FINITELY GENERATED MULTIRESOLUTION ANALYSIS IN SEVERAL VARIABLES

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Abstract. Let Γ be a lattice in $\mathbb{R}^n$ and A a dilation matrix such that $A\Gamma \subset \Gamma$. Let φ be a localized square integrable vector function and assume that the lattice translates of φ are orthonormal. We give necessary and sufficient conditions on φ in order that it generates a Multiresolution Analysis in $\mathbb{R}^n$ with respect to the lattice Γ and the dilation A. This characterization extends previous results to the case of regular non-compactly supported functions.

1. Introduction

The concept of Multiresolution Analysis (MRA) due to Mallat [Mal89] and Meyer [Mey92] provided the first systematic way to construct orthonormal wavelet bases of $\mathcal{L}^2(\mathbb{R})$. The structure of a MRA is generated by a function (the *scaling function*) that satisfies a certain self-similarity condition. The problem of constructing orthonormal wavelets was then shifted to the problem of constructing MRA's.
The theory was extended to several variables. To take full advantage of the higher dimensionality it is important to consider arbitrary dilation matrices (not only dyadic dilations). This has proved to be useful in applications to image representation where the geometry of the picture is better described with matrices that adapt better to the situation. The side effect is that the theory becomes much more complicated and the results are not a straightforward generalization of the 1-dimensional case.
Another important generalization is the case in which a finite number of generators for the MRA are allowed [Alp93] [GLT93] [GHM94] [CH96] [HSS96] [CDP97] [Ald97] [JRZ99] [Cal99] [CHM99]. This is known in the literature as MRA with multiplicity, and the associated wavelets as Multiwavelets. The framework of multiple generators provides much more flexibility to construct bases with predetermined properties. The characterization of MRA in this generality was done in [CHM99] for compactly supported functions.
In the present article we work in the following context: let Γ be an arbitrary lattice in $\mathbb{R}^n$, and A a dilation matrix compatible with the lattice Γ, (i.e. $A(\Gamma) \subset \Gamma$ and every eigenvalue λ of A satisfies $|\lambda| > 1$). Let $\varphi = (\varphi_1, ..., \varphi_r)$, $\varphi \in \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^r)$ and $\widehat{\varphi}_i$ belongs to the Sobolev space $\mathcal{H}^m(R^n)$, $\forall m \in \mathbb{N}$. Assume that the lattice translates of φ are orthonormal. We give necessary and sufficient conditions on

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the vector function φ in order that it generates a Multiresolution Analysis of $\mathbb{R}^n$. (Theorem 3.1).

These conditions were obtained by Albert Cohen for the 1-dimensional case, scalar functions ($r = 1$) and later extended to the multidimensional setting for the case that the dilation matrix is $2I$ [Coh90]. In [CHM99], Cohen's theorem was extended to include the case of arbitrary dilations matrices and a finite number of generators with compact support.

The contribution of this paper is to show that these conditions can be extended to a much wider class of generators. We were able to proof that the hypothesis of the generators to be compactly supported can be relaxed. We assume instead certain decay of φ. More precisely we require that for $i = 1, \dots, r$, each $\widehat{\varphi}_i$ belongs to the Sobolev space $\boldsymbol{\mathcal{H}}^m(R^n)$, $\forall m \in \mathbb{N}$.

The proof, as the ones in [Coh90] and [CHM99], is a "time-domain" proof in the sense that it doesn't use the Fourier Transform. The main argument is based on a counting technique related to the geometric properties of the tiling associated with the dilation matrix. In the case in which the dilation matrix is $A = 2I$, the tile element is a cube; then the geometry is simple and the integrals that have to be estimated are integrals over cubes in $\mathbb{R}^n$. When one allows arbitrary dilation matrices, the associated tile can be of a very complicated geometry and also have fractal boundary. This makes the estimation of the integrals much more involved, and the counting results are more complicated to obtain. The removal of the assumption of compact support for the scaling vector, requires a refinement of the techniques in order that the counting results can be applied to this more general case.

Necessary and sufficient conditions for when a nested sequence of 2^k-dilated principal shift invariant space (PSI), has dense union and zero intersection where obtained in [BDR93] for the one dimensional case. A PSI is a shift invariant space generated by a single function. The generator in this case doesn't need to be an orthonormal basis neither a Riesz basis of the closure of the span of its integer translates. This general condition is expresed in terms of the zeroes of the Fourier transform of the generator. This setting differs from Cohen approach in the sense that Cohen's Theorem characterizes exactly orthonormal MRA's. The PSI here is generated by a function that has orthonormal translates. Later, Jia and Shen [JS94] formulated the conditions on [BDR93], (see also [Shen98]) for the finitely generated case (FSI), given some indication of the proof.

The characterization in our paper is in the direction of the approach of Cohen. We characterize orthonormal MRA's for the case of a general dilation matrix, compatible with an arbitrary lattice, in higher dimensions, for several functions.

The organization of the paper is as follows. In Section 2, we briefly review the concepts of lattices, tiles, Multiresolution Analysis and the relation between them in terms of the generalized Haar's MRA. In Section 3 we state our main result in Theorem 3.1. For a better organization of the proof, in the following subsections, we discuss and prove in Propositions 3.2, 3.4 and 3.5 the necessary and sufficient conditions that have to be satisfied by the localized vector scaling function to generate a MRA. Finally, in subsection 3.4 we combine the results of the Propositions to prove Theorem 3.1.

2. Lattices, Tiles and Multiresolution Analysis

Let Γ be an arbitrary lattice in $\mathbb{R}^n$ (i.e., $\Gamma = R(\mathbb{Z}^n)$ with R any invertible $n \times n$ matrix with real entries). Let now $\boldsymbol{\mathcal{K}}$ be a fundamental domain for this lattice e.g., $\boldsymbol{\mathcal{K}} = R([0,1)^n)$ and set $\kappa = |det(R)|$.
Let A be a *dilation matrix for* Γ, i.e., $A(\Gamma) \subset \Gamma$ and every eigenvalue λ of A satisfies $|\lambda| > 1$. The determinant of a dilation matrix for a lattice is always an integer and its absolute value is the number of cosets of the quotient group $\Gamma/A(\Gamma)$. A *digit set for* A *and* Γ is any set of representatives of this group.
Let $q = |det(A)|$. We assume that there exist a digit set $D \doteq \{d_0, \dots, d_{q-1}\}$ for A and Γ such that the set $\boldsymbol{\mathcal{Q}} \doteq \{\sum_{k=1}^{\infty} A^{-k}\xi_k \ : \xi_k \in D\}$ has n-dimensional Lebesgue measure κ. Without loss of generality we will assume that $d_0 = 0$. For a general dilation matrix it is not always true that such digit set exists. (A counterexample was found in [Pot97]). If this set of digits exists, we will say that A is an *admissible dilation matrix.*
The set $\boldsymbol{\mathcal{Q}}$ is compact and tiles the space by $\Gamma-$translates in the sense that the $\Gamma-$translates $\{\boldsymbol{\mathcal{Q}} + k\}_{k\in\Gamma}$ cover $\mathbb{R}^n$ with overlaps of measure zero. Moreover, they satisfy the following self-similar condition (See [GM92],[Hut81]):

$$A(\boldsymbol{\mathcal{Q}}) = \bigcup_{s=0}^{q-1} \boldsymbol{\mathcal{Q}} + d_s.$$

Given a function $g : \mathbb{R}^n \to \mathbb{C}^r$, A a dilation matrix, $q = |det\ A|$, $j \in \mathbb{Z}$ and $k \in \Gamma$, we will write $g^{j,k}(x) = q^{j/2} g(A^j x - k)$, to denote a translation of g by $A^{-j}k$ followed by an $\boldsymbol{\mathcal{L}}^2$-normalized dilation by A^j.

2.1. Multiresolution Analysis.

A *Multiresolution Analysis* (MRA) *of multiplicity* r *associated to a dilation matrix* A *and a lattice* Γ is a sequence of closed subspaces $\{\boldsymbol{\mathcal{V}}_j\}_{j\in\mathbb{Z}}$ of $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$ which satisfy:

P1 $\boldsymbol{\mathcal{V}}_j \subset \boldsymbol{\mathcal{V}}_{j+1}$ for each $j \in \mathbb{Z}$,
P2 $g(x) \in \boldsymbol{\mathcal{V}}_j \iff g(Ax) \in \boldsymbol{\mathcal{V}}_{j+1}$ for each $j \in \mathbb{Z}$,
P3 $\bigcap_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j = \{0\}$,
P4 $\bigcup_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j$ is dense in $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$, and
P5 there exist functions $\varphi_1, \dots, \varphi_r \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$ such that the collection of lattice translates $\{\varphi_i(x-k)\}_{k\in\Gamma,\, i=1,\dots,r}$ forms an orthonormal basis for $\boldsymbol{\mathcal{V}}_0$.

If these conditions are satisfied, then the vector function $\varphi = (\varphi_1, \dots, \varphi_r)^T$ is referred to as a *scaling vector* for the MRA.
Gröchenig and Madych [GM92] established a connection between self-similar tilings and multiresolution analysis that have a characteristic function for a scaling function. They showed that there is a Haar-like multiresolution analysis associated to each choice of dilation matrix A and a digit set D for which the set $\boldsymbol{\mathcal{Q}}$ is a tile. In particular, they proved that if $\boldsymbol{\mathcal{Q}}$ is a tile, then the scalar-valued function $\mathcal{X}_{\boldsymbol{\mathcal{Q}}}$ generates a multiresolution analysis of $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$ of multiplicity 1. Note that the fact that $\{\mathcal{X}_{\boldsymbol{\mathcal{Q}}}(x-k)\}_{k\in\Gamma}$ forms an orthonormal basis for $\boldsymbol{\mathcal{V}}_0$ is a restatement of the assumption that the lattice translates of the tile $\boldsymbol{\mathcal{Q}}$ have overlaps of measure zero.
An immediate consequence of Gröchenig and Madych's generalization of the Haar's multiresolution analysis is the following:

Lemma 2.1. *The collection*

$$\{\mathcal{X}_{\mathcal{Q}}^{j,k}\}_{j\in\mathbb{Z},k\in\Gamma} = \{q^{j/2}\mathcal{X}_{\mathcal{Q}}(A^j x - k)\}_{j\in\mathbb{Z},k\in\Gamma}$$

is complete in $\mathcal{L}^2(\mathbb{R}^n)$ i.e., its finite linear span is dense in $\mathcal{L}^2(\mathbb{R}^n)$.

2.2. Localized vector scaling functions. We say that a MRA in $\mathcal{L}^2(\mathbb{R}^n)$ is *localized* or *regular* if the scaling vector $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ is localized in the sense that for $i = 1, \ldots, r$, each $\widehat{\varphi}_i$ belongs to the Sobolev space $\mathcal{H}^m(R^n)$, $\forall m \in \mathbb{N}$. This condition is equivalent to: for each $i = 1, ..., r$,

$$\int (1 + |x|)^m |\varphi_i(x)|^2 dx < \infty. \tag{2.1}$$

Using Cauchy-Schwarz's inequality and taking into account that the function $\frac{1}{(1+|x|)^m}$ belongs to $\mathcal{L}^1(\mathbb{R}^n)$ with $m > 1$, it is easy to prove that if a function f satisfies $\widehat{f} \in \mathcal{H}^m(R^n)$ for all $m \in \mathbb{N}$, then:

$$\int_{\|x\|\geq M} |f(x)|^2 dx \leq \frac{C_m}{(1 + M)^m}, \tag{2.2}$$

$$f \in \mathcal{L}^1(\mathbb{R}^n) \ \textit{and} \tag{2.3}$$

$$\int_{\|x\|\geq M} |f(x)| dx \leq \frac{C_m}{(1 + M)^m}, \ m > 1. \tag{2.4}$$

For simplicity, we shall from now on write that the vector function φ has *orthonormal lattice translates* when we mean to say that $\{\varphi_i(x - k)\}_{k\in\Gamma,i=1,\ldots,r}$ is an orthonormal system in $\mathcal{L}^2(\mathbb{R}^n)$.

Definition 2.2. Assume that $\varphi \in \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^r)$ has orthonormal lattice translates. Let $\mathcal{V}_0$ be the closed linear subspace generated by the lattice translates of φ_i, i.e.,

$$\mathcal{V}_0 = \overline{span}\{\varphi_i(x - k)\}_{k\in\Gamma,i=1,\ldots,r}. \tag{2.5}$$

For each $j \in \mathbb{Z}$ let $\mathcal{V}_j$ be the set of all the dilations of $\mathcal{V}_0$ by A^j, i.e.,

$$\mathcal{V}_j = \{g(A^j x) : g \in \mathcal{V}_0\} = \overline{span}\{\varphi_i^{j,k} : i = 1, ..., r\}_{k\in\Gamma}. \tag{2.6}$$

If $\{\mathcal{V}_j\}_{j\epsilon\mathbb{Z}}$ is a MRA for $\mathcal{L}^2(\mathbb{R}^n)$, then we say that the MRA is *generated* by φ.

Remark 2.3. In the characterization of MRA due to A. Cohen [Coh90], he uses a localized generator, but the dilation matrix is the uniform one: $A = 2I$. In the proof, he uses the essential fact that $2I$ maps dyadic cubes into dyadic cubes. This is not possible in the case of arbitrary dilation matrix.

3. Necessary and sufficient conditions.

We now are ready to state the main result in the paper:

Theorem 3.1. *Let $\varphi = (\varphi_1, \cdots, \varphi_r)^T \in L^2(R^n, C^r)$ such that for each $i = 1, \cdots, r$, $\widehat{\varphi}_i \in \mathcal{H}^m(R^n)$ for all $m \in \mathbb{N}$ and that the set $\{\varphi_i(\cdot - k)\}_{k\in\Gamma,\ i=1,\ldots,r}$ is an orthonormal system. Let A be an admissible dilation matrix for the lattice Γ. Then φ generates a multiresolution analysis with multiplicity r associated to (Γ, A) if and only if:*

a) φ satisfies a refinement equation of the form

$$\varphi(x) = \sum_{k\in\Gamma} c_k \varphi(Ax - k)$$

for some $r \times r$ matrices $c_k = (c_{ij}^k)$, such that for each $i, j = 1, ..., r$, $\{c_{ij}^k\}_{k\in\mathbb{Z}^n}, \in \ell^2(\Gamma)$ and

b) $\|\widehat{\varphi}(0)\|^2 = \sum_{i=1}^r |\widehat{\varphi}_i(0)|^2 = |\boldsymbol{\mathcal{Q}}|.$

To prove this theorem, in the next propositions we will give necessary and sufficient conditions on the vector function φ, in order that the subspaces $\boldsymbol{\mathcal{V}}_j$ will satisfy properties P1, P3 and P4 of the definition of MRA. Property P2 is satisfied from the definition of $\boldsymbol{\mathcal{V}}_j$ and P5 is assumed.

3.1. Property (P1): $\boldsymbol{\mathcal{V}}_j \subset \boldsymbol{\mathcal{V}}_{j+1}$.

Proposition 3.2. *Let $\varphi = (\varphi_1, ..., \varphi_r)^T \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n, \mathbb{C}^r)$ with orthonormal lattice translates. Let A be a dilation matrix and define $\boldsymbol{\mathcal{V}}_j$ as in (2.5) and (2.6). Then, the following conditions are equivalent:*

(1) $\boldsymbol{\mathcal{V}}_j \subset \boldsymbol{\mathcal{V}}_{j+1}$ for all $j \in \mathbb{Z}$.

(2) The vector function φ is refinable*, i.e. it satisfies the* refinement equation*:* $\varphi(x) = \sum_{k\in\Gamma} c_k \varphi(Ax - k)$ *for some $r \times r-$matrices c_k, such that for each $i, j = 1, ..., r$, the sequence of coefficients $\{c_{i,j}^k\}_{k\in\Gamma}$ is in $\ell^2(\Gamma)$.*

Proof: If (1) is satisfied, then $\varphi_i \in \boldsymbol{\mathcal{V}}_0 \subset \boldsymbol{\mathcal{V}}_1$ for $i = 1, \cdots, r$. The definition of the subspaces $\boldsymbol{\mathcal{V}}_j$ implies that $\{q^{1/2}\varphi_j(Ax - k)\}_{k\in\Gamma, i,j=1,\cdots,r} = \{\varphi_j^{1,k}(x)\}_{k\in\Gamma}$ is an orthonormal basis for $\boldsymbol{\mathcal{V}}_1$, then the representation of each φ_i respect to the orthonormal basis of $\boldsymbol{\mathcal{V}}_1$ will be:

$$\varphi_i = \sum_{j=1}^r \sum_{k\in\Gamma} c_{i,j}^k \varphi_j^{1,k} \ (in\ \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)), \tag{3.1}$$

where $c_{ij}^k := <\varphi_i, \varphi_j^{1,k}>$. For $i, j = 1, ..., r$, the sequence of coefficients $\{c_{i,j}^k\}_{k\in\Gamma}$ belongs to $\ell^2(\Gamma)$. Let us call c_k the $r \times r-$matrix whose columns are $(c_{i1}^k, ..., c_{ir}^k)$. Considering that $\varphi = (\varphi_1, ..., \varphi_r)^T$ then, from (3.1) we have $\varphi = \sum_{k\in\Gamma} c_k \varphi^{1,k}$ *in* $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n, \mathbb{C}^r)$, or equivalently

$$\varphi(x) = q^{1/2} \sum_{k\in\Gamma} c_k \varphi(Ax - k) \ in\ \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n, \mathbb{C}^r),$$

and condition (2) is satisfied.
For the converse, if φ is refinable, then $\varphi_i \in \boldsymbol{\mathcal{V}}_1, i = 1, ..., r$ so $\boldsymbol{\mathcal{V}}_0$ is included in $\boldsymbol{\mathcal{V}}_1$. □

3.2. Property (P3): $\bigcap_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j = \{0\}$. We shall prove that (P3) is a consequence of the orthonormal $\Gamma-$translates of φ and the localization of each φ_i. To do this, we will use the following lemma (we omit the proof because it is like in the classical 1-dimensional case with dyadic dilations [Woj97]):

Lemma 3.3. *Consider $\boldsymbol{\mathcal{V}}_j$ as in (2.6). Let P_j be the orthogonal projection of $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$ onto $\boldsymbol{\mathcal{V}}_j$. Suppose that for all $g \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$, $\lim_{j\to-\infty} \|P_j g\|_2 = 0$.*

Then $\bigcap_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j = \{0\}$.

Proposition 3.4. *Let $\varphi \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n, \mathbb{C}^r)$ be a localized vector function in the sense of (2.1). Suppose that φ has orthonormal lattice translates, A is a dilation matrix and consider the subspaces $\boldsymbol{\mathcal{V}}_j$ as defined in (2.6). Then $\bigcap_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j = \{0\}$.*

Proof: Using Lemma 3.3, it suffices to show that $\lim_{j\to-\infty} \|P_j g\|_2 = 0, \ \ \forall g \in \boldsymbol{\mathcal{L}}^2$. Moreover, it suffices to establish this limit for g contained in a subset whose finite linear span is dense in $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$. We will use the complete set given in Lemma 2.1, i.e. we will prove that

$$\forall s \in \mathbb{Z},\ \forall \ell \in \Gamma,\ \lim_{j\to-\infty} \|P_j \mathcal{X}^{s,\ell}_{\boldsymbol{\mathcal{Q}}}\|_2 = 0. \tag{3.2}$$

Fix any $s \in \mathbb{Z}$ and $\ell \in \Gamma$. Since $q = |det(A)|$, we have for every $j \in \mathbb{Z}$ that

$$|A^{j-s}(\boldsymbol{\mathcal{Q}} + \ell)| = q^{j-s}|\boldsymbol{\mathcal{Q}} + \ell| = q^{j-s}|\boldsymbol{\mathcal{Q}}|.$$

Since $\{\varphi_i^{j,k}\}_{k\in\Gamma, i=1,\dots,r}$ is an orthonormal basis for the subspace $\boldsymbol{\mathcal{V}}_j$ then,

$$\|P_j \mathcal{X}^{s,\ell}_{\boldsymbol{\mathcal{Q}}}\|_2^2 = \frac{1}{q^{j-s}} \sum_{i=1}^{r} \sum_{k\in\Gamma} \left| \int_{A^{j-s}(\boldsymbol{\mathcal{Q}}+\ell)} \overline{\varphi_i(x-k)}dx \right|^2 .$$

Using Cauchy-Schwarz's inequality, we therefore compute that

$$\begin{aligned} \|P_j \mathcal{X}^{s,\ell}_{\boldsymbol{\mathcal{Q}}}\|_2^2 &\le \frac{|A^{j-s}(\boldsymbol{\mathcal{Q}}+\ell)|}{q^{j-s}} \sum_{i=1}^{r} \sum_{k\in\Gamma} \int_{A^{j-s}(\boldsymbol{\mathcal{Q}}+\ell)} |\varphi_i(x-k)|^2 dx \\ &= |\boldsymbol{\mathcal{Q}}| \sum_{i=1}^{r} \sum_{k\in\Gamma} \int_{A^{j-s}(\boldsymbol{\mathcal{Q}}+\ell)-k} |\varphi_i(x)|^2 dx, \end{aligned} \tag{3.3}$$

where the last sum is finite. To see this, note that for a fixed s and $j < s$, using that $\boldsymbol{\mathcal{Q}}$ is a tile for $\mathbb{R}^n$ and A^{-1} is contractive, we have that the lattice translates of $A^{j-s}\boldsymbol{\mathcal{Q}}$ have overlaps of measure zero. To simplify the notation, write $E := A^{-s}(\boldsymbol{\mathcal{Q}} + \ell)$; then $A^{j-s}(\boldsymbol{\mathcal{Q}} + \ell) - k = A^j E - k$. Choose an integer $J < 0$ small enough such that $|A^j E - k \cap A^j E - k'| = 0$ for all $j \le J$; $k, k' \in \Gamma$, $k \ne k'$. Then, for $j \le J$,

$$\begin{aligned} \sum_{k\in\Gamma} \mathcal{X}_{A^j E-k}(x)|\varphi_i(x)|^2 &= \mathcal{X}_{\bigcup_{k\in\Gamma} A^j E-k}(x)|\varphi_i(x)|^2 \\ &\le |\varphi_i(x)|^2. \end{aligned}$$

And since $|\varphi_i(x)|^2 \in \boldsymbol{\mathcal{L}}^1(\mathbb{R}^n)$, then

$$\sum_{k\in\Gamma} \int_{A^j E-k} |\varphi_i(x)|^2 dx = \int_{\mathbb{R}^n} \sum_{k\in\Gamma} \mathcal{X}_{A^j E-k}(x)|\varphi_i(x)|^2 dx < \infty.$$

Now, using (3.2) and (3.3), it suffices to prove that for $i = 1, \dots, r$ $\sum_{k\in\Gamma} \int_{A^j E-k} |\varphi_i(x)|^2 dx$

goes to 0 when $j \to -\infty$. Consider the same integer $J \in \mathbb{Z}$ as before and $j \leq J$; and define $f_j(x) := \sum_{k \in \mathbb{Z}^n} \mathcal{X}_{A^j E - k}(x) |\varphi_i(x)|^2$. So $f_j(x) = |\varphi_i(x)|^2 \mathcal{X}_{\cup_{k\in\Gamma} A^j E - k}(x)$. Then:

$$\begin{aligned}
\sum_{k \in \Gamma} \int_{A^j E - k} |\varphi_i(x)|^2 dx &= \int_{\mathbb{R}^n} f_j(x) dx \\
&= \int_{[-p,p]^n} f_j(x) dx \ + \ \int_{\mathbb{R}^n \setminus [-p,p]^n} f_j(x) dx \\
&= I_1 \ + \ I_2.
\end{aligned}$$

Write $B_j := \bigcup_{k \in \Gamma} (A^j E - k) \cap [-p, p]^n$, then

$$I_1 = \int |\varphi_i(x)|^2 \mathcal{X}_{B_j}(x) \ dx. \tag{3.4}$$

We can see that $|B_j| \to 0$ as $j \to -\infty$. In fact: consider $t_j := \text{card}(\{k \in \mathbb{Z}^n : A^j E - k \cap [-p, p]^n \neq \emptyset\})$ and write $\delta(\cdot)$ as the diameter of a certain set. Because the spectral radius ρ of A^{-1} is less than 1, we have that $\|A^j\|_\infty \to 0$ when $j \to -\infty$ (see [HJ]) i.e. $\|A^j\|_\infty < \epsilon$ for j small enough. Then, considering the metric $d(x, y) = \|x - y\|_\infty = max\{|x_i - y_i| : \ i = 1, ..., n\}$ we have that for $x, \ y \in A^j E$:

$$\begin{aligned}
d(x, y) &= \|A^j (A^{1-j} x - A^{1-j} y)\|_\infty \\
&\leq \|A^j\|_\infty \|A^{1-j} x - A^{1-j} y\|_\infty \\
&< d(A^{1-j} x, A^{1-j} y) \\
&\leq \delta(AE).
\end{aligned}$$

Taking the supremum over $A^j E$, we have that $\delta(A^j E) \leq \delta(AE)$ and then $t_j < t_1$, for all $j < 0$. Now, for $j \leq J$, the overlaps of the lattice translates of $A^j E$ have measure zero, so:

$$\begin{aligned}
|B_j| &= t_j \left| A^j E - k \cap [-p, p]^n \right| \\
&\leq t_1 \left| A^j E \right| \\
&= t_1 \, |E| \, q^j \\
&< \varepsilon.
\end{aligned}$$

Since $|B_j| \to 0$ as $j \to 0$ then $f_j \mathcal{X}_{[-p,p]^n} \to 0$, moreover $|f_j \mathcal{X}_{[-p,p]^n}(x)| \leq |\varphi_i(x)|^2$. From this fact, equality (3.4) and the Dominated Convergence theorem we have that

$$I_1 = \int_{[-p,p]^n} f_j(x) dx \longrightarrow 0 \text{ when } j \to -\infty. \tag{3.5}$$

For the integral I_2, take $\varepsilon > 0$ and $j \leq J$. Then $0 \leq f_j(x) \leq |\varphi_i(x)|^2$. Using this and property (2.2) for some m, we have

$$\begin{aligned}
I_2 &\leq \int_{\|x\| \geq p} |\varphi_i(x)|^2 dx \\
&\leq \frac{C_m}{(1 + p)^m}.
\end{aligned}$$

Considering p large enough such that $\frac{C_m}{(1+p)^m} < \varepsilon$, we will have $I_2 < \varepsilon$. Finally:

$$\int_{\mathbb{R}^n} f_j(x)dx = I_1 + I_2 \leq I_1 + \varepsilon.$$

Taking limit for $j \to -\infty$, from (3.5) we conclude that for all $s \in \mathbb{Z}$ and $\ell \in \Gamma$, $\lim_{j\to-\infty} \|P_j \mathcal{X}^{s,\ell}_{\boldsymbol{\mathcal{Q}}}\|_2 = 0$. Hence $\bigcap_{j\epsilon\mathbb{Z}} \boldsymbol{\mathcal{V}}_j = \{0\}$.

□

3.3. Property (P4): $\overline{\bigcup_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j} = \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$.

Proposition 3.5. *Let $\varphi = (\varphi_1, ..., \varphi_r)^T \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n, \mathbb{C}^r)$ a localized vector function. Suppose that φ has orthonormal lattice translates. Let A be a dilation matrix and let $\boldsymbol{\mathcal{V}}_0$ and $\boldsymbol{\mathcal{V}}_j$ be defined as (2.5) and (2.6) respectively. If*

$$\sum_{i=1}^{r} |\widehat{\varphi}_i(0)|^2 = \sum_{i=1}^{r} |\int \varphi_i(x)dx|^2 = |\boldsymbol{\mathcal{Q}}| \tag{3.6}$$

Then $\bigcup_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j$ is dense in $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$. Reciprocally, if $\bigcup_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j$ is dense in $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$ and φ is refinable, then (3.6) is satisfied.

Before proving this proposition, we are going to present some auxiliary results with respect to the decomposition of $\mathbb{R}^n$ by the tiles $\{\boldsymbol{\mathcal{Q}} + k\}_{k\in\Gamma}$. First, note that the fact that $\boldsymbol{\mathcal{Q}}$ is self-similar together with the fact that the translates of $\boldsymbol{\mathcal{Q}}$ tile $\mathbb{R}^n$ with overlaps of measure zero, implies that the dilated tile $A^j\boldsymbol{\mathcal{Q}}$, $j \geq 1$ is a union of exactly q^j translates of $\boldsymbol{\mathcal{Q}}$, with each of the translates lying entirely inside $A^j\boldsymbol{\mathcal{Q}}$. Following the idea in [CHM99], for $j \geq 1$ we are going to split the lattice Γ into a finite set containing those elements that translate $\boldsymbol{\mathcal{Q}}$ entirely inside $A^j\boldsymbol{\mathcal{Q}}$, and a finite set containing the elements that translate $\boldsymbol{\mathcal{Q}}$ to the boundary of $A^j\boldsymbol{\mathcal{Q}}$. More precisely, for each $j \geq 1$ let us consider the following finite subsets of Γ :

$$\begin{aligned} N_j &= \{k \in \Gamma : \boldsymbol{\mathcal{Q}} + k \subset A^j\boldsymbol{\mathcal{Q}}\}, \\ N_j^\circ &= \{k \in N_j : \boldsymbol{\mathcal{Q}} + k \subset (A^j\boldsymbol{\mathcal{Q}})^\circ\}, \\ N_j^\partial &= \{k \in N_j : (\boldsymbol{\mathcal{Q}} + k) \cap \partial(A^j\boldsymbol{\mathcal{Q}}) \neq \emptyset\}. \end{aligned} \tag{3.7}$$

These sets satisfy the following relations: $A^j\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{Q}} + N_j$, $\operatorname{card}(N_j) = q^j$, $N_j^\circ \cup N^\partial = N_j$ and $N_j^\circ \cap N_j^\partial = \emptyset$.

Let $\Omega = \{k \in \Gamma : (\boldsymbol{\mathcal{Q}} + k) \cap B \neq \emptyset\}$. The following technical lemma (see [CHM99] for a proof) characterizes those translates $\boldsymbol{\mathcal{Q}} + \gamma$ of $\boldsymbol{\mathcal{Q}}$ for which it is possible to translate $\boldsymbol{\mathcal{Q}} + \gamma$ by elements of Ω so that one translate $\boldsymbol{\mathcal{Q}} + \gamma + k$ with $k \in \Omega$ lies entirely within $A^j\boldsymbol{\mathcal{Q}}$ and another translate $\boldsymbol{\mathcal{Q}} + \gamma + k'$ with $k' \in \Omega$ lies entirely outside of $A^j\boldsymbol{\mathcal{Q}}$ (neglecting its boundary). This lemma also tells us that the ratio of the number of those translates $\boldsymbol{\mathcal{Q}} + k$ that intersect the boundary of $A^j\boldsymbol{\mathcal{Q}}$ to the total number lying inside $A^j\boldsymbol{\mathcal{Q}}$ converges to zero:

Lemma 3.6. *Let B, Ω, N_j, N_j° and N_j^∂ defined as before, then:*

a) $\displaystyle\lim_{j\to\infty} \frac{card(N_j^\circ)}{q^j} = 1$ *and* $\displaystyle\lim_{j\to\infty} \frac{card(N_j^\partial)}{q^j} = 0.$

b) $\lim_{j\to\infty} \dfrac{card(N_j^\circ \setminus ((N_j^\partial - \Omega)\cap N_j))}{q^j} = 1.$

c) Let $\gamma \in \Gamma$. *If there exist* $k,\ k' \in \Omega$ *such that* $\boldsymbol{\mathcal{Q}} + k + \gamma \subset A^j \boldsymbol{\mathcal{Q}}$ *and* $\boldsymbol{\mathcal{Q}} + k' + \gamma \subset \mathbb{R}^n \setminus (A^j\boldsymbol{\mathcal{Q}})^\circ$, *then* $\gamma \in N_j^\partial - \Omega = \{\ell - \omega : \ell \in N_j{}^\partial, \omega \in \Omega\}$.

Proof of proposition 3.5: Suppose that φ is refinable, then we have to prove that $\overline{\bigcup_{j\in\mathbb{Z}} \boldsymbol{\mathcal{V}}_j} = \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n) \iff \sum_{i=1}^r |\widehat{\varphi}_i(0)|^2 = |\boldsymbol{\mathcal{Q}}|$.

Note that if for all $g \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$

$$\lim_{j\to\infty} \|P_j g - g\|_2 = 0, \tag{3.8}$$

then Property (P4) is satisfied. Further, if φ is refinable, then by Proposition 3.2, $\boldsymbol{\mathcal{V}}_j \subset \boldsymbol{\mathcal{V}}_{j+1}$ and therefore (3.8) is equivalent to Property (P4). Now, by orthogonality, $\|P_j g - g\|_2^2 = \|P_j g\|_2^2 - \|g\|_2^2$, then we can rewrite equation (3.8) as:

$$\forall g \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n),\ \lim_{j\to\infty} \|P_j g\|_2^2 = \|g\|_2^2. \tag{3.9}$$

This expression is true for all $g \in \boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$ if and only if is true in a dense subset of $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^n)$. So we will use the set of functions considered in Lemma 2.1. Then,

$$\|P_j(\mathcal{X}_{\boldsymbol{\mathcal{Q}}})\|_2^2 = \frac{1}{q^j} \sum_{i=1}^r \sum_{k\in\Gamma} \left| \int_{A^j\boldsymbol{\mathcal{Q}}} \varphi_i(x-k)dx \right|^2. \tag{3.10}$$

On the other hand, let $s \in \mathbb{Z}$, $\ell \in \Gamma$, and $j \geq s$. By a change of variable and taking into account that $A\Gamma \subset \Gamma$, we have

$$\|P_j(\mathcal{X}_{\boldsymbol{\mathcal{Q}}}^{s,\ell})\|_2^2 = \|P_{j-s}(\mathcal{X}_{\boldsymbol{\mathcal{Q}}})\|_2^2. \tag{3.11}$$

Comparing (3.10) and (3.11), we conclude that (3.9) is equivalent to the statement:

$$\lim_{j\to\infty} \|P_j(\mathcal{X}_{\boldsymbol{\mathcal{Q}}})\|_2^2 = \|\mathcal{X}_{\boldsymbol{\mathcal{Q}}}\|_2^2 = |\boldsymbol{\mathcal{Q}}|. \tag{3.12}$$

Since statements (3.8), (3.9) and (3.12) are equivalent, we conclude that it suffices to prove that $\lim_{j\to\infty} \|P_j(\mathcal{X}_{\boldsymbol{\mathcal{Q}}})\|_2^2 = \sum_{i=1}^r |\widehat{\varphi}_i(0)|^2$, or equivalently, to prove that for each $i = 1, ..., r$:

$$\lim_{j\to\infty} \frac{1}{q^j} \sum_{k\in\Gamma} \left| \int_{A^j\boldsymbol{\mathcal{Q}}} \varphi_i(x-k)dx \right|^2 = |\widehat{\varphi}_i(0)|^2. \tag{3.13}$$

To do this, fix i, consider a constant $M_i > 0$ and define $K_i = \{x \in \mathbb{R}^n : \|x\| \leq M_i\}$. Using the property of unconditional convergence of orthonormal bases, we will split the summation over Γ into three disjoint regions related to the subset K_i. The idea behind this is that the first region should contain only elements k of the lattice Γ such that $K_i + k$ is sure to lie in the interior of $A^j\boldsymbol{\mathcal{Q}}$, the second region should contain those k for which this translation will intersect the boundary of $A^j\boldsymbol{\mathcal{Q}}$, and the last region should be the complement of the first two. More precisely, let B_i be any open ball in $\mathbb{R}^n$ which contains both $\boldsymbol{\mathcal{Q}}$ and K_i, and define

$$\Omega = \{k \in \Gamma : (\boldsymbol{\mathcal{Q}} + k) \cap B_i \neq \emptyset\}.$$

Note that Ω is finite and $K_i \subset \Omega + \boldsymbol{Q}$. For each $j \geq 1$, define:

$$\Gamma_{1,j} = N_j^\circ \backslash ((N_j^\partial - \Omega) \cap N_j),$$
$$\Gamma_{2,j} = N_j^\partial - \Omega,$$
$$\Gamma_{3,j} = \Gamma \backslash (\Gamma_{1,j} \cup \Gamma_{2,j}),$$

where the sets N_j, N_j° and N_j^∂ are as in (3.7). Note that for each j, the sets $\Gamma_{1,j}$, $\Gamma_{2,j}$, $\Gamma_{3,j}$ partition Γ. Further, by Lemma 3.6 a) and b), we have:

$$\lim_{j\to\infty} \frac{\operatorname{card}(\Gamma_{1,j})}{q^j} = 1 \textit{ and } \lim_{j\to\infty} \frac{\operatorname{card}(\Gamma_{2,j})}{q^j} = 0. \tag{3.14}$$

Now define

$$R_{sj} = \frac{1}{q^j} \sum_{k\in\Gamma_{s,j}} \left| \int_{A^j \boldsymbol{Q}} \varphi_i(x-k)dx \right|^2, \ s = 1,2,3$$

We will show that: $\lim_{j\to\infty} R_{1j} = |\widehat{\varphi}_i(0)|^2$, $\lim_{j\to\infty} R_{2j} = 0$ *and* $\lim_{j\to\infty} R_{3j} = 0$.
Let us begin with R_{2j}.

$$\begin{aligned} R_{2j} &\leq \frac{1}{q^j} \sum_{k\in\Gamma_{2,j}} \left(\int_{A^j \boldsymbol{Q}} |\varphi_i(x-k)|dx \right)^2 \\ &\leq \frac{1}{q^j} \sum_{k\in\Gamma_{2,j}} \left(\int_{\mathbb{R}^n} |\varphi_i(x)|dx \right)^2 \\ &= \frac{C \operatorname{card}(\Gamma_{2,j})}{q^j}. \end{aligned}$$

By (3.14), the last term is arbitrarily small for j large enough. Then $R_{2j} \to 0$ when $j \to \infty$.
To analyze R_{3j}, let us write $\tilde{\varphi}_i(x) = \mathcal{X}_{K_i}(x)\varphi_i(x)$. Then $\varphi_i(x) = \tilde{\varphi}_i(x) + \mathcal{X}_{K_i^c}(x)\varphi_i(x)$. Note that $\tilde{\varphi}_i$ has compact support.

$$\begin{aligned} R_{3j} &\leq \frac{1}{q^j} \sum_{k\in\Gamma_{3,j}} \left(\int_{A^j \boldsymbol{Q}} |\tilde{\varphi}_i(x-k)|dx + \int_{A^j \boldsymbol{Q}} |\mathcal{X}_{K_i^c}(y-k)\varphi_i(y-k)|dy \right)^2 \\ &= A + B + C, \end{aligned}$$

where

$$A = \frac{1}{q^j} \sum_{k\in\Gamma_{3,j}} \left(\int_{A^j \boldsymbol{Q}} |\tilde{\varphi}_i(x-k)|dx \right)^2,$$

$$B = \frac{2}{q^j} \sum_{k\in\Gamma_{3,j}} \left(\int_{A^j \boldsymbol{Q}} |\tilde{\varphi}_i(x-k)|dx \right) \left(\int_{A^j \boldsymbol{Q}} |\mathcal{X}_{K_i^c}(y-k)\varphi_i(y-k)|dy \right) \text{ and}$$

$$C = \frac{1}{q^j} \sum_{k\in\Gamma_{3,j}} \left(\int_{A^j \boldsymbol{Q}} |\mathcal{X}_{K_i^c}(y-k)\varphi_i(y-k)|dy \right)^2.$$

We will show that $A = B = 0$ and $C \to 0$ as $j \to \infty$. Suppose that $A \neq 0$, then $\int_{A^j \boldsymbol{Q}} |\tilde{\varphi}_i(x-\gamma)|dx \neq 0$ for some $\gamma \in \Gamma_{3,j}$. Then $(K_i + \gamma) \cap A^j\boldsymbol{Q}$ has to have positive Lebesgue measure. Since $K_i \subset B_i \subset Q + \Omega$, then $(K_i + \gamma) \subset (Q + \Omega + \gamma)$ and $(Q + \Omega + \gamma) \cap A^j\boldsymbol{Q}$ will have positive measure. Because $A^j\boldsymbol{Q}$ is the exact union of

q^j translates of $\boldsymbol{\mathcal{Q}}$ that do not overlap, then the only translates of $\boldsymbol{\mathcal{Q}}$ that intersects $A^j\boldsymbol{\mathcal{Q}}$ in sets of positive measure, are the translates that are completely inside of $A^j\boldsymbol{\mathcal{Q}}$. Hence:

$$\boldsymbol{\mathcal{Q}} + k + \gamma \subset A^j\boldsymbol{\mathcal{Q}} \text{ for some} k \in \Omega. \tag{3.15}$$

Since $0 \in \Omega$ and $N_j^\partial \subset N_j$, then $N_j^\partial \subset (N_j^\partial - \Omega) \cap N_j$. Hence $N_j = N_j^\circ \cup N_j^\partial \subset \Gamma_{1,j} \cup \Gamma_{2,j}$. Since $\gamma \in \Gamma_{3,j} = \Gamma \backslash (\Gamma_{1,j} \cup \Gamma_{2,j})$, then $\gamma \notin \Gamma_{2,j}$, so $\gamma \notin N_j$. This implies that $\boldsymbol{\mathcal{Q}} + \gamma$ is not contained in $A^j\boldsymbol{\mathcal{Q}}$. Then $\boldsymbol{\mathcal{Q}} + \gamma \subset \mathbb{R}^n \backslash (A^j\boldsymbol{\mathcal{Q}})^\circ$. Consequently

$$\boldsymbol{\mathcal{Q}} + 0 + \gamma \subset \mathbb{R}^n \backslash (A^j\boldsymbol{\mathcal{Q}})^\circ, \tag{3.16}$$

and since $0 \in \Omega$, then from Lemma 3.6 c), applied to (3.15) and (3.16) we have $\gamma \in N_j^\partial - \Omega = \Gamma_{2,j}$, and this is a contradiction. Then $A = 0$. By a similar reason, $B = 0$.

To prove that $C \to 0$ as $j \to \infty$, we write $\Gamma_{3,j}$ as the union of disjoint sets as follows:

$$\Gamma_{3,j} = \bigcup_{s=1}^{\infty} D_s^j \tag{3.17}$$

where for each $j \in \mathbb{Z}$, $D_s^j := \{k \in \Gamma_{3,j} : s \le dist(A^j\boldsymbol{\mathcal{Q}} - k, 0) < s + 1\}$.
After the change of variable $x = y - k$, we have:

$$\begin{aligned} C &= \frac{1}{q^j} \sum_{k \in \Gamma_{3j}} \left(\int_{A^j\boldsymbol{\mathcal{Q}} - k \cap K_i^c} |\varphi_i(x)| dx \right)^2 \\ &\le \frac{1}{q^j} \sum_{s=1}^{\infty} \sum_{k \in D_s^j} \left(\int_{A^j\boldsymbol{\mathcal{Q}} - k} |\varphi_i(x)| dx \right)^2 \\ &\le \frac{1}{q^j} \sum_{s=1}^{\infty} \sum_{k \in D_s^j} \left(\int_{\|x\| \ge s} |\varphi_i(x)| dx \right)^2 . \end{aligned}$$

For a purpose that will become clear later consider $m > \frac{1+n}{2}$. Using that $\widehat{\varphi}_i \in \boldsymbol{\mathcal{H}}^m(\mathbb{R}^n)$ and property (2.4), then

$$C \le \frac{1}{q^j} \sum_{s=1}^{\infty} \sum_{k \in D_s^j} \frac{c_m}{(1+s)^{2m}} \le \frac{1}{q^j} \sum_{s=1}^{\infty} \operatorname{card}(D_s^j) \frac{c_m}{(1+s)^{2m}}, \tag{3.18}$$

with c_m a constant that depends on m. Let us find an upper bound for $\operatorname{card}(D_s^j)$. Let us note that if $\gamma \in D_s^j$ then $s \le \operatorname{dist}(A^j\boldsymbol{\mathcal{Q}} - \gamma, 0) < s + 1$. Since $A^j\boldsymbol{\mathcal{Q}} - \gamma$ is compact, then there exists $x \in \partial(A^j\boldsymbol{\mathcal{Q}} - \gamma)$ where the distance is attained. On the other hand $A^j\boldsymbol{\mathcal{Q}} - \gamma$ is the union of exactly q^j tiles that do not overlap, hence $x \in \boldsymbol{\mathcal{Q}} - t$ for some t. Then $\boldsymbol{\mathcal{Q}} - t \subset A^j\boldsymbol{\mathcal{Q}} - \gamma$ and $\boldsymbol{\mathcal{Q}} - (t+\gamma) \cap \partial(A^j\boldsymbol{\mathcal{Q}}) \neq \emptyset$. Finally $t + \gamma \in N_j^\partial$ and $D_s^j \subset \{\gamma \in \Gamma_{3,j} : \exists t \in D_s^0,\ \textit{such that}\ \gamma + t \in N_j^\partial\}$. It follows that

$$\operatorname{card}(D_s^j) \le \operatorname{card}(D_s^0) \cdot \operatorname{card}(N_j^\partial). \tag{3.19}$$

Moreover $\operatorname{card}(D_s^0) \le c(s + 1 + d)$, with $d = \operatorname{diam}(\boldsymbol{\mathcal{Q}})$. To see that, take $\gamma \in D_s^0$ then $\boldsymbol{\mathcal{Q}} - \gamma \subset B(0, s+1+d)$ (the open ball centered at zero, and radius $s+1+d$). Then $D_s^0 \subset L := \{\gamma \in \Gamma : \boldsymbol{\mathcal{Q}} - \gamma \subset B(0, s+1+d)\}$ and $\operatorname{card}(D_s^0) \le \operatorname{card}(L)$. Now, since the lattice translates of $\boldsymbol{\mathcal{Q}}$ do not overlap, we have:

$$\operatorname{card}(L)|\boldsymbol{\mathcal{Q}}| = \sum_{\gamma\in L} |\boldsymbol{\mathcal{Q}} - \gamma| = \left|\bigcup_{\gamma\in L} (\boldsymbol{\mathcal{Q}} - \gamma)\right|. \tag{3.20}$$

On the other hand

$$\left|\bigcup_{\gamma\in L} (\boldsymbol{\mathcal{Q}} - \gamma)\right| \leq |B(0, s+1+d)| = \tilde{c}(s+1+d)^n. \tag{3.21}$$

From (3.20) and (3.21) we have $\operatorname{card}(L) \leq c(s+1+d)^n$, and finally $\operatorname{card}(D_s^0) \leq c(s+1+d)^n$. By (3.19) it follows that $\operatorname{card}(D_s^j) \leq \operatorname{card}(N_j^\partial) \cdot c(s+1+d)^n$, with c a constant that does not depend on j. Then, from (3.18) we have

$$C \leq \frac{\operatorname{card}(N_j^\partial)}{q^j} \sum_{s=1}^{\infty} \frac{\tilde{c}_m (s+1+d)^n}{(1+s)^{2m}}.$$

Here, the summation is finite because $m > \frac{1+n}{2}$, then $C \leq \frac{\operatorname{card}(N_j^\partial)}{q^j} C(m)$, with $C(m)$ a constant that depends on m. Finally, by Lemma 3.6, $C \to 0$ when $j \to \infty$.
It only remains to prove that for each $i = 1, \cdots, r$, $R_{1j} \to |\widehat{\varphi}_i(0)|^2$. Fix i, then

$$\begin{aligned}
\left|\int_{A^j\boldsymbol{\mathcal{Q}}} \varphi_i(x-k)dx\right|^2 &= \left|\int_{\mathbb{R}^n} \varphi_i(x)dx - \int_{(A^j\boldsymbol{\mathcal{Q}})^c - k} \varphi_i(x)dx\right|^2 \\
&\leq \left(|\widehat{\varphi}_i(0)| + \int_{(A^j\boldsymbol{\mathcal{Q}})^c - k} |\varphi_i(x)|dx\right)^2 \\
&= |\widehat{\varphi}_i(0)|^2 + 2|\widehat{\varphi}_i(0)| \int_{(A^j\boldsymbol{\mathcal{Q}})^c - k} |\varphi_i(x)|dx + \\
&\quad + \left(\int_{(A^j\boldsymbol{\mathcal{Q}})^c - k} |\varphi_i(x)|dx\right)^2.
\end{aligned}$$

Summing over $\Gamma_{1,j}$ and dividing by q^j, we have:

$$\begin{aligned}
\frac{1}{q^j}\sum_{k\in\Gamma_{1,j}} \left|\int_{A^j\boldsymbol{\mathcal{Q}}} \varphi_i(x-k)dx\right|^2 &\leq \frac{\operatorname{card}(\Gamma_{1,j})}{q^j} |\widehat{\varphi}_i(0)|^2 \\
&\quad + 2|\widehat{\varphi}_i(0)| \frac{1}{q^j} \sum_{k\in\Gamma_{1,j}} \int_{(A^j\boldsymbol{\mathcal{Q}})^c - k} |\varphi_i(x)|dx \\
&\quad + \frac{1}{q^j} \sum_{k\in\Gamma_{1,j}} \left(\int_{(A^j\boldsymbol{\mathcal{Q}})^c - k} |\varphi_i(x)|dx\right)^2
\end{aligned} \tag{3.22}$$

Now, by definition of $\Gamma_{1,j}$ and Lemma 3.6 c), it can be shown that $K_i + k \subset (A^j\boldsymbol{\mathcal{Q}})^\circ \subset A^j\boldsymbol{\mathcal{Q}}$ for $k \in \Gamma_{1,j}$. Hence $(A^j\boldsymbol{\mathcal{Q}})^c - k \subset K_i^c$. From this and property (2.4), we have:

$$\begin{aligned}
\frac{1}{q^j}\sum_{k\in\Gamma_{1,j}} \int_{(A^j\boldsymbol{\mathcal{Q}})^c - k} |\varphi_i(x)|dx &\leq \frac{1}{q^j}\sum_{k\in\Gamma_{1,j}} \int_{K_i^c} |\varphi_i(x)|dx \\
&\leq \frac{\operatorname{card}(\Gamma_{1,j})}{q^j} \frac{c_m}{(1+M_i)^m}.
\end{aligned}$$

Now, consider $\varepsilon > 0$, using (3.14) we have that $\frac{\operatorname{card}(\Gamma_{1,j})}{q^j} < \varepsilon + 1$ for j large enough. Moreover, in the definition of K_i, we can choose a constant M_i such that $\frac{c_m}{(1+M_i)^m} < \varepsilon$ (for a fixed m). Then, we will have: $\displaystyle\frac{1}{q^j}\sum_{k\in\Gamma_{1,j}}\int_{(A^j\boldsymbol{Q})^c-k}|\varphi_i(x)|dx < \varepsilon$. Using this in (3.22) we can conclude that for $\varepsilon > 0$ and j large enough:

$$\frac{1}{q^j}\sum_{k\in\Gamma_{1j}}\left|\int_{A^j\boldsymbol{Q}}\varphi_i(x-k)dx\right|^2 < |\hat{\varphi}_i(0)|^2 + \varepsilon. \tag{3.23}$$

On the other hand:

$$\begin{aligned}\left|\int_{A^j\boldsymbol{Q}}\varphi_i(x-k)dx\right|^2 &= \left|\int_{\mathbb{R}^n}\varphi_i(x)dx - \int_{(A^j\boldsymbol{Q})^c-k}\varphi_i(x)dx\right|^2\\ &\geq \left(|\widehat{\varphi}_i(0)| - \left|\int_{(A^j\boldsymbol{Q})^c-k}\varphi_i(x)dx\right|\right)^2\\ &\geq |\widehat{\varphi}_i(0)|^2 - 2|\widehat{\varphi}_i(0)|\left|\int_{(A^j\boldsymbol{Q})^c-k}\varphi_i(x)dx\right|.\end{aligned}$$

Remember that from (3.14) we have that $1-\varepsilon < \frac{\operatorname{card}(\Gamma_{1,j})}{q^j}$ and $\frac{1}{q^j}\sum_{k\in\Gamma_{1,j}}\int_{(A^j\boldsymbol{Q})^c-k}|\varphi_i(x)|dx < \varepsilon$. Then, summing over $\Gamma_{1,j}$ and dividing by q^j :

$$\begin{aligned}\frac{1}{q^j}\sum_{k\in\Gamma_{1,j}}\left|\int_{A^j\boldsymbol{Q}}\varphi_i(x-k)dx\right|^2 &\geq \frac{\operatorname{card}(\Gamma_{1,j})}{q^j}|\widehat{\varphi}_i(0)|^2\\ &\quad - 2|\widehat{\varphi}_i(0)|\frac{1}{q^j}\sum_{k\in\Gamma_{1,j}}\left|\int_{(A^j\boldsymbol{Q})^c-k}\varphi_i(x)dx\right|\\ &> (1-\varepsilon)|\widehat{\varphi}_i(0)|^2 - 2|\widehat{\varphi}_i(0)|\varepsilon\\ &> |\widehat{\varphi}_i(0)|^2 - \varepsilon. \qquad (3.24)\end{aligned}$$

Finally, from (3.23) y (3.24), we conclude that for $i = 1, ..., r$, $R_{1j} \to |\widehat{\varphi}_i(0)|^2$ when $j \to \infty$.

□

3.4. Proof of Theorem 3.1. The proof of this result, is a direct consequence of Propositions 3.2, 3.4 and 3.5.
Suppose that φ generates a MRA $(\boldsymbol{\mathcal{V}}_j)_{j\in\mathbb{Z}}$. Then Properties (P1)-(P5) of the multiresolution analysis are satisfied. Statement a) of the theorem is an immediate consequence of Property (P1) and Proposition 3.2. Using Properties (P4), (P1) and Proposition 3.5, then b) is verified.
Now, suppose that φ verifies a) and b) of the theorem. To prove that φ generates a MRA, define $\boldsymbol{\mathcal{V}}_0 = \overline{span}\{\varphi_i(\cdot - k)\}_{k\in\mathbb{Z}^n,\ i=1\ldots r}$ and $\boldsymbol{\mathcal{V}}_j = \{g(A^j x)/\ g \in \boldsymbol{\mathcal{V}}_0\}$. Then Property (P1) is a consequence of a) and Proposition 3.2. Properties (P2) is trivial due to the definition of $\boldsymbol{\mathcal{V}}_0$ and $\boldsymbol{\mathcal{V}}_j$, and (P5) is assumed. Property (P3) is a consequence of the hipothesis of orthonormal lattices translates and the localization property of each φ_i as was proved in Proposition 3.4. Finally from b) and Proposition 3.5, Property (P4) is satisfied. Hence, φ generates a multiresolution analysis with multiplicity r associated to the dilation matrix A and the lattice Γ.

□

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