# Density of fuzzy attractors: A step towards the solution of the inverse problem for fractals and other sets

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**Abstract** The theory of iterated fuzzy set systems, *IFZS*, was introduced by Cabrelli et al. in [4]. They showed that by combining the idea of representing an image as a fuzzy set with the theory of iterated function systems, it is possible to generate images with grey or colour levels as attractors of IFZS. The purpose of this paper is to show that the class of attractors of IFZS is dense in the class of images, i.e., each image can be approximated with the desired accuracy. A brief review of the main concepts of IFZS is presented first.

## 1. Introduction

We first want to present an overview of the theory of iterated fuzzy set systems (IFZS). Since a complete development of the theory can be found in [4], we are going to omit most of the proofs. We then show that the set of images that can be obtained using this approach, is dense in the set of all images.

The notion of self-similarity and its generalizations[1], has found a natural frame in the theory of iterated function systems (IFS): self-similar sets became attractors of certain systems of maps [10, 1, 8]. The generalization of the concept of self-similarity to a more general class of maps—other than similarities, introduced more flexibility in the model, widening the class of sets that have the property to be expressed as smaller copies of themselves.

[1] A subset S of an arbitrary set X, is said to be self-similar (in the wide sense) if there exist a finite number of maps $f_1, \dots, f_N$, $f_i : X \to X$ such that $S = \bigcup_{1 \le i \le N} f_i(S)$



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On the other hand, the use of IFS enabled the construction of self-similar sets of fractional dimensions, and therefore this theory has found wide applications in computer graphics to generate fractal images on computers (see for example [3, 13]). The ergodicity involved in the process is another advantage that this method provides in image generation and representation, see [7].

One of the major applications of IFS theory in image processing, is in data compression: huge amounts of data can be squeezed into a few number of parameters. Two questions naturally arise:

- Which kind of images can be represented through this model, or, how big is the class of images that can be represented through IFS?
- Is there an efficient algorithm or method to find that representation?

Regarding the first question, in the case that the maps are contractive but not necessarily similarities, it has been shown [9] that this class is dense in the class of compact sets. In image processing language this means, that to any object in a black and white image, one can associate an IFS code. This result shows that the so-called inverse problem for fractals and other sets, that is to find the IFS code associated with any given black and white image, has at least one solution. It is well known however, that in most of the cases the solution that can be constructed from the proof of the theorem does not yield good compression rate. It is a very difficult problem to find an efficient IFS code for a given black and white image. Some results in that direction for the one dimensional case can be found in [2, 5, 16].

In the case of images with grey-levels, the IFS theory provides us with a class of measures that are generated by adding a probability vector to each IFS code. The ergodicity allows one to generate this measure through a random iterative algorithm. This approach however, seems to have two weak points: first, the relation between the parameters and the resulting measure is not straightforward, and this then becomes a serious difficulty for the inverse problem. Secondly, the class of measures that can be obtained through IFS, seems not to be as wide as desirable. The question of how big this class of measures is in relation to a suitable space of measures (here suitable refers to images) seems to be still open.

The IFZS approach to grey-level images considers images as functions rather than measures, and hereby tends to avoid these problems. In that direction, Theorem 3.1 of this paper shows that the class of images that can be generated using IFZS is dense in the class of images, i.e., given a grey-level image, we prove that for a given $\varepsilon$ there exists an IFZS whose attractor is closer than $\varepsilon$ to that image.

## 2. The iterated fuzzy set systems (IFZS)

### 2.1. Iterated function systems (IFS)

s Let us briefly recall the basic notions of IFS. Given a compact metric space $(X, d)$ with distance $d$, let us consider $N$ contraction mappings $w_i : X \to X$. The metric space $X$, together with the $N$ contraction mappings is referred to as an Iterated Function System (IFS) and denoted by $\{X, \mathbf{w}\}$. Usually, in applications, $X$ is a compact subset of $\mathfrak{R}^n$.

If $\mathcal{H}(\mathcal{X})$ denotes the set of all nonempty closed subsets of $X$, we can define $N$ set-valued maps $\hat{w}_i : \mathcal{H}(\mathcal{X}) \to \mathcal{H}(\mathcal{X})$, by $\hat{w}_i(S) = \{w_i(x) : x \in S\}$, e.g. the image of $S$ under the transformation $w_i$, for all $S \in \mathcal{H}(\mathcal{X})$. If $h$ is the Hausdorff distance in $\mathcal{H}(\mathcal{X})$:

$$h(A, B) := \max\{D(A, B), D(B, A)\} \tag{2.1}$$

where

$$D(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y) \tag{2.2}$$

then $(\mathcal{H}(\mathcal{X}), h)$ is a compact metric space, and $\hat{w}_i$ are contraction mappings of $\mathcal{H}(\mathcal{X})$. The map $W : \mathcal{H}(\mathcal{X}) \to \mathcal{H}(\mathcal{X})$ defined by:

$$W(S) = \bigcup_{i=1}^{N} \hat{w}_i(S), \quad \forall S \in \mathcal{H}(\mathcal{X}) \tag{2.3}$$

is also a contraction on $\mathcal{H}(\mathcal{X})$. Therefore it possesses an unique fixed point (or invariant set) $\mathcal{A}$, called the *attractor* of the IFS;

$$\mathcal{A} = W(\mathcal{A}) = \bigcup_{i=1}^{N} \hat{w}_i(\mathcal{A}). \tag{2.4}$$

This shows that $\mathcal{A}$ is self-similar with respect to $w_1, \ldots, w_N$. This property is sometimes referred to as the *self-tiling* property of IFS attractors, meaning that $\mathcal{A}$ can be built with smaller copies of itself. As well, the name *attractor* is justified by the following property:

$$h(W^n(S), \mathcal{A}) \longrightarrow 0 \qquad \text{as } n \to \infty, \forall S \in \mathcal{H}(\mathcal{X}). \tag{2.5}$$

### 2.2. Fuzzy sets as generalization of sets

The notion of fuzzy sets introduced by Zadeh in 1965 [17], has been widely used in different contexts. We want to use it here in the sense of a generalization of the concept of *set*: If $X$ is an arbitrary (non empty) set, a *fuzzy set* (in $X$) is a function $\mathbf{u}$ with domain $X$ and values in $[0, 1]$, i.e., $\mathbf{u} : X \to [0, 1]$.

In particular, if $S$ is an ordinary subset of $X$, its characteristic function $\chi_S$ is a fuzzy set. To relate this concept with images, we think of a digitized picture as a set of pixels, each of which has associated a *grey-level;* the value 1 representing *black* or the foreground, the value 0 representing *white* or the background. The value $\mathbf{u}(x)$ then corresponds to the grey-level of the pixel $x$. If the image is black and white, we only have two values: 0 or 1, and therefore we can represent it by a characteristic function, or a "set."

If $\mathcal{F}(X)$ denotes the class of all fuzzy sets in a metric space $(X, d)$, i.e., all functions $\mathbf{u} : X \to [0, 1]$, we are going to restrict ourselves to a subclass $\mathcal{F}^*(X) \subset \mathcal{F}(X)$: namely, $\mathbf{u} \in \mathcal{F}^*(X)$ if and only if:

1) $\mathbf{u} \in \mathcal{F}(X)$,
2) $\mathbf{u}$ is uppersemicontinuous (u.s.c) on $(X, d)$,
3) $\mathbf{u}$ is *normal*, that is $\mathbf{u}(x_0) = 1$ for some $x_0 \in X$.

These properties yield the following results:

**a:** For each $0 < \alpha \leqslant 1$, the $\alpha$-level set, defined as $[\mathbf{u}]^\alpha := \{x \in X : \mathbf{u}(x) \geqslant \alpha\}$ is a nonempty compact subset of $X$,

**b:** The closure of $\{x \in X : \mathbf{u}(x) > 0\}$, denoted by $[\mathbf{u}]^0$, is also a nonempty compact subset of $X$.

Note that the characteristic function of a closed set is in $\mathcal{F}^*(X)$. We also want to point out here that the level sets of the fuzzy set $\mathbf{u}$ completely characterize $\mathbf{u}$, i.e., knowing $\mathbf{u}(x), \forall x \in X$, is equivalent to knowing $[\mathbf{u}]^\alpha, 0 \leqslant \alpha \leqslant 1$.

By the above properties, $[\mathbf{u}]^\alpha \in \mathcal{H}(\mathcal{X}), 0 \leqslant \alpha \leqslant 1$. We now introduce the metric $d_\infty$ on $\mathcal{F}^*(X)$ (see [6]), which has been used in many applications of fuzzy set theory [11, 12, 15]:

$$d_\infty(\mathbf{u}, \mathbf{v}) = \sup_{0 \leqslant \alpha \leqslant 1} \{h([\mathbf{u}]^\alpha, [\mathbf{v}]^\alpha)\} \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{F}^*(X). \tag{2.6}$$

Here $h$ is the Hausdorff metric introduced in (2.1). The metric space $(\mathcal{F}^*(X), d_\infty)$ is complete. This space represents the generalization of the space $(\mathcal{H}(\mathcal{X}), h)$ to fuzzy sets.

At this point we want to incorporate the IFS theory into the fuzzy set frame. Therefore, we first use the *extension principle* for fuzzy sets [18, 14] in order to extend the set-valued maps $\hat{w}_i$ defined in Section 2.1 to maps between fuzzy sets, i.e., we want to define a map from $\mathcal{F}^*(X)$ to $\mathcal{F}^*(X)$ which is equal to $\hat{w}_i$ (with the earlier mentioned identification) when its domain is restricted to the characteristic function of a set. Therefore we define for each $\mathbf{u} \in \mathcal{F}^*(X)$ and each subset $B$ of $X$,

$$\tilde{\mathbf{u}}(B) := \sup\{\mathbf{u}(y) : y \in B\}, \qquad \text{if } B \neq \emptyset$$

$$\tilde{\mathbf{u}}(\emptyset) := 0, \tag{2.7}$$

which implies, in particular, $\tilde{\mathbf{u}}(\{x\}) = \mathbf{u}(x)$ at each $x \in X$.

For each $w_i, i = 1, 2, \ldots, N$, and each $x \in X$ we now define

$$\tilde{\mathbf{u}}_i(x) := \tilde{\mathbf{u}}(w_i^{-1}(\{x\})), \tag{2.8}$$

where, of course, $w_i^{-1}(\{x\}) = \emptyset$ if $x \notin w(X)$. If $\mathbf{u} \in \mathcal{F}^*(X)$, then each of these functions $\tilde{\mathbf{u}}_i : X \to [0,1]$ is a fuzzy set in $\mathcal{F}^*(X)$ (see [4]).

In fuzzy set theory, the union of two fuzzy sets **u**,**v** is usually defined as the fuzzy set $\sup(\mathbf{u}, \mathbf{v})$. We could then generalize the contraction mapping W given by equation (2.3) to a map $\tilde{\mathbf{w}} : \mathcal{F}^*(X) \to \mathcal{F}^*(X)$ defined by:

$$\tilde{\mathbf{w}}(\mathbf{u})(x) = \sup_{1 \leqslant i \leqslant N} \tilde{\mathbf{u}}_i(x), \qquad \text{for each } \mathbf{u} \in \mathcal{F}^*(X). \tag{2.9}$$

In [4] it is shown that this is a contraction mapping on $\mathcal{F}^*(X)$ with the $d_\infty$-metric. Therefore it has a unique fuzzy attractor $\mathbf{u}^* \in \mathcal{F}^*(X)$, e.g., $\tilde{w}(\mathbf{u}^*) = \mathbf{u}^*$. It turns out however, that this fuzzy attractor is the characteristic function of the attractor of the IFS $\{X, \mathbf{w}\}$. This means that the direct generalization of the IFS theory to Fuzzy Sets, does not provide us with a bigger class of attractors. We will see in the next section, how this class can be enlarged without losing the contractivity of the map $\tilde{\mathbf{w}}$.

### 2.3. Modification of the grey-levels of the attractor

In order to gain more generality with the fuzzy set model, the "grey-level maps" are introduced. To each $\tilde{\mathbf{u}}_i(x)$ defined in (2.8), a *grey-level map* $\varphi_i : [0,1] \to [0,1]$ is associated, in order to modify the values of $\tilde{\mathbf{u}}_i$, that is the grey-levels.

Now the supremum of (2.9) is taken over the functions $\tilde{\mathbf{u}}_i$ modified by the functions $\varphi_i$; e.g.,

$$\mathbf{u} \mapsto \sup_{1 \leqslant i \leqslant N} \varphi_i \circ \tilde{\mathbf{u}}_i. \tag{2.10}$$

In other words, an operator $T_s : \mathcal{F}^*(X) \to \mathcal{F}^*(X)$ is introduced:

$$\begin{aligned}(T_s\mathbf{u})(x) &:= \sup\{\varphi_1(\tilde{\mathbf{u}}_1(x)), \ldots, \varphi_N(\tilde{\mathbf{u}}_N(x))\} \\ &= \sup\{\varphi_1(\tilde{\mathbf{u}}(w_1^{-1}(x))), \ldots, \varphi_N(\tilde{\mathbf{u}}(w_N^{-1}(x)))\}.\end{aligned} \tag{2.11}$$

In order for the operator $T_s$ to be well defined, the grey-level functions $\varphi_i$ have to satisfy certain conditions, namely: for $i = 1, 2, \ldots, N$,

1) $\varphi_i : [0,1] \longrightarrow [0,1]$ is non-decreasing,
2) $\varphi_i$ is right continuous on $[0,1)$,
3) $\varphi_i(0) = 0$, and
4) for at least one $j \in \{1, 2, \ldots, N\}$, $\varphi_j(1) = 1$.

The fact that $\varphi_i$ are non-decreasing and right continuous, guarantees the uppersemicontinuity of $\varphi_i \circ \mathbf{u}$ for any $\mathbf{u}$ in $\mathcal{F}^*(X)$, moreover they are necessary and sufficient conditions [4]. Property 3) is a natural assumption in the consideration of grey level functions: if the grey level of a point (pixel) $x \in X$ is zero (the pixel is in the background), then it should remain zero after being acted upon by the $\varphi_i$ maps.

The set of maps $\Phi = \{\varphi_i, i = 1, 2, \ldots, N\}$, satisfying the above conditions, together with the N contraction maps $w_i$ (which then yield $\tilde{\mathbf{u}}_i$) form the Iterated Fuzzy Set System (IFZS) denoted $\{X, \mathbf{w}, \Phi\}$.

In [4] it is shown that the operator $T_s$ as defined in (2.11) is indeed a contraction mapping on $(\mathcal{F}^*(X), d_\infty)$, i.e., $T_s$ maps $\mathcal{F}^*(X)$ into itself and there exists an $s$, $0 \leqslant s < 1$, such that

$$d_\infty(\mathbf{T}_s\mathbf{u}, \mathbf{T}_s\mathbf{v}) \leqslant s\, d_\infty(\mathbf{u}, \mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v} \in \mathcal{F}^*(X). \tag{2.12}$$

Therefore, by the Contraction Mapping Principle, $T_s$ possesses an unique fixed point $\mathbf{u}^*$, that is:

$$\mathbf{T}_s\mathbf{u}^* = \mathbf{u}^*. \tag{2.13}$$

This implies that there exists a unique solution to the functional equation in the unknown $\mathbf{u} \in \mathcal{F}^*(X)$,

$$\begin{aligned} \mathbf{u}(x) = \sup\{&\varphi_1(\tilde{\mathbf{u}}(w_1^{-1}(x))), \varphi_2(\tilde{\mathbf{u}}(w_2^{-1}(x))), \ldots, \\ &\varphi_N(\tilde{\mathbf{u}}(w_N^{-1}(x)))\}, \end{aligned} \tag{2.14}$$

for all $x \in X$. The fuzzy set solution, $\mathbf{u}^*$, will be called the *attor* or *fuzzy attractor* of the IFZS, since it follows from the Contraction Mapping Principle that

$$d_\infty((\mathbf{T}_s)^n\mathbf{v}, \mathbf{u}^*) \to 0 \qquad \text{as } n \to \infty, \forall \mathbf{v} \in \mathcal{F}^*(X). \tag{2.15}$$

It is easy to find examples showing that these fuzzy attractors are not longer only characteristic functions of closed sets. Hence, using IFZS, the class of images that can be obtained using IFS has been widened. In section Section 3, we show in fact that any image can be obtained (up to an $\varepsilon$) as a fuzzy attractor of an IFZS. Note that in the case that all $\varphi_i$ are the identity maps, the operator $T_s$ reduces to the one defined in equation (2.9).

### 2.4. Properties of the fuzzy attractors

It is worth mentioning several properties of the *fuzzy attractors*. The proofs can be found in [4].

**Property 2.1.** If $\mathcal{A} \in \mathcal{H}(\mathcal{X})$ is the attractor of the IFS $\{X, \mathbf{w}\}$, and $\mathbf{u}^* \in \mathcal{F}^*(X)$ denotes the fuzzy attractor of the IFZS $\{X, \mathbf{w}, \Phi\}$, then *support*$(\mathbf{u}^*) \subseteq \mathcal{A}$, that is, $[\mathbf{u}^*]^0 \subseteq \mathcal{A}$.

This means, that using the grey-level maps, we are able to modify the support of our attractor, allowing for example a rough approximation through the $w_i$, and then a "fine-tuning" using the $\varphi_i$. This property may be used for applications, if we want to find the IFZS code for an image. Note that *support*$(\mathbf{u}^*)$ is exactly equal to $\mathcal{A}$, in the following two cases:

- For all $i \in \{1, 2, \dots, N\}$, $\varphi_i(1) = 1$, then $\mathbf{u}^* = \chi_{\mathcal{A}}$.
- For all $i \in \{1, 2, \dots, N\}$, $\varphi_i$ are increasing at 0 (i.e., $\varphi_i^{-1}(0) = \{0\}$). Indeed, in this case $[\mathbf{u}^*]^0 = \bigcup_{i=1}^N w_i([\varphi_i \circ \mathbf{u}^*]^0) = \bigcup_{i=1}^N w_i(\mathcal{A}) = W(\mathcal{A}) = \mathcal{A}$.

We should also point out that in the case that $\varphi_j(0) > 0$ for one $j \in \{1, 2, \dots, N\}$, this inclusion is not longer true.

**Property 2.2.** The level sets of the fuzzy attractor satisfy a generalized self-tiling condition:

$$[\mathbf{u}^*]^\alpha = \bigcup_{i=1}^{N} w_i([\varphi_i \circ \mathbf{u}^*]^\alpha), \; 0 \leqslant \alpha \leqslant 1. \tag{2.16}$$

This condition is a consequence of the property of the operator $T_s$:

$$[\mathbf{Tu}]^\alpha = \bigcup_{i=1}^{N} w_i([\varphi_i \circ \mathbf{u}]^\alpha), \;\; \forall v \in \mathcal{F}^*(X) \qquad \text{(see [4])}. \tag{2.17}$$

This property is interesting, since it shows that the fuzzy attractor is no longer self-similar, in the sense, that it is no longer the union of smaller copies of itself, but rather a union of *modified* copies of itself. The modification is given by the grey-level maps.

**Property 2.3 (IFZS Collage Theorem).** Let $\mathbf{u} \in \mathcal{F}^*(X)$ and suppose that there exists an IFZS $\{X, \mathbf{w}, \Phi\}$ so that

$$d_\infty(\mathbf{u}, \mathbf{T}_s\mathbf{u}) < \varepsilon, \tag{2.18}$$

where the operator $\mathbf{T}_s$ is defined by (2.11). Then

$$d_\infty(\mathbf{u}, \mathbf{u}^*) < \frac{\varepsilon}{1 - s}, \tag{2.19}$$

where $\mathbf{u}^* = \mathbf{T}_s\mathbf{u}^*$ is the invariant fuzzy set of the IFZS, and $s$ is the maximum contraction factor of the $w_i$.

This means that if the $w_i$ are very contractive (i.e., $s$ is very small), every fuzzy set that remains relatively unchanged after the application of the operator $T_s$, is close to the fuzzy attractor.

This property, a direct consequence of the contractivity of $T_s$, is (as for IFS) very useful for the inverse problem.

### 3. Density of fuzzy attractors

In this section we will show that the class of fuzzy attractors is dense in $\mathcal{F}^*(X)$ with the $d_\infty$- metric. In other words, given a fuzzy set $\mathbf{u}$ in $\mathcal{F}^*(X)$, and $\varepsilon > 0$, we can always find a natural number N, N contraction mappings $w_i : X \to X$, and N grey-level maps $\varphi_i : [0,1] \to [0,1]$, such that the fuzzy attractor $\mathbf{u}^*$ of the associated IFZS $\{X, \mathbf{w}, \Phi\}$ satisfies: $d_\infty(\mathbf{u}, \mathbf{u}^*) < \varepsilon$. We therefore have the following:

**Theorem 3.1.** If $X \subset \mathfrak{R}^n$ is compact and $(\mathcal{F}^*(X), d_\infty)$ is defined as above, then the class

$$\mathcal{D} = \{\mathbf{u}^* \in \mathcal{F}^*(X) : \mathbf{u}^* \text{ is attractor of some IFZS on } X.\}$$

is dense in $(\mathcal{F}^*(X), d_\infty)$.

**Proof.**

Let $\varepsilon > 0$ and $\mathbf{u} \in \mathcal{F}^*(X)$. The idea of the proof is to find $N \in \mathfrak{N}$, $\mathbf{w} = \{w_1, \ldots, w_N\}$ and $\Phi = \{\varphi_1, \ldots, \varphi_N\}$ such that:

1) $\sup c_i < \frac{1}{2}$ ($c_i$ is the contractivity factor of $w_i$);
2) $d_\infty(T_s\mathbf{u}, \mathbf{u}) < \frac{\varepsilon}{2}$, where $T_s$ is the operator associated to $\{X, \mathbf{w}, \Phi\}$.

Then, using the IFZS collage theorem (Property 2.3) from 1) and 2) we have:

$$d_\infty(\mathbf{u}, \mathbf{u}^*) < \frac{\varepsilon}{2}\left(\frac{1}{1 - 1/2}\right) = \varepsilon,$$

where $\mathbf{u}^*$ is the attractor of the IFZS $\{X, \mathbf{w}, \Phi\}$, i.e., $T_s\mathbf{u}^* = \mathbf{u}^*$. ∎

Let us now find $\mathbf{w}$ and $\Phi$, such that 1) and 2) are satisfied: Let $N \in \mathfrak{N}$, and $x_1, \ldots, x_N$ be an $\frac{\varepsilon}{4}$-net of $[\mathbf{u}]^0$, i.e., $[\mathbf{u}]^0 \subset \bigcup_{i=1}^N B_i$, where $B_i = B(x_i, \frac{\varepsilon}{4})$, are the open balls of radius $\frac{\varepsilon}{4}$ centered at $x_i$.

Let $w_i : X \to X$, $w_i(X) \subset B_i$, $i = 1, \ldots, N$ be contraction mappings with contraction factor $c_i$, with $c_i < \frac{1}{2}$. Choose now $\alpha_0 = 0$ and $\alpha_i = \sup_{x \in \overline{B_i}} \mathbf{u}(x)$.

Then for $0 \leq \alpha \leq 1$ we have $[\mathbf{u}]^\alpha \subset \bigcup_{\{i : \alpha \leq \alpha_i\}} B_i$.

We now choose $\varphi_i$ non-decreasing, right continuous, such that $\varphi_i(x) \leq \alpha_i$, $\forall x \in [0,1]$, and $\varphi_i(1) = \alpha_i$, $i = 1, \ldots, N$. For example, the stepfunctions $\alpha_i \chi_{[\alpha_i, 1]}$ satisfy these conditions.

Then

$$[\varphi_i \circ \mathbf{u}]^\alpha = \begin{cases} \neq \emptyset & \alpha \leq \alpha_i; \\ \emptyset & \alpha > \alpha_i. \end{cases}$$

But using condition (2.17), we have

$$[T_s\mathbf{u}]^\alpha = \bigcup_{i=1}^N w_i([\varphi_i \circ \mathbf{u}]^\alpha) = \bigcup_{\{i : \alpha \leq \alpha_i\}} w_i([\varphi_i \circ \mathbf{u}]^\alpha).$$

Now, if the $\delta$-dilatation $D_\delta(S)$ of a nonempty closed set $S$ is $D_\delta(S) = \{x \in X : d(x, S) < \delta\}$, $\delta > 0$, we can observe that for $0 \leq \alpha \leq 1$

$$[\mathbf{u}]^\alpha \subseteq \bigcup_{\{i : \alpha \leq \alpha_i\}} B_i \subseteq D_{\frac{\varepsilon}{2}}([\mathbf{u}]^\alpha) \tag{3.1}$$

and

$$w_i(S) \subseteq B_i \subseteq D_{\frac{\varepsilon}{2}}(w_i(S)) \quad \text{for } 1 \leq i \leq N, \forall S \text{ closed subset of } X. \tag{3.2}$$

We then have:

$$[T_w\mathbf{u}]^\alpha = \bigcup_{\{\alpha \leq \alpha_i\}} B_i \subseteq D_{\frac{\varepsilon}{2}}([\mathbf{u}]^\alpha)$$

$$[\mathbf{u}]^\alpha \subseteq \bigcup_{\alpha \leq \alpha_i} B_i = D_{\frac{\varepsilon}{2}}([T_w\mathbf{u}]^\alpha).$$

Using the above equations, we then obtain

$$h([T_w\mathbf{u}]^\alpha, [\mathbf{u}]^\alpha) < \frac{\varepsilon}{2}, \quad 0 \leq \alpha \leq 1, \tag{3.3}$$

and hence

$$d_\infty(T_w\mathbf{u}, \mathbf{u}) < \frac{\varepsilon}{2}. \tag{3.4}$$

∎

## 4. Conclusions

The IFZS model represents a different and promising approach to the inverse problem for fractal construction and image encoding. The introduction of the grey-level maps allows one to enlarge the class of attractors. We prove that this class is dense in $\mathcal{F}^*(X)$, the space of uppersemicontinuous normal functions, a space which is large enough for image representation. Again, the proof of the density does not give an efficient algorithm to find the appropriate code, but it provides a theoretical justification for the fuzzy set approach.

We believe, that we might be able to relax several conditions of the model presented here, in order to efficiently solve the inverse problem. We have experimental results comforting our intuition.

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