi\| \leq \|\hat{f} - \hat{g}\| + \|\hat{g} - h\| + \|h - u_h\| < \varepsilon$$

which shows that \mathcal{W}_O is dense in $\mathcal{L}^2(\mathbb{R}^d)$. \square

5.1. Proof of Theorem 5.1.

Proof. We will divide the proof in several steps. Let us denote the unit cube by $Q = [0, 1]^d$. We will denote by $\|\cdot\|_p$ the p -norm in $\mathcal{L}^p(\mathbb{R}^d)$, and $\|\cdot\|_{\mathcal{L}^p(Q)}$ the p -norm in $\mathcal{L}^p(Q)$.

- We first prove the theorem for the case that $f = \chi_I$ where I is a finite d -dimensional interval. In this case, we compute

$$f_b(\omega) = |\det b| \sum_s \chi_I(b(\omega + s)) = |\det b| \sum_s \chi_{b^{-1}I}(\omega + s). \quad (24)$$

Let

$$N_i(b) := \{s \in \mathbb{Z}^d : bQ + bs \subset I\}$$

$$N_o(b) := \{s \in \mathbb{Z}^d : \mu((bQ + bs) \cap I) > 0 \text{ and } \mu((bQ + bs) \cap I^c) > 0\},$$

and call $n_i = \#N_i$ and $n_o = \#N_o$. Then,

$$\sum_s \chi_{b^{-1}I}(\omega + s) = (n_i + m(b, \omega)), \quad m(b, \omega) \in \mathbb{N}, 0 \leq m(b, \omega) \leq n_o. \quad (25)$$

Note that $|\det b| n_i(b) \leq \mu(I) \leq |\det b| (n_i(b) + n_o(b))$ and hence

$$0 \leq \mu(I) - |\det b| n_i(b) \leq |\det b| n_o(b). \quad (26)$$

We will now see that $|\det b| n_o(b) \rightarrow 0$ for $\|b\| \rightarrow 0$. For this take $\varepsilon > 0$, and let $b \in GL_d(\mathbb{R})$ be such that $\|b\| < \varepsilon/\text{diam}(Q)$. Since $\|b(x - y)\| \leq \|b\| \|x - y\|$ then $\text{diam}(bQ) \leq \|b\| \text{diam}(Q) < \varepsilon$. Therefore, if $s \in N_o(b)$, $bQ + bs \subset \{\omega \in \mathbb{R}^d : \text{dist}(\omega, \partial I) < \varepsilon\}$. Now

$$|\det b| n_o(b) = \sum_{s \in N_o} \mu(bQ + bs) \leq \mu(\{\omega \in \mathbb{R}^d : \text{dist}(\omega, \partial I) < \varepsilon\}) < \varepsilon c_1. \quad (27)$$

Therefore we see from (26) that

$$|\det b| n_i(b) \rightarrow \mu(I) \quad \text{when} \quad \|b\| \rightarrow 0, \quad (28)$$

and using (26) we have $|\det b| \sum_s \chi_{b^{-1}I}(\omega + s) \rightarrow \mu(I)$ a.e. Furthermore, $f_{b,p} \rightarrow \mu(I)^{1/p} = \|f\|_p$.

In addition, since $|\det b| \sum_s \chi_{b^{-1}I}(\omega + s) \leq |\det b|(n_i(b) + n_o(b)) < C$ for all b such that $\|b\| < \delta < 1$ we can apply dominated convergence to obtain

$$\int_Q \left| f_{b,p} - \mu(I)^{1/p} \chi_Q \right|^p \longrightarrow 0 \quad \text{when} \quad \|b\| \rightarrow 0. \quad (29)$$

• Now we will prove the theorem for the case that f is a finite linear combination of characteristic functions of intervals, i.e. $f = \sum_{j=1}^k \alpha_j \chi_{I_j}$, where I_j are almost disjoint d -dimensional intervals (here by almost, we mean that the intersection can have at most measure 0). By the disjointness of the I_j we immediately obtain

$$f_{b,p}(\omega) = \left(\sum_{j=1}^k |\alpha_j|^p (\chi_{I_j})_{b,p}^p \right)^{1/p} \quad (30)$$

and by the previous item, this converges to $\|f\|_p$.

In addition, because of the uniform boundedness of $f_{b,p}$ with respect to b , we obtain convergence in $\mathcal{L}^p(Q)$ by the dominated convergence theorem.

• Assume now that $f > 0$, $f \in \mathcal{L}^p(\mathbb{R}^d)$. Let $\varepsilon > 0$ and $0 \leq g_n \nearrow f$ a.e., with g_n as in step 2.

$$\begin{aligned} |f_{b,p}(\omega) - \|f\|_p| &\leq \\ &|f_{b,p}(\omega) - (g_n)_{b,p}(\omega)| + |(g_n)_{b,p}(\omega) - \|g_n\|_p| + \left| \|g_n\|_p - \|f\|_p \right| \end{aligned} \quad (31)$$

First note that, as in Lemma 5.3 for any $f, g \in \mathcal{L}^p(\mathbb{R}^d)$ we have $\|f_{b,p} - g_{b,p}\|_{\mathcal{L}^p(Q)} \leq \|f - g\|_p$, and therefore

$$(g_n)_{b,p} \xrightarrow{\|\cdot\|_{\mathcal{L}^p(Q)}} f_{b,p}. \quad (32)$$

We can therefore choose a subsequence g_{n_j} , such that $(g_{n_j})_{b,p}(\omega) \rightarrow f_{b,p}(\omega)$ a.e. in Q when $j \rightarrow \infty$. Hence, for a large enough j , the first and third term in (31) are each less than $\varepsilon/4$.

Now, for a fixed j , we can choose δ such that, if $\|b\| < \delta$, then $|(g_{n_j})_{b,p}(\omega) - \|g_{n_j}\|_p| < \varepsilon/2$ which gives the point-wise convergence.

For the convergence in $\mathcal{L}^p(\mathbb{R}^d)$, using the previous results, we write

$$\begin{aligned} \|f_{b,p} - \|f\|_p\|_{\mathcal{L}^p(Q)} &\leq \\ &\|f_{b,p} - (g_n)_{b,p}\|_{\mathcal{L}^p(Q)} + \|(g_n)_{b,p} - \|g_n\|_p\|_{\mathcal{L}^p(Q)} + \left| \|g_n\|_p - \|f\|_p \right|_{\mathcal{L}^p(Q)}. \end{aligned} \quad (33)$$

By the particular choice of the sequence g_n , we have that $g_n \xrightarrow{\|\cdot\|_p} f$. Therefore the first and last term of (33) go to 0 when $n \rightarrow \infty$. The middle term goes to 0 by the previous step.

Since $f_{b,p} = |f|_{b,p}$ the result holds for arbitrary $f \in \mathcal{L}^p(\mathbb{R}^d)$. \square

5.2. Hilbert Spaces. The results in this paper carry over to abstract separable Hilbert spaces, via unitary isomorphisms, where the translation and dilation operators are replaced by arbitrary unitary operators. General systems obtained in this context (unitary systems) have been studied in detail in [DL98].

Finally, we would like to mention that M. Bownik [Bow05], independently of our work and using different techniques has obtained for the case of wavelet frames some similar density results as in section 3.

6. ACKNOWLEDGMENTS

The research for this paper was carried during our visit at the ESI (Erwin Schroedinger Institute) and NuHAG (Numeric and Harmonic Analysis Group at the University of Vienna) during the special semester on “Modern Methods on Harmonic Analysis”. We thank H. Feichtinger and K.-H. Gröchenig for the invitation and the ESI and NuHAG for the hospitality during our stay. We also acknowledge support from an Alβan-fellowship (EU:E04E031835AR) and Grants UBACyT X058, X108; PICT 15033, and CONICET 5650.

We thank Yang Wang for fruitful discussions during his visit to the ESI, and David Larson for stimulating conversations that originated this research.

We also thank the anonymous referees for constructive remarks and suggestions that helped to improve the final version of this manuscript.

REFERENCES

- [ACM04] A. Aldroubi, C. A. Cabrelli, and U. Molter, *Wavelets on irregular grids with arbitrary dilation matrices, and frame atoms for $L^2(\mathbb{R}^d)$* , Applied and Computational Harmonic Analysis **17** (2004), no. 2, 171 – 191.
- [Beu66] Arne Beurling, *Local harmonic analysis with some applications to differential operators*, Some Recent Advances in the Basic Sciences, Vol. 1 (Proc. Annual Sci. Conf., Belfer Grad. School Sci., Yeshiva Univ., New York, 1962–1964), Belfer Graduate School of Science, Yeshiva Univ., New York, 1966, pp. 109–125.
- [BF94] John J. Benedetto and Michael W. Frazier, *Introduction*, Wavelets: mathematics and applications, Stud. Adv. Math., CRC, Boca Raton, FL, 1994, pp. 1–20.
- [BL01] J. J. Benedetto and M. Leon, *The construction of single wavelets in d -dimensions*, J. Geometric Analysis (2001), 1–15.
- [BMM99] Lawrence W. Baggett, Herbert A. Medina, and Kathy D. Merrill, *Generalized multi-resolution analyses and a construction procedure for all wavelet sets in \mathbb{R}^n* , J. Fourier Anal. Appl. **5** (1999), no. 6, 563–573.
- [Bow03] Marcin Bownik, *Quasi-affine systems and the Calderón condition*, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), Contemp. Math., vol. 320, Amer. Math. Soc., Providence, RI, 2003, pp. 29–43.
- [Bow05] ———, *Connectivity and density in the set of framelets*, Preprint, 2005.
- [BS02] John J. Benedetto and Songkiat Sumetkijakan, *A fractal set constructed from a class of wavelet sets*, Inverse problems, image analysis, and medical imaging (New Orleans, LA, 2001), Contemp. Math., vol. 313, Amer. Math. Soc., Providence, RI, 2002, pp. 19–35.
- [BS04] J. J. Benedetto and S. Sumetkijakan, *Tight frames and geometric properties of wavelet sets*, preprint, 2004.
- [BW99] J.J. Benedetto and H. C. Wu, *Non-uniform sampling and spiral MRI reconstruction*, Proc. SampTA 99, Trondheim, Norway, 1999.
- [CCMW02] Charles K. Chui, Wojciech Czaja, Mauro Maggioni, and Guido Weiss, *Characterization of general tight wavelet frames with matrix dilations and tightness preserving oversampling*, J. Fourier Anal. Appl. **8** (2002), no. 2, 173–200.
- [CHM04] C. Cabrelli, C. Heil, and U. Molter, *Self-Similarity and Multiwavelets in Higher Dimensions*, Memoirs of the American Mathematical Society, vol. 170, American Mathematical Society, Providence, RI, USA, 2004.
- [Con98] The Wutam Consortium, *Basic properties of wavelets*, J. Fourier Anal. Appl. **4** (1998), no. 4-5, 575–594.
- [CS00] Charles K. Chui and Xianliang Shi, *Orthonormal wavelets and tight frames with arbitrary real dilations*, Appl. Comput. Harmon. Anal. **9** (2000), no. 3, 243–264.
- [Dau88] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. on Pure and Appl. Math. **41** (1988), 909–996.
- [DL98] Xingde Dai and David R. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Mem. Amer. Math. Soc. **134** (1998), no. 640, viii+68.

- [DLS97] Xingde Dai, David R. Larson, and Darrin M. Speegle, *Wavelet sets in \mathbf{R}^n* , J. Fourier Anal. Appl. **3** (1997), no. 4, 451–456. MR **98m**:42048
- [DLS98] ———, *Wavelet sets in \mathbf{R}^n . II*, Wavelets, multiwavelets, and their applications (San Diego, CA, 1997), Amer. Math. Soc., Providence, RI, 1998, pp. 15–40. MR **99d**:42054
- [GLL⁺04] Kanghui Guo, Demetrio Labate, Wang-Q Lim, Guido Weiss, and Edward Wilson, *Wavelets with composite dilations*, Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 78–87 (electronic).
- [GLT93] T. N. T. Goodman, S. L. Lee, and W.-S. Tang, *Wavelets in wandering subspaces*, Trans. Amer. Math. Soc. **338** (1993), 639–654.
- [ILP98] Eugen J. Ionascu, David R. Larson, and Carl M. Pearcy, *On wavelet sets*, J. Fourier Anal. Appl. **4** (1998), no. 6, 711–721.
- [JWW05] D. Jimenez, L. Wang, and Y. Wang, *PCM quantization errors and the white noise hypothesis*, preprint, April 2005.
- [Lar05] David Larson, *Unitary systems and wavelet sets*, Proceedings of the 4 ICWAA, Springer, Macau, 2005, pp. 5–35.
- [Mal89] S. Mallat, *Multiresolution approximations and wavelet orthonormal basis of $L^2(\mathbf{R})$* , Trans. of the Am. Math. Soc. **315** (1989), no. 1, 69–87.
- [Mey88] Y. Meyer, *Ondelettes et operateurs I*, Hermann, 1988.
- [Mey92] ———, *Wavelets and operators*, Cambridge University Press, Cambridge, 1992.
- [Spe99] D. M. Speegle, *The s -elementary wavelets are path-connected*, Proc. Amer. Math. Soc. **127** (1999), no. 1, 223–233.
- [SW98] Paolo M. Soardi and David Weiland, *Single wavelets in n -dimensions*, J. Fourier Anal. Appl. **4** (1998), no. 3, 299–315.
- [Wan02] Yang Wang, *Wavelets, tiling, and spectral sets*, Duke Math. J. **114** (2002), no. 1, 43–57.
- [Zak96] Victor Zakharov, *Nonseparable multidimensional littlewood-paley like wavelet bases*, Preprint, 1996.

DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES, CIUDAD UNIVERSITARIA, PABELLÓN I, 1428 CAPITAL FEDERAL, ARGENTINA

E-mail address, Carlos Cabrelli: cabrelli@dm.uba.ar

E-mail address, Ursula M. Molter: umolter@dm.uba.ar