Problems (Weibel)

(1) Let $\mathbb{Z}_{(p)}$ be the integers localized at p, and $\mathbb{Z}_p$ the p-adic integers. Show that the abelian categories of torsion modules in these rings are equivalent, and conclude that there is a Mayer–Vietoris sequence

$$\cdots \to K_*(\mathbb{Z}_{(p)}) \to K_*(\mathbb{Q}) \oplus K_*(\mathbb{Z}_p) \to K_*(\mathbb{Q}_p) \to K_{*-1}(\mathbb{Z}_{(p)}) \to \cdots$$

(2) Let R be a Dedekind ring (1-dimensional regular noetherian), such as the ring of integers in a number field or the coordinate ring of a smooth curve, with field of fractions F and maximal ideals m. Show that there is a long exact sequence

$$\cdots K_{*+1}(F) \to \oplus_m K_*(R/m) \to K_*(R) \to K_*(F) \to \cdots$$

ending in $F^\times \to \oplus_m \mathbb{Z} \to K_0(R) \to \mathbb{Z} \to 0$.

(3) When R is a principal ideal domain, the transfer maps $K_*(R/m) \to K_*(R)$ are zero. If all the residue fields R/m are finite (so that $K_{2n}(R/m) = 0$ for $n > 0$), deduce that $K_n(R) \cong K_n(F)$ for odd n and that for even n we have the exact sequence

$$0 \to K_n(R) \to K_n(F) \to \oplus_m K_{n-1}(R/m) \to 0.$$

(4) Given a ring R and a (central) element s, let $(\mathcal{A}, w)$ be the Waldhausen category of bounded chain complexes of fin gen projective R-modules, where weak equivalences are maps $w : A_* \to A'_*$ such that $H_*(A)[1/s] \to H_*(A')[1/s]$ is an isomorphism, and let $\mathcal{B}$ be the Waldhausen category of bounded chain complexes of fin gen projective $R[1/s]$-modules, where weak equivalences are quasi-isomorphisms, and let $\mathcal{B}'$ be the subcategory of chain complexes B_* such that $[B_*]$ is in the image of $K_0(R) \to K_0(R[1/s]$. Show that for every map $f : A_*[1/s] \to B_*$ in $\mathcal{B}'$ with A_* in $\mathcal{A}$ there is an A'_*, a map $A_* \to A'_*$ in $\mathcal{A}$ and a quasi-isomorphism $A'[1/s] \simeq B$ so that f is the composition $A[1/s] \to A'[1/s] \to B$.

This means that the hypotheses of the Approximation Theorem are satisfied, and hence that $K_*(\mathcal{A}) \cong K_*(\mathcal{B})$. By cofinality, $K_*(\mathcal{B}) \cong K_*(R[1/s])$ for $* > 0$ and $K_0(\mathcal{B})$ is a subgroup of $K_0(R[1/s]$.

(5) Recall that $K(R \text{ on } s)$ is the K-theory of the category of bounded complexes of fin gen projective R-modules whose homology is s-torsion. Conclude that there is an exact sequence

$$\cdots K_{*+1}(R[1/s]) \to K_*(R \text{ on } s) \to K_*(R) \to K_*(R[1/s])$$

ending in $K_0(R) \to K_0(R[1/s])$. It continues to negative values of $*$.