ELLIOTT'S CLASSIFICATION OF APPROXIMATELY FINITE C*-ALGEBRAS

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Problem set

*Throughout this problem set all C^*-algebras and *-homomorphisms are assumed to be unital.*

1. Let A be a C^*-algebra. An element $v \in A$ is said to be a *partial isometry* if $v^*v \in \mathcal{P}(A)$.
   (a) Let v be a partial isometry. Show that $v^*(1-vv^*)^2 v = 0$, and conclude that $v = vv^*v$.
   (b) Prove that the following statements are equivalent:
      i. v is a partial isometry.
      ii. $v = vv^*v$.
      iii. v^* is a partial isometry.
2. Let p and q be projections in a unital C^*-algebra A.
   (a) p and q are said to be *orthogonal* if $pq = 0$. Let p and q be orthogonal projections. Show that $p+q$ is a projection and that $[p+q] = [p] + [q]$.
   (b) p is said to be less than or equal to q if $pq = p$. Show that if $p \leq q$, then $q - p$ is a projection and $[q-p] = [q] - [p]$.
3. Let $\tau : M_k(\mathbb{C}) \longrightarrow \mathbb{C}$ be the trace map given by $\tau(M) = \sum_{i=1}^{k} M_{ii}$.
   (a) Let $p, q \in \mathcal{P}(M_n(\mathbb{C}))$. Show that $p \sim q$ if and only if $\tau(p) = \tau(q)$.
   (b) Show that $K_0\big(M_n(\mathbb{C})\big) \longrightarrow \mathbb{Z}$ is an isomorphism of groups.
   (c) Show that $M_n(\mathbb{C})$ has the cancellation property.
   (d) Let A and B be C^*-algebras with the cancellation property. Show that $A \oplus B$ has the cancellation property.
   (e) Show that AF-algebras have the cancellation property.
4. Let p and q be projections in a unital C^*-algebra A such that $\|p - q\| < 1$.

(a) Show that $2p-1$ is a unitary. Set $x = pq + (1-p)(1-q)$, and show that

$$\|1-x\| = \|(2p-1)(p-q)\| \leq \|(p-q)\| < 1.$$

Conclude that x is invertible. (Hint: if $\|a\| < 1$, then $\sum_{0}^{\infty} a^k = (1-a)^{-1}$.)

(b) Let $z = (x^*x)^{1/2}$. Show that z is invertible and that $u = xz^{-1}$ is a unitary.

(c) Show that $px = pq = xq$. Conclude that q commutes with x^*x and hence with z. (Use the following fact: if a is a normal element of a C^*-algebra A, and $b \in A$ is such that $ba = ab$, then $bf(a) = f(a)b$ for all $f \in$C(sp(a)).

(d) Show that $uq = pu$, and conclude that $p \sim q$.

5. Let A be a unital C^*-algebra, and let $a \in A$ be a self-adjoint element such that $\delta = \|a^2 - a\| < \frac{1}{4}$.

(a) Let $t \in \text{sp}(a)$. Show that $|t^2 - t| \leq \delta$, and conclude that

$$\text{sp}(a) \subseteq [-2\delta, 2\delta] \cup [1-2\delta, 1+2\delta].$$

(b) Let f be the characteristic function of the set $\{t \in \text{sp}(a) : t \geq 1-2\delta\}$. Show that $f \in C(\text{sp}(a))$, that $f(a) \in \mathcal{P}(A)$, and that $\|f(a) - a\| \leq 2\delta$.

6. Let $A = \overline{\bigcup_n A_n}$, where $\{A_n\}$ is an increasing sequence of C^*-algebras that contain 1_A. Show that $\big(K_0(A), K_0(j_n)\big)$ is the inductive limit of $\big(K_0(A_n), K_0(i_n)\big)$, where $i_n : A_n \longrightarrow A_{n+1}$ and $j_n : A_n \longrightarrow A$ are the inclusion maps.

7. Let p be a prime number, and let A_p be the direct limit of the sequence

$$M_p(\mathbb{C}) \xrightarrow{\phi_1} M_{p^2}(\mathbb{C}) \xrightarrow{\phi_2} M_{p^3}(\mathbb{C}) \xrightarrow{\phi_3} \cdots,$$

where $\phi_n(a) = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{pmatrix}$.

Use the normalized trace $T(P) = \frac{1}{n}\sum_{i=1}^{n} P_{ii}$ for $P \in M_n(\mathbb{C})$ to find an isomorphism I from $K_0(A_p)$ onto a subgroup of $\mathbb{Q}$. Find $I(K_0(A_p))^+$ and $I([1_{A_p}])$. Show that there is a unital *-isomorphism $\phi : A_p \longrightarrow A_q$ if only if $p = q$.