UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

Propiedades homotópicas de los complejos de p-subgrupos

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

Kevin Iván Piterman

Director de tesis: Elías Gabriel Minian
Consejero de estudios: Jonathan Ariel Barmak

Buenos Aires, Octubre de 2019

Propiedades homotópicas de los complejos de p-subgrupos

Resumen. En esta tesis se investigan las propiedades homotópicas de los posets de p-subgrupos de un grupo finito. Particularmente estudiamos los siguientes problemas: la *conjetura de Quillen* que relaciona la contractibilidad de estos posets con la existencia de p-subgrupos normales no triviales, la *conjetura de Webb* sobre los complejos (y posets) de órbitas, y el grupo fundamental de estos posets. Los métodos desarrollados en este trabajo combinan herramientas de la teoría de grupos finitos, la clasificación de grupos simples y sistemas de fusión, con herramientas topológicas y combinatorias.

A principios de los 70, D. Quillen relacionó la cohomología equivariante módulo p de los G-espacios con los p-subgrupos elementales abelianos de G. El poset $\mathcal{S}_p(G)$ de p-subgrupos no triviales de G fue introducido luego por K. Brown para estudiar la característica de Euler de grupos (no necesariamente finitos), que codifica la presencia de torsión. Unos años más tarde, Quillen introdujo el poset $\mathcal{A}_p(G)$ de p-subgrupos elementales abelianos no triviales de un grupo finito G y estudió las propiedades homotópicas de su complejo de orden asociado $\mathcal{K}(\mathcal{A}_p(G))$ en relación con las propiedades algebraicas p-locales de G. Así, Quillen probó que $\mathcal{K}(\mathcal{A}_p(G))$ y $\mathcal{K}(\mathcal{S}_p(G))$ son homotópicamente equivalentes y que si G posee un p-subgrupo normal no trivial entonces estos complejos son contráctiles. La vuelta a esto último es la bien conocida conjetura de Quillen, que actualmente permanece abierta. El resultado más avanzado en esta dirección se debe a M. Aschbacher y S.D. Smith, quienes establecieron la conjetura si $p > 5$ y los grupos no poseen ciertas componentes unitarias.

En esta tesis adoptamos el punto de vista de R.E. Stong de tratar a los posets $\mathcal{A}_p(G)$ y $\mathcal{S}_p(G)$ como espacios topológicos finitos. Con esta topología intrínseca, estos posets no son homotópicamente equivalentes y la conjetura de Quillen se puede reformular diciendo que si $\mathcal{S}_p(G)$ es homotópicamente trivial como espacio finito entonces es contráctil. En general, hay espacios finitos homotópicamente triviales pero no contráctiles (el teorema de Whitehead no es válido en espacios finitos). Respondimos a una pregunta de Stong mostrando que $\mathcal{A}_p(G)$ puede ser homotópicamente trivial pero no contráctil y describimos la contractibilidad del espacio finito $\mathcal{A}_p(G)$ en términos puramente algebraicos.

En este contexto estudiamos la conjetura de P. Webb que afirma que, en término de espacios finitos, los posets $\mathcal{A}_p(G)'/G$ y $\mathcal{S}_p(G)'/G$ son homotópicamente triviales. La conjetura original de Webb fue probada primero por P. Symonds. En general $\mathcal{S}_p(G)'/G$ puede no ser contráctil como espacio finito, pero $\mathcal{A}_p(G)'/G$ resultó ser contráctil en todos los ejemplos que calculamos, y conjeturamos que esto debe valer siempre (llamamos a esto la versión fuerte de la conjetura de Webb). En la tesis mostramos la validez de la versión fuerte de la conjetura en diversos casos, utilizando para esto herramientas de sistemas de fusión.

El grupo fundamental de los posets de p-subgrupos fue estudiado por varios matemáticos en las últimas tres décadas. Hasta el momento los trabajos más relevantes son los de M.

Aschbacher, quien probó condiciones algebraicas necesarias y suficientes para que $\mathcal{A}_p(G)$ sea simplemente conexo, módulo una conjetura sobre la cual hay considerable evidencia, y los trabajos de Ksontini quien investigó el grupo fundamental de estos posets cuando el grupo G es un grupo simétrico. En todos los casos estudiados los grupos resultaban siempre libres. En esta tesis probamos que el grupo fundamental de estos complejos es libre en *casi* todos los casos. En particular vimos que es libre para ciertas extensiones de grupos simples y para todos los grupos resolubles. En general, asumiendo la conjetura de Aschbacher, mostramos que $\pi_1(\mathcal{A}_p(G)) \cong \pi_1(\mathcal{A}_p(S_G)) * F$, donde F es un grupo libre, S_G es un cociente particular de G y $\pi_1(\mathcal{A}_p(S_G))$ es libre salvo quizás si S_G es casi simple. Además, vimos que $\pi_1(\mathcal{A}_3(\mathbb{A}_{10}))$ no es libre (acá $\mathbb{A}_{10}$ es el grupo alterno en 10 letras), mostrando que la obstrucción a que los complejos de p-subgrupos sean homotópicos a bouquet de esferas puede aparecer también en el π_1. Este es el primer ejemplo en la literatura de un poset de p-subgrupos con grupo fundamental no libre.

Por último, nos centramos en el estudio de la conjetura de Quillen. Demostramos que ésta es cierta si $\mathcal{K}(\mathcal{S}_p(G))$ admite un subcomplejo invariante de dimensión 2 y homotópicamente equivalente a él, probando así nuevos casos de la conjetura que no eran sabidos hasta el momento. También mostramos que la conjetura se puede estudiar bajo la suposición $O_{p'}(G) = 1$ (el subgrupo normal de G más grande de orden coprimo con p), extendiendo varios de los resultados conocidos de Aschbacher y Smith a todo primo p. Esto nos permite concluir que la conjetura es cierta si $\mathcal{K}(\mathcal{A}_p(G))$ tiene dimensión 3.

Palabras clave: p-subgrupos, espacios finitos, clasificación de grupos simples, sistemas de fusión, conjetura de Quillen.

Homotopy properties of the p-subgroup complexes

Abstract. In this thesis we investigate the homotopy properties of the p-subgroup posets of a finite group. Particularly, we study the following problems: *Quillen's conjecture*, which relates the contractibility of these posets with the existence of non-trivial normal p-subgroups, *Webb's conjecture*, on the orbit complexes (and posets), and the fundamental group of these posets. The methods developed in this work combine tools of the theory of finite groups, the classification of finite simple groups and fusion systems, with topological and combinatorial techniques.

At the beginning of the seventies, D. Quillen related the equivariant cohomology modulo p of G-spaces with the elementary abelian p-subgroups of G. The poset $\mathcal{S}_p(G)$ of non-trivial p-subgroups of G was introduced by K. Brown to study the Euler characteristic of groups (not necessary finite), which encodes the presence of torsion. Some years later, Quillen introduced the poset $\mathcal{A}_p(G)$ of non-trivial elementary abelian p-subgroups of a finite group G and studied the homotopy properties of its order complex $\mathcal{K}(\mathcal{A}_p(G))$ in relation with the p-local algebraic properties of G. Quillen proved that $\mathcal{K}(\mathcal{A}_p(G))$ and $\mathcal{K}(\mathcal{S}_p(G))$ are homotopy equivalent and that if G has a non-trivial normal p-subgroup then these complexes are contractible. The reciprocal to this last statement is the well-known Quillen's conjecture, which remains open. The most advanced result on this direction is due to M. Aschbacher and S.D. Smith, which established the conjecture if $p > 5$ and the groups do not have certain unitary components.

In this dissertation we adopt the viewpoint of R.E. Stong of handling the posets $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ as finite topological spaces. With this intrinsic topology, these posets are not homotopy equivalent and Quillen's conjecture can be reformulated by saying that if $\mathcal{S}_p(G)$ is a homotopically trivial finite space then it is contractible. In general, there are homotopically trivial finite spaces which are not contractible (Whitehead's theorem is no longer true in this context). We answer a question raised by Stong by showing that $\mathcal{A}_p(G)$ may be homotopically trivial but non-contractible, and describe the contractibility of the finite space $\mathcal{A}_p(G)$ in purely algebraic terms.

In this context we study Webb's conjecture which states that, in terms of finite spaces, the posets $\mathcal{A}_p(G)'/G$ and $\mathcal{S}_p(G)'/G$ are homotopically trivial. The original Webb's conjecture was proved first by P. Symonds. In general $\mathcal{S}_p(G)'/G$ may be non-contractible as a finite space, but $\mathcal{A}_p(G)'/G$ turned out to be contractible in all the examples that we computed, and we conjecture that this should always hold (we call this the strong version of Webb's conjecture). We prove some cases of the strong version of the conjecture by using tools of fusion systems.

The fundamental group of the posets of p-subgroups was studied by several mathematicians in the last decades. So far, the most relevant works are those of M. Aschbacher, who proved necessary and sufficient algebraic conditions for $\mathcal{A}_p(G)$ to be simply connected, modulo a conjecture for which there is considerable evidence, and the works of Ksontini who investigated

the fundamental group of these posets when G is the symmetric group. In all the cases studied, the groups turned out to be free. In this thesis we show that the fundamental group of these complexes is free in *almost* all cases. In particular we prove that it is free for certain extensions of simple groups and for any solvable group. In general, assuming Aschbacher's conjecture, we show that $\pi_1(\mathcal{A}_p(G)) \cong \pi_1(\mathcal{A}_p(S_G)) * F$, where F is a free group, S_G is a particular quotient of G and $\pi_1(\mathcal{A}_p(S_G))$ is free except perhaps if S_G is almost simple. Moreover, we prove that $\pi_1(\mathcal{A}_3(\mathbb{A}_{10}))$ is non-free (here, $\mathbb{A}_{10}$ is the alternating group in 10 letters), showing that the obstruction for the p-subgroup complexes to be homotopy equivalent to a bouquet of spheres can also rely on the π_1. This is the first example in the literature of a p-subgroup poset with non-free fundamental group.

Finally, we focus on the study of Quillen's conjecture. We prove that the conjecture holds if $\mathcal{K}(\mathcal{S}_p(G))$ admits an invariant 2-dimensional homotopy equivalent subcomplex, showing new cases of the conjecture. We also prove that the conjecture can be studied under the supposition $O_{p'}(G) = 1$ (the largest normal subgroup of G of order prime to p), extending some known results of Aschbacher and Smith to every prime p. This allows us to conclude that the conjecture holds if $\mathcal{K}(\mathcal{A}_p(G))$ has dimension 3.

Key words: p-subgroups, finite spaces, classification of finite simple groups, fusion systems, Quillen's conjecture.

Introducción

El objetivo principal de esta tesis es estudiar las propiedades homotópicas de los posets de p-subgrupos tanto desde el punto de vista de espacios finitos como desde el punto de vista clásico por medio de la topología de sus complejos de órdenes. Dado un grupo finito G y un primo p que divide a su orden, consideramos el poset $\mathcal{S}_p(G)$ de p-subgrupos no triviales de G y el subposet $\mathcal{A}_p(G)$ de p-subgrupos elementales abelianos no triviales de G.

El estudio de estos posets comenzó en la década del 70 con los artículos fundacionales de D. Quillen [Qui71], quien relacionó ciertas propiedades de la cohomología equivariante módulo p de los G-espacios con los p-subgrupos elementales abelianos de G. El grupo G actúa en estos posets vía conjugación en los p-subgrupos, y por lo tanto obtenemos G-espacios cuyas propiedades homotópicas están estrechamente ligadas con G. Por ejemplo, en [Web87] se relaciona la cohomología p-ádica de G con la de los grupos de isotropía de los símplices del complejo de orden $\mathcal{K}(\mathcal{S}_p(G))$, y el teorema de amplitud de Brown establece que la cohomología equivariante módulo p de $|\mathcal{K}(\mathcal{S}_p(G))|$ es isomorfa a la cohomología equivariante módulo p de G (ver [Bro94, Smi11]). Recordar que si X es un poset finito, su complejo de orden $\mathcal{K}(X)$ consiste de las cadenas no vacías de elementos de X. Si Y es un G-espacio entonces $EG\times_G Y$ es su construcción de Borel, y la cohomología equivariante de Y es la cohomología de su construcción de Borel. Cuando $\mathcal{K}(\mathcal{S}_p(G))$ es conexo, se tiene una fibración $|\mathcal{K}(\mathcal{S}_p(G))| \to EG\times_G |\mathcal{K}(\mathcal{S}_p(G))| \to BG$ que induce una sucesión exacta corta en los grupos fundamentales, mostrando que $\pi_1(EG\times_G |\mathcal{K}(\mathcal{S}_p(G))|)$ es en general un grupo infinito (ver Teorema 3.4.2).

Desde un punto de vista algebraico, la estructura como G-poset de $\mathcal{S}_p(G)$ guarda la información p-local de G, es decir, la estructura de los normalizadores de los p-subgrupos no triviales de G. Esto está fuertemente relacionado con la fusión del grupo. El estudio general de los sistemas de fusión y los grupos p-locales comenzó como generalización de esta idea para abstraerse de la estructura global del grupo y tratar de entender sus propiedades p-locales de una manera más sistemática: cómo son los morfismos de conjugación entre p-subgrupos de un p-subgrupo de Sylow fijo. Desde un punto de vista topológico, la estructura p-local del grupo codifica la misma información que la p-completación $BG^\wedge_p$ de su espacio clasificante BG. Más relaciones aparecen en la teoría de representación de grupos finitos. Ver [AKO11, Gro16, Qui78, Smi11, Web87].

En [Bro75], K. Brown trabajó con la parte racional de la característica de Euler de un grupo (no necesariamente finito), la cual guarda relación con la torsión del grupo. Introdujo el poset $\mathcal{S}_p(G)$ de p-subgrupos no triviales y mostró que, cuando G es finito, $\chi(\mathcal{S}_p(G))$ es 1 módulo $|G|_p$ (la potencia más grande de p que divide al orden de G). Esto es comúnmente denominado *Homological Sylow Theorem*.

Unos años más tarde, D. Quillen estudió más en profundidad las propiedades homotópicas de estos posets por medio de sus complejos de órdenes [Qui78]. Él introdujo el poset $\mathcal{A}_p(G)$ y mostró que la inclusión $\mathcal{K}(\mathcal{A}_p(G)) \subseteq \mathcal{K}(\mathcal{S}_p(G))$ es una equivalencia homotópica. También relacionó algunas propiedades homotópicas de estos complejos con propiedades algebraicas de G. Por ejemplo, la desconexión de $\mathcal{K}(\mathcal{S}_p(G))$ se traduce algebraicamente en la existencia de un subgrupo de G fuertemente p-embebido. En [Qui78] se muestra que si G posee un p-subgrupo normal no trivial entonces $\mathcal{K}(\mathcal{S}_p(G))$ es contráctil. La vuelta a esta proposición es la bien-conocida *conjetura de Quillen* [Qui78, Conjecture 2.9]. Quillen estableció la conjetura para grupos resolubles, grupos de p-rango 2 (es decir, $\mathcal{A}_p(G)$ tiene altura 1) y grupos finitos de tipo Lie en característica p (porque en este caso $\mathcal{K}(\mathcal{S}_p(G))$ es homotópico al Tits building del grupo). Actualmente la conjetura permanece abierta pero han habido importantes avances. El resultado más general se encuentra en el famoso artículo de M. Aschbacher y S.D. Smith [AS93]. Ellos utilizan fuertemente la clasificación de los grupos finitos simples para probar la conjetura si $p > 5$ y los grupos no poseen ciertas componentes unitarias. Ver también [AK90, HI88, PSV19, Rob88, Smi11].

En la década de los 80, R.E. Stong consideró los posets de p-subgrupos como espacios topológicos finitos por primera vez. Si X es un poset finito entonces posee una topología intrínseca cuyos abiertos son los downsets (o sea los conjuntos $U \subseteq X$ tales que si $x \in U$ e $y \leq x$ entonces $y \in U$). Esta construcción da lugar a un isomorfismo entre la categoría de posets finitos con funciones que preservan el orden y la categoría de espacios finitos T_0 con funciones continuas. Cuando X es un poset finito, también tenemos la topología de su complejo de orden $\mathcal{K}(X)$. La relación entre estas dos topologías está dada por el teorema de McCord que afirma que existe un equivalencia débil natural $\mu_X : |\mathcal{K}(X)| \to X$, es decir, una función continua que induce isomorfismos en todos los grupos de homotopía y de homología (ver [McC66]). Con la topología intrínseca de espacios finitos, un poset finito X homotópicamente trivial (todos sus grupos de homotopía, y en particular de homología, son triviales) podría no ser contráctil y, más en general, hay equivalencias débiles entre espacios finitos que no son equivalencias homotópicas. Es decir, el teorema de Whitehead no es válido en el contexto de espacios topológicos finitos. Ver [Ale37, Bar11a, Sto66] para más detalles. En [Sto84] Stong consideró los posets $\mathcal{A}_p(G)$ y $\mathcal{S}_p(G)$ como espacios topológicos finitos y probó que, como espacios finitos, no tienen el mismo tipo homotópico (aunque la inclusión $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ es una equivalencia débil por el teorema de McCord y el resultado de Quillen). Más aún, mostró que $\mathcal{S}_p(G)$ es contráctil como espacio finito si y solo si G posee un p-subgrupo normal no trivial. De esta manera, la conjetura de Quillen se puede reformular diciendo que si $\mathcal{S}_p(G)$ es un espacio finito homotópicamente trivial entonces es contráctil (como espacio finito). Como $\mathcal{A}_p(G)$ y $\mathcal{S}_p(G)$ no tienen el mismo tipo homotópico como espacios finitos en general, Stong preguntó si la misma reformulación de la conjetura de Quillen puede ser establecida en términos de $\mathcal{A}_p(G)$.

Nuestro estudio sobre los posets de p-subgrupos comenzó motivado por esta pregunta de Stong y los resultados obtenidos por J. Barmak relacionando los distintos tipos homotópicos de espacios finitos [Bar11a, Chapter 8]. En mi Tesis de Licenciatura [Pit16], respondí por la negativa a la pregunta de Stong exhibiendo un grupo G tal que para $p = 2$, el espacio finito $\mathcal{A}_p(G)$ es homotópicamente trivial pero no contráctil (ver Ejemplo 1.3.17). De esta manera, la conjetura de Quillen en términos de espacios finitos no significa lo mismo para $\mathcal{A}_p(G)$ y $\mathcal{S}_p(G)$. Más aún, como para $\mathcal{S}_p(G)$ hay una descripción puramente algebraica de lo que significa ser contráctil como espacio finito, hicimos lo mismo para el poset $\mathcal{A}_p(G)$ usando la noción de *homotopía en pasos*. Básicamente una homotopía entre funciones continuas de espacios finitos puede describirse combinatoriamente y uno puede definir una longitud $n \geq 0$ de la homotopía. De esta manera, decimos que un poset finito es *contráctil en n pasos* si existe una homotopía de longitud n entre la función identidad del poset y una función constante. Para el caso del poset $\mathcal{A}_p(G)$, esta longitud define un invariante algebraico que se traduce en la existencia de cierto p-subgrupo elemental abeliano de G. Esto permite describir la contractibilidad de $\mathcal{A}_p(G)$ en términos algebraicos (aunque para determinar estos subgrupos se necesita conocer parte de la combinatoria del poset $\mathcal{A}_p(G)$). Estos resultados pueden encontrarse en el artículo escrito en colaboración con E.G. Minian [MP18]. En el Capítulo 1 exhibimos algunos de estos resultados. También estudiamos estas preguntas en relación con otros posets de p-subgrupos que surgen en la literatura. Considere el poset $\mathcal{B}_p(G) = \{P \in \mathcal{S}_p(G) : P = O_p(N_G(P))\}$ de p-subgrupos radicales no triviales de G, introducido por Bouc y comúnmente llamado *poset de Bouc*. Aquí, $O_p(H)$ denota al p-subgrupo normal más grande de H, y $N_G(P)$ es el normalizador de P en G. Es sabido que $\mathcal{K}(\mathcal{B}_p(G)) \hookrightarrow \mathcal{K}(\mathcal{S}_p(G))$ es una equivalencia homotópica (ver [Bou84, TW91]). En términos de espacios finitos, probamos que $\mathcal{B}_p(G)$ puede tener distinto tipo homotópico a $\mathcal{S}_p(G)$ y a $\mathcal{A}_p(G)$ (aunque tienen el mismo tipo homotópico débil por el teorema de McCord). Se puede ver que si $O_p(G) \neq 1$ entonces $O_p(G)$ es un mínimo de $\mathcal{B}_p(G)$ y por lo tanto, $\mathcal{B}_p(G)$ es contráctil como espacio finito si y solo si G posee un p-subgrupo normal no trivial. Así, la conjetura de Quillen (en términos de espacios finitos) se reformula de la misma manera para $\mathcal{B}_p(G)$ que para $\mathcal{S}_p(G)$. En términos de homotopía simple equivariante de espacios finitos, mostramos que $\mathcal{S}_p(G) \searrow^{G} \mathcal{B}_p(G)$, $\mathcal{S}_p(G) \searrow^{G} \mathcal{A}_p(G)$ y $\mathcal{B}_p(G) \nearrow\!\!\!\!\searrow^{G} \mathcal{A}_p(G)$. También consideramos el complejo de Robinson $\mathcal{R}_p(G) \subseteq \mathcal{K}(\mathcal{S}_p(G))$, introducido por R. Kn´orr y G. Robinson [KR89], cuyos símplices son las cadenas de p-subgrupos $(P_0 < \ldots < P_n)$ de manera que P_i es normal en P_n para todo i. La inclusión $\mathcal{R}_p(G) \hookrightarrow \mathcal{K}(\mathcal{S}_p(G))$ es una equivalencia homotópica (ver [TW91]). A diferencia de los otros complejos de p-subgrupos, en general $\mathcal{R}_p(G)$ no proviene de un poset y por lo tanto consideramos su poset de caras de $\mathcal{R}_p(G)$ para estudiar sus propiedades homotópicas como espacio finito. Si K es un complejo simplicial finito, su poset de caras $\mathcal{X}(K)$ es el poset finito cuyos elementos son los símplices no vacíos de K ordenados por inlcusión. Si X es un poset finito, entonces $\mathcal{X}(\mathcal{K}(X)) = X'$ es la *primera subdivisión* de X. Notar que la primera subdivisión baricéntrica de K es $K' = \mathcal{K}(\mathcal{X}(K))$. En vista de estas observaciones, es más natural considerar las relaciones homotópicas entre el espacio finito $\mathcal{X}(\mathcal{R}_p(G))$ y los posets $\mathcal{S}_p(G)'$, $\mathcal{A}_p(G)'$ y $\mathcal{B}_p(G)'$. En general, $\mathcal{X}(\mathcal{R}_p(G))'$ no es homotópicamente equivalente a ninguno de los otros posets y puede ser homotópicamente trivial pero no contráctil (ver Ejemplo 1.3.17), pero $\mathcal{X}(\mathcal{R}_p(G)) \nearrow\!\!\searrow^{G} \mathcal{S}_p(G)$.

En el Capítulo 2 estudiamos la conjetura de P. Webb en términos de espacios finitos. En [Web87] se conjeturó que el espacio de órbitas $|\mathcal{K}(\mathcal{S}_p(G))|/G$ es contráctil. La conjetura de Webb fue probada primero por P. Symonds [Sym98] utilizando herramientas básicas de topología algebraica. Más tarde surgieron otras demostraciones y generalizaciones de este problema utilizando teoría de fusión de grupos y teoría de Morse de Bestvina-Brady (ver [Bux99, Gro16, Lib08, Lin09]). En general, la conjetura se prueba usando el complejo de Robinson. En [Pit16] probamos que, en términos de espacios finitos, la conjetura de Webb afirma que los posets de órbitas $\mathcal{S}_p(G)'/G$, $\mathcal{A}_p(G)'/G$, $\mathcal{B}_p(G)'/G$ y $\mathcal{X}(\mathcal{R}_p(G))/G$ son homotópicamente triviales. La acción de G en estos posets es la inducida por conjugación en las cadenas de p-subgrupos. De esto nace naturalmente la pregunta de si en verdad son contráctiles como espacios finitos. En el artículo [Pit19] mostramos que $\mathcal{S}_p(G)'/G$ y $\mathcal{B}_p(G)'/G$ pueden no ser contráctiles. Sin embargo, no sabemos si $\mathcal{A}_p(G)'/G$ es siempre contráctil o no. Hasta ahora las evidencias sugieren que siempre es contráctil y en [Pit19] conjeturamos que esta versión más fuerte de la conjetura de Webb debe valer. En [Pit19] se muestran varios casos para los que $\mathcal{A}_p(G)'/G$ es un espacio finito contráctil utilizando herramientas básicas de fusión de grupos finitos como el teorema de fusión de Alperin. En este capítulo recordamos los resultados de este artículo y probamos más casos de esta conjetura más fuerte. Los métodos que usamos dependen fuertemente de que estamos lidiando con cadenas de p-subgrupos abelianos y por lo tanto no pueden ser aplicados de la misma manera a los posets $\mathcal{S}_p(G)'/G$ y $\mathcal{B}_p(G)'/G$. En el siguiente teorema resumimos todos los casos en que probamos que $\mathcal{A}_p(G)'/G$ es contráctil como espacio finito. Notamos por $\mathrm{Syl}_p(G)$ al conjunto de p-subgrupos de Sylow de G, $|G|$ al orden de G, $\Omega_1(G) = \langle x \in G : x^p = 1\rangle$ y $Z(G)$ al centro de G. El p-rango de G es $m_p(G) = \max\{r : A \in \mathcal{A}_p(G), |A| = p^r\}$.

Theorem 2.5.12. *Sea G un grupo finito, $S \in \mathrm{Syl}_p(G)$ y $\Omega = \Omega_1(Z(S))$. En los siguientes casos $\mathcal{A}_p(G)'/G$ es un espacio finito contráctil.*

1. *$\Omega_1(S)$ es abeliano,*
2. *$\mathcal{A}_p(G)$ es contráctil,*
3. *$|G| = p^\alpha q$, con q primo,*

4. *Los p-subgrupos de Sylow de G se intersecan trivialmente,*

5. *La fusión de los p-subgrupos elementales abelianos de S está controlada por* $N_G(O)$ *para algún* $1 \neq O \leq \Omega_1(Z(\Omega_1(S)))$,

6. $m_p(G) - m_p(\Omega) \leq 1$,

7. $m_p(G) - m_p(\Omega) = 2$ *y* $m_p(G) \geq \log_p(|G|_p) - 1$,

8. $|G|_p \leq p^4$,

9. $G = M_{11}$, M_{12}, M_{22}, J_1, J_2, HS, *o p es impar y G es un grupo de Mathieu, un grupo de Janko,* He, O'N, *o* Ru, *o* $p = 5$ *y* $G = Co_1$,

10. $\mathcal{A}_p(G)$ *es disconexo.*

La dificultad para probar que $\mathcal{A}_p(G)'/G$ es contráctil si G es p-resoluble recae en el hecho de que $\mathcal{A}_p(G)$ puede ser homotópicamente trivial pero no contráctil como espacio finito. Es decir, $O_p(G) \neq 1$ no garantiza que $\mathcal{A}_p(G)$ sea contráctil. Esto no sucede con $\mathcal{S}_p(G)'/G$.

En el siguiente teorema resumimos los casos en los que hemos probado que el espacio finito $\mathcal{S}_p(G)'/G$ es contráctil. Recordar que $O_{p'}(G)$ es el subgrupo normal de G más grande de orden coprimo con p.

Theorem 2.5.11. *Sea G un grupo finito y* $S \in \mathrm{Syl}_p(G)$*. En los siguientes casos* $\mathcal{S}_p(G)'/G$ *es un espacio finito contráctil:*

1. $O_p(G/O_{p'}(G)) \neq 1$*; en particular esto vale para grupos p-constrained (y por lo tanto para p-resolubles) o si* $O_p(G) \neq 1$,

2. $\Omega_1(S)$ *es abeliano,*

3. $|G| = p^{\alpha}q$*, con q primo,*

4. *Los p-subgrupos de Sylow de G se intersecan trivialmente,*

5. *Existe* $1 \neq O \leq Z(S)$ *tal que* $N_G(O)$ *controla la G-fusión en S.*

El teorema anterior nos permite deducir que el grupo más chico para el cual $\mathcal{S}_p(G)'/G$ no es contráctil es el grupo simple $\mathrm{PSL}_2(7)$ para $p = 2$, y, más en general, si $\mathcal{S}_p(G)'/G$ no es contráctil entonces $G/O_{p'}(G)$ es una extensión de un producto directo de grupos simples por automorfismos externos del producto (ver Observación 2.5.9 y Proposición 2.5.10).

También probamos que el poset de órbitas $\mathcal{A}_p(G)/G$ (sin subdividir) es siempre contráctil como espacio finito.

Theorem 2.4.1. *El espacio finito* $\mathcal{A}_p(G)/G$ *es contráctil.*

Para $\mathcal{B}_p(G)/G$ y $\mathcal{S}_p(G)/G$ esto es inmediato porque tienen un máximo: la clase de conjugación de un p-subgrupo de Sylow. Sin embargo, $\mathcal{A}_p(G)/G$ no tiene un máximo en general pues $\mathcal{A}_p(G)$ podría tener elementos maximales que no sean todos conjugados entre sí e incluso de distintos órdenes.

En el Capítulo 3 nos ocupamos de estudiar aspectos generales sobre el tipo homotópico de los complejos de p-subgrupos, enfocándonos principalmente en su grupo fundamental. Por mucho tiempo se pensó que $\mathcal{K}(\mathcal{A}_p(G))$ tenía siempre el tipo homotópico de un bouquet de esferas (de dimensiones posiblemente distintas). De hecho, Quillen probó esto para ciertos grupos resolubles y grupos de tipo Lie [Qui78]. Más tarde, J. Pulkus y V. Welker dieron una descomposición wedge de $\mathcal{K}(\mathcal{A}_p(G))$ de donde se deduce que si G es resoluble entonces $\mathcal{K}(\mathcal{A}_p(G))$ es un bouquet de esferas si los intervalos superiores $\mathcal{K}(\mathcal{A}_p(G/O_{p'}(G))_{>A})$ lo son ($A \in \mathcal{A}_p(G/O_{p'}(G))$). Ver [PW00]. Sin embargo, J. Shareshian mostró que en general los complejos de p-subgrupos no tienen el tipo homotópico de un bouquet de esferas pues hay torsión en el segundo grupo de homología de $\mathcal{A}_3(\mathbb{A}_{13})$, donde $\mathbb{A}_{13}$ es el grupo alterno en 13 letras [Sha04]. No obstante, nada estaba dicho sobre el grupo fundamental, el cual debería ser libre si fueran homotópicos a bouquet de esferas. M. Aschbacher fue uno de los primeros matemáticos en investigar el grupo fundamental en búsqueda de condiciones puramente algebraicas necesarias y suficientes para que $\mathcal{A}_p(G)$ sea simplemente conexo [Asc93]. Así, Aschbacher probó que, módulo una conjetura sobre la cual hay considerable evidencia [Asc93, p. 2], si $m_p(G) \geq 3$ entonces $\mathcal{A}_p(G)$ es simplemente conexo si y solo si los links $\mathcal{A}_p(G)_{>A}$ son conexos para todo $|A| = p$, salvo quizás si $G/O_{p'}(G)$ es un grupo casi simple u otros dos grupos excepcionales que surgen de los grupos simples. Recordemos que G es denominado *casi simple* si existe un grupo simple no abeliano L tal que $L \leq G \leq \mathrm{Aut}(L)$. Tanto las excepciones como el uso de la conjetura corresponde a la parte del si del teorema. Siguiendo esta línea, K. Das estableció la simple conexión de $\mathcal{A}_p(G)$ para algunos grupos G de tipo Lie [Das95, Das98, Das00]. Luego R. Ksontini trabajó con los grupos simétricos $\mathbb{S}_n$, describiendo los pares (p,n) para los que $\mathcal{A}_p(\mathbb{S}_n)$ es simplemente conexo y mostrando que $\pi_1(\mathcal{A}_p(\mathbb{S}_n))$ es libre salvo quizás si $n = 3p$ o $3p+1$ (p impar) [Kso03, Kso04]. Poco más tarde, J. Shareshian probó que $\pi_1(\mathcal{A}_p(\mathbb{S}_n))$ es libre si $n = 3p$ [Sha04]. Hasta ese momento no se sabía qué sucedía con el caso $n = 3p+1$. Referimos a [Smi11, Section 9.3] para un resumen sobre las diferentes geometrías simplemente conexas para grupos simples, muy relacionadas con los complejos de p-subgrupos.

En esta tesis probamos que $\pi_1(\mathcal{A}_p(G))$ es un grupo libre en *casi* todos los casos. De hecho probamos que es libre para varias familias de grupos casi simples y para todos los grupos resolubles. Sin embargo, encontramos que $\pi_1(\mathcal{A}_3(\mathbb{A}_{10}))$ no es libre y que $\mathbb{A}_{10}$ (el grupo alterno en 10 letras) es el grupo más chico que da lugar a un poset de p-subgrupos con grupo fundamental no libre. Más aún, la homología de $\mathcal{A}_3(\mathbb{A}_{10})$ es libre abeliana. De esta manera, la obstrucción a que $\mathcal{K}(\mathcal{A}_p(G))$ sea un bouquet de esferas también puede recaer en el grupo fundamental y podría no ser detectada con la homología. Usualmente, el estudio de los problemas asociados a los complejos de p-subgrupos es por medio de su homología, y nuestro ejemplo muestra que en general esto no va a ser suficiente para determinar su tipo homotópico. Observar que $\mathcal{A}_3(\mathbb{A}_{10}) = \mathcal{A}_3(\mathbb{S}_{10})$ es uno de los casos excluidos en los cálculos de Ksontini y Shareshian. Estos resultados pueden encontrarse en nuestro artículo [MP19].

Sin embargo, nuestro ejemplo es bastante excepcional y hemos probado que en general el grupo fundamental sí es libre, y que las posibles excepciones surgen esencialmente de los grupos simples (como en el caso de $\mathbb{A}_{10}$). Para probar esto tuvimos que asumir la conjetura de Aschbacher [Asc93, p.2], sobre la cual, como mencionamos antes, hay considerable evidencia.

Theorem 3.4.2. *Sea G un grupo finito y p un primo que divide a $|G|$. Asuma la conjetura de Aschbacher. Entonces existe un isomorfismo $\pi_1(\mathcal{A}_p(G)) \cong \pi_1(\mathcal{A}_p(S_G)) * F$, donde F es un grupo libre y $S_G = \Omega_1(G)/O_{p'}(\Omega_1(G))$. Además, $\pi_1(\mathcal{A}_p(S_G))$ es un grupo libre (y por lo tanto $\pi_1(\mathcal{A}_p(G))$ es libre) excepto posiblemente si S_G es casi simple.*

Para la parte del Además no necesitamos asumir la conjetura. Para grupos p-resolubles $O_p(S_G) \neq 1$, o sea que $\mathcal{S}_p(S_G)$ es contráctil y así obtenemos grupo fundamental libre, módulo la conjetura de Aschbacher. Para grupos resolubles o para $p = 2$ la conjetura no es necesaria.

Corollary 3.0.1. *Asuma la conjetura de Aschbacher. Si $O_p(S_G) \neq 1$ entonces $\pi_1(\mathcal{A}_p(G))$ es libre. En particular, esto vale para grupos p-resolubles y, más en general, para grupos p-constrained.*

Corollary 3.0.3. *Si G es resoluble entonces $\pi_1(\mathcal{A}_p(G))$ es libre.*

Más aún, probamos que $\pi_1(\mathcal{A}_p(G))$ es libre para algunas familias de grupos casi simples G.

Theorem 3.0.4. *Supongamos que $L \leq G \leq \mathrm{Aut}(L)$, donde L es un grupo simple no abeliano. Entonces $\pi_1(\mathcal{A}_p(G))$ es un grupo libre en los siguientes casos:*

1. *$m_p(G) \leq 2$,*
2. *$\mathcal{A}_p(L)$ es disconexo,*
3. *$\mathcal{A}_p(L)$ es simplemente conexo,*
4. *L es simple de tipo Lie en característica p y $p \nmid (G : L)$ cuando L tiene rango Lie 2,*
5. *$p = 2$ y L tiene 2-subgrupos de Sylow abelianos,*
6. *$p = 2$ y $L = \mathbb{A}_n$ (el grupo alterno en n letras),*
7. *L es un grupo de Mathieu, J_1 o J_2,*
8. *$p \geq 3$ y $L = J_3$, McL, O'N.*

Por ejemplo, S.D. Smith comenta en [Smi11, p.290] que para muchos grupos simples L con $m_p(L) \geq 3$ es de esperarse que $\mathcal{A}_p(L)$ sea simplemente conexo.

Las técnicas utilizadas para probar estos resultados involucran herramientas básicas de topología algebraica combinadas con reducciones de espacios finitos y la clasificación de los grupos finitos simples. También usamos los resultados de Aschbacher [Asc93].

En el Capítulo 4 estudiamos en profundidad la conjetura de Quillen. Recordemos que la conjetura afirma que si $\mathcal{K}(\mathcal{A}_p(G))$ es contráctil entonces G posee un p-subgrupo normal no trivial, o sea $O_p(G) \neq 1$. En general se trabaja con una siguiente versión más fuerte de la conjetura.

Strong Quillen's conjecture. *Si $O_p(G) = 1$ entonces $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.*

En las primeras secciones del Capítulo 4 recordamos los resultados conocidos sobre la conjetura (fuerte) junto con breves ideas de sus demostraciones, incluyendo el resultado de M. Aschbacher y S.D. Smith [AS93, Main Theorem].

Luego, utilizando las ideas de B. Oliver y Y. Segev [OS02], probamos el siguiente teorema sobre la conjetura de Quillen.

Theorem 4.3.1 (con I. Sadofschi Costa y A. Viruel)**.** *Si K es un subcomplejo de dimensión 2, $\mathbb{Z}$-acíclico y G-invariante de $\mathcal{K}(\mathcal{S}_p(G))$ entonces $O_p(G) \neq 1$.*

Del cual deducimos inmediatamente:

Corollary 4.3.2. *Sea G un grupo finito. Supongamos que $\mathcal{K}(\mathcal{S}_p(G))$ admite un subcomplejo 2-dimensional y G-invariante homotópicamente equivalente a él mismo. Si $O_p(G) = 1$ entonces $\tilde{H}_*(\mathcal{S}_p(G), \mathbb{Z}) \neq 0$.*

Observar que el teorema no está enunciado para la versión fuerte de la conjetura.

Por ejemplo, el corolario anterior puede ser aplicado si $m_p(G) \leq 3$ o $\mathcal{B}_p(G)$ tiene altura 2.

Otro subposet que podemos considerar para aplicar el teorema anterior es el poset $\mathfrak{i}(\mathcal{A}_p(G))$ de intersecciones no triviales de p-subgrupos elementales abelianos maximales. Este subposet es G-invariante y homotópicamente equivalente a $\mathcal{A}_p(G)$ (como espacio finito), por lo que $\mathcal{K}(\mathfrak{i}(\mathcal{A}_p(G)))) \subseteq \mathcal{K}(\mathcal{S}_p(G))$ es una equivalencia homotópica. Ver también [Smi11] para una lista más extensa de complejos de p-subgrupos homotópicamente equivalentes a $\mathcal{K}(\mathcal{S}_p(G))$.

En aplicación de nuestro teorema, damos algunos ejemplos de grupos G los cuales no entran en las hipótesis de los teoremas de [AS93] pero que aún así verifican la conjetura por el Corolario 4.3.2. Mostramos que es posible construirse un subcomplejo de $\mathcal{K}(\mathcal{S}_p(G))$ homotópicamente equivalente, G-invariante y de dimensión 2. Estos resultados aparecen en el artículo escrito en colaboración con I. Sadofschi Costa y A. Viruel [PSV19].

Culminamos este capítulo mostrando que es posible estudiar la conjetura fuerte de Quillen bajo la suposición $O_{p'}(G) = 1$. En [AS93, Proposition 1.6], se muestra que esta suposición es posible provisto de que $p > 5$. Utilizando técnicas de espacios finitos y el caso p-resoluble de la conjetura de Quillen, probamos que esta reducción es posible para todo primo p. Precisamente, probamos el siguiente teorema.

Theorem 4.5.1. *Sea G un grupo finito tal que $O_p(G) = 1$, $\tilde{H}_*(\mathcal{A}_p(G),\mathbb{Q}) = 0$ y sus subgrupos propios satisfacen la conjetura fuerte de Quillen. Entonces $O_{p'}(G) = 1$. En particular, un contraejemplo minimal G a la conjetura fuerte de Quillen satisface $O_{p'}(G) = 1$.*

Este teorema no solo es interesante por la reducción que nos permite hacer, sino también por el método de su demostración. El uso de la clasificación de los grupos finitos simples en la demostración de este teorema es considerablemente menor que en la del resultado más débil [AS93, Proposition 1.6]. De hecho solo la usamos para invocar el caso p-resoluble de la conjetura, dentro del cual el uso de la clasificación es solo para la estructura de los automorfismos externos de los grupos simples.

La demostración de nuestro teorema también provee una técnica para encontrar ciclos no triviales en la homología de $\mathcal{A}_p(G)$, generalizando la idea original de [AS93, Lemma 0.27].

Aplicando los Teoremas 3.4.2 y 4.5.1, y Corolario 4.3.2, obtenemos los siguientes corolarios.

Corollary 4.5.13. *Si las subgrupos propios de G satisfacen la conjetura fuerte de Quillen pero $O_p(G) = 1$ y $\tilde{H}_*(\mathcal{A}_p(G),\mathbb{Q}) = 0$ entonces $\mathcal{K}(\mathcal{S}_p(G))$ no tiene subcomplejos de dimensión 2, G-invariantes y homotópicamente equivalentes. En particular $m_p(G) \geq 4$.*

Notar que este resultado es una ligera mejora a coeficientes racionales del Corolario 4.3.2.

Corollary 4.5.14. *La conjetura fuerte de Quillen vale para grupos de p-rango a lo sumo 3.*

Como aplicación final de nuestros métodos y resultados, deducimos la conjetura fuerte para grupos de p-rango 4.

Theorem 4.6.8. *La conjetura fuerte de Quillen vale para grupos de p-rango a lo sumo 4.*

Estos resultados serán parte de un nuevo artículo más general sobre la conjetura de Quillen, que actualmente está en preparación.

Muchos de los ejemplos que presentamos en esta tesis fueron calculados en GAP [GAP18] con un paquete de posets desarrollado en colaboración con X. Fernández e I. Sadofschi Costa [FPSC19].

Introduction

The main objective of this thesis is to study the homotopy properties of the posets of p-subgroups both from the point of view of finite topological spaces and from the classical viewpoint by means of the topology of their order complexes. Given a finite group G and a prime p dividing its order, we consider the poset $\mathcal{S}_p(G)$ of nontrivial p-subgroups of G and the poset $\mathcal{A}_p(G)$ of nontrivial elementary abelian p-subgroups of G.

The study of these posets began at the seventies with the foundational articles of D. Quillen [Qui71], who related certain properties of the modulo p equivariant cohomology of G-spaces with the elementary abelian p-subgroups of G. The group G acts on these posets via conjugation of the p-subgroups, and therefore we obtain G-spaces whose homotopy properties are closely related with G. For example, in [Web87] the p-adic cohomology of G is related with that of the isotropy groups of the simplices of the order complex $\mathcal{K}(\mathcal{S}_p(G))$, and Brown's ampleness theorem states that the modulo p equivariant cohomology of $|\mathcal{K}(\mathcal{S}_p(G))|$ is isomorphic to the modulo p cohomology of G (see [Bro94, Smi11]). Recall that if X is a finite poset, its order complex $\mathcal{K}(X)$ consists of the nonempty chains of elements of X. If Y is a G-space then $EG\times_G Y$ is its Borel construction, and the equivariant cohomology of Y is the cohomology of the Borel construction. When $\mathcal{K}(\mathcal{S}_p(G))$ is connected, we have a fibration $|\mathcal{K}(\mathcal{S}_p(G))|\to EG\times_G|\mathcal{K}(\mathcal{S}_p(G))|\to BG$ which induces a short exact sequence between the fundamental groups, showing that $\pi_1(EG\times_G|\mathcal{K}(\mathcal{S}_p(G))|)$ is in general an infinite group (see Theorem 3.4.2).

From an algebraic point of view, the structure of $\mathcal{S}_p(G)$ as a G-poset keeps the p-local information of G, that is, the structure of the normalizers of the nontrivial p-subgroups of G. This is strongly related with the fusion of the group. The general study of the fusion systems and the p-local groups began as a generalization of this idea to get abstracted from the global structure of the group and try to understand the p-local properties in a more systematic way: how the conjugation morphisms between p-subgroups of a fixed Sylow p-subgroup are. From a topological point of view, the p-local structure of the group encodes the same information as the p-completion $BG^\wedge_p$ of its classifying space BG. More relations appears in representation theory of finite groups. See [AKO11, Gro16, Qui78, Smi11, Web87].

In [Bro75], K. Brown worked with the rational part of the Euler characteristic of a group (not necessarily finite), which keeps relation with the torsion of the group. He introduced the poset $\mathcal{S}_p(G)$ of nontrivial p-subgroups and showed that, when G is finite, $\chi(\mathcal{S}_p(G))$ is 1 modulo $|G|_p$ (the greatest power of p dividing the order of G). This is usually called the *Homological Sylow Theorem*.

A few years later, D. Quillen studied in more depth the homotopy properties of these posets by means of their order complexes [Qui78]. He introduced the poset $\mathcal{A}_p(G)$ and showed that the inclusion $\mathcal{K}(\mathcal{A}_p(G)) \subseteq \mathcal{K}(\mathcal{S}_p(G))$ is a homotopy equivalence. He also related some homotopy properties of these complexes with algebraic properties of G. For example, the disconnectedness of $\mathcal{K}(\mathcal{S}_p(G))$ translates algebraically into the existence of a strongly p-embedded subgroup in G [Qui78, Proposition 5.2]. In [Qui78] it is shown that if G has a nontrivial normal p-subgroup then $\mathcal{K}(\mathcal{S}_p(G))$ is contractible. The converse of this proposition is the well-known *Quillen's conjecture* [Qui78, Conjecture 2.9]. Quillen established the conjecture for solvable groups, groups of p-rank 2 (i.e. $\mathcal{A}_p(G)$ has height 1) and finite groups of Lie type in characteristic p (because in this case $\mathcal{K}(\mathcal{S}_p(G))$ has the homotopy type of the Tits building of G). The conjecture remains open so far but there have been important advances. The most general result can be found in the famous article of M. Aschbacher and S.D. Smith [AS93]. They strongly use the Classification of Finite Simple Groups to establish the conjecture if $p > 5$ and the groups do not have certain unitary components. See also [AK90, HI88, PSV19, Rob88, Smi11].

In the eighties, R.E. Stong considered the posets of p-subgroups as finite topological spaces for the first time. If X is a finite poset, then it has an intrinsic topology whose open sets are the downsets (i.e. the subsets $U \subseteq X$ such that if $x \in U$ and $y \leq x$ then $y \in U$). This construction gives rise to an isomorphism between the category of finite posets with order preserving maps and the category of finite T_0-spaces with continuous maps. When X is a finite poset, we also have the topology of its order complex $\mathcal{K}(X)$. The relation between these two topologies is given by McCord's Theorem which states that there exists a natural weak equivalence $\mu_X : |\mathcal{K}(X)| \to X$, i.e. a continuous map inducing isomorphisms in all homotopy groups (and homology groups) (see [McC66]). With the intrinsic topology of finite spaces, a homotopically trivial finite poset X (all its homotopy groups, and in particular homology groups, are trivial) could be non-contractible and, more generally, there are weak equivalences between finite spaces which are not homotopy equivalences. That is, Whitehead's theorem is no longer true in the context of finite spaces. See [Ale37, Bar11a, Sto66] for more details. In [Sto84] Stong considered the posets $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ as finite spaces and proved that, as finite spaces, they do not have the same homotopy type (but the inclusion $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ is a weak equivalence by McCord's theorem and Quillen's results). Moreover, Stong showed that $\mathcal{S}_p(G)$ is a contractible finite space if and only if G has a nontrivial normal p-subgroup. Hence, Quillen's conjecture can be restated by saying that if $\mathcal{S}_p(G)$ is a homotopically trivial finite space then it is contractible (as finite space). Since $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ do not have the same homotopy type in general, Stong asked whether the same reformulation of Quillen's conjecture can be stated in terms of $\mathcal{A}_p(G)$.

We began the study of the posets of p-subgroups motivated by Stong's question and the results obtained by J. Barmak relating the different homotopy types of finite spaces [Bar11a, Chapter 8]. In my Undergraduate Thesis [Pit16], I answered Stong's question by the negative by exhibiting a finite group G such that for $p=2$, the finite space $\mathcal{A}_p(G)$ is homotopically trivial but non-contractible (see Example 1.3.17). Therefore, Quillen's conjecture in terms of finite spaces does not mean the same for $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$. Further, since the contractibility of $\mathcal{S}_p(G)$ is described in purely algebraic terms, we did the same for the poset $\mathcal{A}_p(G)$ by using the notion of *homotopy in steps*. Basically, a homotopy between continuous functions of finite spaces can be described in combinatorial terms and one can define the length $n \geq 0$ of the homotopy. In this way, we say that a finite poset is *contractible in n steps* if there exists a homotopy of length n between the identity map of the poset and a constant map. For the poset $\mathcal{A}_p(G)$, this length defines an algebraic invariant that translates into the existence of certain elementary abelian p-subgroup of G. This allows to describe the contractibility of $\mathcal{A}_p(G)$ in algebraic terms (but some of the combinatorial of the poset $\mathcal{A}_p(G)$ is needed to determine these subgroups). These results can be found in the paper written in collaboration with E.G. Minian [MP18]. In Chapter 1 we exhibit some of these results. We also study these questions in relation with other posets of p-subgroups that appear in the literature. Consider the poset $\mathcal{B}_p(G) = \{P \in \mathcal{S}_p(G) : P = O_p(N_G(P))\}$ of nontrivial radical p-subgroups of G, introduced by Bouc and commonly called *Bouc poset*. Here, $O_p(H)$ denotes the largest normal p-subgroup of H, and $N_G(P)$ is the normalizer of P in G. It is known that $\mathcal{K}(\mathcal{B}_p(G)) \hookrightarrow \mathcal{K}(\mathcal{S}_p(G))$ is a homotopy equivalence (see [Bou84, TW91]). In terms of finite spaces, we proved that $\mathcal{B}_p(G)$ may not be homotopy equivalent to $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ (although they have the same weak homotopy type by McCord's theorem). It can be shown that if $O_p(G) \neq 1$ then $O_p(G)$ is a minimum of $\mathcal{B}_p(G)$ and hence, $\mathcal{B}_p(G)$ is a contractible finite space if and only if G has a nontrivial normal p-subgroup. Therefore, Quillen's conjecture (in terms of finite spaces) reformulates in the same way for $\mathcal{B}_p(G)$ as for $\mathcal{S}_p(G)$. In terms of equivariant simple homotopy of finite space, we show that $\mathcal{S}_p(G) \searrow\!\!\searrow^G \mathcal{B}_p(G)$, $\mathcal{S}_p(G) \searrow\!\!\searrow^G \mathcal{A}_p(G)$ and $\mathcal{B}_p(G) \diagup\!\!\!\!\searrow^G \mathcal{A}_p(G)$. We also consider Robinson complex $\mathcal{R}_p(G) \subseteq \mathcal{K}(\mathcal{S}_p(G))$, introduced by R. Knörr and G. Robinson [KR89], whose simplices are the chains of p-subgroups $(P_0 < \ldots < P_n)$ such that P_i is normal in P_n for all i. The inclusion $\mathcal{R}_p(G) \hookrightarrow \mathcal{K}(\mathcal{S}_p(G))$ is a homotopy equivalence (see [TW91]). Unlike the other p-subgroup complexes, the complex $\mathcal{R}_p(G)$ does not come from a poset, and therefore we consider its face poset to study its homotopy properties as finite space. If K is a finite simplicial complex, its face poset $\mathcal{X}(K)$ is the finite poset whose elements are the nonempty simplices of K ordered by inclusion. If X is a finite poset, then $\mathcal{X}(\mathcal{K}(X)) = X'$ is the *first subdivision* of X. Note that the first barycentric subdivision of K is $K' = \mathcal{K}(\mathcal{X}(K))$. In light of these observations, it is more natural to consider the homotopy relations between the finite space $\mathcal{X}(\mathcal{R}_p(G))$ and the posets $\mathcal{S}_p(G)'$, $\mathcal{A}_p(G)'$ and $\mathcal{B}_p(G)'$. In general, $\mathcal{X}(\mathcal{R}_p(G))$ is not homotopy equivalent to any of the previous posets and it can be homotopically trivial and non-contractible (see Example 1.3.17), but $\mathcal{X}(\mathcal{R}_p(G)) \nearrow\!\!\searrow^{G} \mathcal{S}_p(G)$.

In Chapter 2 we study P. Webb's conjecture in terms of finite spaces. In [Web87] it was conjectured that the orbit space $|\mathcal{K}(\mathcal{S}_p(G))|/G$ is contractible. Webb's conjecture was proved first by P. Symonds [Sym98] by using basic tools of algebraic topology. Later, other proofs and generalizations of this problem arose by using fusion theory of groups and Bestvina-Brady approach to Morse theory (see [Bux99, Gro16, Lib08, Lin09]). In general, the conjecture is proved by using Robinson complex. In [Pit16] we proved that, in terms of finite spaces, Webb's conjecture asserts that the orbit posets $\mathcal{S}_p(G)'/G$, $\mathcal{A}_p(G)'/G$, $\mathcal{B}_p(G)'/G$ and $\mathcal{X}(\mathcal{R}_p(G))/G$ are homotopically trivial. The action of G on these posets is the induced by conjugation on the chains of p-subgroups. From this observation it is natural to ask if they are in fact contractible as finite spaces. In [Pit19] we showed that $\mathcal{S}_p(G)'/G$ and $\mathcal{B}_p(G)'/G$ may be non-contractible. However, we do not know if $\mathcal{A}_p(G)'/G$ is always contractible or not. So far, the evidences suggest that it is always contractible and in [Pit19] we conjecture that this stronger version of Webb's conjecture should hold. In [Pit19] several cases for which $\mathcal{A}_p(G)'/G$ is a contractible finite space are shown, by using basic tools of fusion in finite groups such as Alperin's fusion theorem. In this chapter we recall the results of this article and prove more cases of this stronger conjecture. The methods that we use deeply depend on the fact that we are dealing with chains of abelian p-subgroups and therefore, they cannot be carry out in the same way for the posets $\mathcal{S}_p(G)'/G$ and $\mathcal{B}_p(G)'/G$. In the following theorem we summarize all the cases that we have shown that $\mathcal{A}_p(G)'/G$ is a contractible finite space. Denote by $\mathrm{Syl}_p(G)$ the set of Sylow p-subgroups of G, $|G|$ the order of G, $\Omega_1(G) = \langle x \in G : x^p = 1\rangle$ and $Z(G)$ the center of G. The p-rank of G is $m_p(G) = \max\{r : A \in \mathcal{A}_p(G), |A| = p^r\}$.

Theorem 2.5.12. *Let G be a finite group, $S \in \mathrm{Syl}_p(G)$ and $\Omega = \Omega_1(Z(S))$. In the following cases $\mathcal{A}_p(G)'/G$ is a contractible finite space.*

1. *$\Omega_1(S)$ is abelian,*
2. *$\mathcal{A}_p(G)$ is contractible,*
3. *$|G| = p^\alpha q$, with q prime,*
4. *The Sylow p-subgroups of G intersect trivially,*
5. *The fusion of elementary abelian p-subgroups of S is controlled by $N_G(O)$ for some $1 \neq O \leq \Omega_1(Z(\Omega_1(S)))$,*
6. *$m_p(G) - m_p(\Omega) \leq 1$,*
7. *$m_p(G) - m_p(\Omega) = 2$ and $m_p(G) \geq r_p(G) - 1$,*

8. *$r_p(G) \leq 4$,*

9. *$G = M_{11}$, M_{12}, M_{22}, J_1, J_2,* HS*, or p is odd and G is any Mathieu group, Janko group,* He*,* O'N*, or* Ru*, or $p = 5$ and $G = Co_1$,*

10. *$\mathcal{A}_p(G)$ is disconnected.*

The difficulty for showing that $\mathcal{A}_p(G)'/G$ is contractible if G is a p-solvable group relies on the fact that $\mathcal{A}_p(G)$ may be a homotopically trivial but non-contractible finite space. That is, $O_p(G) \neq 1$ does not guarantee that $\mathcal{A}_p(G)$ is contractible. This does not happen with $\mathcal{S}_p(G)'/G$.

In the following theorem we summarize the cases for which we have proved that the finite space $\mathcal{S}_p(G)'/G$ is contractible. Recall that $O_{p'}(G)$ is the largest normal subgroup of G of order prime to p.

Theorem 2.5.11. *Let G be a finite group and $S \in \mathrm{Syl}_p(G)$. In the following cases $\mathcal{S}_p(G)'/G$ is a contractible finite space:*

1. *$O_p(G/O_{p'}(G)) \neq 1$; in particular it holds for p-constrained groups (and therefore for p-solvable groups) or if $O_p(G) \neq 1$,*

2. *$\Omega_1(S)$ is abelian,*

3. *$|G| = p^\alpha q$, with q prime,*

4. *The Sylow p-subgroups of G intersect trivially,*

5. *There exists $1 \neq O \leq Z(S)$ such that $N_G(O)$ controls G-fusion in S.*

From the above theorem we deduce that the smallest group for which $\mathcal{S}_p(G)'/G$ is non-contractible is the simple group $\mathrm{PSL}_2(7)$ with $p = 2$, and, more general, if $\mathcal{S}_p(G)'/G$ is non-contractible then $G/O_{p'}(G)$ is an extension of a direct product of simple groups by outer automorphisms of the product (see Remark 2.5.9 and Proposition 2.5.10).

We also prove that the orbit poset $\mathcal{A}_p(G)/G$ (without subdividing) is always contractible as finite space.

Theorem 2.4.1. *The finite space $\mathcal{A}_p(G)/G$ is contractible.*

For $\mathcal{B}_p(G)/G$ and $\mathcal{S}_p(G)/G$ this is immediate since they have a maximum: the conjugation class of a Sylow p-subgroup. However, $\mathcal{A}_p(G)/G$ may have no maximum in general since $\mathcal{A}_p(G)$ may have non-conjugate maximal elements and even of different orders.

In Chapter 3 we deal with general aspects of the homotopy type of the p-subgroup complexes, focusing primarily on their fundamental group. For a long time it was believed that $\mathcal{K}(\mathcal{A}_p(G))$ always had the homotopy type of a bouquet of spheres (of possibly different dimensions). In fact, Quillen proved this for some classes of solvable groups and groups of Lie type [Qui78]. Later, J. Pulkus and V. Welker gave a wedge decomposition of $\mathcal{K}(\mathcal{A}_p(G))$ from which it is deduced that if G is solvable then $\mathcal{K}(\mathcal{A}_p(G))$ is a bouquet of spheres if the upper intervals $\mathcal{K}(\mathcal{A}_p(G/O_{p'}(G))_{>A})$ are $(A \in \mathcal{A}_p(G/O_{p'}(G)))$. See [PW00]. However, J. Shareshian showed that in general the p-subgroup complexes do not have the homotopy type of a bouquet of spheres since there is torsion in the second homology group of $\mathcal{A}_3(\mathbb{A}_{13})$, where $\mathbb{A}_{13}$ is the alternating group on 13 letters [Sha04]. Nevertheless, nothing was said about the fundamental group, which should be free if they were homotopic to a bouquet of spheres. M. Aschbacher was one of the first mathematicians who studied the fundamental group in the search of necessary and sufficient purely algebraic conditions for $\mathcal{A}_p(G)$ to be simply connected [Asc93]. Thus, Aschbacher proved that, modulo a conjecture for which there is a considerable evidence [Asc93, p. 2], if $m_p(G) \geq 3$ then $\mathcal{A}_p(G)$ is simply connected if and only if the links $\mathcal{A}_p(G)_{>A}$ are connected for all $|A| = p$, except perhaps if $G/O_{p'}(G)$ is an almost simple group or another two exceptional groups arising from simple groups. Recall that G is termed *almost simple* if there exists a non-abelian simple group L such that $L \leq G \leq \mathrm{Aut}(L)$. Both the exceptions and the use of the conjecture corresponds to the “if” part of the theorem. Following this line, K. Das established simple connectivity of $\mathcal{A}_p(G)$ for some groups G of Lie type [Das95, Das98, Das00]. Later, R. Ksontini worked with the symmetric groups $\mathbb{S}_n$, describing the pairs (p,n) for which $\mathcal{A}_p(\mathbb{S}_n)$ is simply connected and showing that $\pi_1(\mathcal{A}_p(\mathbb{S}_n))$ is a free group in the remaining cases except perhaps if $n = 3p$ or $n = 3p+1$ (p odd) [Kso03, Kso04]. Shortly after, J. Shareshian proved that $\pi_1(\mathcal{A}_p(\mathbb{S}_n))$ is a free group if $n = 3p$ [Sha04]. Until that moment it was not known the case $n = 3p+1$. We refer to [Smi11, Section 9.3] for a summary on different simply connected geometries for simple groups, closely related with the p-subgroup complexes.

In this thesis we prove that $\pi_1(\mathcal{A}_p(G))$ is a free group in *almost* all cases. In fact, we show that it is a free group for various families of almost simple groups and for every solvable group. However, we found that $\pi_1(\mathcal{A}_3(\mathbb{A}_{10}))$ is not free and that $\mathbb{A}_{10}$ (the alternating group in 10 letters) is the smallest group giving rise a p-subgroup complex with non-free fundamental group. Moreover, the homology of $\mathcal{A}_3(\mathbb{A}_{10})$ is free abelian. In this way, the obstruction for $\mathcal{K}(\mathcal{A}_p(G))$ to be a bouquet of spheres can also rely on its fundamental group and may not be detected with the homology. Usually, the study of the problems associated to the p-subgroup complexes is by means of their homology, and our example shows that in general it will not be sufficient to determine their homotopy type. Note that $\mathcal{A}_3(\mathbb{A}_{10}) = \mathcal{A}_3(\mathbb{S}_{10})$ is one of the case excluded in the calculations of Ksontini and Shareshian. These results can be found in our article [MP19].

Nevertheless, our example is rather exceptional and we have proved that in general the fundamental group is free, and that the possible exceptions arise essentially from simple groups (like in the case of $\mathbb{A}_{10}$). In order to prove this, we had to assume Aschbacher's conjecture [Asc93, p.2], for which, as we mentioned before, there is considerable evidence.

Theorem 3.4.2. *Let G be a finite group and p a prime number dividing $|G|$. Assume that Aschbacher's conjecture holds. Then there is an isomorphism $\pi_1(\mathcal{A}_p(G)) \cong \pi_1(\mathcal{A}_p(S_G)) * F$, where F is a free group and $S_G = \Omega_1(G)/O_{p'}(\Omega_1(G))$. Moreover, $\pi_1(\mathcal{A}_p(\mathcal{S}_G))$ is a free group (and therefore $\pi_1(\mathcal{A}_p(G))$ is free) except possible if S_G is almost simple.*

For the "Moreover" part we do not need to assume the conjecture. For p-solvable groups $O_p(S_G) \neq 1$, so $\mathcal{S}_p(S_G)$ is contractible and hence we obtain free fundamental group, modulo Aschbacher's conjecture. For solvable groups or $p = 2$ the conjecture is not needed.

Corollary 3.0.1. *Assume that Aschbacher's conjecture holds. If $O_p(S_G) \neq 1$, then $\pi_1(\mathcal{A}_p(G))$ is free. In particular, this holds for p-solvable groups and, more generally, for p-constrained groups.*

Corollary 3.0.3. *If G is solvable then $\pi_1(\mathcal{A}_p(G))$ is a free group.*

Moreover, we proved that $\pi_1(\mathcal{A}_p(G))$ is free for some families of almost simple groups G.

Theorem 3.0.4. *Suppose that $L \leq G \leq \mathrm{Aut}(L)$, with L a simple group. Then $\pi_1(\mathcal{A}_p(G))$ is a free group in the following cases:*

1. *$m_p(G) \leq 2$,*
2. *$\mathcal{A}_p(L)$ is disconnected,*
3. *$\mathcal{A}_p(L)$ is simply connected,*
4. *L is simple of Lie type in characteristic p and $p \nmid (G : L)$ when L has Lie rank 2,*
5. *$p = 2$ and L has abelian Sylow 2-subgroups,*
6. *$p = 2$ and $L = \mathbb{A}_n$ (the alternating group),*
7. *L is a Mathieu group, J_1 or J_2,*
8. *$p \geq 3$ and $L = J_3$,* McL, O'N.

For example, S.D. Smith mentions in [Smi11, p.290] that in general $\mathcal{A}_p(L)$ is expected to be simply connected for simple groups L with $m_p(L) \geq 3$.

The techniques used to prove these results involves basic tools of algebraic topology combined with reductions of finite spaces and the Classification of the Finite Simple Groups. We have also used Aschbacher's results [Asc93].

In Chapter 4 we focus on the study of Quillen's conjecture. Recall that the conjecture says that if $\mathcal{K}(\mathcal{A}_p(G))$ is contractible then G has a nontrivial normal p-subgroup, that is $O_p(G) \neq 1$. In general a stronger version of the conjecture is considered.

Strong Quillen's conjecture. *If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.*

In the first sections of Chapter 4 we recall the known results on the (strong) conjecture together with brief ideas of their proofs, including the result of M. Aschbacher and S.D. Smith [AS93, Main Theorem].

Then, by using the ideas of B. Oliver and Y. Segev [OS02], we prove the following theorem on Quillen's conjecture.

Theorem 4.3.1 (with I. Sadofschi Costa and A. Viruel)**.** *If K is a $\mathbb{Z}$-acyclic and 2-dimensional G-invariant subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$, then $O_p(G) \neq 1$.*

From which we immediately deduce:

Corollary 4.3.2. *Let G be a finite group. Suppose that $\mathcal{K}(\mathcal{S}_p(G))$ admits a 2-dimensional and G-invariant subcomplex homotopy equivalent to itself. If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{S}_p(G), \mathbb{Z}) \neq 0$.*

Note that the theorem is not stated for the strong version of the conjecture.

For example, the above corollary can be applied if $m_p(G) \leq 3$ or $\mathcal{B}_p(G)$ has height 2.

Another useful subposet we can consider to apply the above theorem is the poset $\mathfrak{i}(\mathcal{A}_p(G))$ of nontrivial intersections of maximal elementary abelian p-subgroups. This subposet is G-invariant and homotopy equivalent to $\mathcal{A}_p(G)$ (as finite space), so $\mathcal{K}(\mathfrak{i}(\mathcal{A}_p(G))) \subseteq \mathcal{K}(\mathcal{S}_p(G))$ is a homotopy equivalence. See also [Smi11] for a longer list of p-subgroup complexes homotopy equivalent to $\mathcal{K}(\mathcal{S}_p(G))$.

In application of our theorem we give some examples of groups G which do not satisfy the hypotheses of the theorems of [AS93] but they do satisfy the conjecture by Corollary 4.3.2. We show that it is possible to construct a homotopy equivalent 2-dimensional and G-invariant subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$. These results appear in the article written in collaboration with I. Sadofschi Costa and A. Viruel [PSV19].

We culminate this chapter by showing that it is possible to study the strong conjecture under the assumption $O_{p'}(G) = 1$, and apply this reduction to yield new cases of the conjecture. In [AS93, Proposition 1.6] it is shown that this assumption is valid provided that $p > 5$. By using techniques of finite spaces and the p-solvable case of Quillen's conjecture, we prove that this reduction is possible for every prime p. Concretely, we prove the following theorem.

Theorem 4.5.1. *Let G be a finite group such that $O_p(G) = 1$, $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) = 0$ and its proper subgroups satisfy the strong Quillen's conjecture. Then $O_{p'}(G) = 1$. In particular, a minimal counterexample G to the strong Quillen's conjecture has $O_{p'}(G) = 1$.*

This theorem is interesting not only by the reduction which allows us to do, but also by the method of its proof. The use of the Classification of the Finite Simple Groups in the proof of this theorem is considerable minor than in the weaker result [AS93, Proposition 1.6]. In fact, we only use it to invoke the p-solvable case of the conjecture, in which the use of the Classification is just for the structure of the outer automorphisms group of simple groups.

The proof of our theorem also provides a technique to find nontrivial cycles in the homology of $\mathcal{A}_p(G)$, generalizing the original idea of [AS93, Lemma 0.27] (see Lemma 4.5.10).

Applying Theorems 3.4.2 and 4.5.1, and Corollary 4.3.2, we obtain the following corollaries.

Corollary 4.5.13. *If the proper subgroups of G satisfy the strong Quillen's conjecture but $O_p(G) = 1$ and $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) = 0$, then $\mathcal{K}(\mathcal{S}_p(G))$ has no 2-dimensional G-invariant homotopy equivalent subcomplex. In particular, $m_p(G) \geq 4$.*

Note that this result is a slight improvement of Corollary 4.3.2 to rational coefficients.

Corollary 4.5.14. *The strong Quillen's conjecture holds for groups of p-rank at most* 3.

Finally, as application of our methods and results, we deduce the strong conjecture for groups of p-rank 4.

Theorem 4.6.8. *The strong Quillen's conjecture holds for groups of p-rank at most* 4.

We also eliminate the possibility of components of p-rank 1, which allows us to extend the main result of [AS93] to $p = 5$.

Theorem 4.6.3. *Let $L \leq G$ be a component such that $L/Z(L)$ has p-rank* 1. *If the strong Quillen's conjecture holds for proper subgroups of G then it holds for G.*

Corollary 4.6.5. *The conclusions of the Main Theorem of [AS93] hold for $p = 5$.*

These results will be part of a new and more general article on Quillen's conjecture, which is now in preparation.

Most of the examples we present in this dissertation were computed with GAP [GAP18] with a package of posets developed in collaboration with X. Fernández and I. Sadofschi Costa [FPSC19]. In Appendix A.2 can be found some of the codes we have used to compute the examples presented here.

Contents

Chapter 1

Finite spaces and the p-subgroup posets

In this chapter, we study the posets of p-subgroups from the point of view of finite topological spaces. Recall that D. Quillen studied the homotopy properties of the order complexes $\mathcal{K}(\mathcal{A}_p(G))$ and $\mathcal{K}(\mathcal{S}_p(G))$. He showed that the inclusion $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ induces a homotopy equivalence at the level of their order complexes, and that $\mathcal{K}(\mathcal{S}_p(G))$ is contractible when G has a nontrivial normal p-subgroup. The converse to this last statement is Quillen's conjecture, which remains open so far, and it is studied in Chapter 4. The standard way to investigate these posets is by means of the topology of their associated order complexes $\mathcal{K}(\mathcal{S}_p(G))$ and $\mathcal{K}(\mathcal{A}_p(G))$.

Throughout this dissertation we regard finite posets as finite T_0-spaces using an intrinsic topology, in which open sets are the *downsets*. The intrinsic topology and the topology of their order complexes are related by McCord's Theorem 1.2.2: for a finite poset X there exists a weak homotopy equivalence $|\mathcal{K}(X)| \to X$. However, weak equivalences between finite spaces are not homotopy equivalences in general, and Whitehead's theorem is no longer true in this context. There are finite spaces which are homotopically trivial but non-contractible (see for instance [Bar11a, Example 4.2.1]). Recall that a topological spaces is called homotopically trivial if all of its homotopy groups are trivial.

R.E. Stong, who had worked with finite spaces in [Sto66], studied the posets $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ with this intrinsic topology in [Sto84]. Stong proved that the inclusion $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ is not a homotopy equivalence of finite spaces in general (but it is a weak homotopy equivalence by McCord's theorem). Moreover, he showed that $\mathcal{S}_p(G)$ is a contractible finite space if and only if G has a nontrivial normal p-subgroup and that Quillen's conjecture can be restated in terms of finite spaces by saying that if $\mathcal{S}_p(G)$ is homotopically trivial, then $\mathcal{S}_p(G)$ is contractible. Stong also proved that the contractibility of $\mathcal{A}_p(G)$ as a finite space implies that of $\mathcal{S}_p(G)$ and left open the question of whether the converse holds. This amounts to asking whether Quillen's conjecture can be rephrased by saying that if $\mathcal{A}_p(G)$ is homotopically trivial, then it is a contractible finite space.

We answered Stong's question by the negative in [Pit16] (see also [MP18]). In Example 1.3.17 we exhibit a finite group G such that $\mathcal{A}_p(G)$ is homotopically trivial but non-contractible for some prime p, and that $\mathcal{S}_p(G)$ is contractible. Our example shows that the contractibility of $\mathcal{A}_p(G)$, as finite space, is not the same as that of $\mathcal{S}_p(G)$. Since Stong described the contractibility of $\mathcal{S}_p(G)$ in algebraic terms, we describe the contractibility of $\mathcal{A}_p(G)$ in algebraic-combinatorial terms in Theorem 1.3.32 by using the notion of *contractibility in steps*.

The chapter is organized as follows. In the first two sections we recall some basic definitions, notations and results on finite groups and finite topological spaces. We also introduce the Bouc poset $\mathcal{B}_p(G)$ (consisting of the nontrivial radical p-subgroups of G) and the Robinson complex $\mathcal{R}_p(G)$ (see Definition 1.1.10). In Section 1.3 we compare the equivariant homotopy type of the different posets of p-subgroups regarded as finite topological spaces. Recall that G acts on its posets of p-subgroups via conjugation. In Subsection 1.3.1 we give some particular conditions under which the p-subgroup posets of a finite group have the same homotopy type. In Subsection 1.3.3 we describe the contractibility of $\mathcal{S}_p(G)$, $\mathcal{B}_p(G)$ and $\mathcal{A}_p(G)$ in algebraic terms. The results of this chapter are extensions of the results of [Pit16] and the article written in collaboration with E.G. Minian [MP18] to the posets $\mathcal{B}_p(G)$ and $\mathcal{X}(\mathcal{R}_p(G))$ (the face poset of the Robinson complex).

Throughout this dissertation we will talk about the homotopy properties of a finite poset regarded always as a finite topological space. For example, if X and Y are finite posets, we write $X \simeq Y$ if X and Y are homotopy equivalent as finite spaces.

1.1 Finite groups

In this section, we recall some basic facts, definitions and results on finite groups that will be useful in the following chapters. We also exhibit the different posets of p-subgroups arising from a finite group at the end of the section. The main references for the classical results on finite group theory are the books of M. Aschbacher [Asc00] and I.M. Isaacs [Isa08].

For a group G, we write $H \leq G$ if H is a subgroup of G, $H < G$ if H is a proper subgroup of G, and $1 \neq H$ if H is nontrivial. Denote by $N \trianglelefteq G$ a normal subgroup N of G, and $N \triangleleft G$ if N is also proper. A subgroup N of G is said to be characteristic in G if $\phi(N) = N$ for any automorphism ϕ of G. We write $N \operatorname{char} G$ in this case. If $K, H \leq G$, then $C_K(H) = \{g \in K : gh = hg \text{ for all } h \in H\}$ and $N_K(H) = \{g \in K : H^g = H\}$ are the centralizer and normalizer, respectively, of H in K. Here, $H^g = g^{-1}Hg$. Write $[H,K]$ for the subgroup of G generated by the commutators elements $[h,k] = hkh^{-1}k^{-1}$ for $h \in H$ and $k \in K$.

For a prime number p, $O_p(G)$ is the largest normal p-subgroup of G. This group is also called the *p-core* of G. By Sylow's theorem, it is equal to the intersection of all Sylow p-

subgroups of G. Denote by $\mathrm{Syl}_p(G)$ the set of Sylow p-subgroups of G. By a p'-group we mean a group of order prime to p. Let $O_{p'}(G)$ be the largest normal p'-subgroup of G. This group is commonly called the *p'-core* of G. Analogously, $O^p(G)$ (resp. $O^{p'}(G)$) is the smallest normal subgroup of G such that the quotient $G/O^p(G)$ (resp. $G/O^{p'}(G)$) is a p-group (resp. p'-group).

Denote by $Z(G)$, $[G,G] = G'$, $\Phi(G)$ and $F(G)$ the center, the derived, the Frattini and the Fitting subgroup of G, respectively. Recall that $\Phi(G)$ equals the intersection of all maximals subgroups of G, or equivalently, the set of nongenerators of G. The Fitting subgroup $F(G)$ is the largest normal nilpotent subgroup of G. Equivalently, $F(G)$ is the product of the subgroups $O_p(G)$, for $p \mid |G|$. Note that all these subgroups are characteristic in G.

For a fixed prime number p, denote by $\Omega_1(G)$ the subgroup of G generated by the elements of order p. The *p-rank of* G is the non-negative integer $m_p(G) = \max\{r : A \leq G$ elementary abelian p-subgroup, $|A| = p^r\}$. Let $r_p(G)$ be $\log_p(|G|_p)$.

Let S be a p-group. If $1 \neq N \trianglelefteq S$, then $N \cap Z(S) \neq 1$. If $H < S$ is a proper subgroup, then $H < N_S(H)$. The Frattini subgroup $\Phi(S)$ equals the smallest normal subgroup of S such that $S/\Phi(S)$ is elementary abelian. In particular, S is elementary abelian if and only if $\Phi(S) = 1$.

Denote by $\mathrm{Aut}(G)$ the automorphisms group of a finite group G. There is a map $G \to \mathrm{Aut}(G)$ defined by $g \mapsto c_{g^{-1}}$, where $c_g(h) = h^g = g^{-1}hg$ is the conjugation morphism. The kernel of this map is the center of G, and the image is $G/Z(G) = \mathrm{Inn}(G) \leq \mathrm{Aut}(G)$, the inner automorphisms group of G. The outer automorphisms group of G is $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$.

Given a short exact sequence $1 \to N \to G \to H \to 1$ of finite groups, we say that G is an extension of N by H. When the extension splits, we write $G = N \rtimes H$ or $G = N : H$. When the extension does not split, we write $G = N \cdot H$. The direct product of N by H is denoted by $N \times H$.

We recall now some classical theorems of finite group theory.

A p-subgroup Q of G is termed *radical* if $Q = O_p(N_G(Q))$. We have the following property on radical p-subgroups.

Proposition 1.1.1. *Let G be a finite group and p a prime number dividing the order of G. Let $Q \leq G$ be a radical p-subgroup and let $P \leq G$ be a p-subgroup such that $N_G(Q) \leq N_G(P)$. Then $P \leq Q$. In particular, by taking $P = O_p(G)$ we get that $O_p(G) \leq Q$.*

Proof. We argue by contradiction. Suppose $P \not\leq Q$. Since $Q \leq N_G(P)$, PQ is a p-subgroup and $Q < PQ$. Therefore, $Q < N_{PQ}(Q)$. On the other hand, if $x \in N_G(Q)$ then $N_{PQ}(Q)^x = (PQ \cap N_G(Q))^x = (PQ)^x \cap N_G(Q)^x = PQ \cap N_G(Q) = N_{PQ}(Q)$. That is, $N_{PQ}(Q) \leq O_p(N_G(Q)) = Q$, a contradiction. □

A finite group G is solvable if it has a composition series whose factors are simple abelian groups. We recall some classical results that ensure the solvability of a group.

Theorem 1.1.2 (Burnside)**.** *If $|G| = p^\alpha q^\beta$ with p and q primes, then G is a solvable group.*

The following theorem is one of the first big steps in the classification of the finite simple groups (CFSG for short).

Theorem 1.1.3 (Feit-Thompson)**.** *Finite groups of odd order are solvable. In particular, any non-abelian simple group has even order.*

Theorem 1.1.4 (Schur-Zassenhaus)**.** *Let $1 \to N \to G \to H \to 1$ be an extension of finite groups with $\gcd(|N|,|H|) = 1$. Then the extension splits, i.e. $G = N : H$ is a semidirect product. Moreover, the group H is a complement of N in G, and it is unique up to conjugacy.*

Definition 1.1.5. A group G is called *p-nilpotent* if $G = O_{p'}(G)S$ for S a Sylow p-subgroup of G. Equivalently, G is an extension of a p'-group by a p-group (and it splits).

The p-nilpotent groups are closely related with the control of fusion of the Sylow p-subgroups. We will use this notion in our treatment of Webb's conjecture in Chapter 2.

Finally, we state the well-known Hall-Higman Lemma 1.2.3.

Theorem 1.1.6 (Hall-Higman)**.** *Let G be a π-separable group such that $O_{\pi'}(G) = 1$. Then $C_G(O_\pi(G)) \leq O_\pi(G)$.*

Recall that a π-separable group, for π a set of prime numbers, is a finite group G such that it has a composition series whose factors are π-groups or π'-groups. Here, a π-group is a finite group such that any prime dividing its order is in the set π. A π'-group is a finite group whose order is not divisible by any prime of the set π.

The above theorem will be used for p-solvable groups. Recall that a p-solvable group is just a π-separable group for $\pi = \{p\}$. Every solvable group is p-solvable. We will use the following property which is weaker than being p-solvable. Let $O_{p',p}(G)$ denotes the unique normal subgroup of G containing $O_{p'}(G)$ such that $O_{p',p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G))$. It is a characteristic subgroup of G.

Definition 1.1.7. A group G is *p-constrained* if

$$C_G(S \cap O_{p',p}(G)) \leq O_{p',p}(G),$$

where $S \in \mathrm{Syl}_p(G)$. If $O_{p'}(G) = 1$, it says that $C_G(O_p(G)) \leq O_p(G)$.

By the Hall-Higman theorem 1.1.6, every p-solvable group is p-constrained.

Throughout the classification of the finite simple groups, there is a characteristic subgroup of the finite groups which plays a key role: *the generalized Fitting subgroup*. If G is a finite group, the Fitting subgroup $F(G)$ is the largest normal nilpotent subgroup of G. When G is a solvable group, this subgroup is self-centralizing, that is, $C_G(F(G)) \leq F(G)$. However, this property does not hold for general groups and it leads to the definition of the generalized Fitting subgroup $F^*(G)$. This characteristic subgroup is self-centralizing and it is equal to $F(G)$ when G is solvable. We refer to [Asc00, Section 31] for more details.

From now on, by a simple group we will mean a *non-abelian* simple group. A finite group K is called *quasisimple* if it is a perfect group and $K/Z(K)$ is simple. Equivalently, it is a perfect central extension of a simple group. A subgroup $L \leq G$ is *subnormal* if there exist subgroups $L_0, L_1, \ldots, L_n \leq G$ such that $L_0 = L$, $L_n = G$ and $L_i \trianglelefteq L_{i+1}$ for all i. A *component* of G is a subnormal quasisimple subgroup. Denote by $\mathcal{C}(G)$ the set of components of G. The *layer* of G is the subgroup generated by the components of G, and we denote it by $E(G)$. It is well-known that $E(G)$ is in fact the central product of the components of G. This means that if $L, K \in \mathcal{C}(G)$ are different components, then $[L,K] = 1$. Note that $Z(E(G))$ is the product of the centres of the components of G. In particular it is a normal nilpotent subgroup of G and $Z(E(G)) \leq F(G)$. The *generalized Fitting subgroup* of G is the product $F^*(G) = F(G)E(G)$. It can be shown that $[F(G),E(G)] = 1$ and $Z(E(G)) = F(G) \cap E(G) \leq Z(F(G))$. The subgroups $F(G)$, $E(G)$ and $F^*(G)$ are characteristic in G. Moreover, $F^*(G)$ is self-centralizing, i.e. $C_G(F^*(G)) = Z(F^*(G))$. Note that $Z(F^*(G)) = Z(F(G)) \geq Z(E(G))$. In consequence, it yields an exact sequence

$$1 \to \mathrm{Inn}(F^*(G)) = F^*(G)/Z(F(G)) \to G/Z(F(G)) \to \mathrm{Out}(F^*(G)).$$

Therefore, one can study the structure of G via the representation $G/Z(F(G)) \hookrightarrow \mathrm{Aut}(F^*(G))$.

Remark 1.1.8. Assume that $F(G) = 1$. Then, $Z(E(G)) \leq Z(F(G)) = 1$ and every component of G is in fact a simple group. In this way, we have that $F^*(G) = E(G)$ is a direct product of simple groups and by the self-centralizer condition, $C_G(E(G)) = C_G(F^*(G)) = Z(F^*(G)) = Z(F(G)) = 1$. Thus, we have a representation of G as a subgroup of $\mathrm{Aut}(E(G))$. Moreover, suppose that we write $E(G) \cong \prod_{i=1}^n L_i^{n_i}$, where the L_is are non-isomorphic simple groups. From [GLS96, Section B] it can be shown that

$$\mathrm{Aut}\,(E(G)) \cong \mathrm{Aut}\left(\prod_{i=1}^n L_i^{n_i}\right) \cong \prod_{i=1}^n (\mathrm{Aut}(L_i) \wr \mathbb{S}_{n_i})$$

where the wreath product $\mathrm{Aut}(L_i) \wr \mathbb{S}_{n_i}$ is taken with respect to the natural permutation of $\mathbb{S}_{n_i}$ on the set of n_i elements. Here $\mathbb{S}_n$ denotes the symmetric group on n letters.

Since the automorphisms of simple groups are well-understood by the classification of the finite simple groups, a group with $F(G) = 1$ can be studied via the representation as subgroup of $\mathrm{Aut}(E(G))$.

This condition holds for example if $O_p(G) = 1 = O_{p'}(G)$ since $F(G) \leq O_p(G)O_{p'}(G)$.

Definition 1.1.9. A finite group G is termed *almost simple* if it is an extension of a simple group by outer automorphisms. Equivalently, $F^*(G)$ is a simple group and hence, $F^*(G) \leq G \leq \mathrm{Aut}(F^*(G))$.

The almost simple groups are important in the classification of finite simple groups since they are, roughly, the first groups one can construct from a simple group. Recall that every finite group has a subnormal series whose factors are simple groups. In the case of almost simple groups, it turns out that this series has a unique simple group in the bottom (i.e. $F^*(G)$) and the rest of the groups are cyclic. This follows by Schreier Conjecture (actually proved as a consequence of the Classification), since the outer automorphisms group of a simple group is solvable.

In the subsequent chapters, we will use the classification of finite simple groups to construct examples. In Chapters 3 and 4, we will use it to study the fundamental group of the p-subgroup complexes and Quillen's Conjecture.

Recall that the Classification states that every finite simple group is either an Alternating Group $\mathbb{A}_n$ with $n \geq 5$, a finite simple group of Lie type or one of the 26 sporadic groups. We refer to the Appendix A.1 for more information on finite simple groups. We include there the description of the different families of simple groups, their orders, their outer automorphisms group structure and some theorems of the Classification that will be useful.

We end up this section by introducing the different posets and complexes of p-subgroups which will be used in this thesis.

Definition 1.1.10. Let G be a finite group and p a prime number dividing its order. Consider the following posets of subgroups of G ordered by inclusion.

$$\mathcal{S}_p(G) = \{P \leq G : P \text{ is a nontrivial } p\text{-subgroup}\}$$

$$\mathcal{A}_p(G) = \{E \in \mathcal{S}_p(G) : E \text{ is elementary abelian}\}$$

$$\mathcal{B}_p(G) = \{P \in \mathcal{S}_p(G) : P = O_p(N_G(P))\}$$

$$X_p(G) = \{P \in \mathcal{S}_p(G) : P \trianglelefteq S \text{ if } S \in \mathrm{Syl}_p(G) \text{ and } P \leq S\}$$

We also have the following finite simplicial complexes. Denote by $K_p(G)$ the commuting complex of G at p, with simplices the sets $\{E_1, \ldots, E_n\}$ of subgroups of order p of G such that $[E_i, E_j] = 1$ for all i, j (i.e. they generate an elementary abelian p-subgroup).

Denote by $\mathcal{R}_p(G)$ the Robinson subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$, whose simplices consist of the chains $(P_0 < \ldots < P_n)$ with $P_i \trianglelefteq P_n$ for all i.

The posets $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$ and $\mathcal{B}_p(G)$ were first considered by K. Brown in [Bro75], D. Quillen in [Qui78] and S. Bouc [Bou84], respectively. That is why their order complexes are commonly called Brown complex, Quillen complex and Bouc complex, respectively. These posets are also called Brown poset, Quillen poset and Bouc poset, respectively.

The commuting complex $K_p(G)$ was used by M. Aschbacher for the study of the fundamental group of the posets of p-subgroups (see [Asc93]).

The Robinson complex $\mathcal{R}_p(G)$ was first considered by G.R. Robinson in his reformulation of Alperin's conjecture [KR89] and it was used in the proof of Webb's conjecture (see Chapter 2).

The poset $X_p(G)$ does not seem to have been considered before. We used this poset in [Pit19] to study Webb's conjecture in Chapter 2.

1.2 Finite topological spaces

The theory of finite topological spaces began with the works of P.S. Alexandroff [Ale37], and later continued by M. McCord [McC66] and R.E. Stong [Sto66]. More recently, J. Barmak and E.G. Minian deepened in the study of finite spaces [Bar11a, Bar11b, BM08b, BM08a, BM12b]. For any finite simplicial complex, its face poset is a finite space with the same homotopy groups and homology groups, and conversely, any finite space has an associated simplicial complex. We can eliminate certain kind of points in a finite spaces (called *beat points* and *weak points*) preserving its (weak) homotopy type. The key point here is that the elimination of a single *beat* or *weak* point in a finite space translates in many simplicial collapses in its associated simplicial complex, which do not change the homotopy type. This allows us to manipulate these objects algorithmically and combinatorially and to find weak equivalent finite spaces with a less number of points.

In this section, we recall the basic facts that we will need concerning finite topological spaces. We refer the reader to the book of J. Barmak for further details [Bar11a].

From now on, by a finite topological space we will mean a finite T_0-space. See [Bar11a, Proposition 1.3.1] to see that there is no loss of generality.

Given a finite poset X, there is an intrinsic topology which makes it a finite topological space. For each element $x \in X$ consider the set $U_x = \{y \in X : y \leq x\}$. Then, the collection $\{U_x : x \in X\}$ is a basis for a topology in X. Note that the open sets are the downsets of the poset and that the resulting topological space is T_0.

Conversely, if X is a finite T_0-space, then for each $x \in X$ we can consider the minimal open set U_x containing x. Since X is finite, U_x is the intersection of all open sets containing x. Now put $x \leq y$ if $U_x \subseteq U_y$. With this order, X becomes a finite poset. It is easy to check that these constructions are inverses to each other. Moreover, a function $f : X \to Y$ between finite posets is a continuous map if and only if it is an order preserving map.

Proposition 1.2.1. *Let X and Y be two finite spaces. A function $f : X \to Y$ is a continuous map if and only if it is an order preserving map.*

Therefore, the categories of finite posets and finite T_0-spaces are isomorphic.

In general, given a finite poset X we can consider its associated *order complex* $\mathcal{K}(X)$. It is the simplicial complex whose vertices are the elements of X and whose simplices are the nonempty chains of elements of X. Any order preserving map $f : X \to Y$ between finite posets induces a simplicial map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$ defined in a vertex $x \in X$ as $\mathcal{K}(f)(x) = f(x)$. In this way, there are two natural topologies arising from a finite poset X. Namely, the intrinsic topology of X as a finite topological space, and the topology of its order complex $\mathcal{K}(X)$. The relation between these two spaces is given by McCord's theorem [McC66, Theorem 1]. For a simplicial complex K, denote by $|K|$ its geometric realization. If $K = \mathcal{K}(X)$ for a finite poset X, we will also write $|X|$ to denote $|\mathcal{K}(X)|$. If $\varphi : K \to L$ is a simplicial map, denote by $|\varphi| : |K| \to |L|$ the induced continuous map in the geometric realizations.

Theorem 1.2.2 (McCord)**.** *Let X be a finite space. There is a natural continuous map $\mu_X : |\mathcal{K}(X)| \to X$ defined by*

$$\mu_X\left(\sum_{i=0}^{r} t_i x_i\right) = \min\{x_i : i = 0, \ldots, r\}$$

which is a weak equivalence. This map is called McCord's map.

Recall that a *weak homotopy equivalence* (or *weak equivalence* for short) between two topological spaces is a continuous map which induces isomorphisms in all homotopy groups.

The naturality of McCord's map in Theorem 1.2.2 implies that, for a continuous map $f : X \to Y$ between finite spaces, $|\mathcal{K}(f)| \circ \mu_X = \mu_Y \circ f$. In particular, $|\mathcal{K}(f)|$ is a homotopy equivalence if and only if f is a weak equivalence.

Remark 1.2.3. In fact, McCord's map is not a homotopy equivalence in general. Weak equivalences between finite spaces may not be homotopy equivalences, and homotopically trivial finite topological spaces may not be contractible (see [Bar11a, Example 4.2.1] or Example 1.3.17). That is, Whitehead's theorem is no longer true in the context of finite topological spaces. The failure of this important theorem is one of the keys for which we study finite posets with this topology: at the level of their intrinsic topology, there could be homotopy differences which may not be perceived with the topology of their order complexes.

For a finite simplicial complex K, denote by $\mathcal{X}(K)$ its *face poset*, whose elements are the simplices of K ordered by inclusion. A simplicial map $\phi : K \to L$ between finite simplicial complexes induces a continuous map $\mathcal{X}(\phi) : \mathcal{X}(K) \to \mathcal{X}(L)$ between their face posets. It can be shown that there is an analogous McCord's map $\mu_K : |K| \to \mathcal{X}(K)$ (see [Bar11a, Theorem 1.4.12]).

If X is a finite poset, denote by X' the poset of nonempty chains of X ordered by inclusion. Note that $X' = \mathcal{X}(\mathcal{K}(X))$. We say that X' is the (first) subdivision of X. Write $X^{(n)}$ for the n-th iterated subdivision of X, i.e. $X^{(n)} = (X^{(n-1)})'$. If K is a simplicial complex, then K' denotes the simplicial complex whose vertices are the simplices of K and whose simplices are the chains of simplices of K ordered by inclusion. Therefore, $K' = \mathcal{K}(\mathcal{X}(K))$. We say that K' is the (first) barycentric subdivision of K. Recall that $|K|$ and $|K'|$ are homeomorphic.

For a finite poset X, its opposite poset X^{op} has the same elements of X but with the opposite order. Note that $(X^{\mathrm{op}})' = X'$ and $\mathcal{K}(X) = \mathcal{K}(X^{\mathrm{op}})$. Moreover, $f : X \to Y$ is an order preserving map (i.e. a continuous map) if and only if $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is. By McCord's theorem X and X^{op} have the same homotopy groups and homology groups. However, it may not exist any weak equivalence between them.

From now on, we will make no distinction between the category of finite posets and the category of finite spaces. We will treat finite posets as finite topological spaces by means of their intrinsic topology, and vice versa.

The homotopy type of finite spaces

In the following we recall the basic facts that we need to compute the homotopy type of finite spaces. The homotopy theory of finite spaces can be studied in combinatorial terms by means of their intrinsic order. This idea is due to Stong [Sto66].

The proposition below relates the connectedness of a finite space with the connectedness of its intrinsic poset structure, i.e. the connectedness of its associated Hasse diagram.

Proposition 1.2.4. *If X is a finite space, then X is locally arc-connected. Moreover, there is a path between two elements $x, y \in X$ if and only if there exist $x_0, \dots, x_n \in X$ such that $x_0 = x$, $x_n = y$ and for all $0 \leq i < n$, $x_i \leq x_{i+1}$ or $x_i \geq x_{i+1}$.*

The homotopies between continuous maps of finite topological spaces can be described by means of the combinatorial of their intrinsic order. In certain way, it is a generalization of the previous proposition.

Definition 1.2.5. Let X and Y be finite spaces. Consider the set Y^X of all continuous maps from X to Y. For $f, g \in Y^X$, define $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

Proposition 1.2.6. *If X and Y are finite spaces then Y^X is a finite space (with the order given as in the above definition).*

Proposition 1.2.7. *Let $f, g : X \to Y$ be two continuous maps between finite spaces X and Y. Then f and g are homotopic in the classical sense, and we write $f \simeq g$, if and only if $f, g \in Y^X$ are arc-connected. That is, $f \simeq g$ if and only if there exist continuous maps $f_0, \dots, f_n : X \to Y$ such that $f_0 = f$, $f_n = g$ and for all $0 \leq i < n$, $f_i \leq f_{i+1}$ or $f_i \geq f_{i+1}$.*

Let $f : X \to Y$ be a map between finite spaces. Then, f induces a simplicial map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$. If $f \simeq g$ then $\mathcal{K}(f)$ and $\mathcal{K}(g)$ are contiguous maps in the sense of simplicial complexes (see [Bar11a, Chapter 2, section 1]). In particular $|\mathcal{K}(f)|$ and $|\mathcal{K}(g)|$ are homotopic in the classical sense. However, if $|\mathcal{K}(f)|$ and $|\mathcal{K}(g)|$ are homotopic, f and g may not be homotopic (see Remark 1.2.3).

Now we describe a method of R.E. Stong [Sto66] to compute the homotopy type of a finite poset. Roughly, the homotopy type of the finite spaces is determined by the minimal finite spaces (in the sense we define below), which are unique up to homeomorphism.

Let X be a finite space. Then $\operatorname{Max}(X)$ denotes the set of maximal elements of X. Similarly we denote by $\operatorname{Min}(X)$ the set of minimal elements. If $x \in X$, we denote by $\operatorname{Max}(x)$ the set of maximal elements over x and by $\operatorname{Min}(x)$ the set of minimal elements below x. Let $F_x = \{y \in X : y \geq x\}$, $\hat{F}_x = \{y \in X : y > x\}$, $U_x = \{y \in X : y \leq x\}$ and $\hat{U}_x = \{y \in X : y < x\}$. We write F_x^X to emphasize that F_x is taken in the poset X. We also denote $X_{\leq x} = U_x^X$, $X_{<x} = \hat{U}_x^X$, $X_{\geq x} = F_x^X$ and $X_{>x} = \hat{F}_x^X$. If $y \in X$, write $x \prec y$ if $x < y$ and there is no element $z \in X$ such that $x < z < y$. If $x \prec y$ then we say that x is *covered* by y and that y *covers* x.

Definition 1.2.8. Let X be a finite space and let $x \in X$. We say that x is an *up beat point* if $\hat{F}_x$ has a minimum, and that x is a *down beat point* if $\hat{U}_x$ has a maximum. We say that x is a *beat point* if it is either up or down beat point.

Remark 1.2.9. Note that $x \in X$ is an up beat point if and only if there exists a unique $y \in X$ such that $x \prec y$. Analogously, x is a down beat point of and only if there exists a unique $y \in X$ such that $y \prec x$.

If X is a finite space and $x \in X$ is a beat point, then $X - x \subseteq X$ is a strong deformation retract. For example, if x is an up beat point covered by $y \in X$, then we may define the retraction $r : X \to X - x$ by $r(x) = y$.

Suppose we start with a finite space and remove beat points one by one. At a certain point, we will reach a space without beat points. A finite space without beat points is called a *minimal finite space*. It turns out that, no matter in what order we remove the beat points of a finite space, the minimal finite spaces we can reach are all homeomorphic. Therefore, for each finite space X there is a strong deformation retract $X_0 \subseteq X$ which is a minimal finite space. We call X_0 the *core* of X and it is unique up to homeomorphism. Moreover, the possible cores of finite spaces classify the homotopy type of finite spaces. That is, two finite spaces X and Y have the same homotopy type if and only if their cores are homeomorphic. We state this Classification Theorem below.

Theorem 1.2.10 ([Sto66, Section 4]). *A homotopy equivalence between minimal finite spaces is a homeomorphism. In particular, the core of a finite space is unique up to homeomorphism and two finite spaces are homotopy equivalent if and only if they have homeomorphic cores.*

As corollary, the core of a contractible finite space is the singleton.

Corollary 1.2.11. *A finite space X is contractible if and only if its core is a point.*

By removing beat points in a finite space X we reach subspaces which are strong deformation retracts of X. It turns out that this kind of subspaces can always be obtained in this way.

Proposition 1.2.12. *Let X be a finite space and let $Y \subseteq X$ be a subspace. Then $Y \subseteq X$ is a strong deformation retract of X if and only if Y is obtained from X by removing beat points.*

We write $X \searrow\searrow Y$ when $Y \subseteq X$ is a strong deformation retract.

The following theorem proved by Barmak and Minian in [BM12b] relates the contractibility of X with that of X'.

Theorem 1.2.13 (Barmak-Minian)**.** *A finite space X is contractible if and only if its subdivision, X', is contractible.*

Proof. See [BM12b, Corollary 4.18]. □

This theorem allows to characterize the contractibility of a finite space in terms of its iterated subdivisions.

The proof roughly shows that, when the core of X is nontrivial, the core X' is larger than the core of X. Therefore, we deduce that X and X' are homotopy equivalent only when their connected components are contractible.

Theorem 1.2.14. *Let X be a finite space. If X is homotopy equivalent to some subdivision $X^{(n)}$ then X, and hence all its subdivisions, have the homotopy type of a discrete space.*

Simple homotopy type for finite spaces

The failure of Whitehead's theorem on finite spaces is encoded in the fact that removing or adding beat points in a finite space is a rather restrictive condition.

In some cases, we want to find a relation between two finite spaces X and Y that guarantees that $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ have the same homotopy type, but less restrictive than the beat points condition. In this way, Barmak and Minian [BM08b] translated the notion of simple homotopy type of simplicial complexes to the context of finite spaces. In simplicial complexes, the simple homotopy type is stronger than the usual homotopy type, but for finite spaces it turns out to be weaker.

Definition 1.2.15. Let X be a finite space. We say that x is an *up weak point* if $\hat{F}_x$ is contractible, and a *down weak point* if $\hat{U}_x$ is contractible. We say that x is a *weak point* if it is either up or down.

Remark 1.2.16. If $x \in X$ is a weak point, the inclusion $X - x \subseteq X$ is a weak equivalence (see [Bar11a, Proposition 4.2.4]).

Let X be a finite space. If $Y \subseteq X$, we say that there is an *elementary collapse* from X to Y, and write $X \overset{e}{\searrow} Y$, if $Y = X - x$ for some weak point $x \in X$. We also say that Y *elementary expands* to X and write $Y \overset{e}{\nearrow} X$. Write $X \searrow Y$ (or $Y \nearrow X$) when Y is obtained from X by removing weak points. In this case we say that X *collapses* to Y and that Y *expands* to X. We say that X is *collapsible* if it collapses to a point.

If Z is other finite space, we say that X is *simple homotopy equivalent* to Z, or that X and Z *have the same simple homotopy type*, if there is a sequence of finite spaces $X_0, X_1, \dots, X_n$ such that $X_0 = X$, $X_n = Z$ and for each i, either X_i collapses or it expands to X_{i+1}. We denote it by $X \diagup\!\!\searrow Z$.

This defines a notion of simple homotopy theory of finite spaces. It can be shown that homotopy equivalent finite spaces are simple homotopy equivalent (see [Bar11a, Lemma 4.2.8]). The converse does not hold. Moreover, there are collapsible finite spaces which are not contractible (see [Bar11a, Example 4.3.3]).

Two finite spaces which are simple homotopy equivalent have the same homotopy and homology groups. Moreover, this notion of simple homotopy type corresponds to the same notion at the level of simplicial complexes.

Theorem 1.2.17 (see [Bar11a, Theorem 4.2.11])**.** 1. *Let X and Y be finite spaces. Then, X and Y are simply homotopy equivalent if and only if $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are simply homotopy equivalent. Moreover, if $X \searrow Y$ then $\mathcal{K}(X) \searrow \mathcal{K}(Y)$.*

2. *Let K and L be finite simplicial complexes. Then, K and L are simply homotopy equivalent if and only if $\mathcal{X}(K)$ and $\mathcal{X}(L)$ are simply homotopy equivalent. Moreover, if $K \searrow L$ then $\mathcal{X}(K) \searrow \mathcal{X}(L)$.*

Note that a finite space X is simple homotopy equivalent to a point if and only if $X \to \{*\}$ is a weak equivalence, if and only if X is homotopically trivial (i.e. it has trivial homotopy groups). However, this notion is strictly weaker than being collapsible. That is, if X is collapsible then it is simple homotopy equivalent to a point but the converse does not hold. For example, take K to be a any triangulation of the Dunce hat. It is known that K is contractible and hence it has the same simple homotopy type of a point. In particular, $X = \mathcal{X}(K)$ is simple homotopy equivalent to a point. On the other hand, no triangulation of the Dunce hat is collapsible, so $\mathcal{X}(K)$ cannot be collapsible.

The join of finite posets

If X and Y are finite spaces, then we define its join $X * Y$ to be the finite poset whose underlying set is the disjoint union of X and Y with the following order. It keeps the same order in X and in Y, and put $x \leq y$ for any $x \in X$ and $y \in Y$. If K and L are finite simplicial complexes, their join $K * L$ is the simplicial complex with vertices the disjoint union of the vertices of K and L. The simplices of $K * L$ are the simplices of K, the simplices of L and the unions $\sigma \cup \tau$ with $\sigma \in K$ and $\tau \in L$. Therefore, $\mathcal{K}(X * Y) = \mathcal{K}(X) * \mathcal{K}(Y)$ and $|K * L|$ is homeomorphic to the classical topological join $|K| * |L|$.

The contractibility of the join of posets is described in the following proposition.

Proposition 1.2.18 (see [Bar11a, Proposition 2.7.3]). *Let X and Y be two finite spaces. Then $X*Y$ is contractible if and only if X or Y is contractible.*

Note that there are finite simplicial complexes K and L such that $K*L$ is contractible but K and L are not contractible (see comment below [Bar11a, Proposition 6.2.12]). Therefore, by taking $X = \mathcal{X}(K)$ and $Y = \mathcal{X}(L)$ we may find finite spaces which are not homotopically trivial but their join is.

We have an analogous result to the above proposition for simple homotopy type.

Proposition 1.2.19 (see [Bar11a, Proposition 4.3.4]). *Let X and Y be two finite spaces. Then $X*Y$ is collapsible if and only if X or Y is collapsible.*

Equivariant homotopy type

We state here the main definitions and results on equivariant homotopy theory of finite spaces. We follow the ideas of [Bre72], [Sto84] and, more recently, [Bar11a, Chapter 8].

We consider actions at the right. For a G-set X, write x^g for the action of $g \in G$ on $x \in X$. If $H \leq G$ is a subgroup and $Y \subseteq X$, denote by $Y^H = \{y^h : y \in Y,\ h \in H\}$ and by $\mathrm{Fix}_H(Y) = \{y \in Y : y^h = y \text{ for all } h \in H\}$ the fixed points set. The group G_x is the stabilizer (or isotropy group) in G of the element $x \in X$. The orbit of $x \in X$ by the action of G is denoted by $\mathcal{O}_x$. By a G-space we mean a topological space with an action of a group G by homeomorphisms.

A map $f : X \to Y$ between G-spaces is said to be a *G-equivariant map* or a *G-map* if it preserves the action of G. That is, $f(x^g) = f(x)^g$ for all $x \in X$ and $g \in G$. A homotopy $X \times [0,1] \to Y$ between G-spaces X and Y is said to be *G-equivariant* if it is an equivariant map when we see $[0,1]$ with the trivial action of G. Two G-maps are said to be *G-homotopy equivalent* if there is a G-equivariant homotopy between them. We write $f \simeq_G g$ if f and g are G-homotopy equivalent maps. The G-map $f : X \to Y$ is said to be a *G-equivalence* or a *G-equivariant homotopy equivalence* if there is a G-map $g : Y \to X$ such that $f \circ g \simeq_G \mathsf{Id}_Y$ and $g \circ f \simeq_G \mathsf{Id}_X$. Two G-spaces X and Y are *G-homotopy equivalent* or they have *the same G-equivariant homotopy type* if there are G-equivariant maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq_G \mathsf{Id}_Y$ and $g \circ f \simeq_G \mathsf{Id}_X$. We write $X \simeq_G Y$ in this case. A subspace $Y \subseteq X$ of the G-space X is said to be *invariant* if $Y^G = Y$. We say that $Y \subseteq X$ is an *equivariant strong deformation retract* if Y is an invariant subspace of X and there is an equivariant retraction $r : X \to Y$ such that $i \circ r \simeq_G \mathsf{Id}_X$ relative to Y, where $i : Y \hookrightarrow X$ is the inclusion.

By a G-poset we mean a poset with an action of a group G by poset isomorphisms. If we have a finite G-poset, then its intrinsic topology inherits the G-structure and it becomes a finite G-spaces in the topological sense. Conversely, if we start with a finite G-space then it becomes a G-poset with its intrinsic order structure. Moreover, G-maps of finite spaces corresponds to order preserving maps of finite G-posets which preserves the action. Therefore, the categories of finite G-posets and finite G-spaces are isomorphic. In the context of finite spaces, it can be shown that two equivariant maps $f,g: X \to Y$ between finite G-spaces are G-homotopy equivalent if and only if there is a sequence of G-maps $f_0, f_1, \dots, f_n \in Y^X$ such that $f_0 = f$, $f_n = g$ and $f_i \leq f_{i+1}$ or $f_i \geq f_{i+1}$ for each i.

If X is an arbitrary G-space, then we can ask whether its homotopy type and equivariant homotopy type coincide. For arbitrary spaces this is false. For instance, there are contractible G-complexes which do not have a fixed point by the action of G. This implies that they are not G-contractible.

This situation does not arise in the context of finite spaces. We have seen that if X is a finite space, then we can remove beat points of X until reach a minimal finite space (a finite space without beat points), which is the core of X. This core determines univocally the homotopy type of X. That is, two finite spaces have the same homotopy type if and only if they have isomorphic cores. It turns out that among all the possibles cores of X, there is at least one which is G-invariant if X is a G-space.

Lemma 1.2.20 ([Sto84, Lemma p.96])**.** *Let X be a finite G-space. Then there exists a core of X which is G-invariant and it is an equivariant strong deformation retract of X.*

The idea of the proof of this lemma is that, instead of removing a single beat point in a finite space, we can remove its whole orbit and obtain an equivariant strong deformation retract. That is, if the finite G-space X has a beat point $x \in X$, then $X - \mathcal{O}_x \hookrightarrow X$ is an equivariant strong deformation retract. It follows from the fact that if x and x^g are comparable for some $g \in G$ then $x = x^g$ (see [Bar11a, Lemma 8.1.1]). In particular, a contractible finite G-space has a fixed point.

Corollary 1.2.21 (Cf. [Sto84])**.** *If X is a contractible finite G-space, then it has a fixed point.*

Moreover, this invariant core allows us to prove a stronger version of the well-known Bredon's theorem for G-CW-complexes. We recall it below.

Theorem 1.2.22 (see [Bre67, II] or [tD87, II.2.7])**.** *Let $f: X \to Y$ be a G-map between G-CW-complexes. Then f is a G-homotopy equivalence if and only if, for all $H \leq G$ the induced map $f_H: \mathrm{Fix}_H(X) \to \mathrm{Fix}_H(Y)$ is a homotopy equivalence.*

Note that, in particular, f is a homotopy equivalence by taking $H = 1$.

For finite spaces, we do not need the condition of being a homotopy equivalence at each fixed point subspace: we just require to f to be a homotopy equivalence.

Theorem 1.2.23 (Finite space version [Sto84])**.** *Let $f: X \to Y$ be a G-map between finite G-spaces. Then f is a G-homotopy equivalence if and only if f is a homotopy equivalence. In particular, two different G-invariant cores of X are G-homeomorphic.*

Proof. The original proof of this theorem is due to Stong [Sto84]. We reproduce it here. Take $X_0 \subseteq X$ and $Y_0 \subseteq Y$ to be G-invariants core which are equivariant strong deformation retract of X and Y respectively. Let $i_X : X_0 \hookrightarrow X$ and $i_Y : Y_0 \hookrightarrow Y_0$ be the inclusions, and let $r_X : X \to X_0$ and $r_Y : Y \to Y_0$ be the equivariant retractions such that $r_X \circ i_X = \mathsf{Id}_{X_0}$, $i_X \circ r_X \simeq_G \mathsf{Id}_X$, $r_Y \circ i_Y = \mathsf{Id}_{Y_0}$ and $i_Y \circ r_Y \simeq_G \mathsf{Id}_Y$. Then $r_Y \circ f \circ i_X$ is a G-map. If f is a homotopy equivalence, then $r_Y \circ f \circ i_X$ is and hence, it is a G-equivariant homeomorphism. Therefore, f is a G-equivalence since $f = i_Y \circ (r_Y \circ f \circ i_X) \circ r_X$ is a composition of G-equivalences. □

Recall that if X is a finite space, then any strong deformation retract subspace of X can be obtained by removing beat points (see Proposition 1.2.12). In the equivariant context, we have a similar result. Instead of removing a single beat point, we can remove its whole orbit and obtain an equivariant strong deformation retract. The following proposition asserts that this is the only way to obtain invariant subspaces of finite G-spaces which are also equivariant strong deformation retract. Write $X \searrow\!\!\searrow^G Y$ if X is a finite G-space and $Y \subseteq X$ is an invariant subspace obtained from X by removing orbits of beat points.

Proposition 1.2.24 ([Bar11a, Proposition 8.3.1])**.** *Let X be a finite G-space and let $Y \subseteq X$ be an invariant subspace. The following are equivalent:*

1. *$X \searrow\!\!\searrow^G Y$,*
2. *$Y \subseteq X$ is an equivariant strong deformation retract,*
3. *$Y \subseteq X$ is a strong deformation retract.*

If X is a G-space, define X/G to be the *orbit space* with elements the classes of elements of X under the equivalence relation $x \sim x^g$ for $g \in G$ and $x \in X$. Denote by $\bar{x}$ the class of the element $x \in X$ under this relation. Endowed with the quotient topology of the map $x \in X \mapsto \bar{x} \in X/G$, X/G is a topological space.

In the context of finite spaces, X/G turns to be a finite space. Recall we are assuming that our finite spaces are T_0. It can be shown that if X is a finite T_0-G-space then X/G is also a T_0-space. In particular, X/G is a poset with the order relation $\bar{x} \leq \bar{y}$ if $x^g \leq y$ for some $g \in G$.

For a finite G-space X, we can relate the extraction of beat points of X with the fixed point subspace $\mathrm{Fix}_G(X)$ and the orbit space X/G.

Theorem 1.2.25 ([Bar11a, Proposition 8.3.14])**.** *Let X be a finite G-space. If $Y \subseteq X$ is an equivariant strong deformation retract, then $\mathrm{Fix}_G(X) \searrow\!\!\searrow \mathrm{Fix}_G(Y)$ and $X/G \searrow\!\!\searrow Y/G$. In particular, if X is contractible then $\mathrm{Fix}_G(X)$ and X/G are, and hence, $\mathrm{Fix}_G(X) \neq \varnothing$, i.e. there is a fixed point.*

As corollary, if two finite G-spaces are G-homotopy equivalent, then their fixed points subspaces and their orbit spaces are homotopy equivalent.

This holds in general for G-spaces: if $X \simeq_G Y$ then $\mathrm{Fix}_G(X) \simeq \mathrm{Fix}_G(Y)$ and $X/G \simeq Y/G$. For finite spaces it also says that it is compatible with the extraction of beat points.

Note that in general, if a G-space X is contractible then $\mathrm{Fix}_G(X)$ and X/G may not be. For example, take $X = \mathbb{R}$ with the action of $G = \mathbb{Z}$ by translation. Then X is contractible but $\mathrm{Fix}_G(X) = \varnothing$ and $X/G = \mathbb{S}^1$ are not. Even if the G-space X is compact and G is finite, there may be no fixed points. For example, the Casacuberta-Dicks conjecture asserts that if X is a 2-dimensional contractible G-CW-complex, then it has a fixed point by the action of G (see [CD92]). In Chapter 4, Section 4.3 we prove this conjecture when X is some p-subgroup complex of the finite group G.

There is also an equivariant version for simple homotopy types of finite spaces introduced by Barmak [Bar11a, Chapter 8 Section 3]. Barmak uses this equivariant version of simple homotopy type to give equivalent statements of Quillen's conjecture [Bar11a, Theorem 8.4.3].

Let X be a finite G-space. If $x \in X$ is a weak point, then so is x^g for all $g \in G$. It can be shown that $X - \mathcal{O}_x \hookrightarrow X$ is a weak equivalence. In this case we say that there is a *G-elementary collapse* from X to $X - \mathcal{O}_x$ and we denote it by $X \searrow^{Ge} X - \mathcal{O}_x$. Note that $X - \mathcal{O}_x$ is an invariant subspace of X. We say that X *G-collapses* to an invariant subspace Y if Y can be obtained from X by removing orbits of weak points. That is, if there is a series of G-elementary collapses starting at X and ending at Y. We denote it by $X \searrow^{G} Y$. We say that X is *G-collapsible* if $X \searrow^{G} *$. Analogously we can define *G-elementary expansions* and *G-expansions*. If Y is an arbitrary finite G-space, then we say that X and Y *have the same equivariant simple homotopy type*, and we denote it by $X \diagup\!\!\!\searrow^{G} Y$, if there exists a series of G-collapses and G-expansions starting at X and ending at Y.

Remark 1.2.26. We have seen that there are finite spaces which are collapsible but not contractible. By putting the trivial action on such spaces, we obtain examples of G-collapsible but non-contractible finite spaces. On the other hand, we have seen that any contractible finite G-space is G-contractible. However, this does not hold for simple homotopy type. Namely, a finite G-space may be collapsible and non-G-collapsible. See [Bar11a, Example 8.3.3].

Recall that a G-complex is a simplicial complex K together with an action of the group G on its set of vertices by simplicial isomorphisms.

There is an analogue version of equivariant simple homotopy type for finite simplicial complexes (see [Bar11a, Definition 8.3.5]). Note that if X is a finite G-space, then $\mathcal{K}(X)$ is a finite G-complex. Analogously, if K is a finite G-complex, then $\mathcal{X}(K)$ is naturally a finite G-space. The relation between the G-collapses in finite G-spaces and G-complexes is given by the following theorem.

Theorem 1.2.27 ([Bar11a, Theorem 8.3.11])**.**

1. *Let X be a finite G-space and let $Y \subseteq X$ be an invariant subspace. If $X \searrow^{G} Y$ then $\mathcal{K}(X) \searrow^{G} \mathcal{K}(Y)$.*

2. *Let K be a finite G-complex and let $L \subseteq K$ be an invariant subcomplex. If $K \searrow^{G} L$ then*

$\mathcal{X}(K) \searrow^G \mathcal{X}(L)$.
In particular, if X and Y are finite G-spaces, then $X \nearrow\searrow^G Y$ if and only if $\mathcal{K}(X) \nearrow\searrow^G \mathcal{K}(Y)$.

We have an analogue of Theorem 1.2.25 for equivariant simple homotopy type, but only for the fixed point subspace.

Theorem 1.2.28 ([Bar11a, Proposition 8.3.15]). *Let X be a finite G-space and $Y \subseteq X$ an invariant subspace such that $X \searrow^G Y$. Then* $\operatorname{Fix}_G(X) \searrow \operatorname{Fix}_G(Y)$. *In particular, if X is G-collapsible then X^G is collapsible (and therefore nonempty).*

Corollary 1.2.29 ([Bar11a, Corollary 8.3.17]). *Let X and Y be finite G-spaces. If $X \nearrow\searrow^G Y$ then* $\operatorname{Fix}_G(X) \nearrow\searrow \operatorname{Fix}_G(Y)$.

For orbit spaces, it is not true that $X \searrow^G Y$ implies $X/G \searrow Y/G$. See [Bar11a, Example 8.3.16]. In the cited example, $X \searrow^G Y$ and not only X/G does not collapse to Y/G, but also they have different homotopy groups.

We end this section by remarking that a G-collapse in finite G-complexes induces an equivariant strong deformation retract. That is, if $K \searrow^G L$ then $|L| \subseteq |K|$ is an equivariant strong deformation retract. In particular, if $K \nearrow\searrow^G L$ then $|K| \simeq_G |L|$. We refer to [Pit16, Teorema 2.4.20] for a proof of this fact.

Theorem 1.2.30. *Let K and L be finite G-complexes. If $K \searrow^G L$ then $|L| \subseteq |K|$ is an equivariant strong deformation retract. Moreover, if $K \nearrow\searrow^G L$ then $|K|$ and $|L|$ has the same equivariant homotopy type. In particular, it holds if X and Y are finite G-spaces, $K = \mathcal{K}(X)$ and $L = \mathcal{K}(Y)$, and $X \searrow^G Y$ or $X \nearrow\searrow^G Y$.*

1.3 The posets of p-subgroups as finite spaces

In this section, we study the relations between the different posets of p-subgroups as finite topological spaces. It is known that $|\mathcal{K}(\mathcal{A}_p(G))|$, $|\mathcal{K}(\mathcal{S}_p(G))|$, $|\mathcal{K}(\mathcal{B}_p(G))|$, $|\mathcal{R}_p(G)|$ and $|K_p(G)|$ have the same equivariant homotopy type (see for example [Asc93, Qui78, Smi11, TW91]). However, at the level of finite spaces Stong showed for instance that $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ do not have the same homotopy type [Sto84]. We provide proofs and examples showing the different relations between these finite spaces in terms of equivariant (strong) homotopy type and equivariant simple homotopy type. We show that in general $\mathcal{A}_p(G)$, $\mathcal{S}_p(G)$, $\mathcal{B}_p(G)$ and $\mathcal{X}(\mathcal{R}_p(G))$ may have different homotopy type as finite spaces. Here, for the simplicial complexes $K_p(G)$ and $\mathcal{R}_p(G)$ we consider their face posets. In Subsection 1.3.1 we study sufficient conditions on the group G that guarantee that two of them have the same homotopy type as finite spaces. In Subsection 1.3.2 we analyze the contractibility as finite spaces of the different poset of p-subgroups. In Example 1.3.17 we show that $\mathcal{A}_p(G)$ may be homotopically trivial and noncontractible, answering by the negative a question raised by Stong at the end of [Sto84]. The

$X \setminus Y$	$\mathcal{S}_p(G)$	$\mathcal{A}_p(G)$	$\mathcal{B}_p(G)$	$\mathcal{X}(K_p(G))$
$\mathcal{S}_p(G)$	$=$	$\searrow^{G}$	$\searrow^{G}$	$\diagup\!\!\!\searrow^{G}$
$\mathcal{A}_p(G)$	${}^{G}\nearrow$	$=$	$\diagup\!\!\!\searrow^{G}$	${}^{G}\nearrow\!\!\nearrow$
$\mathcal{B}_p(G)$	${}^{G}\nearrow$	$\diagup\!\!\!\searrow^{G}$	$=$	$\diagup\!\!\!\searrow^{G}$
$\mathcal{X}(K_p(G))$	$\diagup\!\!\!\searrow^{G}$	$\searrow\!\!\searrow^{G}$	$\diagup\!\!\!\searrow^{G}$	$=$

Table 1.1: Homotopy relations as finite spaces.

same example also shows that $\mathcal{X}(\mathcal{R}_p(G))$ may be homotopically trivial and non-contractible as finite space. Finally, in Subsection 1.3.3 we give an algebraic-combinatorial description of what the contractibility of $\mathcal{A}_p(G)$ means.

Recall that we have the following implications concerning equivariant homotopy types of finite spaces:

$$\begin{tikzcd}
\searrow\!\!\searrow^{G} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \simeq_G \arrow[d, Rightarrow] \\
\searrow^{G} \arrow[r, Rightarrow] & \diagup\!\!\!\searrow^{G}
\end{tikzcd}$$

With these implications in mind, we give the relations between the different posets of p-subgroups in Table 1.1. In each cell we put the relation between X and Y, where X corresponds to a poset of the first column and Y corresponds to a poset of the first row.

The poset $\mathcal{X}(\mathcal{R}_p(G))$ is closely related with the subdivision posets $\mathcal{S}_p(G)'$ and $\mathcal{A}_p(G)'$. In general, we have that $\mathcal{X}(\mathcal{R}_p(G)) \diagup\!\!\!\searrow^{G} \mathcal{S}_p(G)$.

Proposition 1.3.1 (Cf. [Bou84, Qui78, TW91])**.** *The following relations hold in terms of finite spaces:*

1. $\mathcal{S}_p(G) \searrow^{G} \mathcal{A}_p(G)$,
2. $\mathcal{S}_p(G) \searrow^{G} \mathcal{B}_p(G)$,
3. $\mathcal{X}(K_p(G)) \searrow\!\!\searrow^{G} \mathcal{A}_p(G)$,
4. $\mathcal{X}(\mathcal{R}_p(G)) \diagup\!\!\!\searrow^{G} \mathcal{A}_p(G)$.

In particular, the inclusions of $|\mathcal{A}_p(G)|$, $|\mathcal{B}_p(G)|$, $|\mathcal{R}_p(G)|$ in $|\mathcal{S}_p(G)|$ are G-homotopy equivalences.

Proof. The last part of the statement follows from Theorem 1.2.30.

We prove first that $\mathcal{S}_p(G) \searrow^{G} \mathcal{A}_p(G)$ and $\mathcal{S}_p(G) \searrow^{G} \mathcal{B}_p(G)$. The original idea of this proof is due to Bouc [Bou84].

We show that $\mathcal{A}_p(G)$ is obtained from $\mathcal{S}_p(G)$ by removing orbits of weak points from top to bottom. Take $P \in \mathcal{S}_p(G) - \mathcal{A}_p(G)$. Then, $1 \neq \Phi(P)$ and $\Phi(P)Q < P$ for any $Q < P$ since P is not elementary abelian. Therefore, $\mathcal{S}_p(G)_{<P}$ is contractible via $Q \leq Q\Phi(Q) \geq \Phi(Q)$, P is a down weak point and $\mathcal{S}_p(G) \searrow^{G} \mathcal{S}_p(G) - \{P^g : g \in G\}$. By extracting from top to bottom the orbits of the elements in $\mathcal{S}_p(G) - \mathcal{A}_p(G)$ we get that $\mathcal{S}_p(G) \searrow^{G} \mathcal{A}_p(G)$.

An analogue reasoning is obtained with $\mathcal{B}_p(G)$ in place of $\mathcal{A}_p(G)$, but in this case we extract from bottom to top. If $P \in \mathcal{S}_p(G) - \mathcal{B}_p(G)$, then $P < O_p(N_G(P))$ and $Q \leq QO_p(N_G(P)) \geq O_p(N_G(P))$ is a well-defined homotopy for $Q \in \mathcal{S}_p(G)_{>P}$. Therefore, $\mathcal{S}_p(G)_{>P}$ is contractible and P is an up weak point.

Now we show that $\mathcal{X}(K_p(G)) \searrow\!\!\searrow^{G} \mathcal{A}_p(G)$. An element c of $\mathcal{X}(K_p(G))$ is a set of subgroups of G of order p which commute pairwise. This means that they generate a nontrivial elementary abelian p-subgroup. Thus, we have a map $r : \mathcal{X}(K_p(G)) \to \mathcal{A}_p(G)$ defined by $r(c) = \langle X : X \in c \rangle$. In the opposite direction, we have the map $i : \mathcal{A}_p(G) \to \mathcal{X}(K_p(G))$ defined by $i(E) = \{X : X \leq E, |X| = p\}$. Clearly, r and i are order preserving maps, $ri = \mathrm{Id}_{\mathcal{A}_p(G)}$ and $ir \geq \mathrm{Id}_{\mathcal{X}(K_p(G))}$. In consequence, i is an embedding and $\mathcal{A}_p(G)$ is a strong deformation retract of $\mathcal{X}(K_p(G))$. Since $\mathcal{A}_p(G)$ is G-invariant, $\mathcal{X}(K_p(G)) \searrow\!\!\searrow^{G} \mathcal{A}_p(G)$ by Proposition 1.2.24.

Finally, we prove that $\mathcal{X}(\mathcal{R}_p(G)) \nearrow\!\!\searrow^{G} \mathcal{A}_p(G)$. This proof is due to Thévenaz-Webb [TW91]. Consider the map $\phi : \mathcal{X}(\mathcal{R}_p(G)) \to \mathcal{A}_p(G)^{\mathrm{op}}$ defined by $\phi(P_0 < \ldots < P_r) = \Omega_1(Z(P_r)) \cap P_0$. It can be shown that $\Omega_1(Z(P_r)) \cap P_0 = \bigcap_i \Omega_1(Z(P_i))$. Therefore, ϕ is a well-defined and order preserving map. Moreover, $\phi(\mathcal{A}_p(G)^{\mathrm{op}}_{\leq A}) = \{(P_0 < \ldots < P_r) : \Omega_1(Z(P_r)) \cap P_0 \geq A\}$ is contractible via $(P_0 < \ldots < P_r) \leq (A \leq P_0 < \ldots < P_r) \geq (A)$. Hence, by [Bar11a, Proposition 8.3.21], $\mathcal{X}(\mathcal{R}_p(G)) \nearrow\!\!\searrow^{G} \mathcal{A}_p(G)$. $\square$

Remark 1.3.2. To complete the proof of Table 1.1, note that $\nearrow\!\!\searrow^{G}$ is a transitive relation. For instance, $\mathcal{S}_p(G) \searrow^{G} \mathcal{A}_p(G)$ and $\mathcal{S}_p(G) \searrow^{G} \mathcal{B}_p(G)$ implies $\mathcal{A}_p(G) \nearrow\!\!\searrow^{G} \mathcal{B}_p(G)$.

Now we provide examples showing that the relations of Table 1.1 are strict.

Recall that Stong showed that $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ do not have the same homotopy type in general (see [Sto84]). Concretely, for $G = \mathbb{S}_5$ and $p = 2$ he showed that $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ are not homotopy equivalent. In the following example we give the smallest possible group with this property.

Remark 1.3.3. In the subsequent examples we use the fact that $\mathcal{B}_p(G) \subseteq \mathfrak{i}(\mathcal{S}_p(G))$ (see Lemma 1.3.10). Here, $\mathfrak{i}(\mathcal{S}_p(G))$ is the subposet of nontrivial intersections of Sylow p-subgroups of G and it is an equivariant strong deformation retract of $\mathcal{S}_p(G)$ (see [Bar11a, Chapter 9] or Subsection 1.3.3 below).

Example 1.3.4. Let $G = (\mathbb{S}_3 \times \mathbb{S}_3) : C_2$ where C_2 acts permuting the copies of $\mathbb{S}_3$. This group has order 72 and for $p = 2$, the posets $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ do not have the same homotopy type. We have verified it by computing their cores, which have 21 and 39 elements respectively.

In Proposition 1.3.14 we show that this is the smallest groups for which $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ do not have the same homotopy type.

On the other hand, $\mathcal{B}_p(G)$ has no beat point and $\mathrm{i}(\mathcal{S}_p(G)) = \mathcal{B}_p(G)$. Therefore, $\mathcal{B}_p(G)$ and $\mathcal{A}_p(G)$ do not have the same homotopy type but $\mathcal{B}_p(G) \subseteq \mathcal{S}_p(G)$ is a strong deformation retract. We also have that $\mathcal{X}(\mathcal{R}_p(G)) \simeq \mathcal{A}_p(G)'$ and $\mathcal{X}(\mathcal{R}_p(G)) \not\simeq \mathcal{S}_p(G)'$ since their cores have 93 and 57 points respectively.

Example 1.3.5. Let $G = L_3(2) = \mathrm{PSL}_3(2)$ and $p = 2$. Then $\mathcal{S}_p(G)$ and $\mathcal{B}_p(G)$ are not homotopy equivalent since their cores have 56 and 35 elements respectively. This is the smallest group for which the Brown poset of p-subgroups is not homotopy equivalent to the Bouc poset of radical p-subgroups.

The algorithm applied to find this example is the following. We have that $\mathcal{S}_p(G)$ and $\mathcal{B}_p(G)$ have the same homotopy type when $O_p(G) \neq 1$, the Sylow p-subgroups are abelian, $m_p(G) = 1$ or $|G| = p^\alpha q$ for q prime (see Subsection 1.3.1). Then, there are just 4 groups G not satisfying these properties of order at most 168. Namely, $G = (\mathbb{S}_3 \times \mathbb{S}_3) : C_2$, $\mathbb{S}_5$, $(C_3 \times C_3) : QD_{16}$ and $\mathrm{PSL}_3(2)$. In all cases $p = 2$, and $\mathcal{S}_p(G) \simeq \mathcal{B}_p(G)$ except in the latter one.

For $G = \mathrm{PSL}_3(2)$ and $p = 2$, the core of $\mathcal{S}_p(G)$ has 56 points and $\mathcal{B}_p(G)$ and $\mathcal{A}_p(G)$ are minimal finite spaces of with 35 points each. Moreover, $\mathcal{A}_p(G)$ and $\mathcal{B}_p(G)$ are not homotopy equivalent but $\mathcal{A}_p(G)^{\mathrm{op}}$ and $\mathcal{B}_p(G)$ are homeomorphic. In particular, $\mathcal{A}_p(G)'$ and $\mathcal{B}_p(G)'$ are homeomorphic. The core of $\mathcal{X}(\mathcal{R}_p(G))$ have 77 points and it is homotopy equivalent to $\mathcal{A}_p(G)'$ but not to $\mathcal{S}_p(G)'$, whose core has 287 points.

Example 1.3.6. Let G be the group isomorphic to

$$C_2^6 : (C_3^2 : C_3)$$

which has id [1728,47861] in the Small Groups library of GAP. Its order is $1728 = 2^6 3^3$ and it is the smallest group for which $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ do not have the same homotopy type with a prime $p \neq 2$ ($p = 3$ in this case). The cores of $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ have 256 and 512 elements respectively. Moreover, $\mathrm{i}(\mathcal{S}_p(G)) = \mathcal{B}_p(G)$, so $\mathcal{B}_p(G)$ is a strong deformation retract of $\mathcal{S}_p(G)$. The core of $\mathcal{X}(\mathcal{R}_p(G))$ has 1536 points and it is homeomorphic to the core of $\mathcal{A}_p(G)'$. Hence, $\mathcal{X}(\mathcal{R}_p(G)) \simeq \mathcal{A}_p(G)'$. On the other hand, the core of $\mathcal{S}_p(G)'$ has 1024 elements, so $\mathcal{X}(\mathcal{R}_p(G))$ and $\mathcal{S}_p(G)'$ are not homotopy equivalent.

Example 1.3.7. In all examples above, $\mathcal{A}_p(G)'$ and $\mathcal{X}(\mathcal{R}_p(G))$ are homotopy equivalent. However, this does not hold in general. Let $p = 2$ and G be a group isomorphic to

$$H \times \mathbb{S}_3$$

where $H \cong (\mathbb{S}_3 \times \mathbb{S}_3) : C_2$ is the group of Example 1.3.4. The cores of $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ have 87 and 159 elements respectively, so they are not homotopy equivalent. Moreover, $\mathcal{B}_p(G) = \mathfrak{i}(\mathcal{S}_p(G))$ which is a strong deformation retract of $\mathcal{S}_p(G)$. On the other hand, the cores of $\mathcal{S}_p(G)'$, $\mathcal{A}_p(G)'$ and $\mathcal{X}(\mathcal{R}_p(G))$ have 789, 1257 and 1149 elements respectively, so they are not pairwise homotopy equivalent.

Example 1.3.8. The poset $X_p(G)$ is considered in [Pit19] in the study of Webb's conjecture from the point of view of finite spaces (see also Chapter 2 and Proposition 2.3.6). In general, the subposet $X_p(G) \subseteq \mathcal{S}_p(G)$ is not weak equivalent to $\mathcal{S}_p(G)$.

Let $G = (\mathbb{S}_3 \times \mathbb{S}_3) : C_2$ be the group of Example 1.3.4 with $p = 2$. The poset $X_p(G)$ has the homotopy type of a discrete space of 9 points (as finite space), while $\mathcal{S}_p(G)$ is connected and has the weak homotopy type of a wedge of 16 1-spheres. In particular, $\mathcal{S}_p(G)$ could be connected but $X_p(G)$ may not be.

It is easy to see that $X_p(G)$ is invariant under the action of G, $\mathcal{S}_p(G)$ is connected if $X_p(G)$ is, and $\mathcal{S}_p(G)$ is contractible if and only if $X_p(G)$ is contractible.

We do not know if there is a stronger relation or even a version of Quillen's conjecture for this poset (cf. Proposition 1.3.15).

1.3.1 Some cases for which $\mathcal{A}_p(G) \simeq \mathcal{S}_p(G)$

In general, to find the purely algebraic necessary and sufficient conditions on a finite group so that two of its posets of p-subgroups are homotopy equivalent, could be quite difficult since there is a lot of combinatorics involved. The idea of this subsection is to establish some sufficient simple algebraic conditions for two posets of p-subgroups of a group G to be homotopy equivalent. We do not know if some of them are also necessary conditions. Most of these conditions were used to find the examples presented in this chapter. They will also be useful in the subsequent chapters.

The following theorem describes the algebraic condition for $\mathcal{A}_p(G)$ to be a strong deformation retract of $\mathcal{S}_p(G)$. This was partially done by Stong at the end of [Sto84].

Theorem 1.3.9 (Cf. [MP18, Proposition 3.2])**.** *Let G be a finite group and p a prime number dividing its order. The following conditions are equivalent:*

1. *$\Omega_1(S)$ is abelian for $S \in \mathrm{Syl}_p(G)$.*
2. *$\mathcal{A}_p(G) \subseteq \mathcal{S}_p(G)$ is an equivariant strong deformation retract.*
3. *$\mathcal{A}_p(G) \subseteq \mathcal{S}_p(G)$ is a retract.*

Moreover, if these conditions hold, then $\mathcal{A}_p(G)' \subseteq \mathcal{X}(\mathcal{R}_p(G))$ is an equivariant strong deformation retract.

Proof. If $\Omega_1(S)$ is abelian for $S \in \mathrm{Syl}_p(G)$, then $r : \mathcal{S}_p(G) \to \mathcal{A}_p(G)$ defined by $r(Q) = \Omega_1(Q)$ is a strong deformation retract. Conversely, if $r : \mathcal{S}_p(G) \to \mathcal{A}_p(G)$ is a retraction, then for each $Q \in \mathcal{S}_p(G)$ and each subgroup $X \leq Q$ of order p we have that $X = r(X) \leq r(Q) \in \mathcal{A}_p(G)$. In particular, $\Omega_1(Q) \leq r(Q)$ is elementary abelian.

For the moreover part, consider the map $r : \mathcal{X}(\mathcal{R}_p(G)) \to \mathcal{A}_p(G)'$ defined by $r(P_0 < P_1 < \ldots < P_r) = (\Omega_1(P_0) \leq \Omega_1(P_1) \leq \ldots \leq \Omega_1(P_r))$. Set $r_p(G) = \log_p(|G|_p)$. For each $0 \leq i \leq r_p(G)$, consider the following maps $f_i, g_i : \mathcal{X}(\mathcal{R}_p(G)) \to \mathcal{X}(\mathcal{R}_p(G))$.

$$f_i(P_0 < P_1 < \ldots < P_r) = \{\Omega_1(P_j) : |P_j| \leq p^i\} \cup \{P_j : |P_j| \geq p^i\}$$

$$g_i(P_0 < P_1 < \ldots < P_r) = \{\Omega_1(P_j) : |P_j| \leq p^i\} \cup \{P_j : |P_j| > p^i\}$$

Clearly, f_i and g_i are well-defined and order preserving maps which fix $\mathcal{A}_p(G)'$. The fact that $f_i(c), g_i(c) \in \mathcal{X}(\mathcal{R}_p(G))$ if $c \in \mathcal{X}(\mathcal{R}_p(G))$ follows from that $P \trianglelefteq Q$ implies $\Omega_1(P) \trianglelefteq Q$ since $\Omega_1(P) \operatorname{char} P$.

Let $i : \mathcal{A}_p(G)' \hookrightarrow \mathcal{X}(\mathcal{R}_p(G))$ be the inclusion map. Then $ri = \mathrm{Id}_{\mathcal{A}_p(G)'}$. On the other hand, $ir = g_{r_p(G)}$, $\mathrm{Id}_{\mathcal{X}(\mathcal{R}_p(G))} = f_0 = g_0$ and for all $0 \leq i \leq r_p(G)$, $g_{i-1} \leq f_i \geq g_i$. Therefore, $ir \simeq \mathrm{Id}_{\mathcal{X}(\mathcal{R}_p(G))}$ and $\mathcal{A}_p(G)' \subseteq \mathcal{X}(\mathcal{R}_p(G))$ is an equivariant strong deformation retract (see Theorem 1.2.24). $\square$

Recall that $\mathfrak{i}(\mathcal{S}_p(G))$ is the subposet of nontrivial intersections of Sylow p-subgroups of G and it is a strong deformation retract of $\mathcal{S}_p(G)$ (ver Subsection 1.3.3).

Lemma 1.3.10. *For a finite group G, $\mathcal{B}_p(G) \subseteq \mathfrak{i}(\mathcal{S}_p(G))$. That is, every radical p-subgroup of G is an intersection of Sylow p-subgroups of G.*

Proof. Let $Q \in \mathcal{B}_p(G)$ and let P be the intersection of all Sylow p-subgroups of G containing Q. Clearly, $Q \leq P$, and if $Q^g = Q$ then $P^g = P$. Therefore, $N_G(Q) \leq N_G(P)$. By Proposition 1.1.1, $P \leq Q$. In consequence, $Q = Q \in \mathfrak{i}(\mathcal{S}_p(G))$. $\square$

Proposition 1.3.11. *If S is abelian for $S \in \mathrm{Syl}_p(G)$, then $\mathcal{B}_p(G) \subseteq \mathcal{S}_p(G)$ is an equivariant strong deformation retract.*

Proof. By the above lemma it remains to show that $\mathfrak{i}(\mathcal{S}_p(G)) \subseteq \mathcal{B}_p(G)$. Let $Q \in \mathfrak{i}(\mathcal{S}_p(G))$. Then, if $S_1, \ldots, S_n$ are the Sylow p-subgroups of G containing Q, we have $Q = \bigcap_{i=1}^n S_i$. Moreover, $S_i \leq N_G(Q)$ since S_i is abelian. Hence, $\mathrm{Syl}_p(N_G(Q)) = \{S_1, \ldots, S_n\}$ and $O_p(N_G(Q)) = \bigcap_i S_i = Q$, i.e. $Q \in \mathcal{B}_p(G)$. $\square$

Proposition 1.3.12. *Suppose that two different Sylow p-subgroups of G intersect trivially. Then $\mathcal{S}_p(G)$, $\mathcal{B}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{X}(\mathcal{R}_p(G))$, $X_p(G)$ are G-homotopy equivalent to* $\mathrm{Max}(\mathcal{S}_p(G)) = \mathrm{Syl}_p(G)$. *In particular, this holds if G has a unique Sylow p-subgroup.*

Proof. Let n_p be the number of Sylow p-subgroups of G. It is easy to see that each space has $|\operatorname{Syl}_p(G)|$ connected components.

For $\mathcal{S}_p(G)$ and $X_p(G)$, the connected components are $\mathcal{S}_p(S)$ and $X_p(S)$ respectively, for $S \in \operatorname{Syl}_p(G)$. Since they have a maximum, they are all contractible.

The poset $\mathcal{B}_p(G)$ equals $\mathfrak{i}(\mathcal{S}_p(G))$, which is $\operatorname{Syl}_p(G)$ in this case (see Lemma 1.3.10).

The connected components of $\mathcal{A}_p(G)$ are $\mathcal{A}_p(S)$ for $S \in \operatorname{Syl}_p(G)$. Each one of them is contractible via $E \leq E\Omega_1(Z(S)) \geq \Omega_1(Z(S))$ for $E \in \mathcal{A}_p(S)$.

Finally, $\mathcal{X}(\mathcal{R}_p(G))$ has n_p connected components and each one of them has the form $\mathcal{X}(\mathcal{R}_p(S))$ for $S \in \operatorname{Syl}_p(G)$. It remains to show they are all contractible. We can perform a homotopy similar to that of $\mathcal{A}_p(S)$. Let $Z = Z(S)$. Consider the maps $f, g, f_i, g_i : \mathcal{X}(\mathcal{R}_p(S)) \to \mathcal{X}(\mathcal{R}_p(S))$ defined as follows.

$$f(P_0 < \ldots < P_r) = (Z \leq P_0Z \leq \ldots \leq P_rZ)$$

$$g(P_0 < \ldots < P_r) = (Z)$$

$$f_i(P_0 < \ldots < P_r) = \{P_j : |P_j| \leq p^i\} \cup \{P_jZ : |P_j| \geq p^i\}$$

$$g_i(P_0 < \ldots < P_r) = \{P_j : |P_j| < p^i\} \cup \{P_jZ : |P_j| \geq p^i\}$$

They are all well-defined continuous maps and $g_{i+1} \leq f_i \geq g_i$, $g_1 \leq f \geq g$ and $g_{r_p(G)+1} = \mathrm{Id}_{\mathcal{X}(\mathcal{R}_p(S))}$. Therefore, $\mathcal{X}(\mathcal{R}_p(S))$ is contractible. $\square$

The following proposition deals with the homotopy type of a particular family of solvable groups. It is useful when looking for examples.

Proposition 1.3.13. *If $|G| = p^\alpha q$ for p, q primes, then the posets of p-subgroups $\mathcal{S}_p(G)$, $\mathcal{B}_p(G)$, $\mathcal{A}_p(G)$, $\mathcal{X}(\mathcal{R}_p(G))$ and $X_p(G)$ are pairwise G-homotopy equivalent. Moreover, all of them are contractible or else G-homotopy equivalent to* $\operatorname{Max}(\mathcal{S}_p(G)) = \operatorname{Syl}_p(G)$.

Proof. It follows from [MP18, Proposition 3.2] and its proof, Proposition 1.3.12 and Proposition 1.3.15, that $\mathcal{S}_p(G)$ is G-homotopy equivalent to any of the other posets with the exception perhaps of $\mathcal{X}(\mathcal{R}_p(G))$. We prove that $\mathcal{X}(\mathcal{R}_p(G)) \simeq \mathcal{S}_p(G)$. By the proof of [MP18, Proposition 3.2] and Proposition 1.3.12, we may suppose that $1 \neq O_p(G) = S_1 \cap S_2$ for any two distinct Sylow p-subgroups $S_1, S_2 \in \operatorname{Syl}_p(G)$. Therefore, we need to show that $\mathcal{X}(\mathcal{R}_p(G))$ is contractible. We are going to prove that $\mathcal{X}(\mathcal{R}_p(O_p(G))) \subseteq \mathcal{X}(\mathcal{R}_p(G))$ is a strong deformation retract. The result will follow since $\mathcal{X}(\mathcal{R}_p(O_p(G)))$ is contractible by Proposition 1.3.12.

For any $Q \in \mathcal{S}_p(G)$, either $Q \leq O_p(G)$ or there is a unique Sylow p-subgroup $S(Q)$ of G such that $Q \leq S(Q)$. Now, for any chain $c \in \mathcal{X}(\mathcal{R}_p(G))$ either $c \in \mathcal{X}(\mathcal{R}_p(O_p(G)))$ or there exists $Q \in c$ with $Q \not\leq O_p(G)$. If $P \geq Q$ and $P \in c$, then $P \not\leq O_p(G)$ and $S(P) = S(Q)$. Write $c = c_1 \cup c_2$ where $c_1 \in \mathcal{X}(\mathcal{R}_p(O_p(G)))$ and $c_2 \subseteq \mathcal{S}_p(G) - \mathcal{S}_p(O_p(G))$. Note that $P < Q$ if $P \in c_1$

and $Q \in c_2$. Let $S(c_2) = S(Q)$ if $c_2 \neq \varnothing$ and $Q \in c_2$. Consider the following maps defined in $\mathcal{X}(\mathcal{R}_p(G))$.

$$f_i(c) = c_1 \cup \{QZ(S(c_2)) : Q \in c_2, |Q| \geq p^i\} \cup \{Q : Q \in c_2, |Q| \leq p^i\}$$

$$g_i(c) = c_1 \cup \{QZ(S(c_2)) : Q \in c_2, |Q| \geq p^i\} \cup \{Q : Q \in c_2, |Q| < p^i\}$$

$$h_i(c) = c_1 \cup \{(QZ(S(c_2))) \cap O_p(G) : Q \in c_2, |Q| \leq p^i\} \cup \{QZ(S(c_2)) : Q \in c_2, |Q| \geq p^i\}$$

$$e_i(c) = c_1 \cup \{(QZ(S(c_2))) \cap O_p(G) : Q \in c_2, |Q| \leq p^i\} \cup \{QZ(S(c_2)) : Q \in c_2, |Q| > p^i\}$$

All these maps are well-defined and order preserving. For all i we have that $g_i \leq f_i \geq g_{i+1}$, $\mathrm{Id}_{\mathcal{X}(\mathcal{R}_p(G))} = f_{r_p(G)+1}$, $g_0 = e_0$, $e_i \leq h_i \geq e_{i-1}$ and $\mathrm{Im}(h_{r_p(G)+1}) = \mathcal{X}(\mathcal{R}_p(O_p(G)))$. Furthermore, these maps are the identity when restricted to $\mathcal{X}(\mathcal{R}_p(O_p(G)))$. In consequence, we have found a homotopy between the identity of $\mathcal{X}(\mathcal{R}_p(G))$ and a retraction of $\mathcal{X}(\mathcal{R}_p(G))$ onto $\mathcal{X}(\mathcal{R}_p(O_p(G)))$. That is, $\mathcal{X}(\mathcal{R}_p(O_p(G))) \subseteq \mathcal{X}(\mathcal{R}_p(G))$ is a strong deformation retract. □

These results allow us to prove theoretically that the minimal order of a group G for which $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ do not have the same homotopy type is at least 72. The group of Example 1.3.4 realizes this bound.

Proposition 1.3.14. *If $|G| < 72$ then $\mathcal{S}_p(G)$, $\mathcal{B}_p(G)$ and $\mathcal{A}_p(G)$ have the same G-homotopy type for each prime p. Also, $\mathcal{X}(\mathcal{R}_p(G))$ and $\mathcal{A}_p(G)'$ have the same G-homotopy type.*

Proof. Let $1 \leq n < 72$ and let G be a group of order n. If $p \nmid |G|$ all posets are empty and there is nothing to say. Otherwise, $n = p^\alpha m$ with $\alpha \geq 1$ and $(m : p) = 1$. If $\alpha = 1$ or 2, the Sylow p-subgroups are abelian and by Theorem 1.3.9 and Proposition 1.3.11, $\mathcal{A}_p(G) \subseteq \mathcal{S}_p(G)$, $\mathcal{A}_p(G)' \subseteq \mathcal{X}(\mathcal{R}_p(G))$ and $\mathcal{B}_p(G) \subseteq \mathcal{S}_p(G)$ are strong deformation retracts. If $\alpha \geq 3$ then $2^3 3^2 = 72 > n = p^\alpha m \geq 2^3 m$, and thus $1 \leq m < 9$. For $m = 1$ or prime, the result follows from Proposition 1.3.13. So it remains to show that $m \neq 4$, 6 and 8. If $m = 4$, 6 or 8, as $(p : m) = 1$, $p \geq 3$. But then $p^\alpha m \geq 3^3 4 = 108 > 72$. □

1.3.2 Contractibility of the posets of p-subgroups

In this section, we study the contractibility of the posets of p-subgroups. The aim is to find the algebraic necessary and sufficient conditions which characterized it on each poset of p-subgroups.

Proposition 1.3.15 (Cf. [Sto84])**.** *Let G be a finite group and p a prime number. The following conditions are equivalent:*

1. *$O_p(G) \neq 1$,*
2. *$\mathcal{S}_p(G)$ is contractible,*

3. *$\mathcal{B}_p(G)$ is contractible,*

4. *$X_p(G)$ is contractible.*

In particular, if $O_p(G) \neq 1$, $\mathcal{A}_p(G)$ and $\mathcal{X}(\mathcal{R}_p(G))$ are homotopically trivial.

Proof. The proof is essentially the same for each poset. The original idea is due to Stong [Sto84]. If X is any of the posets of the statement and it is contractible, then it has a fixed point (see Theorem 1.2.25). Therefore $O_p(G) \neq 1$.

Conversely, assume $O_p(G) \neq 1$. Then $\mathcal{S}_p(G)$ and $X_p(G)$ are contractible via the same homotopy $P \leq PO_p(G) \geq O_p(G)$ with $P \in \mathcal{S}_p(G)$ or $X_p(G)$ respectively. On the other hand, $O_p(G)$ is the minimum of $\mathcal{B}_p(G)$, so it is contractible (see Proposition 1.1.1). □

Following Stong [Sto84], Quillen's conjecture can be restated in terms of the intrinsic topology of the posets in the following way.

Quillen's conjecture: If $\mathcal{S}_p(G)$ is homotopically trivial then it is contractible.

The same holds for $\mathcal{B}_p(G)$ in place of $\mathcal{S}_p(G)$.

Remark 1.3.16. If X is one of the poset of p-subgroups of a finite group G and it is contractible, then G fixes a point of X and therefore, $O_p(G) \neq 1$ (see Corollary 1.2.21). In particular, if $\mathcal{A}_p(G)$ or $\mathcal{X}(\mathcal{R}_p(G))$ is contractible then $\mathcal{S}_p(G)$ is contractible.

The question of whether $\mathcal{S}_p(G)$ is contractible implies $\mathcal{A}_p(G)$ is contractible was raised by Stong at the end of [Sto84]. We have shown in [MP18] that the answer to Stong's question is by the negative. That is, there exist finite groups G for which $\mathcal{A}_p(G)$ is homotopically trivial but it is not contractible. This highlights the powerful of finite spaces: with the intrinsic topology of the posets, there are homotopic differences between $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ in such a way that Quillen's conjecture does not mean the same in both posets.

Example 1.3.17. Let G be the group of order $576 = 2^6 3^2$ and index 5684 in the library of small groups in Gap [GAP18]. This group has the following structure. Let $H_1 = \langle a, b\rangle = C_2 \times C_2$, $H_2 = \langle c, d\rangle = C_3 \times C_3$ and $N = \langle e, f, g, h\rangle = C_2 \times C_2 \times C_2 \times C_2$. Then $G = (N : H_2) : H_1$ and the actions are given as follows:

$$c^a = c^2, d^a = d^2, e^a = f, f^a = e, g^a = h, h^a = g$$

$$c^b = cb, d^b = d^2, e^b = g, f^b = h, g^b = e, h^b = f$$

$$e^c = e, f^c = e, g^c = h, h^c = gh$$

$$e^d = f, f^d = fe, g^d = gh, h^d = g.$$

In particular, $O_2(G) = N$ is a nontrivial normal 2-subgroup. Thus, $\mathcal{S}_2(G)$ is contractible. However, the poset $\mathcal{A}_2(G)$ is not contractible since it has a core of size 100. Moreover, G is a solvable group, $O_{2'}(G) = 1$ and $C_G(O_2(G)) \leq O_2(G)$ is self-centralizing. Note that $Z(G) = 1$.

This is the smallest example of a finite group with contractible $\mathcal{S}_p(G)$ but non-contractible $\mathcal{A}_p(G)$. The algorithm to find this example was described in [Pit16, Ejemplo 3.7.4]. See also [MP18, Example 3.7].

It can also be shown by using GAP package [FPSC19] that $\mathcal{X}(\mathcal{R}_2(G))$ is a homotopically trivial but non-contractible finite space (its core has 2065 points), and that it is the smallest example with this property. Moreover, the core of $\mathcal{A}_2(G)'$ has 631 points and hence $\mathcal{X}(\mathcal{R}_2(G))$ and $\mathcal{A}_2(G)'$ are not homotopy equivalent as finite spaces (although they are both homotopically trivial). Even more, by Theorem 1.2.13, $\mathcal{S}_2(G)'$ and $\mathcal{B}_2(G)'$ are both contractible.

In particular, $O_p(G) \neq 1$ does not imply that neither $\mathcal{A}_p(G)$ nor $\mathcal{X}(\mathcal{R}_p(G))$ are contractible. Therefore, the natural question of what the contractibility of $\mathcal{A}_p(G)$ means in purely algebraic terms arise. We have answered this question by using an algebraic-combinatorial description of the poset $\mathcal{A}_p(G)$ (see [MP18] and [Pit16]). We do not give a description of the contractibility of $\mathcal{X}(\mathcal{R}_p(G))$ in algebraic terms, but we do give some sufficient conditions for this to be.

In the next section we will develop the results we need to describe the contractibility of $\mathcal{A}_p(G)$ in algebraic-combinatorial terms.

The following proposition provides some particular cases for which the contractibility of $\mathcal{S}_p(G)$ implies that of $\mathcal{A}_p(G)$.

Proposition 1.3.18. *Let G be a finite group and p a prime number. In any of the following cases, the contractibility of $\mathcal{S}_p(G)$ implies that of $\mathcal{A}_p(G)$:*

1. *G acts transitively on* $\mathrm{Max}(\mathcal{A}_p(G))$*,*

2. *$m_p(G) \leq 2$,*

3. *$r_p(G) \leq 3$.*

Proof. Suppose first that all maximal elementary abelian p-subgroups are conjugate. We claim that the intersection of all of them is nontrivial. Indeed, if $\Omega_1(Z(O_p(G))) \leq A$, where A is a maximal elementary abelian p-subgroup, then $\Omega_1(Z(O_p(G))) = \Omega_1(Z(O_p(G)))^g \leq A^g$ for all $g \in G$ and therefore $\Omega_1(Z(O_p(G)))$ is a nontrivial elementary abelian p-subgroup contained in the intersection of all maximal elementary abelian p-subgroups. By Proposition 1.3.25, $\mathcal{A}_p(G)$ is contractible.

If $m_p(G) \leq 2$, then $\mathcal{K}(\mathcal{A}_p(G))$ is a graph. This implies that, in this case, $\mathcal{A}_p(G)$ is homotopically trivial if and only if it is contractible.

If $r_p(G) \leq 3$, $m_p(G) = 1$, 2, or 3, and in the last case $\mathcal{A}_p(G) = \mathcal{S}_p(G)$. □

Example 1.3.19. Let $G = ((\mathbb{Z}_3 \times \mathbb{Z}_3) : \mathbb{Z}_8) : \mathbb{Z}_2$ be the group with id [144,182] in the Small Groups library of GAP. Note that $|G| = 2^4 3^2$. If we take $p = 2$, the cores of the finite spaces $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ have 21 and 39 elements respectively. In particular, they are not homotopy equivalent. It can be shown that all maximal elementary abelian p-subgroups in a same Sylow p-subgroup of this group are conjugate, so in particular all maximal elementary abelian p-subgroups are conjugate in G. This example shows that the condition of item (1) in Proposition 1.3.18 is not sufficient for $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ to be homotopy equivalent.

Example 1.3.20. In Example 1.3.4, for $G = (\mathbb{S}_3 \times \mathbb{S}_3) : C_2$ and $p = 2$, $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ do not have the same homotopy type as finite spaces. Note that $m_p(G) = 1$ and $r_p(G) = 3$. In consequence, conditions (2) and (3) of Proposition 1.3.18 are not sufficient for $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ to be homotopy equivalent.

1.3.3 The contractibility of $\mathcal{A}_p(G)$

By Remark 1.3.16 and Example 1.3.17, the contractibility of the finite space $\mathcal{A}_p(G)$ is strictly stronger than the contractibility of $\mathcal{S}_p(G)$. On the other hand, Proposition 1.3.15 describes the contractibility of $\mathcal{S}_p(G)$ in purely algebraic terms. Therefore, it is natural to look for the purely algebraic conditions that describe the contractibility of $\mathcal{A}_p(G)$. In this section, we give a description of what it means but we need to know some combinatorial aspects of the poset $\mathcal{A}_p(G)$. For doing that, we use the notion of contractibility in steps. Most of the content of this section is part of the article [MP18] and [Pit16].

Definition 1.3.21. Let $f, g : X \to Y$ be two maps between finite spaces. We say that f and g are homotopic in n steps (with $n \geq 0$) if there exist $f_0, \dots, f_n : X \to Y$ such that $f = f_0$, $f_n = g$ and f_i, f_{i+1} are comparable for every $0 \leq i < n$. We denote it by $f \sim_n g$.

Two finite spaces X and Y are homotopy equivalent in n steps (denoted by $X \sim_n Y$) if there are maps $f : X \to Y$ and $g : Y \to X$ such that $fg \sim_n \mathrm{Id}_Y$ and $gf \sim_n \mathrm{Id}_X$. We say that X is contractible in n steps if $X \sim_n *$, or equivalently, there exist $x_0 \in X$ and $f_0 = \mathrm{Id}_X, f_1, \dots, f_n = c_{x_0} : X \to X$, where c_{x_0} is the constant map x_0, such that f_i and f_{i+1} are comparable for each i.

Remark 1.3.22. Note that $X \sim_0 Y$ if and only if they are homeomorphic, and that X is contractible in 1 step if and only if it has a maximum or a minimum. Note also that in this case, X can be carried out to a point by only removing up beat points (if it has a maximum), or down beat points (if it has a minimum). Thus, contractibility in 1 step means that only one type of beat point is needed to be removed. Observe also that if $X \sim_n Y$ and $Y \sim_m Z$, then $X \sim_{n+m} Z$.

Suppose X is a contractible finite space. Therefore, there exists an ordering $x_1, \dots, x_r$ of the elements of X such that x_i is a beat point of $X - \{x_1, \dots, x_{i-1}\}$ for $i = 1, \dots, r-1$. In each step, x_i can be an up beat point or a down beat point. We say that the beat points can be *removed with (at most) n changes* if there are $1 < i_1 < i_2 < \dots < i_n \leq r-1$ such that all the beat points between x_1 and x_{i_1-1}, x_{i_1} and x_{i_2-1}, ..., x_{i_n} and x_{r-1} are of the same kind. For example, if the poset X has a maximum or minimum, one can reach the singleton by removing beat points without any changes (all up beat points, if it has a maximum, and all down beat points if it has a minimum).

Roughly, the number of steps needed in a homotopy between the identity map and a constant map corresponds to the number of changes of beat points needed to reach the core of the finite space. That is the content of the following theorem.

Theorem 1.3.23. *The poset X is contractible in n steps if and only if we can remove the beat points with (at most) $n-1$ changes.*

Proof. Assume first that there exists an ordering $\{x_1,\dots,x_k\} = X$ such that x_j is a beat point of $X_j = X - \{x_1,\dots,x_{j-1}\}$ and that there are at most $n-1$ changes of kind of beat points.

If $n = 1$, then they are all down beat points or all up beat points. Suppose the first case. For each j, let $\hat{U}^{X_j}_{x_j} = \{x \in X_j,\ x < x_j\}$ and $y_j \in X_j$ be $y_j = \max \hat{U}^{X_j}_{x_j}$. Let $r_j : X_j \to X_{j+1}$ be the retraction which sends x_j to y_j and fixes the other points, and let $i_j : X_{j+1} \to X_j$ be the inclusion. Then $\alpha_1 := i_1 r_1 \leq \mathsf{Id}_{X_1} = \mathsf{Id}_X$. Let $\alpha_j = i_1 i_2 \dots i_j r_j \dots r_2 r_1 : X \to X$. Since $i_j r_j \leq \mathsf{Id}_{X_j}$ for all j, we conclude that $\alpha_j \leq \mathsf{Id}_X$ for all j. In particular, for $j = k-1$, $\alpha_{k-1} \leq \mathsf{Id}_X$ and α_{k-1} is a constant map given that $r_{k-1} : X_{k-1} \to X_k = \{x_k\}$. Consequently, $X \sim_1 *$.

Now assume $n > 1$ and take an ordering $\{x_1,\dots,x_k\} = X$ of beat points with at most $n-1$ changes. Take the minimum i such that x_i and x_{i+1} are beat points of different kinds. By the same argument used before, it is easy to see that $X \sim_1 X - \{x_1,\dots,x_i\} = X_{i-1}$ because all the beat points removed are of the same type. By induction, X_{i-1} can be carried out to a point by removing beat points with at most $n-2$ changes, and then $X_{i-1} \sim_{n-1} *$. Therefore, by Remark 1.3.22, $X \sim_n *$.

Suppose now that X is contractible in n steps and proceed by induction on n. If $n = 1$, then X has a maximum or a minimum. In that case we can reach the core of X by removing only up beat points in the first case, or only down beat points in the latter case.

Let $n = 2$ and assume, without loss of generality, that $\mathsf{Id}_X \leq g \geq c_{x_0}$, where c_{x_0} is the constant map x_0. We can suppose that X has neither a minimum nor a maximum, and this implies that g is not the identity map. Let $\operatorname{fix}(g)$ denote the subposet of X of points which are fixed by g. Note that $\operatorname{Max}(X) \subseteq \operatorname{fix}(g) \neq X$. Since $\mathrm{Id}_X \leq g$, for any $x \in X$ we have

$$x \leq g(x) \leq g^2(x) \leq g^3(x) \leq \dots$$

and therefore there exists $i \in \mathbb{N}$ such that $g^i(x) \in \operatorname{fix}(g)$.

Take $x \in X - \operatorname{fix}(g)$ a maximal element. If $x < z$, then $z \in \operatorname{fix}(g)$ by maximality. Now, since $g \geq \mathsf{Id}_X$, we have $x < g(x) \leq g(z) = z$. Therefore, x is an up beat point.

Let $\{x_1,\dots,x_k\}$ be a linear extension of $(X - \operatorname{fix}(g))^{\mathrm{op}}$ and let $X_j = X - \{x_1,\dots,x_{j-1}\}$. We affirm that x_j is an up beat point of X_j for each $j \geq 1$. The case $j = 1$ is what we did before. Suppose $j > 1$ and let $y = g^m(x_j) \in \mathrm{fix}(g) \subseteq X_j$. Take $z \in X_j$ such that $z > x_j$. Then $z \in \mathrm{fix}(g)$ and $x_j < y = g^m(x_j) \leq g^m(z) = z$, which shows that x_j is an up beat point of X_j. Hence, $\mathrm{fix}(g)$ can be obtained from X by removing only up beat points. We show now that $\mathrm{fix}(g)$ has a minimum, using the fact that $g \geq c_{x_0}$. This implies that $\mathrm{fix}(g)$ can be carried out to a single point by removing only down beat points, and hence the beat points of X can be removed with 1 change (first up beat points and then down beat points). In order to see that $\mathrm{fix}(g)$ has a minimum, take $m \in \mathbb{N}$ such that $g^m(x_0) \in \mathrm{fix}(g)$. Then, for any $z \in \mathrm{fix}(g)$ we have $z = g^{m+1}(z) \geq g^m(x_0)$.

Suppose now that $n > 2$. Assume that there exists a fence $\mathrm{Id}_X \leq g_1 \geq g_2 \leq g_3 \geq \ldots$, with $g_n = c_{x_0}$. Let $Y = \mathrm{fix}(g_1)$. We may suppose that $X \neq Y$. By the same argument used in the case $n = 2$, Y is obtained from X by removing only up beat points. Let $i : Y \hookrightarrow X$ be the inclusion map and $r : X \to Y$ the retraction given by the extraction of the up beat points. Then

$$\mathrm{Id}_Y \geq rg_2 i \leq rg_3 i \geq \ldots \overset{\leq}{\geq} rg_n i = c_{rg_n(x_0)}.$$

Then $Y \sim_{n-1} *$ and, by induction, the beat points of Y can be removed with at most $n-2$ changes. This concludes the proof. $\square$

The idea of contractibility in steps is that for each non zero integer n, the combinatorial condition $\mathcal{A}_p(G) \sim_n *$ translate into an algebraic condition in the group G. Therefore, if we know what it means for any n, then we can describe the contractibility of $\mathcal{A}_p(G)$ in algebraic terms.

Using this notion, the contractibility of $\mathcal{A}_p(G)$ in few steps can be described in purely algebraic terms. First we need a lemma.

Lemma 1.3.24. *If $f, g : \mathcal{A}_p(G) \to \mathcal{A}_p(G)$ are two maps such that $\mathrm{Id}_{\mathcal{A}_p(G)} \geq f \leq g$, then $\mathrm{Id}_{\mathcal{A}_p(G)} \leq g$.*

Proof. See [MP18, Lemma 4.4]. $\square$

Proposition 1.3.25. *The followings assertions hold:*

1. *$\mathcal{A}_p(G)$ is contractible in 0 steps if and only if G has only one subgroup of order p, i.e. $\Omega_1(G) \cong C_p$,*

2. *$\mathcal{A}_p(G)$ is contractible in 1 step if and only if $\mathcal{A}_p(G)$ has a maximum, if and only if $\Omega_1(G)$ is abelian,*

3. *$\mathcal{A}_p(G)$ is contractible in 2 steps if and only if the intersection of all maximal elementary abelian p-subgroups is nontrivial, if and only if $p \mid |C_G(\Omega_1(G))|$, if and only if $p \mid |Z(\Omega_1(G))|$,*

4. *$\mathcal{A}_p(G)$ is contractible in 3 steps if and only if there exists an elementary abelian p-subgroup subgroup of G which intersects (in a nontrivial way) every nontrivial intersection of maximal elementary abelian p-subgroups.*

If $\mathcal{A}_p(G)$ is contractible in 2 steps, then $\mathcal{X}(\mathcal{R}_p(G))$ is contractible.

Proof. Item (1) is clear and item (2) follows from the previous lemma. We prove (3) and (4). Assume that $\mathcal{A}_p(G)$ is contractible in 2 steps. By the previous lemma we can suppose that there exists a map $f : \mathcal{A}_p(G) \to \mathcal{A}_p(G)$ with $\mathsf{Id}_{\mathcal{A}_p(G)} \leq f \geq c_N$, where c_N is the constant map with value N, for some $N \in \mathcal{A}_p(G)$. In this way, if $A \in \mathcal{A}_p(G)$ is a maximal element, then $A \leq f(A)$ implies $A = f(A)$ and hence, $A \geq N$. It follows that $N \leq Z(\Omega_1(G))$, i.e. $p \mid |C_G(\Omega_1(G))|$. Conversely, if $p \mid |C_G(\Omega_1(G))|$ then there exists $a \in Z(\Omega_1(G))$ of order p. Let $N = \langle a \rangle$. Thus, $N \in \mathcal{A}_p(G)$ and $A \leq AN \geq N$ is a homotopy in 2 steps in $\mathcal{A}_p(G)$. This concludes the proof of (3).

If $\mathcal{A}_p(G)$ is contractible in 3 steps, then by the previous lemma we can take a homotopy $\mathsf{Id}_{\mathcal{A}_p(G)} \leq f \geq g \leq c_N$, where c_N is the constant map with value N. Moreover, $f(B) \leq r(B)$, and thus $r(B) \geq g(B) \leq N$. This means that $r(B) \cap N \geq g(B) > 1$, and therefore

$$B \leq r(B) \geq r(B) \cap N \leq N$$

is a well-defined homotopy between the identity of $\mathcal{A}_p(G)$ and the constant map N. But then N intersects in a nontrivial way every nontrivial intersection of maximal elements of $\mathcal{A}_p(G)$. Note that this also proves the converse.

It remains to show that if $\mathcal{A}_p(G)$ is contractible in two steps then $\mathcal{X}(\mathcal{R}_p(G))$ is contractible. First note that $\mathcal{X}(\mathcal{R}_p(G))$ is homotopy equivalent to the subposet $X = \{c \in \mathcal{X}(\mathcal{R}_p(G)) : Q = \Omega_1(Q)$ for all $Q \in c\}$. The retraction to this subposet is given by taking Ω_1 to the elements of a chain.

Secondly, let $Z = \Omega_1(Z(\Omega_1(G)))$ (which is nontrivial by hypothesis) and define the following maps in X.

$$f_i(c) = \{QZ : Q \in c, |Q| \geq p^i\} \cup \{Q : Q \in c, |Q| \leq p^i\},$$

$$g_i(c) = \{QZ : Q \in c, |Q| \geq p^i\} \cup \{Q : Q \in c, |Q| < p^i\},$$

$$g(c) = \{QZ : Q \in c\} \cup \{Z\},$$

$$h(c) = \{Z\}.$$

Clearly, $f_i, g_i, g, h : X \to X$ are well defined and order preserving maps, $f_i \geq g_i \leq f_{i-1}$ for all i, $f_0 = g_0 \leq g \geq h$, and $f_{r_p(G)+1} = \mathsf{Id}_X$. □

The following example shows that, unlike what happens with $\mathcal{S}_p(G)$ (which is always contractible in 2 steps since it is conically contractible [Qui78, Proposition 2.4]), the poset $\mathcal{A}_p(G)$ may be contractible in more than 2 steps.

Example 1.3.26. Let $G = \mathbb{S}_4$. Then $|G| = 2^3 3$. Since $N = \langle (12)(34), (13)(24) \rangle$ is a nontrivial normal 2-subgroup of G, both posets $\mathcal{S}_2(G)$ and $\mathcal{A}_2(G)$ are contractible by item (3) of Proposition 1.3.18. In fact, $\mathcal{A}_2(G)$ is contractible in 3 steps but it is not contractible in 2 steps since $\Omega_1(G) = G$ has trivial center. The poset $\mathfrak{i}(\mathcal{A}_p(G))$ of nontrivial intersections of maximal elements (see below for a formal definition) is given by

$$\begin{tikzcd}
\langle (12),(34) \rangle \arrow[d, dash] & \langle (12)(34),(13)(24) \rangle \arrow[dl, dash] \arrow[dr, dash] \arrow[drr, dash] & \langle (13),(24) \rangle \arrow[d, dash] & \langle (14),(23) \rangle \arrow[d, dash] \\
\langle (12)(34) \rangle & & \langle (13)(24) \rangle & \langle (14)(23) \rangle
\end{tikzcd}$$

The intersection of all maximal elementary abelian p-subgroups is trivial, but the subgroup N intersects in a nontrivial way each nontrivial intersection of maximal elementary abelian p-subgroups.

We can find larger groups for which $\mathcal{A}_p(G)$ is contractible in strictly more than 3 steps. However, we do not know if more than 4 steps are necessary. That is, if $\mathcal{A}_p(G) \simeq *$ implies $\mathcal{A}_p(G) \sim_4 *$.

The contractibility of $\mathcal{A}_p(G)$ in more than 3 steps can be described in algebraic terms but with the aid of extra combinatorial information about the poset. The methods that we will use are a generalization of those used in the proofs of Lemma 1.3.24 and Proposition 1.3.25.

For a finite lattice L, recall that $L^* = L - \{\hat{0}, \hat{1}\}$ is called the *proper part* of L. We say that a finite poset X is a *reduced lattice* if $X = L^*$ for some lattice L. Equivalently, for every pair of elements $\{x, y\}$ with an upper bound in X there exists the supremum $x \vee y$. This condition is equivalent to saying that for each pair of elements $\{x, y\}$ with a lower bound in X there exists the infimum $x \wedge y$ (see [Bar11a, Chapter 9]). A reduced lattice X is *atomic* if every element is the supremum of the minimal elements below it, i.e. if $x = \bigvee_{y \in \mathrm{Min}(x)} y$ for each $x \in X$. Analogously, X is *coatomic* if X^{op} is atomic, i.e. if $x = \bigwedge_{y \in \mathrm{Max}(x)} y$ for each $x \in X$.

The poset $\mathcal{A}_p(G)$ is an atomic reduced lattice: the infimum of two elementary abelian p-subgroups with nontrivial intersection is their intersection, and the supremum, when they have an upper bound, is the subgroup generated by both subgroups.

Given two order preserving maps $f, g : X \to Y$, where Y ia a reduced lattice, such that $\{f(a), g(a)\}$ is lower bounded (resp. upper bounded) for each $a \in X$, we define the maps $f \wedge g, f \vee g : X \to Y$ by $(f \wedge g)(a) = f(a) \wedge g(a)$ and $(f \vee g)(a) = f(a) \vee g(a)$.

Proposition 1.3.27. *Let X be an atomic reduced lattice. If $\mathrm{Id}_X \sim_n g$, then there exist $f_0, \ldots, f_n : X \to X$ with*

$$\mathrm{Id}_X = f_0 \leq f_1 \geq f_2 \leq \ldots \overset{\geq}{\leq} f_n = g$$

and such that $f_{2k} = f_{2k-1} \wedge f_{2k+1}$ for each $1 \leq k < n/2$ and $f_{2k+1} = f_{2k} \vee f_{2k+2}$ for each $0 \leq k < n/2$.

Proof. See [MP18, Proposition 4.7] $\square$

The following constructions were introduced by J. Barmak in [Bar11a, Chapter 9]. Given a reduced lattice X, let

$$\mathfrak{i}(X) = \left\{ \bigwedge_{x\in S} x : S \subseteq \mathrm{Max}(X),\, S \neq \varnothing \text{ and lower bounded} \right\}$$

$$\mathfrak{s}(X) = \left\{ \bigvee_{x\in S} x : S \subseteq \mathrm{Min}(X),\, S \neq \varnothing \text{ and upper bounded} \right\}.$$

With these notations, X is atomic if and only if $X = \mathfrak{s}(X)$, and it is coatomic if and only if $X = \mathfrak{i}(X)$. Both $\mathfrak{i}(X)$ and $\mathfrak{s}(X)$ are strong deformation retracts of X (see [Bar11a, Chapter 9]). Moreover, $\mathfrak{i}(X)$ can be obtained from X by extracting only up beat points, and $\mathfrak{s}(X)$ by extracting only down beat points. As $\mathfrak{i}\mathfrak{i}(X) = \mathfrak{i}(X)$ and $\mathfrak{s}\mathfrak{s}(X) = \mathfrak{s}(X)$, we can perform these two operations until we obtain a core of X. In particular, the core of X is both an atomic and coatomic reduced lattice. Let $n \geq 0$. If X is atomic and $n \geq 0$, denote by X_n the $(n+1)$-th term in the sequence

$$X \supseteq \mathfrak{i}(X) \supseteq \mathfrak{s}\mathfrak{i}(X) \supseteq \mathfrak{i}\mathfrak{s}\mathfrak{i}(X) \supseteq \ldots.$$

In the same way, when X is coatomic denote by X_n the $(n+1)$-th term in the sequence

$$X \supseteq \mathfrak{s}(X) \supseteq \mathfrak{i}\mathfrak{s}(X) \supseteq \mathfrak{s}\mathfrak{i}\mathfrak{s}(X) \supseteq \ldots.$$

Remark 1.3.28. If X is a G-poset, then $\mathfrak{i}(X)$ and $\mathfrak{s}(X)$ are G-invariant strong deformation retracts of X. In consequence, this method provides an easy tool to find a G-invariant core of X.

Remark 1.3.29. Note that if X is a reduced lattice and $\mathrm{Id}_X \leq f$, then $f(x) \leq \bigwedge_{y\in \mathrm{Max}(x)} y$ for any $x \in X$.

Theorem 1.3.30. *Let X be an atomic reduced lattice. The following conditions are equivalent:*

1. $X \sim_n *$,
2. $\mathfrak{i}(X) \sim_{n-1} *$,
3. $X_i \sim_{n-i} *$ *for all* $i \geq 0$,
4. $X_n = *$.

With the convention that, for a negative number m, $X \sim_m *$ means that $X \sim_0 *$. Analogous equivalences hold when X is a coatomic reduced lattice, with $\mathfrak{s}(X)$ instead of $\mathfrak{i}(X)$.

Proof. The proof is straightforward from Theorem 1.3.23 and Proposition 1.3.27. The idea is that the only way to extract beat points in a atomic reduced lattice X is by extracting the points of $X - \mathfrak{i}(X)$, which are up beat points. Then we iterate the procedure. See the proof of [MP18, Theorem 4.9] for more details. $\square$

Remark 1.3.31. If X is atomic, then X_n is coatomic for n odd and it is atomic for n even. In particular, if $X \sim_n *$, by the previous theorem $X_n = *$, which means that X_{n-1} has a maximum if n is odd, or it has a minimum if n is even. Thus if we let $\mathcal{M}_n$ to be $\text{Min}(X_n)$ for n even and $\text{Max}(X_n)$ for n odd, we conclude that $X \sim_n *$ if and only if $|\mathcal{M}_n| = 1$.

Now we can apply these results to describe the contractibility in steps of $\mathcal{A}_p(G)$ in algebraic terms.

For each set $\mathcal{M}_n$ we have a subgroup of G which describes if it is a point or not. This subgroup is the intersection of elements of $\mathcal{M}_n$ or the subgroup generated by $\mathcal{M}_n$. In each case, this subgroup will determine if $\mathcal{A}_p(G)$ is contractible in $n+1$ steps or not.

Theorem 1.3.32. *The poset $\mathcal{A}_p(G)$ is contractible in n steps if and only if one of the following holds:*

1. *$n = 0$ and $\mathcal{A}_p(G) = \{*\}$,*
2. *$n \geq 1$ is even and $\bigcap_{A\in\mathcal{M}_{n-1}} A > 1$,*
3. *$n \geq 1$ is odd and $\langle A : A \in \mathcal{M}_{n-1}\rangle$ is abelian.*

Proof. By the above remark $\mathcal{A}_p(G) \sim_n *$ if and only if $|\mathcal{M}_n| = 1$.

If n is odd, $\mathcal{A}_p(G)_{n-1}$ has a maximum and $\mathcal{M}_{n-1}$ is the set of minimal elements of $\mathcal{A}_p(G)_{n-1}$. If $B \in \mathcal{A}_p(G)_{n-1}$ is the maximum, then $B \geq A$ for each $A \in \mathcal{M}_{n-1}$ and hence, $\langle A : A \in \mathcal{M}_{n-1}\rangle$ is an abelian subgroup.

If n is even, $\mathcal{A}_p(G)_{n-1}$ has a minimum and $\mathcal{M}_{n-1}$ is the set of maximal elements of $\mathcal{A}_p(G)_{n-1}$. If $B \in \mathcal{A}_p(G)_{n-1}$ is the minimum, $B \leq A$ for each $A \in \mathcal{M}_{n-1}$ and then $1 < B \leq \bigcap_{A\in\mathcal{M}_{n-1}} A$ is a nontrivial subgroup. This proves the "if" part.

For the "only if" part, note $\mathcal{A}_p(G)_{n-1} \sim_1$. Now the result follows from Theorem 1.3.30. $\square$

It would be interesting to find a better description for the subgroups $\bigcap_{A\in\mathcal{M}_{n-1}} A$ (even n) and $\langle A : A \in \mathcal{M}_{n-1}\rangle$ (odd n). Note that the conditions of the above theorem on these groups say that there are some commuting relations between the elements of order p. For instance, when $n = 2$ it means that there is an element of order p commuting with all the elements of order p. For $n = 3$, it means that there is a nontrivial elementary abelian p-subgroup N such that for each $A \in \mathcal{A}_p(G)$ there is a nontrivial element $x \in N$ that commutes with every element of order p commuting with A.

We could continue this reasoning for higher steps but the algebraic description becomes more technical.

Chapter 2

Webb's conjecture

In the survey [Web87], P. Webb considered the p-subgroup complexes together with the conjugation action induced by the underlying group. He related the cohomology of a finite group G over the p-adic integers with the cohomology of certain stabilizers of the simplices of the p-subgroups complexes (see [Web87, Theorem 3.3]). Webb noted that the p-subgroup complexes of G are a geometry for the group encoding its p-local information. As a matter of fact, when G is a group of Lie type in characteristic p, its Tits building (which is a geometry of G in the sense of Tits) is homotopy equivalent to the p-subgroup complex $\mathcal{K}(\mathcal{S}_p(G))$. Hence, these complexes can be seen as a generalization of the buildings for every finite group at the prime p. Moreover, in [Web87, §5] Webb introduced a Steinberg module of G at the prime p by using the chain complexes of the p-subgroup complexes of G. This Steinberg module (which is actually a virtual module), agrees with the classical Steinberg module for finite Chevalley groups.

Webb also showed that $|\mathcal{K}(\mathcal{S}_p(G))|/G$ is mod p acyclic and conjectured that it is in fact contractible. This conjecture was proved first by P. Symonds in [Sym98]. The proof of Symonds consists on showing that $|\mathcal{K}(\mathcal{S}_p(G))|/G$ is simply connected and acyclic. Later, other authors proved Webb's conjecture by using different methods. For example, Assaf Libman [Lib08] and Markus Linckelmann [Lin09] proved a generalization of Webb's conjecture arising from fusion systems. Namely, if $\mathcal{F}$ is a saturated fusion system over a finite p-group S and $\mathcal{C}$ is a nonempty closed $\mathcal{F}$-collection, then we may form the orbit space $|\mathcal{C}|/\mathcal{F}$ which agrees with $|\mathcal{K}(\mathcal{S}_p(G))|/G$ when $\mathcal{F} = \mathcal{F}_S(G)$ and $\mathcal{C} = \mathcal{S}_p(S)$. They proved that this space is contractible. A more recent proof of this result for fusion systems was obtained by J. Grodal in [Gro16]. Another proof of Webb's conjecture is due to Kai-Uwe Bux [Bux99], by using the Bestvina-Brady's approach to Morse Theory.

In this chapter, we study Webb's conjecture from the point of view of finite spaces. The usual way to study Webb's conjecture is by means of the orbit space of the geometric realization of a p-subgroup complex. Since $\mathcal{K}(\mathcal{A}_p(G))$, $\mathcal{K}(\mathcal{S}_p(G))$, $\mathcal{K}(\mathcal{B}_p(G))$ and $\mathcal{R}_p(G)$ are G-

homotopy equivalent (see Proposition 1.3.1), the orbit spaces $|\mathcal{K}(\mathcal{A}_p(G))|/G$, $|\mathcal{K}(\mathcal{S}_p(G))|/G$, $|\mathcal{K}(\mathcal{B}_p(G))|/G$ and $|\mathcal{R}_p(G)|/G$ are homotopy equivalent. By Symonds' result, all of them are contractible. On the other hand, the structures of the orbit spaces of their face posets may be different. If X is a finite G-space, then we can consider the orbit spaces X/G (which is a poset), $|\mathcal{K}(X)/G|$, $|\mathcal{K}(X)|/G$ and X'/G. In general, these spaces are not homotopy equivalent (see Example 2.2.2). In the context of the p-subgroup posets, we will study the relations between these orbit spaces and their homotopy types. We show in Proposition 2.3.1 that if K is one of the p-subgroup complexes of above, then Webb's conjecture asserts that the finite space $\mathcal{X}(K)/G$ is homotopically trivial. This reformulation of Webb's conjecture makes us think that maybe a stronger restatement of this conjecture holds. That is, whether the finite space $\mathcal{X}(K)/G$ is in fact contractible. We show that $\mathcal{S}_p(G)'/G$ and $\mathcal{B}_p(G)'/G$ are not contractible in general (see Examples 2.3.2 and 2.3.3). However, in all the examples that we have computed the poset $\mathcal{A}_p(G)'/G$ turned out to be contractible. We believe that a stronger version of Webb's conjecture should hold:

Strong Webb's conjecture. *The finite space $\mathcal{A}_p(G)'/G$ is contractible.*

We will prove some particular cases of this stronger conjecture (see Theorem 2.5.12). Note that, since the finite spaces $\mathcal{A}_p(G)$, $\mathcal{S}_p(G)$ and $\mathcal{B}_p(G)$ are not homotopy equivalent in general, the orbit spaces of their subdivision posets may have different homotopy type (and in fact they do).

On the other hand, Webb's conjecture is related with the subdivision posets, but we may ask what happen with the orbit spaces of the original posets. That is, what is the homotopy type of X/G, where $X = \mathcal{A}_p(G)$, $\mathcal{S}_p(G)$ or $\mathcal{B}_p(G)$. Observe that $\mathcal{S}_p(G)/G$ and $\mathcal{B}_p(G)/G$ have a maximum (the class of a Sylow p-subgroup) and therefore they are contractible. However, $\mathcal{A}_p(G)/G$ may not have a maximum and we cannot deduce easily its contractibility. Nevertheless, we show in Theorem 2.4.1 that $\mathcal{A}_p(G)/G$ is always contractible by using some basic notions of fusion theory.

The fusion at p of a finite group G is roughly the information of the conjugation on the p-subgroups of G. One defines the category of fusion of G over a fixed Sylow p-subgroup S and this category (called the fusion system of G at S) encodes the p-local structure of G. In some cases (such as for finite simple groups), the p-local structure determines the global structure of the group. Group theorists and algebraic topologists study general fusion systems over p-groups to better understand (and simplify) the classification of the finite simple groups, modular representation theory and even cohomological properties of finite groups over characteristic p fields.

Some of the results of this chapter appear in [Pit19].

2.1 Fusion systems

The study of the orbit spaces of the posets of p-subgroups is strongly related with the fusion of the groups at the prime p. The aim of this section is to give a brief review of some of the main theorems on fusion systems.

Fix a Sylow p-subgroup $S \leq G$. The *fusion category of G over S* is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S and, for $P, Q \leq S$, the set of morphisms from P to Q is

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \{c_g|_P : g \in G, P^g \leq Q\}.$$

That is, the morphisms from P to Q are those induced by conjugation of an element of G which sends P to a subgroup of Q.

A subgroup $H \leq G$ containing S is said to *control fusion in S* if $\mathcal{F}_S(H) = \mathcal{F}_S(G)$. This means that if $P^g, P \leq S$ for $g \in G$, then there exists $h \in H$ such that $c_g|_P = c_h|_P$.

One of the goal on the study of fusion systems is to find a subgroup H controlling the fusion in S which allows to better describe the fusion category. In this way, we have the following well-known theorems.

Theorem 2.1.1 (Burnside's fusion theorem)**.** *If S is abelian then $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(S))$, and every morphism in $\mathcal{F}_S(G)$ extends to an automorphism of S.*

Recall that a finite group G is *p-nilpotent* if it has a normal p-complement, i.e. $G = O_{p'}(G)S$ for $S \in \mathrm{Syl}_p(G)$. A *p-local subgroup* of G is a subgroup of the form $N_G(P)$ for $P \leq G$ a nontrivial p-subgroup.

Theorem 2.1.2 (Frobenius)**.** *The group G is p-nilpotent if and only if $\mathcal{F}_S(G) = \mathcal{F}_S(S)$, if and only if each p-local subgroup of G is p-nilpotent.*

It is surprising that for odd primes, the condition of the above theorem needs only to be checked in a special subgroup of G. Namely, let $J(S)$ be the subgroup of S generated by the elementary abelian p-subgroups of S of rank $m_p(S)$.

Theorem 2.1.3 (Glauberman-Thompson)**.** *Let p be an odd prime. Then $\mathcal{F}_S(G) = \mathcal{F}_S(S)$ if and only if $\mathcal{F}_S(N_G(Z(J(S)))) = \mathcal{F}_S(S)$.*

This theorem says that we only need to check the condition of Frobenius theorem on the subgroup $N_G(Z(J(S)))$. In particular, if $\mathcal{F}_S(N_G(Z(J(S)))) = \mathcal{F}_S(S)$ then $N_G(Z(J(S)))$ controls G-fusion in S. This last situation holds more often than expected. The obstruction to this to happen is given by the presence of the group $Qd(p)$ in some subquotient of G. Recall that $Qd(p)$ is the natural split-extension $C_p^2 : \mathrm{SL}_2(p)$.

Proposition 2.1.4. *If $G = Qd(p)$ and $S \in \mathrm{Syl}_p(G)$ then $\mathcal{F}_S(G) \neq \mathcal{F}_S(N_G(Z(J(S))))$.*

Theorem 2.1.5 (Glauberman *ZJ*-theorem)**.** *Let p be an odd prime, and let G be a finite group with no subquotient isomorphic to $Qd(p)$. Let S be a Sylow p-subgroup of G. Then $\mathcal{F}_S(N_G(Z(J(S)))) = \mathcal{F}_S(G)$.*

The proof of the above theorems can be found in [Cra11].

In the earlier nineties, Puig axiomatized the properties of the G-fusion in a Sylow p-subgroup S and introduced the *Frobenius categories* on a finite p-group S. He used them as a tool in modular representation theory to study the p-blocks of finite groups, in a more general context, motivated by Alperin-Broué's article [AB79]. Puig did not publish his ideas until 2006, in [Pui06]. The Frobenius categories of Puig are called, in our terminology, *saturated fusion systems*, and they encode the same data as the category $\mathcal{F}_S(G)$.

Further relations of fusion systems with homotopy theory appear in the works of Broto-Levi-Oliver [BLO03]. The connection with homotopy theory is partially motivated by Martino-Priddy's conjecture (whose proof was completed first by Oliver). It asserts that for two finite groups G and H, the p-completions of their classifying spaces $BG_p^\wedge$ and $BH_p^\wedge$ are homotopy equivalent if and only if G and H have the same p-local structure, i.e. they have isomorphic fusion systems at a Sylow p-subgroup (in particular they have isomorphic Sylow p-subgroups). Hence, the fusion systems contain the same information as the p-local information of a classifying space of a group (which is its p-completion). This idea translates to the general context of fusion systems, leading to the notion of *finite p-local groups*.

We give the general definition of a fusion system over a p-group and the main properties we are interested in. We follow the conventions of [AKO11].

Definition 2.1.6. A fusion system over a p-group S is a category $\mathcal{F}$ whose objects are the subgroups of S and whose sets of morphisms satisfy the following conditions for all $P,Q \leq S$:

1. $\mathrm{Hom}_{\mathcal{F}_S(S)}(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$,

2. Each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ can be written as a $\mathcal{F}$-isomorphism followed by an inclusion.

When the fusion system $\mathcal{F}$ is *realizable* by a finite group G, i.e. $\mathcal{F} = \mathcal{F}_S(G)$, then we have some additional properties on it arising from Sylow's theorems. We translate these properties to the general context and define *saturated fusion systems*. Recall that if $K,H \leq G$ are groups, then $\mathrm{Aut}_H(K) = N_H(K)/C_H(K)$.

Definition 2.1.7. Let $\mathcal{F}$ be a fusion system over a p-group S and let $P \leq S$.

1. $Q \leq S$ is *$\mathcal{F}$-conjugate* to P if there exists an $\mathcal{F}$-isomorphism $\varphi : P \to Q$. Write $P^{\mathcal{F}}$ for the set of $\mathcal{F}$-conjugate of P.

2. P is *fully centralized* (fc for short) in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(Q)|$ for all $Q \leq S$ $\mathcal{F}$-conjugate to P.

3. P is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \leq S$ $\mathcal{F}$-conjugate to P.

4. P is *fully automized* in $\mathcal{F}$ if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

5. P is *receptive* in $\mathcal{F}$ if for each $Q \leq S$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q,P)$, there exists a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_Q = \varphi$, where

$$N_\varphi = \{g \in N_S(Q) : \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(P)\}$$

Remark 2.1.8. If $\mathcal{F} = \mathcal{F}_S(G)$, by Sylow's theorems it is easy to prove that $Q \leq S$ is fully normalized (resp. centralized) if and only if $N_S(Q) \in \mathrm{Syl}_p(N_G(Q))$ (resp. $C_S(Q) \in \mathrm{Syl}_p(C_G(Q))$).

Definition 2.1.9. A fusion system $\mathcal{F}$ over a p-group S is called *saturated* if the following conditions hold:

1. (Sylow axiom) Each subgroup $P \leq S$ which is fully normalized in $\mathcal{F}$ is also fully centralized and fully automized in $\mathcal{F}$.

2. (Extension axiom) Each subgroup $P \leq S$ which is fully centralized in $\mathcal{F}$ is also receptive in $\mathcal{F}$.

It can be shown that $\mathcal{F}_S(G)$ is a saturated fusion systems [AKO11, Theorem 2.3]. Those saturated fusion systems which are not realizable by a finite group are called *exotic fusion systems*.

2.2 G-posets and G-complexes

In this section, we study the relations between the different orbit spaces arising from G-posets. We follow Bredon's book [Bre72] for the main definitions and properties of G-complexes. We also refer to Subsection 1.2 for the main results and definitions of G-spaces.

Recall that a G-complex is a finite simplicial complex K with an action of G by simplicial automorphisms. A G-poset is a finite poset X together with an action of a group G by poset automorphisms.

If K is a G-complex, then K/G is the simplicial complex whose vertices are the orbits of vertices of K and $\{\overline{v_0}, \overline{v_1}, \dots, \overline{v_r}\}$ a simplex of K/G, with v_i vertices of K, if there exist $w_i \in \overline{v_i}$ such that $\{w_0, w_1, \dots, w_r\}$ is a simplex of K.

Recall from Subsection 1.2 that if X is a G-poset then X/G is a poset.

If $(x_0 < x_1 < \ldots < x_n)$ is a chain in the G-poset X and $g \in G$, put $(x_0 < x_1 < \ldots < x_n)^g = (x_0^g < x_1^g < \ldots < x_n^g)$. This defines an action of G on $\mathcal{K}(X)$ by simplicial automorphisms and on X' by poset automorphisms. Therefore, if X is a G-poset, $\mathcal{K}(X)$ is a G-complex and X' is a G-poset. Analogously, if K is a G-complex, then $\mathcal{X}(K)$ is a G-poset in the obvious way.

Following the terminology of [Bre72, p. 115], a G-complex K is said to satisfy property (B) on $H \leq G$ if each time we have $\{v_0, \dots, v_n\}$ and $\{v_0^{h_0}, \dots, v_n^{h_n}\}$ simplices of K with $h_i \in H$, then there exists $h \in H$ such that $v_i^{h_i} = v_i^h$ for all i. We say that K satisfies (B) if it does on $H = G$, and that K is regular if it satisfies property (B) on each subgroup of G.

It is easy to see that if K is a G-complex, then K'' is a regular complex (see [Bre72, Ch. III, 1.1 Proposition]). Moreover, if $K = \mathcal{K}(X)$ for a G-poset X, then K' is regular.

Proposition 2.2.1. *Let X be a finite G-poset. Then $\mathcal{K}(X)' = \mathcal{K}(X')$ is a regular G-complex.*

Proof. See [Pit19, Proposition 2.3]. □

We say that the G-poset X satisfies (B) on $H \leq G$ if $\mathcal{K}(X)$ does it as G-complex, and that X is regular if $\mathcal{K}(X)$ is regular.

From now on, we will make a distinction between a simplicial complex and its geometric realization. Recall that $|K|$ is the geometric realization of a simplicial complex K.

If K is a G-complex, $|K|$ is a G-space and $|K|/G$ is its orbit space. There is an induced cell structure on $|K|/G$ which makes it a CW-complex. This structure may not be a triangulation for $|K|/G$ as the following example shows.

Example 2.2.2. Let X be the finite model of $\mathbb{S}^1$ with four points. See Figure 2.1.

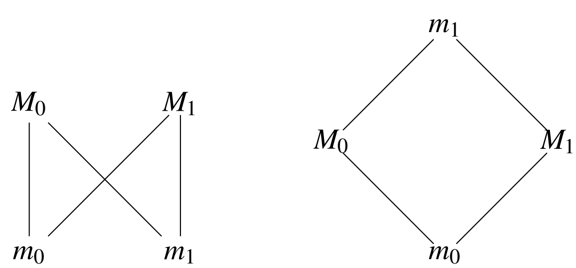

Figure 2.1: Poset X (left) and complex $\mathcal{K}(X)$ (right).

The cyclic group C_2 acts on X by flipping the maximal elements and the minimal elements. The action induced on $|\mathcal{K}(X)|$ is the antipodal action on $\mathbb{S}^1$. The cellular structure induced on $|\mathcal{K}(X)|/C_2$ has two 0-cells and two 1-cells, and it is not a triangulation. See Figure 2.2.

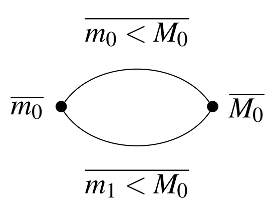

Figure 2.2: Inherited cellular structure on $|\mathcal{K}(X)|/C_2$.

If K is a G-complex, there is a simplicial map $K \to K/G$ which takes a vertex $v \in K$ to its orbit $\overline{v} \in K/G$. The following proposition says that for a regular G-complex K, K/G gives a triangulation for $|K|/G$ (see [Bre72, p. 117]).

Proposition 2.2.3. *If K is a regular G-complex, there is a homeomorphism $\varphi_K : |K|/G \to |K/G|$ induced by the quotient map $|K| \to |K|/G$.*

In general, there is an induced map $\varphi_K : |K|/G \to |K/G|$ defined by

$$\varphi_K\left(\overline{\sum_i t_i v_i}\right) = \sum_i t_i \overline{v_i}.$$

It is just a continuous and surjective map which may not be injective.

If X is a finite G-poset, we consider the orbit spaces $\mathcal{K}(X/G)$, $\mathcal{K}(X)/G$ and $|\mathcal{K}(X)|/G$. We are interested in studying the relationships between them.

Example 2.2.4. Let X be the poset of Example 2.2.2 with $G = C_2$. Then $X/G = \{\overline{m_0}, \overline{M_0}\}$ and $\overline{m_0} < \overline{M_0}$. In particular it is a contractible finite space. The complex $\mathcal{K}(X)/G$ has two vertices $\overline{m_0}, \overline{M_0}$ and a single 1-simplex $\{\overline{m_0}, \overline{M_0}\}$. Consequently, $\mathcal{K}(X)/G$ is contractible. Since $|\mathcal{K}(X)|/G \equiv \mathbb{S}^1$ is not contractible, in general $|K|/G$ and $|K/G|$ do not have the same homotopy type.

It is immediate from the definition that $\mathcal{K}(X/G) = \mathcal{K}(X)/G$ when X is a finite G-poset.

Proposition 2.2.5. *Let X be a G-poset. Then $\mathcal{K}(X)/G$ is exactly the simplicial complex $\mathcal{K}(X/G)$.*

It is easy to see that McCord's map (see Theorem1.2.2) is equivariant and it induces a continuous map on the orbit spaces $\hat{\mu}_X : |\mathcal{K}(X)|/G \to X/G$. We can deduce the following proposition.

Proposition 2.2.6. *If X is a G-poset, we have a commutative diagram*

$$\begin{tikzcd}
|\mathcal{K}(X)|/G \arrow[r, "\hat{\mu}_X"] \arrow[d, "\varphi_{\mathcal{K}(X)}"] & X/G \\
|\mathcal{K}(X)/G| \arrow[r, equal] & |\mathcal{K}(X/G)| \arrow[u, "\approx"', "\mu_{X/G}"]
\end{tikzcd}$$

where $\approx$ stands for weak equivalence. In particular, if $\varphi_{\mathcal{K}(X)}$ is a homeomorphism, $\hat{\mu}_X$ is a weak equivalence.

For a simplicial complex K, let $h : |K'| \to |K|$ be the homeomorphism defined by sending a simplex to its barycentre. If K is a G-complex, then K' is, and h is an equivariant map. In particular $\hat{h} : |K'|/G \to |K|/G$ is a homeomorphism.

Let X be a G-poset and let $K = \mathcal{K}(X)$. The following commutative diagram shows the relationships between the involved maps.

$$
\begin{tikzcd}
X' \arrow[d] & {|K'|} \arrow[l, "\mu_{X'}", "\approx"'] \arrow[r, "h", "\equiv"'] \arrow[d] & {|K|} \arrow[r, "\mu_X", "\approx"'] \arrow[d] & X \arrow[d] \\
X'/G \arrow[ddd, bend right, "\alpha"'] & {|K'|/G} \arrow[l, "\hat{\mu}_{X'}", "\approx"'] \arrow[r, "\hat{h}", "\equiv"'] \arrow[d, "\varphi_{K'}", "\equiv"'] & {|K|/G} \arrow[r, "\hat{\mu}_X"] \arrow[d, "\varphi_K"] & X/G \\
{|\mathcal{K}(X'/G)|} \arrow[u, "\mu_{X'/G}"', "\approx"] \arrow[r, equal] & {|K'/G|} & {|K/G|} \arrow[r, equal] & {|\mathcal{K}(X/G)|} \arrow[u, "\mu_{X/G}"', "\approx"] \\
(X/G)' & & {|\mathcal{K}((X/G)')|} \arrow[ll, "\mu_{(X/G)'}", "\approx"'] \arrow[r, equal] & {|\mathcal{K}(X/G)'|} \arrow[u, "h"', "\equiv"]
\end{tikzcd}
\tag{2.1}
$$

We use the symbol $\equiv$ to denote a homeomorphism.

By Proposition 2.2.1, K' is regular and $\varphi_{K'} : |K'|/G \to |K'/G|$ is a homeomorphism. In particular, $\hat{h} \circ \varphi_{K'}^{-1} : |K'/G| \to |K|/G$ gives a canonical triangulation for $|K|/G$.

In the diagram we have included the map $\alpha : X'/G \to (X/G)'$ defined by

$$\alpha(\overline{(x_0 < x_1 < \ldots < x_n)}) = (\overline{x_0} < \overline{x_1} < \ldots < \overline{x_n}).$$

The following proposition shows that, in a certain way, α is the finite space version of the map $\varphi_K : |K|/G \to |K/G|$.

Proposition 2.2.7. *The map α is injective if and only if X satisfies property (B) on G. Moreover, if α is injective then it is an isomorphism of posets.*

Proof. See [Pit19, Proposition 2.9]. □

We deduce the following corollaries.

Corollary 2.2.8. *If X is a G-poset, for all $n \geq 1$ there is an isomorphism of posets $X^{(n)}/G \equiv (X'/G)^{(n-1)}$. If X is regular, $X^{(n)}/G \equiv (X/G)^{(n)}$ for all $n \geq 0$.*

Proof. Since $\mathcal{K}(X)' = \mathcal{K}(X')$ is a regular G-complex by Proposition 2.2.1, it follows by definition that $X^{(n)}$ is regular for all $n \geq 1$. Assume $n \geq 2$. By the previous proposition, $X^{(n)}/G \equiv (X^{(n-1)}/G)'$, and by induction, $X^{(n-1)}/G \equiv (X'/G)^{(n-2)}$. Thus, $X^{(n)}/G \equiv ((X'/G)^{(n-2)})' \equiv (X'/G)^{(n-1)}$.

If X is regular, then $X'/G \equiv (X/G)'$ and $X^{(n)}/G \equiv (X'/G)^{(n-1)} \equiv (X/G)^{(n)}$ for $n \geq 0$. □

Corollary 2.2.9. *For a G-poset X and $n \geq 1$, $X^{(n)}/G$ is contractible if and only if X'/G is contractible. If X is regular, $X^{(n)}/G$ is contractible if and only if X/G is contractible.*

Proof. It follows from the above corollary and Theorem 1.2.13. □

We consider the action of G by right conjugation on the posets of p-subgroups. That is, $A^g = g^{-1}Ag$ for $A \leq G$ and $g \in G$. The following example taken from [Smi11, Example 3.2.9] shows that $\mathcal{S}_p(G)$ may not be regular.

Example 2.2.10. Let $G = \mathbb{S}_4$, the symmetric group on four letters, and let $X = \mathcal{S}_2(G)$. A Sylow 2-subgroup of G is $S = \langle (13), (1234) \rangle \cong D_8$. The elements $(13)(24)$ and $(12)(34)$ belong to S and they are conjugate by $(23) \in G$. In this way, we have two different subgroups $Q_1 = \langle (13)(24) \rangle$ and $Q_2 = \langle (12)(34) \rangle$ which determine the same point in X/G.

Take the chains $(Q_1 < S)$ and $(Q_2 < S)$. We affirm they have different orbits. Since $Z(S) = Q_1$, if $(Q_1 < S)^g = (Q_2 < S)$, then $g \in N_G(S) \leq N_G(Z(S)) = N_G(Q_1)$ and $Q_2 = Q_1^g = Q_1$, a contradiction.

2.3 Reformulation of Webb's conjecture and a stronger conjecture

In [Web87] P. Webb conjectured that $|\mathcal{K}(\mathcal{S}_p(G))|/G$ is contractible. Since the first proof of this conjecture due to P. Symonds (see [Sym98]), there have been various proofs and generalizations of this conjecture involving fusion systems and Morse Theory (see [Bux99, Gro16, Lib08, Lin09]). In all these articles, the authors work with the homotopy type of the orbit space $|K|/G$, where K is a simplicial complex G-homotopy equivalent to $\mathcal{K}(\mathcal{S}_p(G))$. For example, Symonds and Bux proved that $|\mathcal{R}_p(G)|/G$ is contractible.

By using the results of the previous section, we can restate Webb's conjecture in terms of finite spaces.

Proposition 2.3.1 (Webb's conjecture)**.** *If $K \subseteq \mathcal{K}(\mathcal{S}_p(G))$ is a G-invariant subcomplex which is G-homotopy equivalent to $\mathcal{K}(\mathcal{S}_p(G))$, the finite space $\mathcal{X}(K)/G$ is homotopically trivial. In particular, it holds for $K \in \{\mathcal{K}(\mathcal{S}_p(G)), \mathcal{K}(\mathcal{A}_p(G)), \mathcal{K}(\mathcal{B}_p(G)), \mathcal{R}_p(G)\}$.*

Proof. It follows from Symonds' theorem [Sym98] and Diagram 2.1. See also [Pit19, Proposition 3.1]. ☐

In the context of finite spaces, being contractible is strictly stronger than being homotopically trivial. Hence, we could ask if $\mathcal{X}(K)/G$ is in fact contractible when K is one of the simplicial complexes $\mathcal{K}(\mathcal{S}_p(G))$, $\mathcal{K}(\mathcal{A}_p(G))$, $\mathcal{K}(\mathcal{B}_p(G))$ or $\mathcal{R}_p(G)$. The following examples show that it fails for $K = \mathcal{K}(\mathcal{S}_p(G))$ and $K = \mathcal{K}(\mathcal{B}_p(G))$.

Example 2.3.2. If $G = \mathbb{A}_6$ or $\mathrm{PSL}_2(7)$ and $p = 2$, then $\mathcal{S}_p(G)'/G$ is not a contractible finite space. These examples were tested with GAP and SageMath. In Proposition 2.5.10 we show that $G = \mathrm{PSL}_2(7)$ with $p = 2$ is the smallest configuration for which $\mathcal{S}_p(G)'/G$ is not contractible.

In general, the poset $\mathcal{B}_p(G)'/G$ is not contractible but the example is much larger than that for $\mathcal{S}_p(G)$.

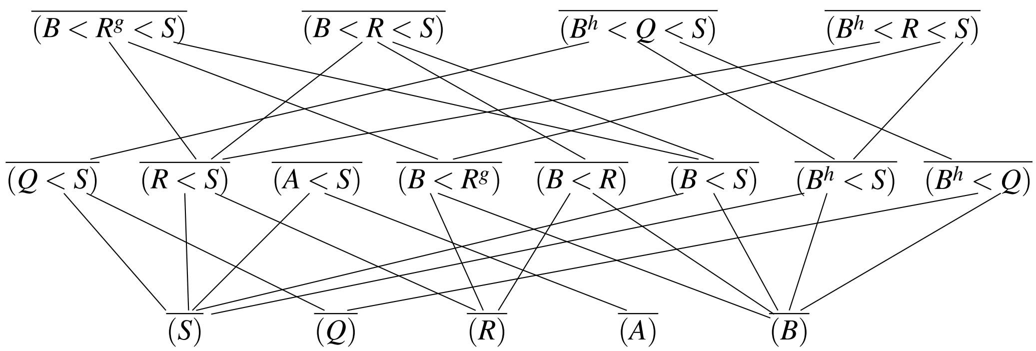

Figure 2.3: Finite space $\mathcal{B}_p(G)'/G$.

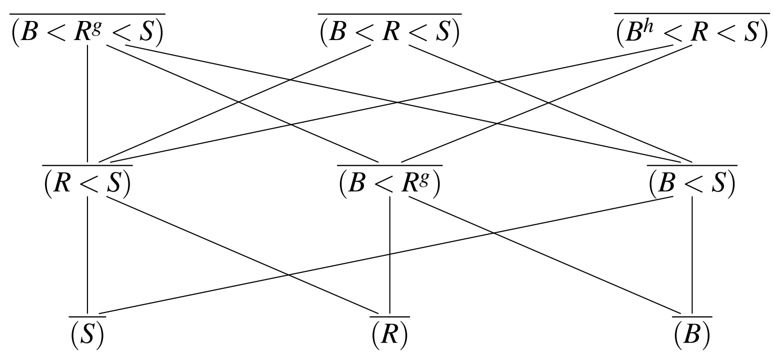

Figure 2.4: Core of the finite space $\mathcal{B}_p(G)'/G$.

Example 2.3.3. Let G be the transitive group of degree 26 and number 62 in the library of transitive groups of GAP. This group can be described as a semidirect product of a non-split extension of $\mathrm{PSL}_2(25)$ by C_2, by C_2, i.e. $G \simeq (\mathrm{PSL}_2(25) \cdot C_2) : C_2$. Here, the dot denotes non-split extension. Its order is $|G| = 2^5.3.5^2.13 = 31200$.

We have computed the poset $\mathcal{B}_p(G)'/G$ by using GAP with the package [FPSC19]. Fix $S \in \mathrm{Syl}_p(G)$. Then $\mathcal{B}_p(G)/G = \{\overline{S},\overline{Q},\overline{R},\overline{A},\overline{B}\}$ with $A,B,R,Q \leq S$ radical p-subgroups of G inside S. For certain $g,h \in G$, we have that $R \neq R^g$, $B \neq B^h$ and $R^g, B^h \leq S$. See Figure 2.3 for the Hasse diagram of the poset.

The finite space $\mathcal{B}_p(G)'/G$ is not contractible since its core has more than one element. For example, we can perform the following extraction of beat points $\overline{(Q<S)}$, $\overline{(B<R)}$, $\overline{(B^h<Q)}$, $\overline{(A)}$, $\overline{(Q)}$, $\overline{(A<S)}$, $\overline{(B^h<Q<S)}$, $\overline{(B^h<S)}$. This leads to the core of $\mathcal{B}_p(G)'/G$ (see Figure 2.4), which has more than one point.

So far, no example of a non-contractible poset $\mathcal{A}_p(G)'/G$ has been found. Recall that Webb's conjecture asserts that $\mathcal{A}_p(G)'/G$ is a homotopically trivial finite space (see Proposition 2.3.1). In Section 2.5 we prove some particular cases for which $\mathcal{A}_p(G)'/G$ is contractible by using both fusion and finite spaces tools. We believe that this stronger property holds in general.

Conjecture 2.3.4. *The poset $\mathcal{A}_p(G)'/G$ is a contractible finite space.*

Remark 2.3.5. By Corollary 2.2.8, for a poset of p-subgroups X, the homotopy type (as finite space) of $X^{(n)}/G$ is determined by X'/G. Moreover, by Corollary 2.2.9, X'/G is contractible if and only if $X^{(n)}/G$ is contractible for some $n \geq 1$.

When trying to prove that $\mathcal{A}_p(G)'/G$ is contractible, the problem of how to control the fusion of chains of elementary abelian p-subgroups appears. This motivated us to work with the poset $X_p(G)$ in which it is easy to control the fusion of its chains. Recall that $X_p(G)$ consists of the nontrivial p-subgroups $Q \leq G$ which are normalized by the Sylow p-subgroup containing it. That is,

$$X_p(G) = \{Q \in \mathcal{S}_p(G) : Q \trianglelefteq S \text{ for all } S \in \mathrm{Syl}_p(G) \text{ such that } Q \leq S\}.$$

In general, the subposet $X_p(G) \subseteq \mathcal{S}_p(G)$ is not weak equivalent to $\mathcal{S}_p(G)$ (see Example 1.3.8).

The following proposition is an easy consequence of Sylow's theorems.

Proposition 2.3.6. *The poset $X_p(G)'/G$ is conically contractible.*

Proof. Fix $S \in \mathrm{Syl}_p(G)$. If $c \in X_p(G)'$ is a chain whose elements are subgroups of S, then $c \cup (S) \in X_p(G)'$. We have the inequalities:

$$\overline{c} \leq \overline{c \cup (S)} \geq \overline{(S)}$$

which will turn into a homotopy of finite spaces after proving that the map $\overline{c} \mapsto \overline{c \cup (S)}$ does not depend on the chosen representative $c \in \mathcal{S}_p(S)' \cap X_p(G)'$.

Assume $c, c^g \in \mathcal{S}_p(S)' \cap X_p(G)'$ and $c = (Q_0 < Q_1 < \ldots < Q_r)$. Then $S, S^g \leq N_G(Q_i^g)$ for all i and S, S^g are Sylow p-subgroups of $\bigcap_i N_G(Q_i^g)$. Take $h \in \bigcap_i N_G(Q_i^g)$ such that $S = S^{gh}$. Thus, $(c \cup (S))^{gh} = (c^{gh} \cup (S^{gh})) = (c^g \cup (S))$. □

Example 2.3.7. A slight modification of the previous idea shows that $\mathcal{N}/G$ is conically contractible, where $\mathcal{N} \subseteq \mathcal{S}_p(G)'$ is the subposet of chains $c \in \mathcal{S}_p(G)'$ such that there exists a Sylow p-subgroup S with $Q \trianglelefteq S$ for all $Q \in c$. We can construct the homotopy in the same way we did in Proposition 2.3.6. First, fix a Sylow p-subgroup S. For $c \in \mathcal{N}$, take $g \in G$ with $Q^g \trianglelefteq S$ for all $Q \in c$. We have a well-defined homotopy $\overline{c} \leq \overline{c^g \cup (S)} \geq \overline{(S)}$ in $\mathcal{N}/G$ (cf. [Lib08, Theorem 3.2]).

2.4 Contractibility of $\mathcal{A}_p(G)/G$

Since G acts transitively on the maximal elements of $\mathcal{S}_p(G)$, the orbit space $\mathcal{S}_p(G)/G$ is contractible because it has a maximum (the orbit of any Sylow p-subgroup). Moreover, the same proof shows that the orbit space of a G-invariant subposet of $\mathcal{S}_p(G)$ containing the Sylow p-subgroups of G is contractible. For example, it holds for the subposets $\mathcal{B}_p(G)$ and $X_p(G)$.

On the other hand, G does not act transitively on the maximal elements of $\mathcal{A}_p(G)$ in general. As a matter of fact, the maximal elements of $\mathcal{A}_p(G)$ could have different orders. Hence, the homotopy type of $\mathcal{A}_p(G)/G$ cannot be deduced easily as in the case of $\mathcal{S}_p(G)/G$. Nevertheless, we show now that $\mathcal{A}_p(G)/G$ is contractible.

Theorem 2.4.1. *The poset $\mathcal{A}_p(G)/G$ is conically contractible.*

Proof. Fix a Sylow p-subgroup $S \leq G$. Given that every orbit of $\mathcal{A}_p(G)/G$ can be represented by a fully centralized element inside S, we define the following homotopy: for $x \in \mathcal{A}_p(G)/G$, take $A \in x$ such that $A \leq S$ is fully centralized, and put

$$x = \overline{A} \leq \overline{\Omega_1(Z(C_S(A)))} \geq \overline{\Omega_1(Z(S))}.$$

We are going to prove that the map $x = \overline{A} \mapsto \overline{\Omega_1(Z(C_S(A)))}$ is well-defined, i.e. it does not depend on the choice of A, and it is order preserving. After proving this, since $\Omega_1(Z(S))$ is always contained in the subgroups of the form $\Omega_1(Z(C_S(A)))$, the result will follow.

Well-defined: take $A, B \in x$ both fully centralized contained in S. We have to see that there exists $k \in G$ such that $\Omega_1(Z(C_S(A)))^k = \Omega_1(Z(C_S(B)))$.

Since $\overline{A} = \overline{B}$, $B = A^g$ for some $g \in G$. Note that $C_S(A) \in \mathrm{Syl}_p(C_G(A))$ implies $C_{S^g}(A^g) \in \mathrm{Syl}_p(C_G(A^g))$. Given that $C_S(A^g) \in \mathrm{Syl}_p(C_G(A^g))$, there exists $h \in C_G(A^g)$ such that $C_S(A^g) = C_{S^g}(A^g)^h = C_S(A)^{gh}$. Thus, conjugation by gh induces an isomorphism between $C_S(A)$ and $C_S(A^g)$. On the other hand, every isomorphism of groups $H_1 \to H_2$ maps $\Omega_1(Z(H_1))$ isomorphically to $\Omega_1(Z(H_2))$. In conclusion, it must be $\Omega_1(Z(C_S(A)))^{gh} = \Omega_1(Z(C_S(A^g)))$.

Order preserving: suppose that $\overline{A} < \overline{B}$ with both $A, B \leq S$ fully centralized. We have to see that

$$\overline{\Omega_1(Z(C_S(A)))} \leq \overline{\Omega_1(Z(C_S(B)))}.$$

Since $\overline{A} < \overline{B}$, there exists $g \in G$ such that $A < B^g$. However, it may happen that $B^g \not\leq S$. We are going to fix this problem by using a trick that will be used repetitively along this chapter. Given that $C_S(A)$ is a Sylow p-subgroup of $C_G(A)$ and $B^g \leq C_G(A)$, there exists $h \in C_G(A)$ such that $B^{gh} \leq C_S(A)$. Moreover, we may choose h in such a way that $B^{gh} \leq C_S(A)$ is fully centralized in $C_G(A)$ with the Sylow p-subgroup $C_S(A)$. This means that $C_{C_S(A)}(B^{gh})$ is a Sylow p-subgroup of $C_{C_G(A)}(B^{gh})$. But $C_{C_S(A)}(B^{gh}) = C_S(B^{gh})$ and $C_{C_G(A)}(B^{gh}) = C_G(B^{gh})$. Therefore, $A \leq B^{gh}$ and $B^{gh} \leq S$ is fully centralized in G.

Now we prove that $\Omega_1(Z(C_S(A))) \leq \Omega_1(Z(C_S(B^{gh})))$. If $x \in Z(C_S(A))$ has order p and $C_S(B^{gh}) \leq C_S(A)$, then $[x, C_S(B^{gh})] \leq [x, C_S(A)] = 1$. Since $B^{gh} \leq C_S(B^{gh})$, we conclude that $x \in Z(C_S(B^{gh}))$. In consequence, by the well-definition we obtain that

$$\Omega_1(Z(C_S(A))) \leq \Omega_1(Z(C_S(B^{gh}))) = \Omega_1(Z(C_S(B)))^k$$

for some $k \in G$. □

Remark 2.4.2. The proof of the above theorem was extracted from [Pit19, Theorem 4.3]. There, we use the subgroup $\Omega_1(Z(\Omega_1(C_S(A))))$ instead of $\Omega_1(Z(C_S(A)))$. The proof is essentially the same, but we decided to change this subgroup in order to give an alternative way to prove it.

Remark 2.4.3. In the previous proof we have used the fact that if $A \leq B$ are elementary abelian p-subgroups of G with $A \leq S$ fully centralized, then there exists $h \in C_G(A)$ such that $B^h \leq S$ is also fully centralized. This works because taking centralizers reverses the inclusion between subgroups.

The other trick used in the proof is that we always take representative elements of an orbit inside a fixed Sylow p-subgroup. This will also be used repetitively.

Remark 2.4.4. For $A \in \mathcal{A}_p(G)$, the subgroup $\Omega_1(Z(\Omega_1(C_G(A))))$ is the intersection of all maximal elementary abelian p-subgroups of G containing A.

Let $M_1, \ldots, M_r$ be the maximal elementary abelian p-subgroups of G containing A. Then $M_i \leq C_G(A)$ for all i, and in particular $M_i \leq \Omega_1(C_G(A))$. If $x \in \bigcap_i M_i$ and $z \in \Omega_1(C_G(A))$ has order p, then the subgroup $\langle A, z\rangle$ is elementary abelian and thus it is contained in some M_i. In particular, x and z commutes. This proves that $\bigcap_i M_i \leq \Omega_1(Z(\Omega_1(C_G(A))))$. Reciprocally, if $y \in \Omega_1(Z(\Omega_1(C_G(A))))$, then y commutes with M_i, and maximality implies $y \in M_i$ for all i. This shows that $\Omega_1(Z(\Omega_1(C_G(A)))) \leq \bigcap_i M_i$.

Remark 2.4.5. Take $X \in \{\mathcal{A}_p(G), \mathcal{S}_p(G), \mathcal{B}_p(G)\}$. If we knew that X is regular, then $X'/G \equiv (X/G)'$ would be contractible by Corollary 2.2.9. However, this may not happen as we have shown in Example 2.2.10.

2.5 Contractibility of $\mathcal{A}_p(G)'/G$

Examples 2.3.3 and 2.3.2 show that $\mathcal{B}_p(G)'/G$ and $\mathcal{S}_p(G)'/G$ may not be contractible in general. However, we have conjectured that $\mathcal{A}_p(G)'/G$ is always contractible (see Conjecture 2.3.4). In this section we prove some particular cases of this stronger conjecture. We also show several cases for which $\mathcal{S}_p(G)'/G$ is contractible and see that the failure of its contractibility arise from the simple groups.

In the previous section we have shown that $\mathcal{A}_p(G)/G$ is contractible by using a trick with the centralizers. This trick cannot be carried out in $\mathcal{S}_p(G)$ or $\mathcal{B}_p(G)$ because not every subgroup in these posets is abelian. The following propositions suggest that the property of being abelian makes things work (cf. Theorem 2.1.1).

Recall from Theorem 1.3.9 that $\mathcal{A}_p(G) \subseteq \mathcal{S}_p(G)$ is a strong deformation retract if and only if $\Omega_1(S)$ is abelian, for $S \in \mathrm{Syl}_p(G)$.

Proposition 2.5.1. *If $\mathcal{A}_p(G) \subseteq \mathcal{S}_p(G)$ is a strong deformation retract, $\mathcal{A}_p(G)'/G$, $\mathcal{S}_p(G)'/G$ and $\mathcal{X}(\mathcal{R}_p(G))/G$ are contractible finite spaces. In particular, this holds when the Sylow p-subgroups are abelian.*

Proof. The hypothesis implies that $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ is an equivariant strong deformation retract. It induces an equivariant strong deformation retract $\mathcal{A}_p(G)' \hookrightarrow \mathcal{S}_p(G)'$ and therefore, $\mathcal{A}_p(G)'/G$ and $\mathcal{S}_p(G)'/G$ have the same homotopy type as finite space. By Theorem 1.3.9, $\mathcal{A}_p(G)'/G \hookrightarrow \mathcal{X}(\mathcal{R}_p(G))/G$ is also a homotopy equivalence. Therefore, it remains to show that $\mathcal{A}_p(G)'/G$ is contractible.

Fix $S \in \mathrm{Syl}_p(G)$. If $A \in \mathcal{A}_p(S)$ and $g \in G$ is such that $A^g \leq S$, then $(A \cap \Omega_1(S))^g \leq A^g \leq \Omega_1(S)$. Hence, $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(\Omega_1(S)))$ by [AKO11, Corollary 4.7] (i.e. $\Omega_1(S)$ is strongly closed, see Remark 2.5.15). In particular $N_G(\Omega_1(S))$ controls fusion in $\mathcal{A}_p(S)'$. The result follows from Theorem 2.5.13 with $O = \Omega_1(S)$. □

Proposition 2.5.2. *If the Sylow p-subgroups of G are abelian then $\mathcal{A}_p(G)'/G$, $\mathcal{S}_p(G)'/G$, $\mathcal{B}_p(G)'/G$ and $\mathcal{X}(\mathcal{R}_p(G))/G$ are contractible finite spaces.*

Proof. By Proposition 1.3.11, $\mathcal{B}_p(G)$ is an equivariant strong deformation retract of $\mathcal{S}_p(G)$. Hence, $\mathcal{B}_p(G)'/G \subseteq \mathcal{S}_p(G)'/G$ is a strong deformation retract. The result now follows from Proposition 2.5.1. □

Remark 2.5.3. If X is a contractible G-poset, X' is contractible by Theorem 1.2.13 and thus, its orbit space X'/G is contractible by Theorem 1.2.25. In particular we have the following proposition.

Proposition 2.5.4. *If $O_p(G) \neq 1$ the finite spaces $\mathcal{S}_p(G)'/G$ and $\mathcal{B}_p(G)'/G$ are contractible.*

Proof. It follows from the above remark and Proposition 1.3.15. □

Remark 2.5.5. By Proposition 1.3.15, $\mathcal{S}_p(G)$ and $\mathcal{B}_p(G)$ are contractible if and only if $O_p(G) \neq 1$. However, by Example 1.3.17, it may be that $O_p(G) \neq 1$ and that $\mathcal{A}_p(G)$ and $\mathcal{A}_p(G)'$ are not contractible. Hence, a priori, we cannot deduce that $\mathcal{A}_p(G)'/G$ is contractible from the above proposition. Nevertheless, if $\mathcal{A}_p(G)$ is contractible then $\mathcal{A}_p(G)'/G$ is (see Remark 2.5.3).

Similarly as in Propositions 1.3.12 and 1.3.13, we can prove the following result.

Proposition 2.5.6 (cf. [Thé92])**.** *If the Sylow p-subgroups of G intersect trivially or $|G| = p^\alpha q$ for q prime, then $\mathcal{A}_p(G)'/G$, $\mathcal{S}_p(G)'/G$, $\mathcal{B}_p(G)'/G$ and $\mathcal{X}(\mathcal{R}_p(G))/G$ are contractible finite spaces.*

Proof. By Propositions 1.3.12 and 1.3.13, these posets have the equivariant homotopy type of a discrete space in which G acts transitively. Therefore, their orbit spaces are contractible. $\square$

Remark 2.5.7. It can be shown that the subgroup $O_{p'}(G)$ does not affect the fusion of the p-subgroups of G. Let $\overline{G} = G/O_{p'}(G)$. In terms of fusion systems, it means that the category $\mathcal{F}_S(G)$ is isomorphic to $\mathcal{F}_S(\overline{G})$ (see [AKO11, Exercise 2.1]). In terms of finite spaces, it means that, for example, $\mathcal{S}_p(G)'/G \equiv \mathcal{S}_p(\overline{G})/\overline{G}$ and $\mathcal{A}_p(G)'/G \equiv \mathcal{A}_p(\overline{G})'/\overline{G}$. Here, $\equiv$ denotes poset isomorphism or homeomorphism of topological spaces.

We also have that $\mathcal{X}(\mathcal{R}_p(G))/G \equiv \mathcal{X}(\mathcal{R}_p(\overline{G}))/\overline{G}$, but it may be less clear. Fix a Sylow p-subgroup $S \leq G$. We can see $\mathcal{X}(\mathcal{R}_p(G))/G$ as the set of equivalence classes of chains $(P_0 < \ldots < P_r) \in \mathcal{X}(\mathcal{R}_p(S))$ with $(P_0 < \ldots < P_r) \sim (P_0 < \ldots < P_r)^g$ if $P_r^g \leq S$. If $(P_0 < \ldots < P_r) \sim (P_0 < \ldots < P_r)^h$ and $h \in \overline{G}$, then for some $g \in G$, $c_g|_{P_r} = c_h|_{P_r}$ (here, c_g is the group morphism induced by conjugation by g at right). Thus, $(P_0 < \ldots < P_r) \sim (P_0 < \ldots < P_r)^h = (P_0 < \ldots < P_r)^g$, and $(P_0 < \ldots < P_r)^h, (P_0 < \ldots < P_r) \in \mathcal{X}(\mathcal{R}_p(S))$.

This homeomorphism may not hold for the poset of radical p-subgroups since the quotient map $G \to \overline{G}$ may send a radical p-subgroup onto a non-radical p-subgroup. For example, let G be the extension of an elementary abelian p-group S of p-rank at least 2 acting faithfully on a solvable p'-group L. Then $\overline{G} \cong S$ is a p-group and $\mathcal{B}_p(\overline{G}) = \{S\}$. In consequence, $\mathcal{B}_p(\overline{G})'/\overline{G}$ consists of a single point. On the other hand, the poset $\mathcal{B}_p(G)$ has the weak homotopy type of a bouquet of spheres of dimension $m_p(S) - 1 \geq 1$ by Theorem 3.1.4. Since the action of S in L is faithful, $\mathcal{B}_p(G)$ is not contractible and it has height $m_p(S) - 1$. Hence, $\mathcal{B}_p(G)'/G$ and $\mathcal{B}_p(\overline{G})'/\overline{G}$ are not homeomorphic since they have different heights.

This remark yields the following proposition. Recall that a group G is *p-constrained* if $C_G(S \cap O_{p',p}(G)) \leq O_{p',p}(G)$, with $S \in \mathrm{Syl}_p(G)$ (see Definition 1.1.7).

Proposition 2.5.8 (cf. [Thé92]). *If $O_{p'}(G) < O_{p',p}(G)$ then $\mathcal{S}_p(G)'/G$ is contractible. In particular it holds for p-solvable groups and, more general, for p-constrained groups.*

Proof. Every p-solvable group is p-constrained, and the condition $O_{p'}(G) < O_{p',p}(G)$ holds for p-constrained groups.

By Remark 2.5.7, if $\overline{G} = G/O_{p'}(G)$, then $\mathcal{S}_p(G)'/G \equiv \mathcal{S}_p(\overline{G})'/\overline{G}$. By Proposition 2.5.4 $\mathcal{S}_p(\overline{G})'/\overline{G}$ is contractible since $O_p(\overline{G}) = O_{p',p}(G)/O_{p'}(G) \neq 1$. $\square$

Remark 2.5.9. The above proposition suggests that the failure to the contractibility of $\mathcal{S}_p(G)'/G$ arise from the extensions of finite simple groups in the following way.

If $O_p(G) \neq 1$ or $O_{p'}(G) < O_{p',p}(G)$, then $\mathcal{S}_p(G)'/G$ is contractible. Therefore, suppose that $O_p(G) = 1 = O_{p'}(G)$ (see Remark 2.5.7). By Remark 1.1.8, $F^*(G) = E(G) = L_1 \times \ldots \times L_n$ is the direct product of simple groups of order divisible by p and $F^*(G) \leq G \leq \mathrm{Aut}(F^*(G))$. Thus, G is an extension of $L_1 \times \ldots \times L_n$ by some outer automorphisms of this direct product of simple groups.

The following proposition shows that in fact, the minimal example for which $\mathcal{S}_p(G)'/G$ is not contractible is a simple group (see Example 2.3.2).

Proposition 2.5.10. *If (G,p), with p a prime number dividing $|G|$, is a minimal configuration for which $\mathcal{S}_p(G)'/G$ is not contractible, then $G = \mathrm{PSL}_2(7)$ and $p = 2$.*

Proof. Take (G,p) a minimal configuration for which $\mathcal{S}_p(G)'/G$ is not contractible. By the above remark, $F^*(G) \le G \le \mathrm{Aut}(F^*(G))$ and $F^*(G)$ is the direct product of simple groups of order divisible by p. The smallest configurations for $F^*(G)$ are $\mathbb{A}_5$, $\mathrm{PSL}_2(7)$ and $\mathbb{A}_6$, and $p = 2$. Therefore, the first possibilities are $(G,p) = (\mathbb{A}_5, 2)$, $(\mathbb{S}_5, 2)$ or $(\mathrm{PSL}_2(7), 2)$. If $(G,p) = (\mathbb{A}_5, 2)$, then the Sylow p-subgroups of G intersects trivially and therefore $\mathcal{S}_p(G)'/G$ is contractible by Proposition 2.5.6. If $(G,p) = (\mathbb{S}_5, 2)$, by analyzing the radical p-subgroups of G it can be shown that $\imath(\mathcal{S}_p(G)) = \mathcal{B}_p(G)$ and that this poset has height 1. By Proposition 2.3.1 $\mathcal{S}_p(G)'/G$ is a homotopically trivial, and by Remark 1.3.28 it is homotopy equivalent to a poset of height 1. Hence, $\mathcal{S}_p(G)'/G$ is contractible. By Example 2.3.2, we conclude that $(G,p) = (\mathrm{PSL}_2(7), 2)$ is the minimal configuration. □

In the following theorem we summarize the case for which $\mathcal{S}_p(G)'/G$ is a contractible finite space.

Theorem 2.5.11. *Let G be a finite group, p a prime number diving $|G|$ and $S \in \mathrm{Syl}_p(G)$. In the following cases $\mathcal{S}_p(G)'/G$ is a contractible finite space:*

1. *$O_p(G/O_{p'}(G)) \neq 1$; in particular it holds for p-constrained groups (and therefore for p-solvable groups) or if $O_p(G) \neq 1$ (Proposition 2.5.8),*
2. *$\Omega_1(S)$ is abelian (Proposition 2.5.1),*
3. *$|G| = p^\alpha q$, with q prime (Proposition 2.5.6),*
4. *The Sylow p-subgroups of G intersect trivially (Proposition 2.5.6),*
5. *There exists $1 \neq O \le Z(S)$ such that $N_G(O)$ controls G-fusion in S (Remark 2.5.15).*

Now we prove some particular cases of Conjecture 2.3.4, which asserts that $\mathcal{A}_p(G)'/G$ is a contractible finite space. Recall that $\mathcal{A}_p(G)'/G$ is homotopically trivial by Proposition 2.3.1. Some of the techniques used here to prove the contractibility of $\mathcal{A}_p(G)'/G$ strongly use the fact that we are dealing with chains of elementary abelian p-subgroups.

In the following theorem, we collect the main results of this section on the contractibility of $\mathcal{A}_p(G)'/G$. Recall that $r_p(G) = \log_p(|G|_p)$.

Theorem 2.5.12. *Let G be a finite group, p a prime number dividing its order, $S \in \mathrm{Syl}_p(G)$ and $\Omega = \Omega_1(Z(S))$. In the following cases $\mathcal{A}_p(G)'/G$ is a contractible finite space.*

1. $\Omega_1(S)$ is abelian (Proposition 2.5.1),

2. $\mathcal{A}_p(G)$ is contractible (Remark 2.5.5),

3. $|G| = p^\alpha q$, with q prime (Proposition 2.5.6),

4. The Sylow p-subgroups of G intersect trivially (Proposition 2.5.6),

5. The fusion of elementary abelian p-subgroups of S is controlled by $N_G(O)$ for some $1 \neq O \leq \Omega_1(Z(\Omega_1(S)))$ (Theorem 2.5.13),

6. $m_p(G) - m_p(\Omega) \leq 1$ (Theorem 2.5.19),

7. $m_p(G) - m_p(\Omega) = 2$ and $m_p(G) \geq r_p(G) - 1$ (Theorem 2.5.27),

8. $r_p(G) \leq 4$ (Corollary 2.5.28),

9. $G = M_{11}$, M_{12}, M_{22}, J_1, J_2, HS, or p is odd and G is any Mathieu group, Janko group, He, $O'N$, or Ru, or $p = 5$ and $G = Co_1$ (see Corollaries 2.5.30 and 2.5.31 and the discussion below them),

10. $\mathcal{A}_p(G)$ is disconnected (Theorem 2.5.29).

From now on, fix a Sylow p-subgroup $S \leq G$ and let $\Omega = \Omega_1(Z(S))$ be the subgroup generated by the central elements in S of order p. We will use this subgroup to construct homotopies and extract beat points. Note that $m_p(G) = m_p(S)$, and if $H \leq G$ then $m_p(H) \leq m_p(G)$.

We deal with orbits of chains $\overline{c} \in \mathcal{A}_p(G)'/G$ and always assume that c is chosen to be a representative of its orbit inside $\mathcal{A}_p(S)'$.

The following theorem shows that if we require that the normalizer of a nontrivial subgroup $O \leq \Omega_1(Z(\Omega_1(S)))$ controls the fusion on the elementary abelian p-subgroups of S, then $\mathcal{A}_p(G)'/G$ is contractible. Since $N_G(S) \leq N_G(\Omega)$, this condition is satisfied if $N_G(S)$ controls the fusion on the p-subgroups of S. In this case G is termed *p-Goldschmidt*.

Theorem 2.5.13. *Assume that $N_G(O)$ controls the fusion of the subgroups in $\mathcal{A}_p(S)$, where $O \in \mathcal{A}_p(S)$ commute with every elementary abelian p-subgroup of S (i.e. $O \leq Z(\Omega_1(S))$). Then $\mathcal{A}_p(G)'/G$ is contractible.*

In particular, by taking $O = \Omega$, and since $S \leq N_G(S) \leq N_G(\Omega)$, the hypothesis holds if $\mathcal{F}_S(G) = \mathcal{F}_S(S)$ (i.e. G is p-nilpotent), $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(S))$ (i.e. G is p-Goldschmidt) or $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(\Omega))$.

Proof. We construct a homotopy in $\mathcal{A}_p(G)'/G$ between the identity map and a constant map.

For each $i \in \{1, \ldots, r_p(G)\}$, let $f, f_i, g_i : \mathcal{A}_p(G)'/G \to \mathcal{A}_p(G)'/G$ be defined as

$$f_i(\overline{(A_0 < \ldots < A_n)}) = \overline{\{A_j : |A_j| \leq p^i\} \cup \{A_j O : |A_j| \geq p^i\}}$$

$$g_i(\overline{(A_0 < \ldots < A_n)}) = \overline{\{A_j : |A_j| < p^i\} \cup \{A_jO : |A_j| \geq p^i\}}$$
$$f(\overline{(A_0 < \ldots < A_n)}) = \overline{(O \leq OA_0 < \ldots < OA_n)}$$

where the representatives are chosen inside $\mathcal{A}_p(S)'$.

We must check they are well-defined. Suppose that $(A_0 < \ldots < A_n)$, $(A_0 < \ldots < A_n)^g \in \mathcal{A}_p(S)'$. By hypothesis, there exists $h \in N_G(O)$ such that $c_h|_{A_n} = c_g|_{A_n}$. Therefore, $(A_0 < \ldots < A_n)^g = (A_0 < \ldots < A_n)^h$ and $(A_0O \leq \ldots \leq A_nO)^h = (A_0^gO \leq \ldots \leq A_n^gO)$. This shows that f_i, g_i and f are well-defined. Clearly they are order preserving. Finally, $f_i \geq g_i \leq f_{i-1}$ for all i, $f_{m_p(G)+1} = \mathsf{Id}_{\mathcal{A}_p(G)'/G}$, and $g_1(x) \leq f(x) \geq \overline{(O)}$ for all $x \in \mathcal{A}_p(G)'/G$. □

Remark 2.5.14. In the hypotheses of the above theorem, if $O^g \leq S$, then there exists $h \in N_G(O)$ such that $c_h|_O = c_g|_O : O \to O^g$. Therefore, $O = O^g$ and O is the unique conjugate to itself which lies in S. In this case O is termed *weakly closed* in $\mathcal{F}_S(G)$.

Remark 2.5.15. The above theorem can be generalized to prove the contractibility of the other posets $\mathcal{S}_p(G)'/G$, $\mathcal{B}_p(G)'/G$ and $\mathcal{X}(\mathcal{R}_p(G))/G$ when there exists a subgroup analogous to O whose normalizer controls the fusion on the elements of the corresponding p-subgroup poset. For example, if for some $1 \neq O \leq Z(S)$, $N_G(O)$ controls the fusion in S, then $\mathcal{X}(\mathcal{R}_p(G))/G$ and $\mathcal{S}_p(G)'/G$ are contractible. The proof works in the same way than that for $\mathcal{A}_p(G)'/G$ by taking chains in the corresponding poset. Moreover, the hypothesis means that each conjugation morphism $\varphi : R \to P$ between subgroups $P, R \leq S$ extends to a morphism $\Phi : RO \to PO$ with $\Phi(O) = O$ and $\Phi|_R = \varphi$.

This condition can be carried out to a general fusion system. It is equivalent to ask for $\mathcal{F} = N_{\mathcal{F}}(O)$, where $N_{\mathcal{F}}(O)$ is the normalizer category for O. For $Q \leq S$, the normalizer of Q in $\mathcal{F}$ is the category $N_{\mathcal{F}}(Q)$ with elements the subgroups of $N_S(Q)$ and morphisms $\varphi : R \to P$ in $\mathcal{F}$, with $R, P \leq N_S(Q)$, such that there exists $\Phi : RQ \to PQ$ in $\mathcal{F}$ with $\Phi(Q) = Q$ and $\Phi|_R = \varphi$. When Q is fully $\mathcal{F}$-normalized, $N_{\mathcal{F}}(Q)$ is a fusion system over $N_S(Q)$, and if $\mathcal{F} = \mathcal{F}_S(G)$, then $N_{\mathcal{F}}(Q) = \mathcal{F}_{N_S(Q)}(N_G(Q))$.

Since O is an abelian normal subgroup of S, $\mathcal{F} = N_{\mathcal{F}}(O)$ if and only if for any subgroup $R \leq S$ and morphism $\varphi : R \to S$ we have that $\varphi(R \cap O) \subseteq O$ (see [AKO11, Corollary 4.7]). In this case, O is termed *strongly closed in* $\mathcal{F}$.

Remark 2.5.16. Proposition 2.5.8 shows that $\mathcal{S}_p(G)'/G$ is contractible when G is p-solvable. The analogue conclusion for $\mathcal{A}_p(G)'/G$ is not immediate since $O_p(G) \neq 1$ does not imply that $\mathcal{A}_p(G)$ is contractible.

The following theorems focus on the study of the contractibility of $\mathcal{A}_p(G)'/G$ when the difference between the p-rank of G and the p-rank of $Z(S)$ is small. We can interpret the difference $m_p(G) - m_p(\Omega)$, which only depends on the Sylow p-subgroup S and its center, as a measures of how many non-central elements of order p in S there are. For example, if $m_p(G) - m_p(\Omega) = 0$ then $\Omega_1(S) = \Omega_1(Z(S))$, i.e. elements of order p in S are central in S.

We will prove first the case $m_p(G) - m_p(\Omega) \leq 1$. Later, we will show some particular cases of the strong conjecture when $m_p(G) - m_p(\Omega) \leq 2$. In particular we will deduce the contractibility of $\mathcal{A}_p(G)'/G$ when $|S| \leq p^4$ and for some sporadic groups.

The proofs of these theorems were extracted from [Pit19].

First, we need a basic tool of the fusion theory of groups: Alperin's Fusion Theorem. We just require a weaker version of this theorem, which says that we can control the fusion inside a fixed Sylow p-subgroup via the normalizers of its nontrivial subgroups.

Theorem 2.5.17. *Let $S \in \mathrm{Syl}_p(G)$ and suppose that $A, A^g \leq G$. Then there exist subgroups $Q_1, \ldots, Q_n \leq S$ and elements $g_i \in N_G(Q_i)$ such that:*

1. $A^{g_1 \cdots g_{i-1}} \leq Q_i$ *for* $1 \leq i \leq n$,
2. $c_g|_A = c_{g_1 \ldots g_n}|_A$.

In particular, $A^g = A^{g_1 \cdots g_n}$.

A more general version of this theorem asserts that the Q_is can be taken to be *essential* subgroups of S or even S. For more details see [AKO11, Part I, Theorem 3.5].

Remark 2.5.18. If $A \in \mathcal{A}_p(S)$ is a maximal elementary abelian p-subgroup of S, then $\Omega \leq A$. Thus, Ω is contained in the intersection of all maximal elementary abelian p-subgroups of S.

Theorem 2.5.19. *If $m_p(G) - m_p(\Omega) \leq 1$, the finite space $\mathcal{A}_p(G)'/G$ is contractible.*

Proof. The case $m_p(G) = m_p(\Omega)$ holds by Proposition 2.5.1 since $\Omega_1(S)$ is abelian. Hence, we may assume that $m_p(G) - m_p(\Omega) = 1$. Consequently, if $A \in \mathcal{A}_p(S)$, then $A\Omega \in \mathcal{A}_p(S)$ is maximal when $\Omega \not\leq A$ and $A \not\leq \Omega$.

Consider the set $\mathcal{A} = \{A \in \mathcal{A}_p(S) : A \text{ is fc}, \Omega \not\leq A \text{ and } \overline{A} \not\leq \overline{\Omega}\}$. The condition $\overline{A} \leq \overline{\Omega}$ is equivalent to $A \leq \Omega$ for $A \in \mathcal{A}_p(S)$ fc. If $A \in \mathcal{A}_p(S)$ is fully centralized and $\overline{A} \leq \overline{\Omega}$, then $A^g \leq \Omega$ for some $g \in G$. Therefore, $|C_S(A)| \geq |C_S(A^g)| = |S|$ and thus, $C_S(A) = S$, i.e. $A \leq \Omega$. In consequence, $\mathcal{A}$ does not contain maximal elements of $\mathcal{A}_p(S)$, nor central subgroups of S, and $A\Omega$ is maximal if $A \in \mathcal{A}$. Moreover, if a maximal element $B \in \mathcal{A}_p(S)$ containing A also contains Ω, then $B = A\Omega$.

Take representatives of conjugacy classes $A_1, \ldots, A_k \in \mathcal{A}$ such that if $A \in \mathcal{A}$ then $\overline{A} = \overline{A_i}$ for some i, and $\overline{A_i} < \overline{A_j}$ implies $j < i$. We will prove first that the subposet of orbits of chains $\overline{c}$ such that $\overline{(A_i)} \not\leq \overline{c}$ for all i is a strong deformation retract of $\mathcal{A}_p(G)'/G$. In order to do that, assuming we have extracted all orbits of chains containing $\overline{(A_j)}$ for $j < i$, we extract first all orbits containing $\overline{(A_i)}$ but not $\overline{(A_i\Omega)}$. Later we will extract those containing both $\overline{(A_i)}$ and $\overline{(A_i\Omega)}$.

For each $1 \leq i \leq k$, let $\mathcal{P}_i = \{\overline{c} \in \mathcal{A}_p(G)'/G : \overline{(A_j)} \not\leq \overline{c} \text{ for } j \leq i\}$. Assume we have shown that $\mathcal{P}_{i-1} \subseteq \mathcal{A}_p(G)'/G$ is a strong deformation retract. We will prove that $\mathcal{P}_i \subseteq \mathcal{P}_{i-1}$ is a strong deformation retract. Let $A = A_i$. Suppose that we have extracted all possible orbits $\overline{c}$ such that $\overline{(A)} \leq \overline{c}$ but $\overline{(A\Omega)} \not\leq \overline{c}$ as up beat points. If one of them still remains, take a maximal one, say $\overline{c}$.

Note that $\overline{c \cup (A\Omega)}$ was not extracted because $\overline{A\Omega} \neq \overline{A_j}$ for all j. We affirm that $\overline{c}$ is an up beat point covered by $\overline{c \cup (A\Omega)}$. Let $c = (B_1 < \ldots < B_s < A)$, where s could be 0. The chain c has A as largest element since any maximal element containing A has the form $A\Omega$. Moreover, if $B \in c$ is fully centralized with $A < B \leq A\Omega$, then $\Omega \leq B$ or $B \leq \Omega$, as $\overline{B} \neq \overline{A_j}$ for all j. Since $A \not\leq \Omega$, $\Omega \leq B$ and in consequence $B = A\Omega$. Let $d = c \cup (A\Omega) = (B_1 < \ldots < B_s < A < A\Omega)$. If $\overline{c} \prec \overline{d'}$, the representative d' can be taken to have the form $d' = (B_1 < \ldots < B_s < A < (A\Omega)^g)$ by maximality of $\overline{c}$. We also may assume $(A\Omega)^g \leq S$ since A is fully centralized (see Remark 2.4.3). Since $(A\Omega)^g \in \mathcal{A}_p(S)$ is maximal and contains both A and Ω, we have that $A\Omega = (A\Omega)^g$. Thus, $\overline{d} = \overline{d'}$ and $\overline{c}$ is an up beat point covered by $\overline{d}$.

We have proved that the subposet $\mathcal{P}_i \cup \mathcal{D}_i$, where $\mathcal{D}_i = \{\overline{c} \in \mathcal{P}_{i-1} : \overline{(A)}, \overline{(A\Omega)} \leq \overline{c}\}$, is a strong deformation retract of $\mathcal{P}_{i-1}$. Note that $\mathcal{P}_i \cap \mathcal{D}_i = \varnothing$.

Define a map $r : \mathcal{P}_i \cup \mathcal{D}_i \to \mathcal{P}_i$ in the following way. Let $r(\overline{c}) = \overline{c - \{(A)\}}$ if $\overline{c} \in \mathcal{D}_i$ and we choose $c \in \mathcal{A}_p(S)'$ such that $(A) \leq c$. Note that if $(A) \leq c$ then $(A\Omega) \leq c$. Define r to be the identity on $\mathcal{P}_i$. It is easy to see that r is a well-defined order preserving map such that $r(x) \leq x$ for all $x \in \mathcal{P}_i \cup \mathcal{D}_i$.

Hence, we have shown that $\mathcal{P}_i$ is a strong deformation retract of $\mathcal{A}_p(G)'/G$, for all $1 \leq i \leq k$.

The elements of $\mathcal{P}_k$ have the following possible representations:

$$\overline{(C_1 < \ldots < C_s < \Omega < B)} \tag{2.2}$$

$$\overline{(C_1 < \ldots < C_s < B)} \tag{2.3}$$

$$\overline{(C_1 < \ldots < C_s < \Omega)} \tag{2.4}$$

$$\overline{(C_1 < \ldots < C_s)} \tag{2.5}$$

with $s \geq 0$ and $C_s < \Omega < B \leq S$. To see that this list is complete, take $\overline{c} \in \mathcal{P}_k$ with $c \in \mathcal{A}_p(S)'$ and such that every $A \in c$ is fully centralized in S (see Remark 2.4.3). If $B \in c$ then $\Omega \leq B$ or $B \leq \Omega$, since $\overline{(A_i)} \not\leq \overline{c}$ for all i.

The aim is to prove that the map that includes Ω between the C_s's and the B's is well-defined and order preserving. That is, if $\overline{(C_1 < \ldots < C_s)} = \overline{(C_1^g < \ldots < C_s^g)}$ then $\overline{(C_1 < \ldots < C_s < \Omega)} = \overline{(C_1^g < \ldots < C_s^g < \Omega)}$, and analogously with the elements of the form $\overline{(C_1 < \ldots < C_s < B)}$.

Since Ω is central in S, if $\Omega^g \leq S$ is fully centralized then $\Omega^g = \Omega$. Hence, if C_s, $C_s^g \leq \Omega$ and $\overline{(C_1 < \ldots < C_s)} = \overline{(C_1^g < \ldots < C_s^g)}$, the subgroups C_s, C_s^g are fully centralized and

$$\begin{aligned}\overline{(C_1 < \ldots < C_s < \Omega)} &= \overline{(C_1^g < \ldots < C_s^g < \Omega^g)} \\ &= \overline{(C_1^g < \ldots < C_s^g < \Omega^{gh})} \\ &= \overline{(C_1^g < \ldots < C_s^g < \Omega)}.\end{aligned}$$

Since Ω^g may not be included in S, we take $h \in C_G(C_s^g)$ such that $\Omega^{gh} \leq C_S(C_s^g) = S$ is fully centralized (see proof of Theorem 2.4.1 and Remark 2.4.3).

For the other case, as before, assume that

$$\overline{(C_1 < \ldots < C_s < B)} = \overline{(C_1^g < \ldots < C_s^g < B^g)}$$

with $C_s < \Omega < B \leq S$ and $C_s^g < \Omega < B^g \leq S$. By conjugating by g^{-1} we obtain

$$\overline{(C_1^g < \ldots < C_s^g < \Omega < B^g)} = \overline{(C_1 < \ldots < C_s < \Omega^{g^{-1}} < B)}$$

Take $h \in C_G(C_s)$ such that $\Omega^{g^{-1}h} \leq C_S(C_s) = S$ is fully centralized. Therefore, $\Omega^{g^{-1}h} = \Omega$. Given that B^h may not be a subgroup of S, we take $h' \in C_G(\Omega)$ such that $B^{hh'} \leq S$. In particular, $h, h' \in C_G(C_s)$ and we have to prove that

$$\overline{(C_1 < \ldots < C_s < \Omega < B)} = \overline{(C_1 < \ldots < C_s < \Omega < B^{hh'})}$$

with $hh' \in C_G(C_s)$. We use Alperin's Fusion Theorem 2.5.17 inside the group $C_G(C_s)$ with $B, B^{hh'} \leq S \in \mathrm{Syl}_p(C_G(C_s))$. Hence, it remains to see the case $hh' \in N_{C_G(C_s)}(Q)$ where $Q \leq S$ and $B, B^{hh'} \leq Q$. Let $k = hh'$. If $k \in N_G(\Omega)$ we are done, so assume $\Omega \neq \Omega^k$. Observe that $\Omega, \Omega^k \leq B^k$. Since $\Omega \leq Z(Q)$, $\Omega^k \leq Z(Q^k) = Z(Q)$ and Ω^k commutes with every element of order p inside Q. In particular it commutes with B, and by maximality of this, $\Omega^k \leq B$. The condition $\Omega \neq \Omega^k$ implies $B = \Omega\Omega^k = B^k$ by an order argument. In any case, we have proved that

$$\overline{(C_1 < \ldots < C_s < \Omega < B)} = \overline{(C_1 < \ldots < C_s < \Omega < B^k)}$$

as desired.

To complete the proof, note that the map that includes Ω inside the orbits of chains $\overline{c}$ represented as (2.2), (2.3), (2.4), (2.5) is an order preserving map that satisfies

$$\overline{c} \leq \overline{c \cup (\Omega)} \geq \overline{(\Omega)}.$$

In consequence, $\mathcal{A}_p(G)'/G$ is contractible because it has a strong deformation retract which is. □

We obtain the following immediate corollaries (without assuming Proposition 2.3.1).

Corollary 2.5.20. *If $\mathcal{A}_p(G)'/G$ has height 1 (i.e. $m_p(G) \leq 2$) then it is a contractible finite space.*

Corollary 2.5.21 (cf. [Thé92])**.** *If the Sylow p-subgroups of G are generalized quaternion or dihedral then $\mathcal{A}_p(G)'/G$ is contractible.*

Proof. In any case, $m_p(G) \leq 2$, and the result follows from Theorem 2.5.19. □

By following the ideas of the above theorem, we focus now in the case $m_p(G) - m_p(\Omega) = 2$. We will not prove this case in general. Nevertheless, the techniques used here allow us to prove that $\mathcal{A}_p(G)'/G$ is contractible if $|G|_p \leq p^4$ or $r_p(G) \leq m_p(G)+1$. Moreover, we find a strong deformation retract of $\mathcal{A}_p(G)'/G$ which can be used to prove it is contractible. We use again Alperin's Fusion Theorem 2.5.17.

We begin with some preliminary lemmas.

Lemma 2.5.22. *Let G, p and S be as above. If $C \leq S$ is fully centralized, $C\Omega \leq S$ is fully centralized and $C_S(C\Omega) = C_S(C)$.*

Proof. If $A, B \subseteq G$, then it holds $C_G(AB) = C_G(A) \cap C_G(B)$. In this way, $C_S(C\Omega) = C_S(C) \cap C_S(\Omega) = C_S(C)$ since $\Omega \leq Z(S)$. Let $g \in G$ such that $(C\Omega)^g \leq S$. Then

$$|C_S((C\Omega)^g)| = |C_S(C^g) \cap C_S(\Omega^g)| \leq |C_S(C^g)| \leq |C_S(C)| = |C_S(C\Omega)|.$$

$\square$

Remark 2.5.23. If $m_p(G) - m_p(\Omega) = 2$, any maximal element of $\mathcal{A}_p(S)$ has p-rank $m_p(G)$ or $m_p(G) - 1$.

Lemma 2.5.24. *Let G, p and S be as above. Let $\mathcal{P} \subseteq \mathcal{A}_p(G)'/G$ be the following subposet.*

$$\mathcal{P} = \{x \in \mathcal{A}_p(G)'/G : \textit{if } x = \overline{c}, \textit{ with } c \in \mathcal{A}_p(S)', \textit{ and } A \in c \textit{ is fc, then } A \leq \Omega \textit{ or } \Omega \leq A\}.$$

If $m_p(G) - m_p(\Omega) \leq 2$, $\mathcal{P} \subseteq \mathcal{A}_p(G)'/G$ is a strong deformation retract.

Proof. If $m_p(G) - m_p(\Omega) \leq 1$, then the proof is similar to the one of Theorem 2.5.19, so we may assume that $m_p(G) - m_p(\Omega) = 2$. Let $r = m_p(G)$ and let

$$\mathcal{A} = \{A \in \mathcal{A}_p(S) : A \text{ is fc}, \Omega \not\leq A \text{ and } \overline{A} \not\leq \overline{\Omega}\}.$$

Clearly, $\mathcal{A} \cap \mathrm{Max}(\mathcal{A}_p(S)) = \varnothing$ and $\Omega^g \notin \mathcal{A}$ for any $g \in G$.

Note that $\mathcal{P}$ is the subposet of orbits of chains not containing $\overline{(A)}$ for $A \in \mathcal{A}$. We want to extract elements containing $\overline{(A)}$ with $A \in \mathcal{A}$, as beat points.

Let $\mathcal{P}_{r'} = \{\overline{c} \in \mathcal{A}_p(G)'/G : \overline{(A)} \not\leq \overline{c}$ for all $A \in \mathcal{A}$ such that $m_p(A) \geq r'+1\}$. Observe that $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{P}_{r-1} = \mathcal{A}_p(G)'/G$. Inductively, suppose we have proved that $\mathcal{P}_{r'} \subseteq \mathcal{P}_{r'+1}$ is a strong deformation retract with $1 \leq r' \leq r-1$. We will show that $\mathcal{P}_{r'-1} \subseteq \mathcal{P}_{r'}$ is a strong deformation retract. Take $A \in \mathcal{A}$ of p-rank r'. Then either $(A\Omega)^g \in \mathcal{A}$ for some $g \in G$, or else $(A\Omega)^g \notin \mathcal{A}$ for all $g \in G$.

Case 1: there exists $g \in G$ with $(A\Omega)^g \in \mathcal{A}$. Thus, $(A\Omega)^g$ is fully centralized and does not contain Ω. Since $m_p(\Omega) = r-2$,

$$m_p((A\Omega)^g\Omega) = m_p((A\Omega)^g) + m_p(\Omega) - m_p((A\Omega)^g \cap \Omega)$$

$$
\begin{aligned}
&\geq m_p((A\Omega)^g) + m_p(\Omega) - (m_p(\Omega) - 1) \\
&= m_p(A\Omega) + 1 \\
&= m_p(A) + m_p(\Omega) - m_p(A \cap \Omega) + 1 \\
&\geq m_p(A) + (r-2) - (m_p(A) - 1) + 1 \\
&= r
\end{aligned}
$$

That is, $(A\Omega)^g\Omega \in \mathcal{A}_p(S)$ is maximal of p-rank r. Given that $((A\Omega)^g\Omega)^{g^{-1}} \geq A$, there exists $x \in C_G(A)$ such that $((A\Omega)^g\Omega)^{g^{-1}x} \leq S$ is fully centralized. Therefore, we have the following inequalities

$$
\begin{aligned}
|C_S((A\Omega)^g\Omega)| = |C_S((A\Omega)^g)| = |C_S(A\Omega)| &= |C_S(A)| \\
&\geq |C_S(((A\Omega)^g\Omega)^{g^{-1}x})| \\
&\geq |C_S((A\Omega)^g\Omega)|
\end{aligned}
$$

which are in fact equalities. Moreover, we deduce that $C_S(A) = C_S(((A\Omega)^g\Omega)^{g^{-1}x})$. Let $D = ((A\Omega)^g\Omega)^{g^{-1}x}$. Observe that $D \in \mathcal{A}_p(S)$ is maximal because of its rank. Hence, $D = \Omega_1(C_S(D)) = \Omega_1(C_S(A))$ and there is a unique maximal element above A.

Now take any orbit $\overline{c} \in \mathcal{P}_{r'}$ containing $\overline{(A)}$. If $\overline{(A < E)} \leq \overline{c}$, for some $E \leq S$ with $m_p(A) < m_p(E) \leq r-1$, by changing E by a conjugate, we may assume that $E \leq S$ is fully centralized. Thus $\overline{E} \notin \overline{\mathcal{A}}$ and it implies $\Omega \leq E$, so that $E \geq A\Omega$. Since $m_p(A\Omega) \geq r-1$, we have the equality $E = A\Omega$, and it is a contradiction. Therefore no orbit of chain over $\overline{(A)}$ can contain an element of rank between $m_p(A)+1$ and $r-1$. Consequently, they have the form $\overline{(A_0 < \ldots < A_s < A)}$ or $\overline{(A_0 < \ldots < A_s < A < E)}$ with $E \in \mathcal{A}_p(S)$ maximal of rank r. By the reasoning above, $E = D$ and $\overline{c}$ is an up beat covered by $\overline{(A_0 < \ldots < A_s < A < D)}$. We can remove all orbits containing $\overline{(A)}$ but not $\overline{(D)}$ from top to bottom since they are up beat points at the moment of their extraction.

After extracting all these elements, the elements containing $\overline{(A)}$ that remain have the form $\overline{(A_0 < \ldots < A_s < A < D)}$. Each one of them is a down beat point covering uniquely the element $\overline{(A_0 < \ldots < A_s < D)}$, if we extract them from bottom to top.

Case 2: $(A\Omega)^g \notin \mathcal{A}$ for all $g \in G$. It is easy to see that $m_p(A\Omega) = r$ or $r-1$. In the former case, we can extract all elements containing $\overline{(A)}$ by using the same reasoning of the Case 1. In the latter case, we want to do something similar, but it may happen that A has more than one maximal element of $\mathcal{A}_p(S)$ above it. If it has just one, it is similar to the proof of Case 1. Assume there are more than one maximal element above A. Note that they have p-rank r because $m_p(A\Omega) = r-1$.

Like before, we extract first from top to bottom all orbits of chains containing $\overline{(A)}$ but not $\overline{(A\Omega)}$. These elements have the forms $\overline{(A_0 < \ldots < A_s < A)}$ and $\overline{(A_0 < \ldots < A_s < A < B)}$ with

$B \in \mathcal{A}_p(S)$ maximal. As $\Omega \le B$,

$$\overline{(A_0 < \ldots < A_s < A < B)} < \overline{(A_0 < \ldots < A_s < A < A\Omega < B)}.$$

If $\overline{(A_0 < \ldots < A_s < A < B)} \prec \overline{d}$ at the moment of its extraction, then, after conjugating, d can be taken to have the form $d = (A_0 < \ldots < A_s < A < A\Omega < B^g)$ for some $g \in C_G(A)$. We apply now Alperin's Fusion Theorem on $C_G(A)$ in order to prove that

$$\overline{(A_0 < \ldots < A_s < A < A\Omega < B)} = \overline{(A_0 < \ldots < A_s < A < A\Omega < B^g)}$$

with the morphism $c_g : C_G(A) \to C_G(A)$. There exist subgroups $Q_1, \ldots, Q_r \le C_S(A)$ and $g_i \in N_{C_G(A)}(Q_i)$ such that $B^{g_1 \cdots g_{i-1}} \le Q_i$ and $c_g|_B = c_{g_r} \circ \ldots \circ c_1|_B$. Let $B_i = B^{g_1 \cdots g_i}$ and $B_0 = B$. Since they are all maximal and $A \le B_i$, we have that $A\Omega \le B_i$ for all i. Therefore it is enough to show that

$$\overline{(A_0 < \ldots < A_s < A < A\Omega < B_i)} = \overline{(A_0 < \ldots < A_s < A < A\Omega < B_{i-1})}$$

for all $i \ge 1$. If $(A\Omega)^{g_i} = A\Omega$ we are done. Otherwise, note that $A\Omega \le \Omega_1(Z(Q_i))$ and so $(A\Omega)^{g_i} \le \Omega_1(Z(Q_i^{g_i})) = \Omega_1(Z(Q_i))$. It implies that $C = (A\Omega)(A\Omega)^{g_i} = \Omega_1(Z(Q_i))$ and in particular $\mathcal{A}_p(Q_i)$ has a unique maximal element which is C. Since $B_i, B_{i-1} \in \mathcal{A}_p(Q_i)$ are maximal elements, we deduce that $B_i = C = B_{i-1}$.

This has shown that $\overline{(A_0 < \ldots < A_s < A < B)}$ is an up beat point covered, at the moment of its extraction, by $\overline{(A_0 < \ldots < A_s < A < A\Omega < B)}$. After extracting it, the element $\overline{(A_0 < \ldots < A_s < A)}$ becomes an up beat point covered by $\overline{(A_0 < \ldots < A_s < A < A\Omega)}$ and we can extract it.

The remaining elements containing $\overline{(A)}$ can be extracted in the same way we did in Case 1.

We have shown that $\mathcal{P}_{r'-1} \subseteq \mathcal{P}_{r'}$ is a strong deformation retract for any $1 \le r' \le r-1$. In particular, $\mathcal{P} = \mathcal{P}_0 \subseteq \mathcal{P}_{r-1} = \mathcal{A}_p(G)'/G$ is a strong deformation retract. □

Remark 2.5.25. If $m_p(G) - m_p(\Omega) = 2$, the non-central and fully centralized elements $A \in \mathcal{A}_p(S)$ that could appear as elements of a chain $c \in \mathcal{A}_p(S)'$ with $\overline{c} \in \mathcal{P}$ have p-rank $m_p(G)$ or $m_p(G) - 1$.

Lemma 2.5.26. *With the notations of the lemma above, assume $m_p(\Omega) = m_p(G) - 2$. If $A \in \mathcal{A}_p(S)$ has rank $m_p(G) - 1$ and it is covered by a unique maximal element in $\mathcal{A}_p(S)$, then $\mathcal{P}$ retracts by strong deformation to the subposet of elements not containing $\overline{(A)}$.*

Proof. Clearly the result holds if such elements were extracted. Therefore, we may assume that every $A^g \le S$ fully centralized contains Ω. We can also take A to be fully centralized. If $B = \Omega_1(C_S(A))$, $B \in \mathcal{A}_p(S)$ is the unique maximal element strictly containing A. We extract first the elements $x = \overline{(C_0 < \ldots < C_s < \Omega < A)}$, with $s \ge -1$, from top to bottom as up beat points. If $x \prec \overline{d}$, then $\overline{d} = \overline{(C_0 < \ldots < C_s < \Omega < A < D)}$ for some $D \in \mathcal{A}_p(S)$, and it has to be $D = B$ by uniqueness. Thus, x is an up beat point. Moreover, any element of the form

$\overline{(C_0^g < \ldots < C_s^g < \Omega^g < A)}$ is equal to $\overline{(C_0 < \ldots < C_s < \Omega < A^h)}$ with $A^h \leq S$ fully centralized. It is easy to see that A^h is also covered by a unique element, so $\overline{(C_0 < \ldots < C_s < \Omega < A^h)}$ is also an up beat point. After extracting all these orbits, the unique elements containing $\overline{(A)}$ are those of the form $\overline{(C_0 < \ldots < C_s < A < B)}$ and $\overline{(C_0 < \ldots < C_s < A)}$, for $s \geq -1$ and $C_s \leq \Omega$. We can extract $\overline{(C_0 < \ldots < C_s < A)}$ from top to bottom since they are up beats points covered by $\overline{(C_0 < \ldots < C_s < A < B)}$ at the moment of their extraction. Now the remaining elements containing $\overline{(A)}$ are $\overline{(C_0 < \ldots < C_s < A < B)}$ and $\overline{(C_0^g < \ldots < C_s^g < \Omega^g < A < B)}$. We extract them from bottom to top since they are down beat points covering $\overline{(C_0 < \ldots < C_s < B)}$ and $\overline{(C_0^g < \ldots < C_s^g < \Omega^g < B)}$, respectively. □

Now we prove a result which roughly says that $\mathcal{A}_p(G)'/G$ is a contractible finite space when the p-rank of G and the rank of Ω are very close to $r_p(G)$. As corollary, we show that $\mathcal{A}_p(G)'/G$ is contractible if S has order at most p^4.

Theorem 2.5.27. *If $m_p(G) - m_p(\Omega) \leq 2$ and $m_p(G) \geq r_p(G) - 1$, then $\mathcal{A}_p(G)'/G$ is a contractible finite space.*

Proof. By Proposition 2.5.1 and Theorem 2.5.19, we may assume $m_p(G) = r_p(G) - 1$ and $m_p(G) - m_p(\Omega) = 2$. By Lemma 2.5.24, we only need to show that $\mathcal{P}$ is contractible.

The idea is to extract beat points in order to reach a subposet with minimum $\overline{(\Omega)}$.

Let $A \in \mathcal{A}_p(S)$ be a non-maximal element of rank $m_p(G) - 1$. There exists $B \in \mathcal{A}_p(S)$ of rank $m_p(G)$ such that $A < B$. It implies $B \leq C_S(A)$. On the other hand, $A \not\leq Z(S)$ given that $m_p(A) > m_p(\Omega) = m_p(Z(S))$, so $C_S(A) < S$. By order, $C_S(A) = B$ and A is covered by a unique maximal element of $\mathcal{A}_p(S)$. In particular A is fully centralized. By Lemma 2.5.26, $\mathcal{P}$ retracts by strong deformation to the subposet of elements not containing $\overline{(A)}$. Hence, we get a strong deformation retract subposet of $\mathcal{P}$ whose elements are $\overline{(C_0 < \ldots < C_s < \Omega)}$, $\overline{(C_0 < \ldots < C_s < \Omega < B)}$ and $\overline{(C_0 < \ldots < C_s < B)}$, for $B \in \mathcal{A}_p(S)$ maximal of rank r or $r-1$ and $s \geq -1$, and $\overline{(C_0 < \ldots < C_s)}$ for $s \geq 0$, with $C_s \leq \Omega$ in all cases.

Suppose that $B \in \mathcal{A}_p(S)$ is maximal of rank $m_p(G) - 1$. The elements containing $\overline{(B)}$ have the form $\overline{(C_0 < \ldots < C_s < B)}$ for $s \geq -1$ and $C_s \leq \Omega$. By repeating the proof of Theorem 2.5.19, they can be extracted from top to bottom as up beat points covered by $\overline{(C_0 < \ldots < C_s < \Omega < B)}$ at the moment of their extraction. Hence, any element containing $\overline{(B)}$ will also contain $\overline{(\Omega)}$.

We extract the elements containing $\overline{(B)}$ and not containing $\overline{(\Omega)}$ for $B \in \mathcal{A}_p(S)$ maximal of rank $m_p(G)$. These elements have the form $\overline{(C_0 < \ldots < C_s < B)}$ for $s \geq -1$ and $C_s < \Omega$, and are covered by $\overline{(C_0 < \ldots < C_s < \Omega < B)}$ and $\overline{(C_0 < \ldots < C_s < \Omega < B^g)}$ at the moment of their extraction, for some $g \in C_G(C_s)$. We need to prove they are equal. By using Alperin's Fusion Theorem, we can assume that $g \in N_{C_G(C_s)}(Q)$ for a subgroup $Q \leq S$ with $B, B^g \leq Q$. If $\Omega^g = \Omega$ we are done. Otherwise, $g \notin N_G(\Omega)$ and so $Q = B$ since it has to be $Q < S$ by order. This yields $B = B^g$.

We have reached a strong deformation retract of $\mathcal{P}$ whose only elements not containing $\overline{(\Omega)}$ are $\overline{(C_0 < \ldots < C_s)}$ with $C_s \leq \Omega$. Again, by repeating the end of the proof of Theorem 2.5.19, we can extract them from top to bottom as up beat points covered by $\overline{(C_0 < \ldots < C_s < \Omega)}$ at the moment of their extraction.

Finally, $\mathcal{P}$ retracts by strong deformation to a subposet with minimum $\overline{(\Omega)}$, and consequently it is contractible. □

We get the following corollaries.

Corollary 2.5.28. *If $r_p(G) \leq 4$, the finite space $\mathcal{A}_p(G)'/G$ is contractible.*

Proof. It follows from Proposition 2.5.1 and Theorems 2.5.19 and 2.5.27. □

By Proposition 3.1.1, the poset $\mathcal{A}_p(G)$ is disconnected if and only if G has a strongly p-embedded subgroup. The proof of the following theorem relies on the classification of the groups with a strongly p-embedded subgroup together with the results obtained above.

Corollary 2.5.29. *If $\mathcal{A}_p(G)$ is disconnected then $\mathcal{A}_p(G)'/G$ is contractible.*

Proof. We may assume that $O_{p'}(G) = 1$ by Remark 2.5.7 and that $O_p(G) = 1$. Hence, $\Omega_1(G)$ is one of the groups in the list of Theorem A.1.1. Note that $m_p(G) = m_p(\Omega_1(G))$.

The cases $m_p(\Omega_1(G)) \leq 2$ hold by Theorem 2.5.19.

If $\Omega_1(G)$ is simple of Lie type of Lie rank 1 and characteristic p, then the Sylow p-subgroups of G intersect trivially by Theorem A.1.3. Therefore, $\mathcal{A}_p(G)'/G$ is contractible by Proposition 2.5.6.

The remaining groups of the list have p-rank 2 by Table A.4. □

We obtain contractibility of $\mathcal{A}_p(G)'/G$ for some sporadic simple groups as application of our results. We require p odd.

Corollary 2.5.30. *If G is a Janko group, a Mathieu group, He, HS, $O'N$ or Ru, and p is odd, then $\mathcal{A}_p(G)'/G$ is contractible.*

Proof. By Table A.6 and Corollary 2.5.20, it only remains to prove the case $G = J_3$ and $G = O'N$, both with $p = 3$.

It can be shown that the center of a Sylow 3-subgroup of J_3 has 3-rank 2. Since $m_3(J_3) = 3$, the result follows from Theorem 2.5.19.

The Sylow 3-subgroups of $O'N$ are elementary abelian of rank 4. Hence, the result follows from Proposition 2.5.2. □

Corollary 2.5.31. *Let G be the sporadic simple group Co_1 and $p = 5$. Then $\mathcal{A}_p(G)'/G$ is contractible.*

Proof. Let S a Sylow 5-subgroup of $G = Co_1$. Then $r_5(G) = 4$, $m_5(G) = 3$ and $m_5(Z(S)) = 1$. By Theorem 2.5.19, $\mathcal{A}_5(G)'/G$ is contractible. □

For G a sporadic group and $p = 2$, the difference $m_p(G) - m_p(\Omega)$ is bigger than 2 in general, and our theorems do not apply. Nevertheless, for the smaller sporadic groups we can use GAP together with package [FPSC19] to carry out the computation of the core of $\mathcal{A}_p(G)'/G$. In this way, for $p = 2$ and $G = M_{11}, M_{12}, M_{22}, M_{23}, J_1, J_2$ or HS, $\mathcal{A}_p(G)'/G$ is contractible. Note that J_1 has abelian Sylow 2-subgroup and we can apply Proposition 2.5.2 (in fact, it is the unique sporadic group with abelian Sylow 2-subgroup by Walter's classification Theorem 3.5.4).

Remark 2.5.32. Almost all the proofs that we have done can be carried out in a general saturated fusion system over a fixed p-group S. If $\mathcal{F}$ is a saturated fusion system over S, we can form the orbit poset $\mathcal{A}_p(S)/\mathcal{F}$ in the following way: if $A, B \in \mathcal{A}_p(S)$ define the relation $A \sim B$ if $\varphi(A) = B$ for some morphism φ in the category $\mathcal{F}$. Then $\mathcal{A}_p(S)/\mathcal{F} := \mathcal{A}_p(S)/\sim$ is contractible with the same homotopy that we have defined in Theorem 2.4.1.

Analogously, we can define the relation $\sim$ in $\mathcal{A}_p(S)'$ by setting $(P_0 < \ldots < P_r) \sim (Q_0 < \ldots < Q_r)$ if there exists a morphism φ in $\mathcal{F}$ such that $\varphi(P_i) = Q_i$ for all i, and set $\mathcal{A}_p(S)'/\mathcal{F} := \mathcal{A}_p(S)'/\sim$.

When $\mathcal{F} = \mathcal{F}_S(G)$, $\mathcal{A}_p(S)/\mathcal{F} = \mathcal{A}_p(G)/G$ and $\mathcal{A}_p(S)'/\mathcal{F} = \mathcal{A}_p(G)'/G$.

Chapter 3

The fundamental group of the posets of p-subgroups

In Chapter 1 we have studied the posets of p-subgroups as finite topological spaces. We have seen that their homotopy type is determined by the core of the poset (up to homeomorphism) and that in general, the posets of p-subgroups do not have the same homotopy type as finite spaces. Moreover, the cores of $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ can be computed algorithmically by taking alternately the subposets $\mathfrak{i}$ and $\mathfrak{s}$ (see Remark 1.3.28). However, when we work with the topology of their order complexes, we know that they are G-homotopy equivalent and there is no algorithm to compute their homotopy type. In general, the homotopy type of the p-subgroup complexes is not known.

In his foundational article [Qui78], Quillen computed the homotopy type of certain families of p-subgroup complexes. For example, he showed that $\mathcal{K}(\mathcal{A}_p(\mathrm{GL}_n(q)))$ (with $q \equiv 1 \mod p$) and $\mathcal{K}(\mathcal{A}_p(LA))$ (with L a solvable p'-group and A an elementary abelian p-group acting on L) are Cohen-Macaulay complexes (see Definition 3.1.3 and Theorem 3.1.4). Moreover, when G is of Lie type in characteristic p, $\mathcal{K}(\mathcal{A}_p(G))$ is homotopy equivalent to the Tits building of G. In any of these cases, the p-subgroup complexes have the homotopy type of a bouquet of spheres.

Along the following decades, it was proved that $\mathcal{K}(\mathcal{A}_p(G))$ have the homotopy type of a bouquet of spheres (of possible different dimensions) for particular classes of groups (see for example [Qui78] and [Smi11, Section 9.4]). In [PW00] Pulkus and Welker obtained a wedge decomposition for $\mathcal{K}(\mathcal{A}_p(G))$ when G is a solvable group, reducing the study of the homotopy type of $\mathcal{K}(\mathcal{A}_p(G))$ for solvable groups to the study of the homotopy type of the upper intervals $\mathcal{A}_p(G)_{>A}, A \in \mathcal{A}_p(G)$ (see Theorem 3.1.7). There is even a question in [PW00], attributed to Thévenaz, of whether the p-subgroup complexes always have the homotopy type of a bouquet of spheres (of possibly different dimensions). In 2004 Shareshian gave the first example of a group whose p-subgroup complex is not homotopy equivalent to a bouquet of spheres. Concretely, he showed that there is torsion in the second homology group of $\mathcal{A}_3(\mathbb{A}_{13})$ (see [Sha04]). Note that the example showing the failure of being homotopy equivalent to a bouquet of sphere arises from a simple group.

In this chapter we focus on the study of the fundamental group of the p-subgroup complexes. It was first investigated by M. Aschbacher, who provided algebraic conditions for $\mathcal{A}_p(G)$ to be simply connected, modulo a well-known conjecture for which there is considerable evidence (see [Asc93, Theorems 1 & 2]). Later, K. Das studied the simple connectivity of the p-subgroups complexes of some groups of Lie type (see [Das95, Das98, Das00]). In [Kso03, Kso04], Ksontini investigated the fundamental group of $\mathcal{A}_p(\mathbb{S}_n)$. He established necessary and sufficient conditions in terms of n and p for $\mathcal{A}_p(\mathbb{S}_n)$ to be simply connected. In the remaining cases he proved that this fundamental group is free except possibly for $n = 3p$ or $n = 3p+1$ (p odd). In [Sha04], Shareshian extended Ksontini's results and showed that the fundamental group of $\mathcal{A}_p(\mathbb{S}_n)$ is also free for $n = 3p$. All these results could suggest that the fundamental group of $\mathcal{A}_p(G)$ is always free. We will show that this holds for solvable groups (see Corollary 3.0.3 below) and, modulo Aschbacher's conjecture, for p-solvable groups (see Corollary 3.0.1 below). In fact, there are only few known examples of p-subgroup complexes which are not homotopy equivalent to a bouquet of spheres, and Shareshian's counterexample $\mathcal{A}_3(\mathbb{A}_{13})$ fails in the second homology group but it does have free fundamental group. Surprisingly we found that the fundamental group of $\mathcal{A}_3(\mathbb{A}_{10})$ is not free. It is isomorphic to a free product of the free group on 25200 generators and a non-free group whose abelianization is $\mathbb{Z}^{42}$. This is the smallest group G with $\pi_1(\mathcal{A}_p(G))$ non-free for some p. Note that the integral homology of $\mathcal{A}_3(\mathbb{A}_{10})$ ($= \mathcal{A}_3(\mathbb{S}_{10})$) is free abelian (cf. [Sha04, p.306]), so in this case the obstruction to being a bouquet of spheres relies on the fundamental group and not on the homology.

We will show that p-subgroup complexes with non-free fundamental group are rather exceptional. The first of our main results asserts that, modulo Aschbacher's conjecture, the study of freeness of $\pi_1(\mathcal{A}_p(G))$ reduces to the almost simple case. Let $S_G = \Omega_1(G)/O_{p'}(\Omega_1(G))$.

Theorem 3.4.2. *Let G be a finite group and p a prime dividing $|G|$. Assume that Aschbacher's conjecture holds. Then there is an isomorphism $\pi_1(\mathcal{A}_p(G)) \cong \pi_1(\mathcal{A}_p(S_G)) * F$, where F is a free group. Moreover, $\pi_1(\mathcal{A}_p(S_G))$ is a free group (and therefore $\pi_1(\mathcal{A}_p(G))$ is free) except possibly if S_G is almost simple.*

We can always assume that $\mathcal{A}_p(G)$ is connected (see Remark 3.1.2). Note that, in that case, $\mathcal{A}_p(S_G)$ is also connected since the induced map $\mathcal{A}_p(G) \to \mathcal{A}_p(S_G)$ is surjective.

In fact, in Theorem 3.4.2 we only need Aschbacher's conjecture to hold for the p'-simple groups L involved in $O_{p'}(\Omega_1(G))$ and for p-rank 3 (see Proposition 3.3.9 below). It is not needed for the 'Moreover' part of the theorem.

We recall now Aschbacher's conjecture [Asc93].

Aschbacher's Conjecture 3.3.8. *Let G be a finite group such that $G = F^*(G)A$, where A is an elementary abelian p-subgroup of rank $r \geq 3$ and $F^*(G)$ is the direct product of the A-conjugates of a simple component L of G of order prime to p. Then $\mathcal{A}_p(G)$ is simply connected.*

Aschbacher proved the conjecture for all simple groups L except for Lie type groups with Lie rank 1 and the sporadic groups which are not Mathieu groups (see [Asc93, Theorem 3]).

Below we list some immediate consequences of Theorem 3.4.2.

If G is p-solvable, $O_p(S_G) \neq 1$, so S_G is not almost simple. In fact, $\mathcal{A}_p(S_G)$ is homotopically trivial by Proposition 1.3.15. Recall that $O_{p',p}(G)$ is the unique normal subgroup of G containing $O_{p'}(G)$ such that $O_{p',p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G))$.

Corollary 3.0.1. *Assume that Aschbacher's conjecture holds. If $O_{p'}(\Omega_1(G)) < O_{p',p}(\Omega_1(G))$, then $\pi_1(\mathcal{A}_p(G))$ is free. In particular, this holds for p-solvable groups and, more generally, for p-constrained groups.*

If we take an abelian simple group L in the hypotheses of Aschbacher's conjecture, then the conjecture holds for L by Theorem 3.1.4. Since there are no non-abelian simple groups involved in a solvable group, Aschbacher's conjecture does not need to be assumed for solvable groups.

By Feit-Thompson Theorem 1.1.3, if $p = 2$ then there are no non-abelian p'-simple group. In consequence, Aschbacher's conjecture does not need to be assumed and we get the following corollaries.

Corollary 3.0.2. *There is an isomorphism $\pi_1(\mathcal{A}_2(G)) \cong \pi_1(\mathcal{A}_2(S_G)) * F$, where F is a free group. Moreover, $\pi_1(\mathcal{A}_2(S_G))$ is a free group (and therefore $\pi_1(\mathcal{A}_2(G))$ is free) except possibly if S_G is almost simple.*

Corollary 3.0.3. *If G is solvable then $\pi_1(\mathcal{A}_p(G))$ is a free group.*

In Section 3.3, we use a variant of Pulkus-Welker's wedge decomposition Theorem 3.1.7 to restrict Aschbacher's conjecture to the p-rank 3 case.

Proposition 3.3.9. *If Aschbacher's conjecture holds for p-rank 3, then it holds for any p-rank $r \geq 3$. Moreover, if the conjecture holds in p-rank 3 for a p'-simple group L then it holds in any p-rank $r \geq 3$ for L.*

In Section 3.5 we study freeness for some particular cases of almost simple groups. We do not need to assume Aschbacher's conjecture for these cases. In the following theorem we collect the results of Section 3.5. Recall that by a simple group we mean a non-abelian simple group.

Theorem 3.0.4. *Suppose that $L \leq G \leq \mathrm{Aut}(L)$, with L a simple group. Then $\pi_1(\mathcal{A}_p(G))$ is a free group in the following cases:*

1. *$m_p(G) \leq 2$,*
2. *$\mathcal{A}_p(L)$ is disconnected,*
3. *$\mathcal{A}_p(L)$ is simply connected,*
4. *L is simple of Lie type in characteristic p and $p \nmid (G:L)$ when L has Lie rank 2,*
5. *$p = 2$ and L has abelian Sylow 2-subgroups,*
6. *$p = 2$ and $L = \mathbb{A}_n$ (the alternating group),*
7. *L is a Mathieu group, J_1 or J_2,*
8. *$p \geq 3$ and $L = J_3$,* McL*,* O'N.

From our base example $\mathbb{A}_{10}$ (with $p = 3$) and Theorem 3.4.2, one can easily construct an infinite number of examples of finite groups G with non-free $\pi_1(\mathcal{A}_3(G))$, by taking extensions of $3'$-groups H whose $3'$-simple groups involved satisfy Aschbacher's conjecture, by $\mathbb{A}_{10}$. However, $\mathbb{A}_{10}$ is the unique known example so far of a simple group with non-free fundamental group. We do not know whether $\pi_1(\mathcal{A}_p(\mathbb{A}_{3p+1}))$ is non-free for $p \geq 5$. It would be interesting to find new examples of simple groups G (other than the alternating groups) with $\pi_1(\mathcal{A}_p(G))$ non-free. Besides the works of Aschbacher, Das, Ksontini and Shareshian mentioned above, we refer the reader to S.D. Smith's book [Smi11, Section 9.3] for more details on the fundamental groups of Quillen complexes and applications to group theory, such as uniqueness proofs. Also a recent work of J. Grodal [Gro16] relates the fundamental group of the p-subgroup complexes with modular representation theory of finite groups via the exact sequence

$$1 \to \pi_1(\mathcal{S}_p(G)) \to \pi_1(\mathscr{T}_p(G)) \to G \to 1$$

(when $\mathcal{S}_p(G)$ is connected). Here $\mathscr{T}_p(G)$ denotes the transport category, whose objects are the nontrivial p-subgroups of G, and with $\mathrm{Hom}_{\mathscr{T}_p(G)}(P,Q) = \{g \in G : P^g \leq Q\}$. It is well-known that the geometric realization of $\mathscr{T}_p(G)$ is homotopy equivalent to the Borel construction $EG \times_G |\mathcal{S}_p(G)|$ (see for example [Gro16, Remark 2.2]), and the exact sequence follows from the fibration sequence $|\mathcal{S}_p(G)| \to EG \times_G |\mathcal{S}_p(G)| \to BG$. Recall that, by Brown's ampleness theorem, the mod-p cohomology of $EG \times_G |\mathcal{S}_p(G)|$ is isomorphic to the mod-p cohomology of G (see [Bro94, Smi11]).

Throughout this chapter, when we talk about the homotopy type of a poset we mean the homotopy type of its intrinsic topology of finite space. Nevertheless, the results of this chapter concern the weak homotopy type of the p-subgroup posets, which is the homotopy type of their order complexes. Most of the new results of this chapter appeared in the article written in collaboration with E.G. Minian [MP19].

3.1 General properties on the homotopy type of the p-subgroup complexes

In this section, we exhibit some results on the general homotopy type of the p-subgroup complexes. We also provide the tools we will need for the study of the fundamental group in the subsequent sections. We refer the reader to Section 1.1 of Chapter 1 for notations and definitions on finite group theory.

The first step on the study of the homotopy type of the p-subgroup complexes could be to determine the purely algebraic conditions characterizing their connectivity as topological spaces. This was done by Quillen [Qui78]. Recall that *a strongly p-embedded subgroup* of G is a proper subgroup $M < G$ such that $|M|_p = |G|_p$ and $M \cap M^g$ is a p'-group for all $g \in G - M$.

Proposition 3.1.1 ([Qui78, Proposition 5.2])**.** *The poset $\mathcal{A}_p(G)$ is disconnected if and only if G has a strongly p-embedded subgroup.*

Remark 3.1.2. Suppose $\mathcal{A}_p(G)$ is disconnected and let C be a connected component. Let $M \leq G$ be the stabilizer of C under the conjugation of G on the connected components of $\mathcal{A}_p(G)$. It can be shown that $C = \mathcal{A}_p(M)$ and that M is a strongly p-embedded subgroup of G (see for example [Asc00, Section 46] or [Qui78, Section 5]). Moreover, since G permutes transitively the connected components of $\mathcal{A}_p(G)$, they are homeomorphic and in particular homotopy equivalent (even in the sense of finite spaces). This allows us to define for $n \geq 1$, $\pi_n(\mathcal{A}_p(G))$ as $\pi_n(\mathcal{A}_p(M))$ (for any connected component C) and $\tilde{H}_n(\mathcal{A}_p(G))$ as $\tilde{H}_n(\mathcal{A}_p(M))$. Here, π_n and $\tilde{H}_n$ denote the n-th homotopy group and reduced homology group respectively. Therefore the study of the homotopy type (and in particular of the homotopy groups and homology groups) of the p-subgroup complexes can be restricted to the connected case.

The groups with a strongly p-embedded subgroup are classified (see Theorem A.1.1) and it is an ingredient of the CFSG. We frequently use this classification.

We show now how the Cohen-Macaulay property arises in the context of the p-subgroup complexes. This was noted first by D. Quillen [Qui78].

Definition 3.1.3. Let X be a finite poset. We say that X is *spherical* if it is $(h(X)-1)$-connected (i.e. $\mathcal{K}(X)$ has the homotopy type of a wedge of spheres of dimension $h(X)$). We say that X is *Cohen-Macaulay* (or that $\mathcal{K}(X)$ is) if it is spherical, and for each $x < y \in X$, $X_{<x}$ is spherical of height $h(x)-1$, $X_{>x}$ is spherical of height $h(X)-h(x)-1$ and $X_{>x} \cap X_{<y}$ is spherical of height $h(y)-h(x)-2$. Note that a simplicial complex K is spherical or Cohen-Macaulay if $\mathcal{X}(K)$ is.

The Cohen-Macaulay property (CM for short) relies on inductive steps since it involves sphericity of links of elements.

The following theorem shows that $\mathcal{A}_p(G)$ is Cohen-Macaulay for certain configuration of solvable groups. This theorem is part of the proof of Quillen's conjecture for solvable groups (see Section 4.1).

Theorem 3.1.4 ([Qui78, Theorem 11.2])**.** *Let $G = LA$ where L is a p'-solvable group and A is an elementary abelian p-group acting on L. Then $\mathcal{A}_p(LA)$ is Cohen-Macaulay.*

Proof. (Sketch) Let $G = LA$, where L is a p'-solvable group on which the elementary abelian p-group A acts. We proceed by induction on the order of LA. Assume that L has a nontrivial proper normal subgroup H which is also A-invariant. Then $\mathcal{A}_p(HA)$ and $\mathcal{A}_p((L/H)A)$ are Cohen-Macaulay. Consider the map $q : \mathcal{A}_p(LA) \to \mathcal{A}_p((L/H)A)$ induced by the quotient $L \to L/H$. We apply [Qui78, Propositions 9.7 and 10.1], which assert that $\mathcal{A}_p(LA)$ is Cohen-Macaulay if $\mathcal{A}_p((L/H)A)$ and $q^{-1}(\mathcal{A}_p((L/H)A)_{\leq B})$ are (the later with height $m_p(B)-1$), for each $B \in \mathcal{A}_p(A)$. Note that if $B \in \mathcal{A}_p(A)$, then $q^{-1}(\mathcal{A}_p((L/H)A)_{\leq B}) = q^{-1}(\mathcal{A}_p(HB/H)) = \mathcal{A}_p(HB)$, which is Cohen-Macaulay of height $m_p(B)-1$ by induction. Therefore, $\mathcal{A}_p(LA)$ is Cohen-Macaulay.

Now suppose that L has no nontrivial proper LA-invariant subgroup. Then L is a characteristically simple solvable group, and it implies that L is an elementary abelian q-group, with q prime. We see L as a vectorial space over $\mathbb{F}_q$, and A acts on L by linear automorphisms. The minimality of L implies that either $C_A(L) = A$, or L is irreducible with the action of $A/C_A(L)$ and $A/C_A(L)$ is cyclic of order p. In the first case $\mathcal{A}_p(LA) = \mathcal{A}_p(A)$ is Cohen-Macaulay. In the second case $LA = LA_0 \times C_A(L)$ for some complement $A_0 \leq A$ of $C_A(L)$. The posets $\mathcal{A}_p(LA_0)$ and $\mathcal{A}_p(C_A(L))$ are Cohen-Macaulay and hence $\mathcal{A}_p(LA)$ is by [Qui78, Proposition 10.3] and Proposition 3.1.16. ☐

The above result is not true if $G = LP$ and P is just a p-group acting on a solvable p'-group L (see [Smi11, Example 9.3.2]). Nevertheless, if $\Omega_1(P)$ is abelian, then $\mathcal{A}_p(LP) = \mathcal{A}_p(L\Omega_1(P))$ is Cohen-Macaulay.

As a matter of fact, Quillen asked if the above result can be extended to the non-solvable case, namely, when L is just a p'-group [Qui78, Problem 12.3]. See also [Smi11, p. 299]. This problem remains open and Aschbacher's conjecture 3.3.8 can be seen as a particular case.

Remark 3.1.5. Consider the configurations $G = LA$, where L is a p'-group on which the elementary abelian p-group A acts. We have that, for these configurations, the posets $\mathcal{A}_p(LA)$ are Cohen-Macaulay if and only if they are spherical.

Assume they are spherical. Fix a configuration LA and let $B \leq A$. Then $\mathcal{A}_p(LA)_{<B} = \mathcal{A}_p(B) - \{B\}$ is Cohen-Macaulay of height $m_p(B)-2$. On the other hand, $N_{LA}(B) = N_L(B)A$ and $N_L(B) = O_{p'}(N_{LA}(B))$. Therefore, by Lemma 3.1.8,

$$\mathcal{A}_p(LA)_{>B} \simeq \mathcal{A}_p(N_{LA}(B)/B) = \mathcal{A}_p(N_L(B)(A/B))$$

is spherical of height $(m_p(A)-m_p(B)-1) = ((m_p(A)-1)-(m_p(B)-1)-1)$. For $B < C \leq A$, the intervals $\mathcal{A}_p(LA)_{>B} \cap \mathcal{A}_p(LA)_{<C} = \mathcal{A}_p(C/B) - \{C/B\}$ are spherical with height $((m_p(C)-1)-(m_p(B)-1)-2)$. Hence, if all the posets $\mathcal{A}_p(LA)$ are spherical, then all of them are Cohen-Macaulay. The reciprocal is immediate from the definitions.

Moreover, the proof of the above theorem works for a non-solvable L up to the point we assume that L has no nontrivial proper LA-invariant subgroups. It implies that L is the direct product of isomorphic simple p'-groups, permuted transitively by A. Therefore, in order to show that $\mathcal{A}_p(LA)$ is always Cohen-Macaulay, we need to show that it is spherical when L is a direct product of isomorphic simple groups permuted transitively by A.

Quillen's problem: Is $\mathcal{A}_p(LA)$ $(m_p(A)-2)$-connected when L is the direct product of simple p'-groups permuted transitively by A, an elementary abelian p-group?

Remark 3.1.6. Aschbacher's conjecture 3.3.8 says that $\mathcal{A}_p(LA)$ is 1-connected when $m_p(A) \geq 3$.

We recall Pulkus-Welker's wedge decomposition.

Theorem 3.1.7 ([PW00, Theorem 1.1])**.** *Let G be a finite group with a solvable normal p'-subgroup N. For $A \in \mathcal{A}_p(G)$, set $\overline{A} = NA/N$. Then $\mathcal{A}_p(G)$ is weak homotopy equivalent to the wedge*

$$\mathcal{A}_p(\overline{G}) \bigvee_{\overline{A} \in \mathcal{A}_p(\overline{G})} \mathcal{A}_p(NA) * \mathcal{A}_p(\overline{G})_{>\overline{A}}$$

where for each $\overline{A} \in \mathcal{A}_p(\overline{G})$ an arbitrary point $c_{\overline{A}} \in \mathcal{A}_p(NA)$ is identified with $\overline{A} \in \mathcal{A}_p(\overline{G})$.

The upper intervals are better understood in the poset $\mathcal{S}_p(G)$. See [Qui78, Proposition 6.1].

Lemma 3.1.8 (Upper intervals)**.** *We have a homotopy equivalence $\mathcal{S}_p(G)_{>P} \simeq \mathcal{S}_p(N_G(P)/P)$ as finite spaces. In particular, if G has elementary abelian Sylow p-subgroups then $\mathcal{A}_p(G)_{>A} = \mathcal{S}_p(G)_{>A} \simeq \mathcal{S}_p(N_G(A)/A) = \mathcal{A}_p(N_G(A)/A)$.*

Proof. (Sketch) Note that $\mathcal{S}_p(N_G(P)/P) = \mathcal{S}_p(N_G(P))_{>P}$, and the maps $f : Q \in \mathcal{S}_p(G)_{>P} \mapsto N_Q(P)/P\mathcal{S}_p(N_G(P)/P)$ and $g : Q \in \mathcal{S}_p(N_G(P))_{>P} = \mathcal{S}_p(N_G(P)/P) \mapsto Q \in \mathcal{S}_p(G)_{>P}$ are well-defined and order preserving with $fg(Q) = Q$ and $gf(Q) = N_Q(P) \leq Q$. □

The following propositions show that the posets of p-subgroups behave like the face-poset of simplicial complexes, in the sense that the inclusion of the subposet of elements of height at most n (the "n-skeleton") induces an n-equivalence. First we recall a generalization of Quillen's Theorem A for posets (see also [Qui78, Proposition 1.6]).

Proposition 3.1.9 ([Bjo03, Theorem 2], see also [Bar11b])**.** *Let $f : X \to Y$ be a map between posets. Assume that $f^{-1}(Y_{\leq a})$ is n-connected for all $a \in Y$. Then f is an $(n+1)$-equivalence.*

Let X^n be the subposet of X of elements of height at most n. Note that $\mathcal{S}_p(G)^n = \{P \in \mathcal{S}_p(G) : |P| \leq p^{n+1}\}$ and $\mathcal{A}_p(G)^n = \mathcal{A}_p(G) \cap \mathcal{S}_p(G)^n = \{A \in \mathcal{A}_p(G) : m_p(A) \leq n+1\}$. Recall that $\mathcal{A}_p(G) \hookrightarrow \mathcal{S}_p(G)$ is a weak homotopy equivalence by Proposition 1.3.1.

Proposition 3.1.10. *The inclusions $\mathcal{A}_p(G)^n \hookrightarrow \mathcal{A}_p(G)^{n+1}$ and $\mathcal{S}_p(G)^n \hookrightarrow \mathcal{S}_p(G)^{n+1}$ are n-equivalences. In particular, the inclusions $\mathcal{A}_p(G)^2 \hookrightarrow \mathcal{A}_p(G)$ and $\mathcal{S}_p(G)^2 \hookrightarrow \mathcal{S}_p(G)$ induce isomorphisms between the fundamental groups.*

Proof. We show that the inclusion $i : \mathcal{S}_p(G)^n \hookrightarrow \mathcal{S}_p(G)^{n+1}$ is an n-equivalence by using the previous proposition. Let $P \in \mathcal{S}_p(G)^{n+1}$. Note that $i^{-1}(\mathcal{S}_p(G)^{n+1}_{\leq P}) \subseteq \mathcal{S}_p(P)$. If $|P| \leq p^{n+1}$, then $P \in \mathcal{S}_p(G)^n$ and $i^{-1}(\mathcal{S}_p(G)^{n+1}_{\leq P}) = \mathcal{S}_p(P)$ is contractible (see Proposition 1.3.15). In particular, it is $(n-1)$-connected. Suppose $|P| = p^{n+2}$. If P is elementary abelian, then $i^{-1}(\mathcal{S}_p(G)^{n+1}_{\leq P}) = \mathcal{A}_p(P) - P$ is the poset of proper subspaces of P, which is a wedge of spheres of dimension $(n-1)$ by the classical Solomon-Tits result. If P is not elementary abelian, $i^{-1}(\mathcal{S}_p(G)^{n+1}_{\leq P}) = \mathcal{S}_p(P) - P$, and $1 \neq \Phi(P) < P$, where $\Phi(P)$ is the Frattini subgroup of P. Then, $Q \leq Q\Phi(P) \geq \Phi(P)$ induces a homotopy between the identity map and the constant map inside $\mathcal{S}_p(P) - P$, and therefore $\mathcal{S}_p(P) - P$ is contractible. This shows that $i : \mathcal{S}_p(G)^n \hookrightarrow \mathcal{S}_p(G)^{n+1}$ is an n-equivalence. A similar proof works for $\mathcal{A}_p(G)$. □

Remark 3.1.11. By Proposition 3.1.10, in order to study the fundamental group of the Quillen complex, we only need to deal with the subposet $\mathcal{A}_p(G)^2$. Note that we could have deduced the isomorphism $i_* : \pi_1(\mathcal{A}_p(G)^2) \to \pi_1(\mathcal{A}_p(G))$ without need of Propositions 3.1.9 and 3.1.10: it follows from van Kampen theorem and the fact that for any $P \in \mathcal{A}_p(G) - \mathcal{A}_p(G)^2$, $\mathcal{A}_p(G)_{<P}$ is a wedge of spheres of dimension greater than or equal to 2.

Remark 3.1.12. For a subgroup $H \leq G$, consider the subposet $\mathcal{N} = \{E \in \mathcal{A}_p(G) : E \cap H \neq 1\} \subseteq \mathcal{A}_p(G)$. Note that the inclusion $\mathcal{A}_p(H) \subseteq \mathcal{N}$ is a strong deformation retract via $E \in \mathcal{N} \mapsto E \cap H \in \mathcal{A}_p(H)$.

Lemma 3.1.13. *Let $H \leq G$ and let $E \in \mathcal{A}_p(G) - \mathcal{A}_p(H)$. Then $i : \mathcal{A}_p(C_H(E)) \to \mathcal{N} \cap \mathcal{A}_p(G)_{>E}$ defined by $i(A) = AE$ is a strong deformation retract.*

Proof. The result is clear if $\mathcal{A}_p(C_H(E))$ is empty. If it is not empty, let $r : \mathcal{N} \cap \mathcal{A}_p(G)_{>E} \to \mathcal{A}_p(C_H(E))$ be the map $r(A) = A \cap H$. Then $ri(A) = A$ by modular law, and $ir(A) \leq A$. □

We will use the following lemma of [Asc93].

Lemma 3.1.14 ([Asc93, (6.9)])**.** *Let $N \trianglelefteq G$ and suppose $\mathcal{A}_p(N)$ is simply connected. If $\mathcal{A}_p(C_N(E))$ is connected for each subgroup $E \leq G$ of order p, then $\mathcal{A}_p(G)$ is simply connected.*

The following proposition describes the fundamental group of a join of two finite posets.

Proposition 3.1.15 ([Bar11a, Lemma 6.2.4])**.** *If X and Y are finite nonempty posets, then $\pi_1(X * Y)$ is a free group of rank $(|\pi_0(X)| - 1).(|\pi_0(Y)| - 1)$.*

When $G = G_1 \times G_2$ is the direct product of two groups, Quillen showed that $\mathcal{K}(\mathcal{A}_p(G))$ is homotopy equivalent to $\mathcal{K}(\mathcal{A}_p(G_1)) * \mathcal{K}(\mathcal{A}_p(G_2))$, where $*$ denotes the join of simplicial complexes (see [Qui78, Proposition 2.6]). In the following proposition we show that there is a stronger relation at the level of finite spaces which implies Quillen's result.

Proposition 3.1.16. *If $G = G_1 \times G_2$ then $\mathcal{A}_p(G)$ has the same G-equivariant simple homotopy type as $\mathcal{A}_p(G_1) * \mathcal{A}_p(G_2)$, where the action of G on the later poset is by conjugation on each factor of the join. In particular, by Theorem 1.2.30, $\mathcal{K}(\mathcal{A}_p(G))$ and $\mathcal{K}(\mathcal{A}_p(G_1)) * \mathcal{K}(\mathcal{A}_p(G_2)) = \mathcal{K}(\mathcal{A}_p(G_1) * \mathcal{A}_p(G_2))$ are G-homotopy equivalent.*

Proof. If X is a finite poset, let $C^+X = X \cup \{1\}$, resp. $C^-X = X \cup \{0\}$, be the poset obtained from X by adding it a maximum, resp. a minimum. Let $X_1 = C^-\mathcal{A}_p(G_1) \times C^-\mathcal{A}_p(G_2) - \{(0,0)\}$. Note that X_1 has an induced action of G and thus, it is G-poset. Let $p_1 : G_1 \times G_2 \to G_1$ and $p_2 : G_1 \times G_2 \to G_2$ be the projections. We have a well-defined order preserving map $f : \mathcal{A}_p(G_1 \times G_2) \to X_1$, $f(H) = (p_1(H), p_2(H))$. The map $g : X_1 \to \mathcal{A}_p(G_1 \times G_2)$ defined by $g(H_1, H_2) = H_1 \times H_2$ satisfies $gf(H) \geq H$ and $fg(H_1 \times H_2) = H_1 \times H_2$. Note that both maps f and g are equivariant. Therefore, $\mathcal{A}_p(G_1 \times G_2) \simeq_G X_1$.

In a similar sense, it is easy to check that $X_2 = C^+\mathcal{A}_p(G_1) \times C^-\mathcal{A}_p(G_2) - \{(1,0)\} \simeq_G \mathcal{A}_p(G_1) * \mathcal{A}_p(G_2)$ (see [Pit16, Proposición 2.2.1]).

We are going to prove that $X_1 \nearrow\!\!\!\searrow^G X_2$. Let $X = \mathcal{A}_p(G_1)$ and $Y = \mathcal{A}_p(G_2)$. Consider $X_3 = (C^+C^-X) \times (C^-Y) - \{(0,0),(1,0)\}$. Note that X_1 and X_2 are subposets of X_3 and that X_3 is a G-poset. We show that $X_1 \;{}^G\!\!\nearrow X_3$.

Let $\{y_1, \ldots, y_r\}$ be representatives of the orbits of the action of G on Y such that $y_i \leq y_j^g$ for some $g \in G$ implies $i \leq j$. Let $Z_i = X_3 - \{(1,y_j)^g : 1 \leq j \leq i, g \in G\}$. Note that each Z_i is G-invariant, $X_1 = X_3 - \{(1,y) : y \in Y\} = Z_r$, $X_3 = Z_0$ and $Z_{i-1} = Z_i \cup \{(1,y_i)^g : g \in G\}$. We have

$$\hat{U}^{Z_{i-1}}_{(1,y_i)^g} = C^-X \times C^-U^Y_{y_i^g} - \{(0,0)\} = C^-\mathcal{A}_p(G_1) \times C^-\mathcal{A}_p(y_i^g) - \{(0,0)\} \simeq \mathcal{A}_p(G_1 \times y_i^g) \simeq *.$$

The poset $\mathcal{A}_p(G_1 \times y_i^g)$ is contractible via the homotopy $A \leq Ay_i^g \geq y_i^g$. Hence, by extracting the whole orbit of $(1, y_i)$, we obtain an equivariant simple collapse $Z_{i-1} \searrow^G Z_i$. Inductively, $X_3 = Z_0 \searrow^G Z_1 \searrow^G \ldots \searrow^G Z_r = X_1$.

An analogous proof shows that $X_3 \searrow^G X_2$. Therefore, $X_1 \nearrow\!\!\!\searrow^G X_2$. $\square$

By Propositions 3.1.15 and 3.1.16 we obtain free fundamental group for direct products.

Corollary 3.1.17. *If $G = G_1 \times G_2$ and $p \mid \gcd(|G_1|, |G_2|)$, then $\mathcal{A}_p(G)$ is connected and $\pi_1(\mathcal{A}_p(G))$ is free. Moreover, $\mathcal{A}_p(G)$ is simply connected if and only if $\mathcal{A}_p(G_i)$ is connected for some $i = 1, 2$.*

The following result follows immediately from [BM12a, Corollary 4.10]. We include here an alternative proof.

Proposition 3.1.18. *Let X be a finite connected poset and let $Y \subseteq X$ be a subposet such that $X - Y$ is an anti-chain (i.e. $\forall\ x, x' \in X - Y$, x and x' are not comparable). If Y is simply connected, then $\pi_1(X)$ is free.*

Proof. Since the inclusion $|\mathcal{K}(Y)| \subseteq |\mathcal{K}(X)|$ is a cofibration and $|\mathcal{K}(Y)|$ is simply connected, by van Kampen theorem there is an isomorphism $\pi_1(|\mathcal{K}(X)|) \cong \pi_1(|\mathcal{K}(X)|/|\mathcal{K}(Y)|)$ induced by the quotient map. Since $X - Y$ is an anti-chain, the space $|\mathcal{K}(X)|/|\mathcal{K}(Y)|$ has the homotopy type of a wedge of suspensions. Therefore, $\pi_1(X) = \pi_1(|\mathcal{K}(X)|) \cong \pi_1(|\mathcal{K}(X)|/|\mathcal{K}(Y)|)$ is a free group. □

3.2 A non-free fundamental group

The fundamental group of the Quillen complex was first investigated by Aschbacher, who analyzed simple connectivity [Asc93]. K. Das studied simple connectivity of the p-subgroup complexes of groups of Lie type (see [Das95, Das98, Das00]). In [Kso03, Kso04], Ksontini investigated the fundamental group of the Quillen complex of symmetric groups. Below we recall Ksontini's results. These results will be used in Proposition 3.5.11.

Theorem 3.2.1 ([Kso03, Kso04])**.** *Let $G = \mathbb{S}_n$ and let p be a prime.*

1. *If p is odd, then $\mathcal{A}_p(\mathbb{S}_n) = \mathcal{A}_p(\mathbb{A}_n)$. In this case, $\mathcal{A}_p(\mathbb{S}_n)$ is simply connected if and only if $3p+2 \leq n < p^2$ or $n \geq p^2 + p$. If $p^2 \leq n < p^2 + p$, then $\pi_1(\mathcal{A}_p(\mathbb{S}_n))$ is free unless $p = 3$ and $n = 10$. If $n < 3p$ then $m_p(\mathbb{S}_n) \leq 2$ and $\pi_1(\mathcal{A}_p(\mathbb{S}_n))$ is free.*
2. *If $p = 2$, then $\mathcal{A}_2(\mathbb{S}_n)$ is simply connected if and only if $n = 4$ or $n \geq 7$. In other cases, $\pi_1(\mathcal{A}_2(\mathbb{S}_n))$ is a free group by direct computation.*

In [Sha04] Shareshian extended Ksontini's results and showed that the fundamental group of $\mathcal{A}_p(\mathbb{S}_n)$ is also free for $n = 3p$.

Theorem 3.2.2 ([Sha04])**.** *$\pi_1(\mathcal{A}_p(\mathbb{S}_n))$ is free when $n = 3p$.*

In [Sha04] Shareshian gave the first example of a group whose p-subgroup complex is not homotopy equivalent to a bouquet of spheres: he showed that there is torsion in the second homology group of $\mathcal{A}_3(\mathbb{S}_{13}) = \mathcal{A}_3(\mathbb{A}_{13})$. However its fundamental group is free. Surprisingly we found that the fundamental group of $\mathcal{A}_3(\mathbb{A}_{10})$ is not free. This is the first concrete known example of a Quillen complex with non-free fundamental group. In fact, $\mathbb{A}_{10}$ is, so far, the unique known example of a simple group whose Quillen complex has non-free fundamental group.

To compute $\pi_1(\mathcal{A}_3(\mathbb{A}_{10}))$ we used the Bouc poset $\mathcal{B}_p(G)$ of nontrivial radical p-subgroups. Recall that $\mathcal{B}_p(G) \hookrightarrow \mathcal{S}_p(G)$ is a weak homotopy equivalence. In particular, $\pi_1(\mathcal{A}_p(G)) = \pi_1(\mathcal{S}_p(G)) = \pi_1(\mathcal{B}_p(G))$.

We calculated $\pi_1(\mathcal{B}_3(\mathbb{A}_{10}))$ using GAP [GAP18] with the package [FPSC19]. See also Appendix A.2

We found that $\pi_1(\mathcal{B}_3(\mathbb{A}_{10}))$ is a free product of the free group on 25200 generators and a non-free group on 42 generators and 861 relators whose abelianization is $\mathbb{Z}^{42}$. It does not have torsion elements but it has commuting relations. Note that the integral homology of $\mathcal{A}_3(\mathbb{A}_{10})$ is free abelian (cf. [Sha04, p.306]). As a consequence of Theorem 3.4.2, one can construct an infinite number of examples of finite groups G with non-free $\pi_1(\mathcal{A}_3(G))$, by taking extensions of $3'$-groups whose $3'$-simple groups involved satisfy Aschbacher's conjecture, by $\mathbb{A}_{10}$. It would be interesting to find other examples of simple groups with non-free fundamental group.

We were able to verify that $\mathbb{A}_{10}$ is the smallest group with a p-subgroup complex with non-free π_1. Note that, by Theorem 3.4.2, we only need to verify freeness in almost simple groups (note also that Aschbacher's conjecture holds for groups of order less than the order of $\mathbb{A}_{10}$). On the other hand, Theorem 3.0.4 allowed us to discard many potential counterexamples. The remaining almost simple groups which are smaller than $\mathbb{A}_{10}$ were checked by computer calculations.

3.3 The reduction $O_{p'}(G)=1$

In this section, we reduce the study of the fundamental group of $\mathcal{A}_p(G)$ to the case $O_{p'}(G)=1$ by using Aschbacher's conjecture. We assume that $\mathcal{A}_p(G)$ is connected and that $G=\Omega_1(G)$ since $\mathcal{A}_p(G)=\mathcal{A}_p(\Omega_1(G))$.

The reduction $O_{p'}(G)=1$ relies on the wedge lemma of homotopy colimits. We will use Pulkus-Welker's result [PW00, Theorem 1.1] (stated as Theorem 3.1.7 above) but for the poset $\mathcal{A}_p(G)^2$ instead of $\mathcal{A}_p(G)$. Recall that $\pi_1(\mathcal{A}_p(G))=\pi_1(\mathcal{A}_p(G)^2)$ by Proposition 3.1.10.

Note that $m_p(G)=m_p(G/O_{p'}(G))$ and that $\mathcal{A}_p(G/O_{p'}(G))$ is connected when $\mathcal{A}_p(G)$ is connected since the induced map $\mathcal{A}_p(G)\to\mathcal{A}_p(G/O_{p'}(G))$ is surjective. The following lemma is a slight variation of Pulkus-Welker's result Theorem 3.1.7. We have replaced the hypothesis of solvability of the normal p'-subgroup $N\leq G$ by simple connectivity of $\mathcal{A}_p(NA)$ for $A\in\mathcal{A}_p(G)$ of p-rank 3.

Lemma 3.3.1. *Let N be a normal p'-subgroup of G such that $\mathcal{A}_p(NA)$ is simply connected for each elementary abelian p-subgroup $A\leq G$ of p-rank 3. Then $\mathcal{A}_p(G)^2$ is weak homotopy equivalent to the wedge*

$$\mathcal{A}_p(G/N)^2 \bigvee_{\overline{B}\in\mathcal{A}_p(G/N)^2} \mathcal{A}_p(NB)*\mathcal{A}_p(G/N)^2_{>\overline{B}}.$$

In particular, for a suitable base point,

$$\pi_1(\mathcal{A}_p(G))\cong\pi_1(\mathcal{A}_p(G/N)) *_{\overline{B}\in\mathcal{A}_p(G/N)^2} \pi_1(\mathcal{A}_p(NB)*\mathcal{A}_p(G/N)^2_{>\overline{B}}).$$

Proof. We essentially follow the proof of Pulkus-Welker [PW00, Theorem 1.1]. Let $N \leq G$ be a normal p'-subgroup of G. Write $\overline{G} = G/N$ and let $f : \mathcal{A}_p(G)^2 \to \mathcal{A}_p(\overline{G})^2$ be the map induced by taking quotient. Note that it is well defined and surjective. We will use [PW00, Corollary 2.4]. For this, we have to verify that the inclusions $f^{-1}(\mathcal{A}_p(\overline{G})^2_{<\overline{B}}) \hookrightarrow f^{-1}(\mathcal{A}_p(\overline{G})^2_{\leq\overline{B}})$ are homotopic to constant maps. Note that $f^{-1}(\mathcal{A}_p(\overline{G})^2_{<\overline{B}}) = \mathcal{A}_p(NB) - \operatorname{Max}(\mathcal{A}_p(NB))$ and $f^{-1}(\mathcal{A}_p(\overline{G})^2_{\leq\overline{B}}) = \mathcal{A}_p(NB)$.

By hypothesis and Remark 3.3.2 below we deduce that $\mathcal{A}_p(NB) - \operatorname{Max}(\mathcal{A}_p(NB))$ and $\mathcal{A}_p(NB)$ are spherical of the corresponding dimension for each $B \leq G$ of p-rank at most 3. For instance, if B has p-rank 3, then $\mathcal{A}_p(NB) - \operatorname{Max}(\mathcal{A}_p(NB))$ is 0-spherical and $\mathcal{A}_p(NB)$ is 1-spherical.

The result now follows from the fact that the inclusion of a sphere of dimension n into a sphere of dimension $m > n$ is homotopic to a constant map, and $\mathcal{A}_p(NB) - \operatorname{Max}(\mathcal{A}_p(NB))$ and $\mathcal{A}_p(NB)$ are spherical. □

Remark 3.3.2. Let A be an elementary abelian p-group of p-rank at least 2 acting on a p'-group N. We affirm that $\mathcal{A}_p(NA)$ is connected. Otherwise, take a minimal counterexample NA. Thus, $1 = O_p(NA)$ and $NA = \Omega_1(NA)$ by minimality. Therefore, $N = O_{p'}(NA) = O_{p'}(\Omega_1(NA))$ and $A \cong NA/N = \Omega_1(NA)/O_{p'}(\Omega_1(NA))$ is one of the groups in the list of Theorem A.1.1. But none of the groups in this list is elementary abelian of p-rank at least 2. Consequently $\mathcal{A}_p(NA)$ is connected.

Lemma 3.3.3. *Let N be a normal p'-subgroup of G and let $A \in \mathcal{A}_p(G)$ of p-rank at most 3. Assume Aschbacher's conjecture for p-rank 3. Then, the fundamental group of $\mathcal{A}_p(NA) * \mathcal{A}_p(G/N)^2_{>\overline{A}}$ is free if $|A| = p$ or p^2 and trivial if $|A| = p^3$.*

Proof. We can suppose $N \neq 1$. We examine the possible ranks of A. If $|A| = p$, $\mathcal{A}_p(NA)$ is a disjoint union of points while $\mathcal{A}_p(G/N)^2_{>A}$ is a nonempty graph. Thus, their join is homotopic to a wedge of 2-spheres and 1-spheres.

If $|A| = p^2$, $\mathcal{A}_p(NA)$ is a connected nonempty graph by the above remark. Note that $\mathcal{A}_p(G/N)^2_{>A}$ may be either empty, if A is maximal, or discrete. Thus, their join is homotopic to a wedge of 2-spheres or 1-spheres.

It remains the case $|A| = p^3$. Here, $\mathcal{A}_p(G/N)^2_{>A}$ is empty. We have to show that $\mathcal{A}_p(NA)$ is simply connected. Since we are assuming Aschbacher's conjecture, $\mathcal{A}_p(NA)$ is simply connected when N is the direct product of simple groups permuted transitively by A. In the general case, $\mathcal{A}_p(NA)$ is simply connected by Remarks 3.1.5 and 3.1.6. □

Remark 3.3.4. Note that in the proof of Lemma 3.3.3, Aschbacher's conjecture only needs to be assumed on the p'-simple groups involved in N.

Remark 3.3.5. Assume the hypotheses of Lemma 3.3.3. If in addition we have that $\mathcal{A}_p(G/N)^2_{>\overline{A}}$ is connected for $|A| = p$ (resp. there are no maximal elements in $\mathcal{A}_p(G)$ of order p^2), then the poset $\mathcal{A}_p(NA) * \mathcal{A}_p(G/N)^2_{>\overline{A}}$ is simply connected for $|A| = p$ (resp. $|A| = p^2$). This conditions hold for instance when G/N is an elementary abelian p-group.

Now we apply these results to reduce the study of the fundamental group of the p-subgroup posets to the groups with trivial p'-core.

Assume $G = \Omega_1(G)$ and that Aschbacher's conjecture holds for p-rank 3. Then, by Lemmas 3.3.1 and 3.3.3 and Remark 3.3.2,

$$\pi_1(\mathcal{A}_p(G)) \cong \pi_1(\mathcal{A}_p(G/O_{p'}(G))) *_{\overline{B}\in\mathcal{A}_p(G/N)^2} \pi_1\left(\mathcal{A}_p(NB) * \mathcal{A}_p(G/N)^2_{>\overline{B}}\right).$$

The groups $\pi_1\left(\mathcal{A}_p(NB) * \mathcal{A}_p(G/N)^2_{>\overline{B}}\right)$ are free by Lemma 3.3.3. In particular, $\pi_1(\mathcal{A}_p(G))$ is free whenever $\pi_1(\mathcal{A}_p(G/O_{p'}(G)))$ is.

We deduce the following corollary, which corresponds to the first part of Theorem 3.4.2. Let $S_G = \Omega_1(G)/O_{p'}(\Omega_1(G))$.

Corollary 3.3.6. *Assume Aschbacher's conjecture for p-rank 3. Then there is an isomorphism $\pi_1(\mathcal{A}_p(G)) \cong \pi_1(\mathcal{A}_p(S_G)) * F$, where F is a free group. In particular, $\pi_1(\mathcal{A}_p(G))$ is free if $\pi_1(\mathcal{A}_p(S_G))$ is free.*

Remark 3.3.7. In Corollary 3.3.6, we only need Aschbacher's conjecture to hold on the p'-simple groups involved in $O_{p'}(\Omega_1(G))$.

We finish this section with some remarks concerning Aschbacher's conjecture. Recall the statement of the conjecture.

Conjecture 3.3.8 (Aschbacher)**.** *Let G be a finite group such that $G = F^*(G)A$, where A is an elementary abelian p-subgroup of rank $r \geq 3$ and $F^*(G)$ is the direct product of the A-conjugates of a simple component L of G of order prime to p. Then $\mathcal{A}_p(G)$ is simply connected.*

Aschbacher showed that the conjecture holds for a wide class of simple groups L. Namely, the alternating groups, the groups of Lie type and Lie rank at least 2, the Mathieu sporadic groups and the groups $L_2(q)$ with q even (see [Asc93, Theorem 3]). The case of the Lyons sporadic group is deduced from [AS92]. Later, Segev dealt with many of the remaining groups of Lie type and Lie rank 1 in [Seg94].

In the following proposition we reduce the study of Aschbacher's conjecture to the p-rank 3 case.

Proposition 3.3.9. *If Aschbacher's conjecture holds for p-rank 3, then it holds for any p-rank $r \geq 3$. Moreover, if the conjecture holds in p-rank 3 for a p'-simple group L then it holds in any p-rank $r \geq 3$ for L.*

Proof. Assume the conjecture holds for p-rank 3 and take $G = F^*(G)A$ as in the conjecture, with $m_p(A) \geq 4$. Since $\pi_1(\mathcal{A}_p(G)) \cong \pi_1(\mathcal{A}_p(G)^2)$, it is enough to show that $\mathcal{A}_p(G)^2$ is simply connected. Let $N = F^*(G)$ and note that $G/N \cong A$. We are in the hypotheses of Lemma 3.3.1 and therefore $\mathcal{A}_p(G)$ is simply connected provided that $\mathcal{A}_p(A)$ and the join posets $\mathcal{A}_p(NB) * \mathcal{A}_p(A)^2_{>B}$, with $B \in \mathcal{A}_p(A)^2$, are. Note that we have made the identification $G/N = A$. Since $\mathcal{A}_p(A)$ is spherical of dimension $m_p(A) - 1 \geq 2$, it is simply connected. Moreover, for $B \in \mathcal{A}_p(A)^2$, $\mathcal{A}_p(NB) * \mathcal{A}_p(A)^2_{>B}$ is simply connected by Remark 3.3.5. □

3.4 Reduction to the almost simple case

In this section, we reduce the study of freeness of the fundamental group to the almost simple case. We prove the following result, for which Aschbacher's conjecture is not needed to be assumed.

Theorem 3.4.1. *Let G be a finite group and p a prime dividing its order. Suppose that $O_{p'}(G) = 1$. Then $\pi_1(\mathcal{A}_p(G))$ is a free group except possibly if G is almost simple.*

It allows us to complete the proof of the main theorem of this chapter Theorem 3.4.2.

Theorem 3.4.2. *Let G be a finite group and p a prime dividing $|G|$. Assume that Aschbacher's conjecture holds. Then there is an isomorphism $\pi_1(\mathcal{A}_p(G)) \cong \pi_1(\mathcal{A}_p(S_G)) * F$, where F is a free group. Moreover, $\pi_1(\mathcal{A}_p(S_G))$ is a free group (and therefore $\pi_1(\mathcal{A}_p(G))$ is free) except possibly if S_G is almost simple.*

Proof. By Corollary 3.3.6, we only need to prove the Moreover part. In that case we may assume that $G = S_G$. Therefore, we are in the hypotheses of Theorem 3.4.1 and the result follows. □

Now we focus on the proof of Theorem 3.4.1. Suppose it does not hold and take a minimal counterexample G. Then G satisfies the following conditions:

(C1) $G = \Omega_1(G)$ and $\mathcal{A}_p(G)$ is connected,

(C2) $O_p(G) = 1$ (otherwise $\mathcal{A}_p(G)$ is homotopically trivial by Proposition 1.3.15),

(C3) $\pi_1(\mathcal{A}_p(G))$ is not a free group. In particular, $\mathcal{A}_p(G)$ is not simply connected and G has p-rank at least 3,

(C4) $O_{p'}(G) = 1$,

(C5) $G \ncong G_1 \times G_2$ (by Proposition 3.1.17).

Remark 3.4.3. From conditions (C2) and (C4) we deduce that $Z(G) = 1$, $Z(E(G)) = 1$, $F(G) = 1$ and $F^*(G) = L_1 \times \ldots \times L_r$ is the direct product of simple components of G, each one of order divisible by p. In particular $C_G(F^*(G)) = Z(E(G)) = 1$, so $F^*(G) \leq G \leq \mathrm{Aut}(F^*(G))$.

Remark 3.4.4. If G satisfies the above conditions, by Remark 3.4.3, $F^*(G) = L_1 \times \ldots \times L_r$. Therefore $\mathcal{A}_p(F^*(G))$ has free fundamental group if $r = 2$, and it is simply connected for $r > 2$ (see Proposition 3.1.17). If $r = 1$, G is almost simple. We deal with the cases $r = 2$ and $r > 2$ separately (see Theorems 3.4.8 and 3.4.9 below).

In what follows, we do not need to assume Aschbacher's conjecture. In [Asc93, Sections 7 & 8], Aschbacher characterized the groups G for which some link $\mathcal{A}_p(G)_{>E}$, with $E \leq G$ of order p, is disconnected. The following proposition deals with the case of connected links. Concretely, [Asc93, Theorem 1] asserts that if $O_{p'}(G) = 1$ and the links $\mathcal{A}_p(G)_{>E}$ are connected for all $E \leq G$ of order p, then either $\mathcal{A}_p(G)$ is simply connected, G is almost simple and $\mathcal{A}_p(G)$ and $\mathcal{A}_p(F^*(G))$ are not simply connected, or else G has certain particular structure. We prove that in the later case, the fundamental group is free.

Proposition 3.4.5. *Suppose G satisfies conditions (C1)...(C5). If the links $\mathcal{A}_p(G)_{>E}$ are connected for all $E \leq G$ of order p, then G is almost simple and $\mathcal{A}_p(F^*(G))$ is not simply connected.*

Proof. We use [Asc93, Theorem 1]. By conditions (C1)...(C5), G corresponds either to case (3) or case (4) of [Asc93, Theorem 1]. Case (4) implies that G is almost simple and $\mathcal{A}_p(F^*(G))$ is not simply connected. If G is in case (3) of [Asc93, Theorem 1], then $\pi_1(\mathcal{A}_p(G))$ is free (which is a contradiction by (C3)). This is deduced from the proof of [Asc93, (10.3)], since under these hypotheses $\mathcal{A}_p(G)$ and $\mathcal{A}_p(F^*(G))$ are homotopy equivalent, and $\pi_1(\mathcal{A}_p(F^*(G)))$ is free by Proposition 3.1.17. □

Remark 3.4.6. In [Asc93, Theorem 1], Aschbacher's conjecture is required. However, since we are assuming $O_{p'}(G) = 1$, we do not need to assume the conjecture in the above proposition.

For the rest of this section we will assume that G is not almost simple, so $F^*(G) = L_1 \times \ldots \times L_r$ with $r > 1$. We deal with the cases $r = 2$ and $r > 2$ separately.

Remark 3.4.7. Let L be a simple group with a strongly p-embedded subgroup, i.e. such that $\mathcal{A}_2(L)$ is disconnected. Then L is one of the simple groups in the list of Theorem A.1.1. By Theorem A.1.3, the Sylow p-subgroups of L have the trivial intersection property (i.e. two different Sylow p-subgroups intersect trivially). Therefore the connected componentes of $\mathcal{A}_p(L)$ have the form $\mathcal{A}_p(S)$ for $S \in \mathrm{Syl}_p(L)$, there are $|\mathrm{Syl}_p(L)|$ components and all of them are contractible by Proposition 1.3.15.

Theorem 3.4.8. *Under conditions (C1)...(C5), if $F^*(G) = L_1 \times L_2$ is a direct product of two simple groups, then $p = 2$, $G \cong L \wr C_2$ (the standard wreath product), with L a simple group of Lie type and Lie rank 1 in characteristic 2 and $L_1 \cong L_2 \cong L$. In this case, $\pi_1(\mathcal{A}_2(G))$ is a free group with $(|\mathrm{Syl}_2(L)| - 1)(|\mathrm{Syl}_2(L)| - 1 + |L|)$ generators.*

Proof. Note that $\mathcal{A}_p(F^*(G))$ is homotopy equivalent to $\mathcal{A}_p(L_1) * \mathcal{A}_p(L_2)$, which is simply connected if and only if $\mathcal{A}_p(L_1)$ or $\mathcal{A}_p(L_2)$ is connected (see Proposition 3.1.17).

Assume $\mathcal{A}_p(F^*(G))$ is simply connected. Since $\mathcal{A}_p(G)$ is not simply connected, by Lemma 3.1.14 there exists some subgroup $E \leq G$ of order p such that $\mathcal{A}_p(C_{F^*(G)}(E))$ is disconnected. Since $F^*(G) = L_1 \times L_2$, $m_p(F^*(G)) > 2$ by simple connectivity. By [Asc93, (10.5)] E acts regularly on the set of components of G and each L_i has a strongly p-embedded subgroup, so $\mathcal{A}_p(L_i)$ is disconnected for $i = 1,2$. In particular $p = 2$ and $L_1 \cong L_2$. Since $\mathcal{A}_p(F^*(G)) \underset{w}{\approx} \mathcal{A}_p(L_1) * \mathcal{A}_p(L_2)$ is simply connected, $\mathcal{A}_p(L_i)$ is connected for some i, a contradiction.

Now suppose $\mathcal{A}_p(F^*(G))$ is not simply connected. Then, $\pi_1(\mathcal{A}_p(F^*(G)))$ is a free group by Proposition 3.1.17, and both L_1 and L_2 are simple groups with strongly p-embedded subgroups. We use [Asc93, (10.3)]. By the above hypotheses, G corresponds to either case (2) or case (3) of [Asc93, (10.3)]. In case (3), as we mentioned in the proof of Proposition 3.4.5, $\mathcal{A}_p(G)$ and $\mathcal{A}_p(F^*(G))$ are homotopy equivalent (which contradicts the conditions on G). Therefore, G is in case (2) of [Asc93, (10.3)], $p = 2$ and $G = L_1 \wr E$ for some $E \leq G$ of order 2. Then, $E = \langle e \rangle$ for an involution $e \in G$, and L_1 is a group of Lie type and Lie rank 1 in characteristic 2 by Remark 3.4.7.

We prove now that $\pi_1(\mathcal{A}_2(G))$ is free. Let $N = F^*(G) = L_1 \times L_2$ and $L = L_1$. Let $\mathcal{N} = \{A \in \mathcal{A}_2(G) : A \cap N \neq 1\}$ and let $\mathcal{S} = \mathcal{A}_2(G) - \mathcal{N}$ be the complement of $\mathcal{N}$ in $\mathcal{A}_2(G)$. If $A \in \mathcal{S}$ then $A \cong NA/N \leq NE/N = E$, so $|A| = 2$. Therefore, $\mathcal{S} = \{A \leq G : |A| = 2, A \not\leq N\}$ consists of some minimal elements of the poset, and we have

$$\mathcal{A}_2(G) = \mathcal{N} \bigcup_{A \in \mathcal{S}} \mathcal{A}_2(G)_{\geq A}.$$

For each $A \in \mathcal{S}$, $\mathcal{N} \cap \mathcal{A}_2(G)_{\geq A} = \{W \in \mathcal{A}_2(G) : W \cap N \neq 1, W \geq A\} \simeq \mathcal{A}_2(C_N(A))$ by Lemma 3.1.13. Note that $C_N(A) \cong L$ since $G = NA$. By Remark 3.4.7, $\mathcal{A}_2(L)$ has $|\mathrm{Syl}_2(L)|$ connected components and each component is simply connected. Since $\mathcal{A}_2(G)_{\geq A}$ is contractible, by the non-connected version of van Kampen theorem (see [Bro06, Section 9.1]), $\pi_1(\mathcal{N} \cup \mathcal{A}_2(G)_{\geq A}) = \pi_1(\mathcal{N}) * F_A$, where F_A is the free group of rank $|\mathrm{Syl}_2(L)| - 1$. Moreover, $\mathcal{A}_2(G)_{\geq A} \cap \mathcal{A}_2(G)_{\geq B} \subseteq \mathcal{N}$ for each $A \neq B \in \mathcal{S}$, and recursively we have $\pi_1(\mathcal{A}_2(G)) \cong \pi_1(\mathcal{N}) * F$, where F is the free group of rank $(|\mathrm{Syl}_2(L)| - 1)|\mathcal{S}|$.

By Remark 3.1.12, $\mathcal{N} \simeq \mathcal{A}_2(N) = \mathcal{A}_2(L_1 \times L_2)$. Therefore, by Proposition 3.1.15, $\pi_1(\mathcal{N})$ is a free group of rank $(|\mathrm{Syl}_2(L)| - 1)^2 = (|\pi_0(\mathcal{A}_2(L_1))| - 1)(|\pi_0(\mathcal{A}_2(L_2))| - 1)$.

Finally we compute $|\mathcal{S}|$. Let $\mathcal{I}(G)$ be the number of distinct involutions in G. Therefore, $\mathcal{I}(G) = \mathcal{I}(N) + s$, where s is the number of involutions not contained in N. Note that $s = |\mathcal{S}|$. If $g \in G - N$ is an involution, then $g = xye$ with $x \in L_1$ and $y \in L_2$. The condition $g^2 = 1$ implies $1 = xyexye = xyx^e y^e = (xy^e)(yx^e)$ with $y^e \in L_1$ and $x^e \in L_2$. Since $L_1 \cap L_2 = 1$, $xy^e = 1$ and $yx^e = 1$, i.e. $y = (x^{-1})^e$. In consecuence, $g = x(x^{-1})^e e = xex^{-1}$ and $s = |\{x(x^{-1})^e e : x \in L_1\}| = |L_1| = |L|$.

In conclusion, $\pi_1(\mathcal{A}_2(G))$ is a free group with $(|\operatorname{Syl}_2(L)|-1)^2+|L|(|\operatorname{Syl}_2(L)|-1)$ generators. $\square$

Now we deal with the case $r>2$.

Theorem 3.4.9. *Under conditions (C1)...(C5), if $F^*(G)=L_1\times\ldots\times L_r$ is a direct product of simple groups with $r>2$, then p is odd, $r=p$, each L_i has a strongly p-embedded subgroup, $\{L_1,\ldots,L_r\}$ is permuted regularly by some subgroup of order p of G, and $\pi_1(\mathcal{A}_p(G))$ is a free group.*

Proof. The hypotheses imply that $\mathcal{A}_p(F^*(G)) \underset{w}{\approx} \mathcal{A}_p(L_1)*\ldots*\mathcal{A}_p(L_r)$ is simply connected by Proposition 3.1.17. Then there exists some subgroup $E\leq G$ of order p such that $\mathcal{A}_p(C_{F^*(G)}(E))$ is disconnected by Lemma 3.1.14. Therefore, we are in case (5) of [Asc93, (10.5)], E permutes regularly the components $\{L_1,\ldots,L_r\}$ and each L_i has a strongly p-embedded subgroup. In particular, $r=p$ is odd and $L_i\cong L_j$ for all i,j.

Set $N=F^*(G)$ and let $H=\bigcap_i N_G(L_i)$. Then $N\trianglelefteq H$. If $A\in\mathcal{A}_p(H)$, then $A\leq\bigcap_i N_G(L_i)$, so $C_N(A)=\prod_i C_{L_i}(A)$. In particular,

$$\mathcal{A}_p(C_N(A))=\mathcal{A}_p\left(\prod_i C_{L_i}(A)\right)\underset{w}{\approx}\mathcal{A}_p(C_{L_1}(A))*\mathcal{A}_p(C_{L_2}(A))*\ldots*\mathcal{A}_p(C_{L_p}(A))$$

is simply connected. Therefore, by Lemma 3.1.14, $\mathcal{A}_p(H)$ is simply connected, hence $\mathcal{N}=\{X\in\mathcal{A}_p(G): X\cap H\neq 1\}$ is also simply connected by Remark 3.1.12. Consider the complement $\mathcal{S}=\mathcal{A}_p(G)-\mathcal{N}$. If $X\in\mathcal{S}$ then $X\cap H=1$. Thus, $X=X_1X_2$ where X_2 permutes regularly the components $\{L_1,\ldots,L_p\}$ and $X_1\leq\bigcap_i N_G(L_i)=H$. Since $X\cap H=1$, we conclude that $X_1=1$ and $|X_2|=p$, i.e. $|X|=p$. Therefore, $\mathcal{S}$ is an anti-chain and, by Proposition 3.1.18, $\pi_1(\mathcal{A}_p(G))$ is free. $\square$

Proof of Theorem 3.4.1. If G is a minimal counterexample to this theorem, then G satisfies conditions (C1)...(C5) and G is not almost simple. By Remark 3.4.4 and Theorems 3.4.8 and 3.4.9, $\pi_1(\mathcal{A}_p(G))$ is free, a contradiction. $\square$

3.5 Freeness in some almost simple cases

In this section we prove that $\pi_1(\mathcal{A}_p(G))$ is free when G is an almost simple group with some extra hypothesis. We will use the structure of the outer automorphisms group of a simple group. We refer the reader to sections 7 and 9 of [GL83] and Chapters 2 to 5 of [GLS98]. Note that [GLS98] uses different definitions of field and graph automorphisms (see [GLS98, Warning 2.5.2]). We follow the definitions given in [GL83]. For the p-rank of simple groups we will use the results of section 10 of [GL83] and in particular [GL83, (10-6)]. See also Appendix A.1.

Consider a finite group G such that $L \leq G \leq \mathrm{Aut}(L)$, where L is a simple group of order divisible by p. We may suppose that $G = \Omega_1(G)$, $m_p(G) \geq 3$ and $\mathcal{A}_p(G)$ is connected.

In the following theorem we will use [Qui78, Theorem 3.1]. Recall that this theorem shows that if H is a group of Lie type and Lie rank n in characteristic p, then $\mathcal{K}(\mathcal{A}_p(H))$ has the homotopy type of the Tits building of H, which is homotopy equivalent to a wedge of spheres of dimension $(n-1)$. This wedge is nontrivial if $O_p(H) = 1$.

Theorem 3.5.1. *Let G and L be as above. Then $\pi_1(\mathcal{A}_p(G))$ is free if $\mathcal{A}_p(L)$ is disconnected or simply connected.*

Proof. We prove first that $\pi_1(\mathcal{A}_p(G))$ is free when $\mathcal{A}_p(L)$ is disconnected. In this case, L has a strongly p-embedded subgroup. We deal with each case of the list of Theorem A.1.1. See also Table A.4.

- If $m_p(L) = 1$, then p is odd and $m_p(G) \leq 2$ by [GL83, (7-13)].
- If L is a simple group of Lie type and Lie rank 1 in characteristic p, then the Sylow p-subgroups of L have the trivial intersection property, i.e. $P \cap P^g = 1$ if $P \in \mathrm{Syl}_p(L)$ and $g \in L - N_L(P)$ (see Theorem A.1.3). The proof is similar to the proofs of Theorems 3.4.8 and 3.4.9. Let $\mathcal{N} = \{X \in \mathcal{A}_p(G) : X \cap L \neq 1\}$ and let $\mathcal{S} = \mathcal{A}_p(G) - \mathcal{N}$. Since $m_p(\mathrm{Out}(L)) \leq 1$, $\mathcal{S}$ consists of subgroups of order p. By Remarks 3.1.12 and 3.4.7, $\mathcal{N} \simeq \mathcal{A}_p(L)$ has simply connected components. If $A \in \mathcal{S}$, then $\mathcal{A}_p(G)_{\geq A} \cap \mathcal{N} \simeq \mathcal{A}_p(C_L(A))$ by Lemma 3.1.13, and the Sylow p-subgroups of $C_L(A)$ intersect trivially, so $\mathcal{A}_p(G)_{\geq A} \cap \mathcal{N}$ has simply connected components. Then $\pi_1(\mathcal{A}_p(G))$ is free by van Kampen theorem.
- $L \ncong {}^2G_2(3) = \mathrm{Ree}(3)$ or $\mathrm{Aut}(\mathrm{Sz}(32))$ since these groups are not simple.
- In the remaining cases, $m_p(G) = 2$ by Table A.4, [GL83, (10-6)] or by direct computation.

Now we prove that $\pi_1(\mathcal{A}_p(G))$ is free when $\mathcal{A}_p(L)$ is simply connected. Note that $m_p(L) \geq 3$ since otherwise $O_p(L) \neq 1$, contradicting that L is simple. By Lemma 3.1.14, we may assume that $\mathcal{A}_p(C_L(E))$ is disconnected for some $E \leq G$ of order p. Therefore, we are in case (1), (2), (3) or (4) of [Asc93, (10.5)]. We deal with each one of them.

1. If L is of Lie type and Lie rank 1 in characteristic p, then $\mathcal{A}_p(L)$ is disconnected, contradicting the hypothesis.
2. If $p = 2$, q is even and $L \cong L_3(q)$, $U_3(q)$ or $Sp_4(q)$, then L is of Lie type and Lie rank at most 2. In any case, $\mathcal{K}(\mathcal{A}_p(L))$ is not simply connected since it has the homotopy type of a nontrivial wedge of spheres of dimension equal to the Lie rank of L minus 1. If $L \cong G_2(3)$, then $\mathrm{Out}(L) \cong \mathrm{Out}(G_2(3)) \cong C_2$. Hence, $\pi_1(\mathcal{A}_p(G))$ is free by Proposition 3.1.18

applied to $Y = \{X \in \mathcal{A}_p(G) : X \cap L \neq 1\} \subseteq \mathcal{A}_p(G)$. Note that Y is simply connected since $Y \simeq \mathcal{A}_p(L)$ by Remark 3.1.12.

3. If $p = 2$ and $L \cong L_3(q^2)$ with q even, then L has Lie rank 2, and thus $\mathcal{K}(\mathcal{A}_p(L))$ is a nontrivial wedge of 1-spheres, contradicting the hypothesis.

4. If $p > 3$, $q \equiv \varepsilon \mod p$ and $L \cong L_p^{\varepsilon}(q)$, then $m_p(\mathrm{Out}(L)) = 1$ by [GL83, (9-3)]. Therefore, $\pi_1(\mathcal{A}_p(G))$ is free by Proposition 3.1.18 applied to $Y = \{X \in \mathcal{A}_p(G) : X \cap L \neq 1\} \subseteq \mathcal{A}_p(G)$.

□

Corollary 3.5.2. *If L is a Lie type group in characteristic p and $p \nmid (G : L)$ when L has Lie rank 2, then $\pi_1(\mathcal{A}_p(G))$ is free.*

Proof. Recall that $\mathcal{K}(\mathcal{A}_p(L))$ is homotopy equivalent to a nontrivial bouquet of spheres of dimension $n-1$, where n is the Lie rank of L. If $n \neq 2$, L is in the hypotheses of Theorem 3.5.1. If $n = 2$, $\mathcal{A}_p(G) = \mathcal{A}_p(L)$ since $p \nmid (G : L)$. In either case, $\pi_1(\mathcal{A}_p(G))$ is a free group. □

Remark 3.5.3. Corollary 3.5.2 does not give information in the case that L has Lie rank 2 and $p \mid (G : L)$. By [Qui78, Theorem 3.1], $\pi_1(\mathcal{A}_p(L))$ is a free group but if $p \mid (G : L)$ then it may happen that $\pi_1(\mathcal{A}_p(G)) \ncong \pi_1(\mathcal{A}_p(L))$. For example, take $L = L_3(4)$ and $p = 2$. Note that $\mathrm{Out}(L) = D_{12}$. We have computed $\pi_1(\mathcal{A}_p(G))$ for all possible groups G, with $L \leq G \leq \mathrm{Aut}(L)$, and they turned out to be free. If $G = \mathrm{Aut}(L)$, $\mathcal{A}_p(G)$ is simply connected, while $\pi_1(\mathcal{A}_p(L))$ is a nontrivial free group.

The simply connectivity of the p-subgroup complex has been widely studied. See for example [Asc93, Das95, Das98, Das00, Kso03, Kso04, Qui78, Sha04, Smi11]. In general, when the group has p-rank at least 3 its p-subgroup complex is expected to be simply connected, and often even Cohen-Macaulay (see [Smi11, p.290]). Therefore, the above theorem do indeed cover a large class of almost simple groups.

Recall the classification of J. Walter of simple groups with abelian Sylow 2-subgroup [Wal69].

Theorem 3.5.4 (Walter's Classification)**.** *Let L be a a simple group with abelian Sylow 2-subgroup S. Then L is isomorphic to one of the following groups:*

1. *$L_2(q)$, $q \equiv 3,5 \mod 8$ (and S is elementary abelian of order 2^2),*

2. *$L_2(2^n)$, $n \geq 2$ (and S is elementary abelian of order 2^n),*

3. *${}^2G_2(3^n)$, n odd (and S is elementary abelian of order 2^3),*

4. *J_1 (and S is elementary abelian of order 2^3).*

In the proof of the next result, we work with Bouc poset $\mathcal{B}_p(G)$ of nontrivial radical p-subgroups instead of $\mathcal{A}_p(G)$. Recall that $\mathcal{A}_p(G)$ and $\mathcal{B}_p(G)$ have the same weak homotopy type, so in particular $\pi_1(\mathcal{A}_p(G)) \cong \pi_1(\mathcal{B}_p(G))$.

Theorem 3.5.5. *Suppose G is almost simple, $p = 2$ and $F^*(G)$ has abelian Sylow 2-subgroups. Then $\pi_1(\mathcal{A}_2(G))$ is free.*

Proof. Let $L = F^*(G)$. By Walter's Theorem 3.5.4, L is one of the groups (1)...(4).

The case (2) follows from the disconnected case of Theorem 3.5.1.

In case (4), $G = J_1$ and $\pi_1(\mathcal{A}_2(G))$ is free on 4808 generators by computer calculation with GAP [GAP18] and package [FPSC19].

In case (3), $L = {}^2G_2(3^n) = \mathrm{Ree}(3^n)$, with n odd. Then $\mathrm{Out}(L) \cong C_n$ has odd order and $\mathcal{A}_2(G) = \mathcal{A}_2(L)$. It suffices to prove that $\mathcal{B}_2(L)$ has height 1. Note that $\mathcal{S}_2(L) = \mathcal{A}_2(L)$. Let $q = 3^n$. By [GLS98, Theorem 6.5.5], the normalizers of the nontrivial 2-subgroups of L have the following forms: $C_2 \times L_2(q)$ for involutions and $(C_2^2 \times D_{(q+1)/2}) : C_3$ for four-subgroups. If $t \in L$ is an involution, $O_2(N_L(t)) = \langle t \rangle$. If X is a order 4 subgroup of L then $X < O_2(N_L(X)) \cong C_2^3$ since $q \equiv 3 \mod 4$. For a Sylow 2-subgroup S of L we have that $S = O_2(N_L(S))$. Therefore, the poset $\mathcal{B}_2(L)$ contains the Sylow 2-subgroups and the subgroups of order 2. In consequence, $\mathcal{B}_2(L)$ has height 1 (and therefore it has free fundamental group).

In case (1), $L = L_2(q)$, $q \equiv 3, 5 \mod 8$, so q is odd and it is not a square. Therefore, $\mathrm{Outdiag}(L)$ has odd order and thus we may assume $L \leq G \leq \mathrm{Inndiag}(L)$. In any case, $m_2(G) = 2$ by [GLS98, Theorem 4.10.5(b)]. □

Now we compute the fundamental group of $\mathcal{A}_p(G)$ for some particular sporadic groups L. Note that $m_p(L) \leq 2$ if $p > 7$. See Table A.6 for the p-rank of the Sporadic simple groups and Table A.5 for their outer automorphisms groups.

Example 3.5.6. By computer calculations, $\pi_1(\mathcal{A}_2(G))$ is free for $L = J_1$ or J_2. Note that $\mathrm{Out}(J_1) = 1$ and $\mathrm{Out}(J_2) = C_2$. If p is odd, $m_p(G) \leq 2$ for $L = J_1$ or J_2.

Example 3.5.7. If $G = J_3$ or O'N and $p = 3$, then $\pi_1(\mathcal{A}_3(G))$ is free. By [Kot97, Proposition 3.1.4] and [UY02, Section 6.1], there are only two conjugacy classes of nontrivial radical 3-subgroups of G. Therefore, $\mathcal{K}(\mathcal{B}_3(G))$ has dimension 1. For $p > 3$, $m_p(G) \leq 2$, so $\pi_1(\mathcal{A}_p(G))$ is free.

Example 3.5.8. If $G = \mathrm{McL}$ and $p = 3$, then $\pi_1(\mathcal{A}_3(G))$ is free. By computer calculations, if S is a Sylow 3-subgroup of G, then there exist three subgroups of S (up to conjugacy) which are nontrivial radical 3-subgroups of G, namely A, B and S. Their orders are $|A| = 81$, $|B| = 243$, $|S| = 729$. Moreover, $A, B \trianglelefteq S$ and $A \not\leq B$. Then $A^g \not\leq B$ for any $g \in G$ such that $A^g \leq S$, and therefore $\mathcal{K}(\mathcal{B}_3(G))$ is 1-dimensional. For $p > 3$, $m_p(G) \leq 2$, so $\pi_1(\mathcal{A}_p(G))$ is free.

Proposition 3.5.9. *Assume L is a Mathieu sporadic group. If p is odd then $m_p(G) \leq 2$ and $\mathcal{A}_p(G)$ has free fundamental group. If $p = 2$, $\mathcal{A}_2(G)$ is simply connected except for $L = M_{11}$, in which case $\pi_1(\mathcal{A}_2(G))$ is a nontrivial free group.*

Proof. Let L be a one of the Mathieu sporadic groups M_{11}, M_{12}, M_{22}, M_{23} or M_{24}. In all cases, $m_p(L) \leq 2$ if p is odd (see Table A.6). Assume that $p = 2$. Recall that $\mathrm{Aut}(L) = L$ for $L = M_{11}$, M_{23} and M_{24}, and $\mathrm{Out}(L) = C_2$ for $L = M_{12}$ and M_{22}.

Note that $m_2(M_{11}) = 2$. For $L = M_{12}$ or M_{22} we checked with GAP that $\mathcal{A}_2(L)$ is simply connected.

If $G = \mathrm{Aut}(M_{22})$, then $\mathcal{S} = \{X \in \mathcal{A}_p(G) : X \cap M_{22} = 1\} \subseteq \mathrm{Min}(\mathcal{A}_p(G))$. Let $\mathcal{N} = \mathcal{A}_p(G) - \mathcal{S}$. Recall that $\mathcal{N} \simeq \mathcal{A}_2(M_{22})$ by Remark 3.1.12 and therefore it is simply connected. Any $A \in \mathcal{S}$ is generated by an involution acting by outer automorphism on M_{22}. By [GLS98, Table 5.3c], its centralizer in M_{22} has a nontrivial normal 2-subgroup. That is, $\mathcal{A}_2(C_{M_{22}}(A))$ is homotopically trivial by Proposition 1.3.15. Then for any $A \in \mathcal{S}$, $\mathcal{N} \cup \mathcal{A}_2(G)_{\geq A}$ is simply connected by van Kampen theorem and, recursively, $\mathcal{A}_2(G)$ is simply connected. A similar reasoning shows that $\mathcal{A}_2(G)$ is simply connected for $G = \mathrm{Aut}(M_{12})$ (see [GLS98, Table 5.3b]).

By [Smi11, p.295], $\mathcal{K}(\mathcal{A}_2(M_{24}))$ is homotopy equivalent to its 2-local geometry, which is simply connected.

It remains to determine the fundamental group of $\mathcal{A}_2(M_{23})$. For this we use the classification of the maximal subgroups of M_{23} and M_{22}. First note that M_{22} is a maximal subgroup of M_{23} of odd index 23. In particular, any elementary abelian 2-subgroup of M_{23} is contained in some conjugate of M_{22}. Therefore, $\mathcal{U} = \{\mathcal{A}_2(M_{22}) \cup \mathcal{A}_2(M_{22}^g) : g \in M_{23}\}$ is a cover of $\mathcal{A}_2(M_{23})$ by subposets. We have computed the intersections between different conjugates of M_{22} with GAP. All the intersections $M_{22} \cap M_{22}^g$, with $g \in M_{23} - M_{22}$, form a subgroup of M_{22} of order 20160. All the maximal subgroups of M_{22} have order less than 20160 except for the maximal subgroup isomorphic to $L_3(4)$ (and all its conjugates) which have order exactly 20160. Thus, $M_{22} \cap M_{22}^g \cong L_3(4)$ and, by van Kampen theorem, each element of $\mathcal{U}$ is simply connected. The triple intersections of different conjugates of M_{22} are all isomorphic to $C_2^2 : \mathbb{A}_8$, and the quadruple intersections of different conjugates of M_{22} are all isomorphic to $C_2^4 : C_3$. This shows that double and triple intersections of elements of $\mathcal{U}$ are connected. In consequence, by van Kampen theorem, $\mathcal{A}_2(M_{23})$ is simply connected. $\square$

We investigate now the fundamental group of the Quillen complex of alternating groups at $p = 2$. We use Ksontini's results on $\pi_1(\mathcal{A}_2(\mathbb{S}_n))$ (see Section 3.2).

Remark 3.5.10. By the list of Theorem 3.5.1, the poset $\mathcal{A}_2(\mathbb{A}_n)$ is disconnected if and only if $n = 5$, since, for $p = 2$, the unique isomorphism of an alternating group with a group of this list is $\mathbb{A}_5 \cong L_2(4)$ (see Theorem A.1.4).

Proposition 3.5.11. *Let $n \geq 4$. The fundamental group of $\mathcal{A}_2(\mathbb{A}_n)$ is simply connected for $n = 4$ and $n \geq 8$. For $n = 5$, each component of $\mathcal{A}_2(\mathbb{A}_5)$ is simply connected, for $n = 6$ it is free of rank 16, and for $n = 7$ it is free of rank 176.*

Proof. If $n = 4$, then $O_2(\mathbb{A}_4) \neq 1$ and $\mathcal{A}_2(\mathbb{A}_4)$ is contractible. If $n = 5$, $\mathbb{A}_5 = L_2(4)$, which has trivial intersections of Sylow 2-subgroups by Theorem A.1.3. Therefore, its connected components are contractible (and un particular simply connected). The cases $n = 6,7,8$ can be obtained directly by computer calculations.

We prove the case $n \geq 9$. We proceed similarly as before. Take $\mathcal{N} = \{A \in \mathcal{A}_2(\mathbb{S}_n) : A \cap \mathbb{A}_n \neq 1\}$. By Remark 3.1.12, $\mathcal{N} \simeq \mathcal{A}_2(\mathbb{A}_n)$, and let $\mathcal{S} = \mathcal{A}_2(\mathbb{S}_n) - \mathcal{N}$ be its complement in $\mathcal{A}_2(\mathbb{S}_n)$. Note that $\mathcal{S}$ consists of the subgroups of order 2 of $\mathbb{S}_n$ generated by involutions which can be written with an odd number of disjoint transpositions. Observe also that for any $A \neq B \in \mathcal{S}$, $\mathcal{A}_2(\mathbb{S}_n)_{\geq A} \cap \mathcal{A}_2(\mathbb{S}_n)_{\geq B} \subseteq \mathcal{N}$. By Ksontini's Theorem 3.2.1, $\pi_1(\mathcal{A}_2(\mathbb{S}_n)) = 1$. Therefore, by van Kampen theorem, in order to prove that $\pi_1(\mathcal{A}_2(\mathbb{A}_n)) = \pi_1(\mathcal{N})$ is trivial, we only need to show that the intersections $\mathcal{N} \cap \mathcal{A}_2(\mathbb{S}_n)_{\geq A} \simeq \mathcal{A}_2(C_{\mathbb{A}_n}(A))$ are simply connected for all $A \in \mathcal{S}$.

We appeal now to the characterization of the centralizers of involutions in $\mathbb{A}_n$ to show that $\mathcal{A}_2(C_{\mathbb{A}_n}(x))$ is simply connected if $\langle x \rangle \in \mathcal{S}$. Let $x \in \mathbb{S}_n - \mathbb{A}_n$ be an involution acting as the product of r disjoint transpositions and with s fixed points. By [GLS98, Proposition 5.2.8], $C_{\mathbb{A}_n}(x) \cong (H_1 \times H_2)\langle t \rangle$, where $H_1 \leq \mathbb{Z}_2 \wr \mathbb{S}_r$ has index 2, and $H_2 \cong \mathbb{A}_s$. Here, the wreath product is taken with respect to the natural permutation of $\mathbb{S}_r$ on the set $\{1,\dots,r\}$. Moreover, $H_1 = E : \mathbb{S}_r$ where $E \leq \mathbb{Z}_2^r$ is the subgroup $\{(a_1,\dots,a_r) \in \mathbb{Z}_2^r : \sum_i a_i = 0\}$. If $s \leq 1$ then $t = 1$. If $s \geq 2$, then $t \neq 1$, $H_1\langle t \rangle = \mathbb{Z}_2 \wr \mathbb{S}_r$ and $H_2\langle t \rangle = \mathbb{S}_s$. In any case, $E \trianglelefteq (H_1 \times H_2)\langle t \rangle$ since $[H_1,H_2] = 1$ and $E \trianglelefteq H_1\langle t \rangle$. Therefore, if $r > 1$, $O_2(C_{\mathbb{A}_n}(x)) \neq 1$ and $\mathcal{A}_2(C_{\mathbb{A}_n}(x))$ is contractible. In particular it is simply connected. If $r = 1$, $C_{\mathbb{A}_n}(x) \cong H_2\langle t \rangle \cong \mathbb{S}_s$, and therefore $\mathcal{A}_2(C_{\mathbb{A}_n}(x)) \simeq \mathcal{A}_2(\mathbb{S}_s)$ is simply connected by Theorem 3.2.1 (note that $s \geq 7$ since $2r + s = n \geq 9$). □

Combining Proposition 3.5.11 with Theorem 3.2.1 we deduce the following corollary.

Corollary 3.5.12. *If $\mathbb{A}_n \leq G \leq \mathrm{Aut}(\mathbb{A}_n)$, then $\pi_1(\mathcal{A}_2(G))$ is a free group.*

Proof. If $n \neq 6$, then $G = \mathbb{A}_n$ or $\mathbb{S}_n$. In any case, $\pi_1(\mathcal{A}_2(G))$ is free by Proposition 3.5.11 and Theorem 3.2.1. For $n = 6$, $\mathrm{Out}(\mathbb{A}_6) = C_2 \times C_2$ and $\mathbb{A}_6 < \mathbb{S}_6 < \mathrm{Aut}(\mathbb{A}_6)$. If $F^*(G) = \mathbb{A}_6$, then $G \cong \mathbb{S}_6$, $\mathbb{A}_6$ or $\mathrm{Aut}(\mathbb{A}_6)$. In either case, $\pi_1(\mathcal{A}_2(G))$ is free by the above results or by computer calculations. □

Chapter 4

Quillen's conjecture

In [Qui78], D. Quillen conjectured that G has a nontrivial normal p-subgroup (i.e. $O_p(G) \neq 1$) if and only if $\mathcal{K}(\mathcal{S}_p(G))$ is contractible. If $O_p(G) \neq 1$ then $\mathcal{S}_p(G)$ is conically contractible via the homotopy $P \leq PO_p(G) \geq O_p(G)$, and hence, $\mathcal{K}(\mathcal{S}_p(G))$ is contractible. Quillen's conjecture is focused on the reciprocal: if $O_p(G) = 1$ then $\mathcal{K}(\mathcal{S}_p(G))$ is not contractible. Quillen showed that his conjecture holds for groups of p-rank at most 2 [Qui78, Proposition 2.10], solvable groups [Qui78, Corollary 12.2] and groups of Lie type in characteristic p [Qui78, Theorem 3.1]. A remarkable progress on this conjecture was done by M. Aschbacher and S.D. Smith in [AS93], based on the Classification of the Finite Simple Groups. They proved that the conjecture holds if $p > 5$ and G does not contain certain unitary groups as components (see Theorem 4.1.2 below).

In general, it is believed that a stronger version of Quillen's conjecture holds. Namely, if $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{S}_p(G), \mathbb{Q}) \neq 0$. Both in [Qui78] and in [AS93], it is shown this stronger version of the conjecture. Most of Quillen's proofs consist on showing that the top level homology group $\tilde{H}_{m_p(G)-1}(\mathcal{A}_p(G), \mathbb{Z})$ does not vanish when $O_p(G) = 1$. Note that the top homology group is a free abelian group. Aschbacher and Smith call this feature the *Quillen dimension property at p*, $(QD)_p$ for short, and it is a central tool in the proof of their main theorem on Quillen's conjecture. Nevertheless, there are groups not satisfying $(QD)_p$. For example, when G is a finite group of Lie type in characteristic p, $\mathcal{K}(\mathcal{A}_p(G))$ has the homotopy type of the Tits building of G, which in general has lower dimension than Quillen complex. In [AS93, Theorem 3.1] there is a list of the simple groups for which its p-extension may not satisfy $(QD)_p$.

In this chapter we review the results on Quillen's conjecture. We briefly explain the ideas behind the proofs of some cases of the conjecture in Section 4.1. We sketch the proof of Aschbacher-Smith's result [AS93, Main Theorem] in Section 4.2.

In Section 4.3, we prove new cases of Quillen's conjecture, which were not known so far.

Theorem 4.3.1. *If K is a $\mathbb{Z}$-acyclic and 2-dimensional G-invariant subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$, then $O_p(G) \neq 1$.*

The previous result provides a useful tool to prove that a group verifies Quillen's conjecture.

Corollary 4.3.2. *Let G be a finite group. Suppose that $\mathcal{K}(\mathcal{S}_p(G))$ admits a 2-dimensional and G-invariant subcomplex homotopy equivalent to itself. If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{S}_p(G), \mathbb{Z}) \neq 0$.*

In particular, it shows that the conjecture holds for groups of p-rank 3 (extending the p-rank 2 case). See Corollary 4.3.3. Our proof of the 2-dimensional case relies on the Classification since it is based on the theory developed by Oliver and Segev in [OS02]. In Section 4.4 we provide examples of groups satisfying the conjecture which are not included in the theorems of [AS93].

The results of Sections 4.3 and 4.4 appeared in a joint work with I. Sadofschi Costa and A. Viruel [PSV19].

In Section 4.5 we work with the strong version of Quillen's conjecture.

Strong Quillen's conjecture. *If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.*

We prove that the strong conjecture can be studied under the assumption $O_{p'}(G) = 1$.

Theorem 4.5.1. *Let G be a finite group such that $O_p(G) = 1$, $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) = 0$ and its proper subgroups satisfy the strong Quillen's conjecture. Then $O_{p'}(G) = 1$. In particular, a minimal counterexample G to the strong Quillen's conjecture has $O_{p'}(G) = 1$.*

This theorem generalizes Aschbacher-Smith result [AS93, Proposition 1.6], in which they prove an analogous statement but for $p > 5$. They strongly use the CFSG in several parts of the proof of their proposition. We only use the Classification to invoke the p-solvable case Theorem 4.1.3.

In combination with Corollary 4.3.2 (which is stated in terms of integral homology and not in rational homology) and Theorem 3.4.2, we get the following results.

Corollary 4.5.13. *If proper subgroups of G satisfy the strong Quillen's conjecture but $O_p(G) = 1$ and $\tilde{H}_*(\mathcal{S}_p(G), \mathbb{Q}) = 0$, then $\mathcal{K}(\mathcal{S}_p(G))$ has no 2-dimensional G-invariant homotopy equivalent subcomplex. In particular, $m_p(G) \geq 4$.*

Corollary 4.5.14. *The strong Quillen's conjecture holds for groups of p-rank at most 3.*

Finally, with an exhaustive use of the results and techniques developed along these chapters, we culminate with the proof of the strong Quillen's conjecture for groups of p-rank at most 4 and reduce the study of the conjecture to groups with components of p-rank at least 2.

Theorem 4.6.8. *The strong Quillen's conjecture holds for groups of p-rank at most 4.*

Theorem 4.6.3. *Let $L \leq G$ be a component such that $L/Z(L)$ has p-rank 1. If the strong Quillen's conjecture holds for proper subgroups of G then it holds for G.*

In particular, it deals with some of the excluded cases of [AS93] and allows us to extend the main theorem of Aschbacher-Smith to $p = 5$.

Corollary 4.6.5. *The conclusions of the Main Theorem of [AS93] hold for $p = 5$.*

4.1 Background on Quillen's conjecture

In this section we summarize the cases in which the conjecture is known to be valid, and illustrate the ideas behind the proof. We work with the following strong version of the conjecture:

Strong Quillen's conjecture. *If $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.*

Theorem 4.1.1. *The strong Quillen's conjecture holds in the following cases:*

1. *$m_p(G) \leq 2$ ([Qui78, Proposition 2.10]),*
2. *G is of Lie Type in characteristic p ([Qui78, Theorem 3.1]),*
3. *G is solvable ([Qui78, Corollary 12.2]),*
4. *$G = \mathrm{GL}_n(q)$ with $q \equiv 1 \mod p$ ([Qui78, Theorem 12.4]),*
5. *G is p-solvable (various authors),*
6. *G is almost simple ([AK90, Theorem 3]).*

Aschbacher-Smith's result [AS93, Main Theorem] has a more technical statement.

Theorem 4.1.2 (Aschbacher-Smith)**.** *Assume that $p > 5$ and that, whenever G has a unitary component $U_n(q)$ with $q \equiv -1 \mod p$ and q odd, $(QD)_p$ holds for all p-extensions of $U_m(q^{p^e})$ with $m \leq n$ and $e \in \mathbb{Z}$. Then G satisfies the strong Quillen's conjecture.*

Here, a p-extension of a simple group L is a semidirect product LA of L by an elementary abelian p-group A inducing outer automorphisms on L. Recall that G is said to satisfy *the Quillen dimension property at p*, $(QD)_p$ for short, if $\tilde{H}_{m_p(G)-1}(\mathcal{A}_p(G), \mathbb{Z}) \neq 0$. Equivalently, since the top homology group is free abelian, $\tilde{H}_{m_p(G)-1}(\mathcal{A}_p(G), \mathbb{Q}) \neq 0$.

In fact, it is believed that the p-extension of unitary groups as in the statement of the theorem of Aschbacher-Smith should satisfy $(QD)_p$ when $p > 3$, so the hypothesis on the unitary components should not be necessary. See [AS93, Conjecture 4.1] and [AS93, Proposition 4.8].

On the other hand, the problem to the extension of Theorem 4.1.2 to the prime $p = 5$ relies on the presence of certain Suzuki groups as components of G (see Theorems 4.2.6 and 4.2.7). For $p = 2$ and $p = 3$, further obstructions arise and the extension to these cases is even more delicate and would require a more thorough analysis. In application of our result on Quillen's conjecture Corollary 4.3.2, we present in Section 4.4 some examples of groups satisfying the conjecture for $p = 2$, 3 and 5 which are not included in the hypotheses of the theorems of [AS93].

Throughout this section we work with homology with coefficients in $\mathbb{Q}$, and by acyclic we mean $\mathbb{Q}$-acyclic.

Cases proved by Quillen

In [Qui78], Quillen showed some cases of his conjecture, which are listed in Theorem 4.1.1.

The p-rank 2 case is a consequence of the following Serre's result: a finite group acting on a tree has a fixed point. Therefore, if $m_p(G) = 2$ and $\mathcal{A}_p(G)$ is acyclic, then it is a tree and it has a fixed point by the conjugation action of G on $\mathcal{A}_p(G)$. It implies that G has a nontrivial normal p-subgroup.

When G is a group of Lie Type in characteristic p, Quillen noted that $\mathcal{K}(\mathcal{A}_p(G))$ is homotopy equivalent to the Tits building of G [Qui78, Theorem 3.1]. By the classical Solomon-Tits result, the building of G has the homotopy type of a bouquet of spheres of dimension $n-1$, where n is the rank of the Lie type group. This result can also be interpreted by considering Bouc poset $\mathcal{B}_p(G)$. In this case, $\mathcal{B}_p(G)$ is the poset of unipotent radical of parabolic subgroups G, and it is isomorphic to the opposite poset of parabolic subgroups of G, whose order complex gives the building of G.

The proof of the solvable case is reduced to a base case. Namely, Quillen first showed that the groups of the form LA have $(QD)_p$, where L is a solvable p'-group on which the elementary abelian p-group A acts faithfully. If G is solvable and $O_p(G) = 1$, by Hall-Higman Theorem 1.1.6, $L := O_{p'}(G) \neq 1$ and $C_G(L) \leq L$. Hence, every $A \in \mathcal{A}_p(G)$ acts faithfully on L (since $O_p(LA) = C_A(L) = 1$). If A has p-rank $m_p(G)$, by the LA case, $0 \neq H_{m_p(G)-1}(\mathcal{A}_p(LA)) \subseteq H_{m_p(G)-1}(\mathcal{A}_p(G))$. This inclusion holds because A is a maximal element of $\mathcal{A}_p(G)$ (see [Qui78, Theorem 12.1]).

Now we show the LA case. The proof of this case is included in the conclusions of [Qui78, Theorem 11.2]. By Theorem 3.1.4, we know that $\mathcal{A}_p(LA)$ is Cohen-Macaulay (with $C_A(L)$ not necessary trivial). If in addition $C_A(L) = 1$, we show that LA has $(QD)_p$. We proceed by induction as in the proof of Theorem 3.1.4.

Let $G = LA$, where L is a p'-solvable group on which the elementary abelian p-group A acts faithfully (i.e. $C_A(L) = 1$). Assume that L has a nontrivial and proper LA-invariant subgroup H. Then, if A acts faithfully on H (reps. L/H), then HA has $(QD)_p$ (resp. $(L/H)A$ has $(QD)_p$). Consider the map $q : \mathcal{A}_p(LA) \to \mathcal{A}_p((L/H)A)$ induced by the quotient $L \to L/H$. Let $B = C_A(L/H)$. We have an isomorphism of the homology group $\tilde{H}_{m_p(A)-1}(\mathcal{A}_p(LA))$ with the group

$$\tilde{H}_{m_p(A)-1}(\mathcal{A}_p((L/H)A)) \bigoplus_{C\in\mathcal{A}_p((L/H)A)} \tilde{H}_{m_p(C)-1}(\mathcal{A}_p(HC)) \otimes \tilde{H}_{m_p(A)-m_p(C)-1}(\mathcal{A}_p((L/H)A)_{>C}).$$

See [Qui78, Theorem 9.1] or [Pit16, Teorema 2.1.28]. If $B = 1$, then $(L/H)A$ has $(QD)_p$ by induction and so does LA by the above homology decomposition. Suppose that $B \neq 1$. Note that $\mathcal{A}_p((L/H)A)_{>B} = \mathcal{A}_p((L/H)(A/B))$ is Cohen-Macaulay and A/B is faithful on L/H. Hence, $\tilde{H}_{m_p(A)-m_p(B)-1}(\mathcal{A}_p((L/H)A)_{>B}) \neq 0$. In order to show that LA has $(QD)_p$, by induction and the above homology decomposition, it is enough to prove that $C := C_B(H) = O_p(HB) = 1$. Observe that $[L,C] \leq [L,B] \leq H$. Let $l \in L$ and $c \in C$. Then $lcl^{-1} = [l,c]c \in HC \simeq H \times C$. Since lcl^{-1} is a p-element, $[l,c] = 1$. In consequence, $[L,C] = 1$ and it implies that $C \leq C_A(L) = 1$.

If L has no nontrivial proper LA-invariant subgroup, then L characteristically simple. Since it is also solvable, L is an elementary abelian q-group and $A/C_A(L)$ acts irreducibly on L by linear automorphisms. The faithful action of A on L implies that A is cyclic, so $\mathcal{A}_p(LA)$ is disconnected of height 0 and $(QD)_p$ holds for LA.

Other proofs of the solvable case can be found in [Smi11] and [PW00].

Quillen's proof of the case $G = \mathrm{GL}_n(q)$ with $q \equiv 1 \mod p$ consists on showing that $\mathcal{A}_p(G)$ is Cohen-Macaulay of dimension $n-1$.

The p-solvable case

We sketch a slight variation of the proof of the p-solvable case suggested by Alperin in unpublished notes during the eighties. The original proof is explained in Smith's book [Smi11, Theorem 8.2.12]. We simplify it by combining it with the proof of the solvable case presented above. Another proof of this case is due to A. Díaz Ramos [DR16].

If G is a p-solvable group with $O_p(G) = 1$, we show that G has $(QD)_p$. Similar to the solvable case, by Hall-Higman Theorem 1.1.6 it remains to prove the case $G = LA$, where L is a p'-group admitting a faithful action of an elementary abelian p-group A. Hence, we prove the following theorem.

Theorem 4.1.3. *Let $G = LA$, where L is a p'-group on which the elementary abelian p-group A acts faithfully. Then G has $(QD)_p$.*

Proof. (Sketch) Take a minimal configuration LA failing on satisfying $(QD)_p$. The idea is to construct a solvable subgroup $K \leq L$ with a faithful action of A. By minimality, it will be $K = L$ and hence, by the solvable case, LA has $(QD)_p$.

Note that the reduction of the solvable configuration LA to the characteristically simple case L works in the same way without the solvability assumption. Therefore by minimality, L is a characteristically simple group which is the direct product of the A-conjugates of a simple component L_0 of L. We are going to find a Sylow subgroup S_0 of L_0 for which $N_A(L_0)/C_A(L_0)$ acts faithfully on S_0, i.e. $N_A(L_0) \leq N_A(S_0)$ and $C_{N_A(L_0)}(S_0) \leq C_A(L_0)$. Once S_0 is chosen, we take K to be the product of the A-conjugates of S_0. Note that K is solvable. From $C_{N_A(L_0)}(S_0) \leq C_A(L_0)$, it follows that $C_A(K) = 1$, and minimality implies $L = K$.

Now we choose the Sylow subgroup $S_0 \leq L_0$. The interest case is $N_A(L_0) > C_A(L_0)$ (otherwise any Sylow subgroup works). By coprime action [Asc00, (18.7)], $N_A(L_0)$ fixes some Sylow q-subgroup of L_0, for each prime q dividing $|L_0|$. Since the action of $N_A(L_0)$ on L_0 is nontrivial and L_0 is generated by these Sylow subgroups, $N_A(L_0)$ must act nontrivially in at least one of these Sylow subgroups, say S_0. By using the classification of the finite simple groups for describing their outer automorphisms groups, it can be shown that $N_A(L_0)/C_A(L_0) \leq \mathrm{Out}(L_0)$ is cyclic of order p. This yields the requirement $C_{N_A(L_0)}(S_0) \leq C_A(L_0)$. $\square$

A well known result of Hawkes-Isaacs on Quillen's conjecture asserts that, for a p-solvable group G with abelian Sylow p-subgroups, $O_p(G) \neq 1$ if and only if $\chi(\mathcal{A}_p(G)) = 1$ (see [HI88]). However, they do not explicit which grade of the homology is nontrivial when $O_p(G) = 1$.

We omit the proof of the almost simple case of the conjecture since it requires a more technical language. See [AK90].

4.2 Sketch of Aschbacher-Smith's methods and proof

In this section we provide an overview of the techniques and the proof of the main theorem of [AS93] (see Theorem 4.1.2 above). We exclude the unitary components $U_n(q)$ with $q \equiv -1 \mod p$ from the analysis to simplify some technical aspects of the proof. Here we also work with rational homology and by acyclic we mean $\mathbb{Q}$-acyclic.

Similar to the solvable and p-solvable cases, the idea is to construct spheres in the homology. In the general setting, it may happen that the groups fail to have $(QD)_p$, so we may have to construct spheres in other homology groups rather than in the top dimensional one. In this way, the key tools are the variant of Robinson method Lemma 4.2.1 and the Homology Propagation Lemma 4.2.5.

Before quoting the main tools of [AS93], we give a very brief explanation of the proof of Theorem 4.1.2. We want to prove that if $O_p(G) = 1$ then $\mathcal{A}_p(G)$ is not acyclic, when $p > 5$ and G has no unitary components $U_n(q)$ with $q \equiv -1 \mod p$. Take a minimal counterexample G subject to these conditions. The first important reduction by using the Homology Propagation Lemma 4.2.5 is that $O_{p'}(G) = 1$. This reduction deeply depends on the fact that $p > 5$, and it cannot be carried out in the same way without this hypothesis.

Once we can suppose that $O_{p'}(G) = 1$, the second reduction consists in assuming that each component of G has some p-extensions inside G not satisfying $(QD)_p$. By contradiction, suppose there is a component L whose p-extensions satisfy $(QD)_p$. Then, there exists a "maximal" semidirect product $LB \leq G$, with B an elementary abelian p-subgroup inducing outer automorphisms on L, with LB satisfying $(QD)_p$. Under a suitable choice of B, the Homology Propagation Lemma 4.2.5 with $H = LB$, $K = C_G(H)$ gives nontrivial homology for $\mathcal{A}_p(G)$. Once again, the hypotheses of Lemma 4.2.5 are guaranteed since $p > 5$.

In the final step, we have that $O_{p'}(G) = 1$ and no component of G satisfy $(QD)_p$ for all its p-extensions. As we have mentioned, Aschbacher and Smith provided a list with the simple groups for which some of its p-extensions may not satisfy $(QD)_p$ (see [AS93, Theorem 3.1]). This result deeply depends on the Classification of the Finite Simple Groups. The final contradiction comes from finding a 2-hyperelementary p'-subgroup $H \leq G$ such that the reduced

Euler characteristic of the fixed point subposet $\mathcal{S}_p(G)^H$, with the aid of the variant of Robinson method Lemma 4.2.1, has two distinct values when we computed it in two different ways.

In what follows, we quote the main results we need to give a more detailed sketch of the proof of Aschbacher-Smith's result.

Recall that a q-hyperelementary group H is a split extension of a normal cyclic group by a q-group. The following lemma is a variant of [Rob88, Proposition 2.3]. Robinson's original proposition asserts that if G contains a q-hyperelementary p'-subgroup H such that $\mathcal{S}_p(G)^H = \varnothing$, then $\tilde{L}(\mathcal{S}_p(G)) \neq 0$ and in particular, $\mathcal{S}_p(G)$ is not acyclic. Here, $\tilde{L}(K)$ denotes the *reduced Lefschetz (virtual) module* of a G-complex K of dimension d. It is defined as an alternating sum of the chain complex groups over $\mathbb{Z}$: $\tilde{L}(K) = \bigoplus_{i=-1}^{d}(-1)^i C_i(K)$. If $\tilde{L}(L) = 0$ then $\tilde{\chi}(K) = 0$.

Lemma 4.2.1 ([AS93, Lemma 0.14])**.** *Suppose that a q-hyperelementary group H acts on an acyclic poset X. Then $\tilde{\chi}(X^H) \equiv 0 \mod q$.*

The use of the CFSG together with Robinson's result allow to prove Quillen's conjecture for simple groups and even to show that some p-extensions of simple groups satisfy $(QD)_p$. In this way, Aschbacher and Kleidman showed that if G is almost simple then it has a q-hyperelementary abelian p'-subgroup H fixing no nontrivial p-subgroup except if $p = 2$ and $F^*(G) = L_3(2^2)$. Therefore, by Robinson's result, such groups satisfy Quillen's conjecture. Moreover, it proves Quillen's conjecture for almost simple groups (that is, if $O_p(G) = 1$ then $\mathcal{A}_p(G)$ is not acyclic).

The following lemmas are useful to make reductions on the groups we want to prove Quillen's conjecture.

Lemma 4.2.2 ([AS93, Lemma 0.11])**.** *If $N \leq Z(G)$ is a p'-group then the quotient map $G \to G/N$ induced a poset isomorphism $\mathcal{A}_p(G) \equiv \mathcal{A}_p(G/N)$.*

Lemma 4.2.3 ([AS93, Lemma 0.12])**.** *If $N \leq G$ is a normal p'-subgroup then $\tilde{H}_*(\mathcal{A}_p(G/N)) \subseteq \tilde{H}_*(\mathcal{A}_p(G))$.*

Lemma 4.2.4 ([AS93, Lemma 0.13])**.** *If $N = Z(G)$ or $Z(E(G))$ then $O_p(G/N) = O_p(G)N/N$.*

Lemma 4.2.5 (Homology Propagation Lemma, [AS93, Lemma 0.27])**.** *Suppose that $H, K \leq G$ with $K \leq C_G(H)$, $H \cap K$ a p'-group and that the following conditions hold:*

1. *for some A exhibiting $(QD)_p$ for H, $\mathcal{A}_p(G)_{\geq A} \subseteq A \times K$,*

2. *$\mathcal{A}_p(K)$ is not acyclic.*

Then $\mathcal{A}_p(G)$ is not acyclic.

Here, by A *exhibiting* $(QD)_p$ for H we mean that H satisfies $(QD)_p$, $m_p(A) = m_p(H)$ and that there is a nontrivial cycle α in the top homology group of $\mathcal{A}_p(H)$ which contains a chain whose largest member is A.

The proofs of the following two theorems strongly depend on the CFSG. They are a crucial tool in the proof of Theorem 4.1.2 as they allow to get the hypotheses of Lemma 4.2.5. Note that both theorems work without additional assumptions for $p > 5$.

Theorem 4.2.6 (Existence of nonconical complement [AS93, Theorem 2.3])**.** *Assume that p is odd and that B is an elementary abelian p-group inducing outer automorphisms on a simple group L. Exclude the cases $L = L_2(2^3)$, $U_3(2^3)$ and $\mathrm{Sz}(2^5)$ with $p = 3,3,5$ respectively. Then there exists a nonconical complement B' to L in LB. That is, $O_p(C_L(B')) = 1$ and no member of $\mathcal{A}_p(\mathrm{Aut}(L))_{>B'}$ centralizes $C_L(B')$.*

Theorem 4.2.7 (Nonconical complements [AS93, Theorem 2.4])**.** *Assume that IB is a semidirect product of a group I by an elementary abelian p-group B and that $F(I) = Z(I)$ is a p'-group and the p-extensions of the components L of I have nonconical complements as in Theorem 4.2.6. Then some complement B' to I in IB satisfies $O_p(C_I(B')) = 1$.*

Now we give a more detailed sketch of Aschbacher-Smith's proof of their main result Theorem 4.1.2. For simplicity, we assume that $p > 5$ and that the unitary components which are not known to satisfy $(QD)_p$ are not involved in G.

Take G a minimal counterexample to the strong Quillen's conjecture, i.e. subject to the conditions $O_p(G) = 1$ and $\tilde{H}_*(\mathcal{A}_p(G)) = 0$. The idea is to make a series of reductions over the group G in order to reach a minimal configuration with additional features from which we can derive a final contradiction. We assume that $Z(G) = Z(E(G)) = 1$ by Lemma 4.2.4. Note that $F(G) \leq O_{p'}(G)$.

First step: we prove that our minimal counterexample G has $O_{p'}(G) = 1$.

Proposition 4.2.8 ([AS93, Proposition 1.6])**.** *We can suppose that $O_{p'}(G) = 1$ if $p > 5$.*

In particular it gives $F(G) = 1$ and hence, $F^*(G) = E(G)$ is the direct product of simple groups each one of order divisible by p.

Let $L = O_{p'}(G)$ and suppose that $L \neq 1$. If every elementary abelian p-subgroup of G centralizes L, then $[L, \Omega_1(G)] = 1$ and by minimality $L \leq G = \Omega_1(G)$. Hence, $L \leq Z(G)$, a contradiction. Therefore, there exists $A \in \mathcal{A}_p(G)$ acting faithfully on L, and we choose it of maximal rank.

Let $H = LA$ and $K = C_G(H)$. The idea is to use Lemma 4.2.5, so we need first $O_p(K) = 1$.

Let $I = C_G(L)$ and note that A acts faithfully on I. It can be checked that $F(I) = Z(I) = Z(L)$, which is a p'-group. Now, the hypothesis $p > 5$ guarantees the existence of a nonconical complement A' to I in IA by Theorems 4.2.6 and 4.2.7. Namely, A' is an elementary abelian p-subgroup of IA with $O_p(C_I(A')) = 1$ and A' has the same order as A. Since A' also acts faithfully on L, we may assume that $A = A'$. Note that $C_I(A) = C_G(LA) = K$ and $O_p(K) = O_p(C_I(A)) = 1$.

We verify the hypotheses of Lemma 4.2.5. Note that $H \cap K = Z(H) = Z(LA) \leq Z(L)$ is a p'-group given that the action of A in L is faithful. Moreover, by coprime action and the p-solvable case, A exhibits $(QD)_p$ for $H = LA$. It can be shown that $\mathcal{A}_p(G)_{>A} \subseteq A \times K$ since A is of maximal rank subject to acting faithfully on L. On the other hand, K is a proper subgroup of G and therefore it satisfies Quillen's conjecture, i.e. $\tilde{H}_*(\mathcal{A}_p(K)) \neq 0$. In conclusion, we are in the hypotheses of Lemma 4.2.5 and then $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$.

Second step: no component of G satisfies $(QD)_p$ for all its p-extensions (see [AS93, Proposition 1.7]).

Suppose that L is a component of G satisfying $(QD)_p$ for all its p-extensions. Note that L is a simple group of order divisible by p. Similar to the previous step, we may take $A \in \mathcal{A}_p(N_G(L))$ maximal subject to acting faithfully on L. Moreover, $A \cap L \neq 1$ and we can choose A maximizing this intersection. The idea is to apply Lemma 4.2.5 with $H = LA$ and $K = C_G(H)$. Decompose $A = (A \cap L) \times B$ and note that $m_p(A) = m_p(LB)$, and $LA = LB \leq \mathrm{Aut}(L)$. By hypothesis, LB satisfies $(QD)_p$, and since A is of maximal rank, we may assume that A exhibits $(QD)_p$ for LB.

Finally, we check that A can be chosen in such a way that $O_p(K) = 1$. Let $I = C_G(L)$ and note that IB is a semidirect product given that A is faithful on I. Since $F(C_G(L))$ is solvable and it is normalized by $E(G)$, it can be proved that $[E(G), F(C_G(L))] = 1$. The condition $C_G(E(G)) = Z(E(G)) = 1$ implies that $Z(I) \leq F(I) = F(C_G(L)) = 1$, which is a p'-group. Again, the hypothesis $p > 5$ guarantees the hypothesis of Theorem 4.2.7 and we can find a nonconical complement B' to I in IB with $O_p(C_I(B')) = 1$. By taking $A' = (A \cap L) \times B'$ and checking that A' could be our A, we can take A' and B' to be A and B. Therefore, $O_p(K) = O_p(C_G(LB)) = O_p(C_I(B)) = 1$.

Analogously to the first step, we can check the hypotheses of Lemma 4.2.5 and hence, $\tilde{H}_*(\mathcal{A}_p(G)) \neq 0$.

Third step: contradiction by computing the Euler characteristic.

By the Second step, the components of $F^*(G)$ are in the list of [AS93, Theorem 3.1] because they may have some p-extensions not satisfying $(QD)_p$. Recall that we are assuming that G does not contain unitary groups $U_n(q)$ with $q \equiv -1 \mod p$ as components. Let $L_1, \ldots, L_n$ be the components of G. Since $p > 3$, we can invoke [AS93, Theorem 5.3] and find in each L_i certain Brauer 2-elementary p'-subgroup H_i (a direct product of a cyclic group $\langle x_i \rangle$ by a 2-group Q_i) such that $\tilde{\chi}(\mathcal{S}_p(L_i)^{H_i}) = \pm 1$. Let $H = (\langle x_1 x_2 \ldots x_n \rangle) \times (Q_1 \ldots Q_n)$ and note that H is a Brauer 2-elementary p'-subgroup of G (and in particular 2-hiperelementary). By using coprime actions and [AS93, Theorem 5.3], it can be shown that $\mathcal{S}_p(G)^H = \mathcal{S}_p(L_1 \times \ldots \times L_n)^H$.

By Proposition 3.1.16 applied to $\mathcal{S}_p(G)$, since $\mathcal{S}_p(L_1 \times \ldots \times L_n) \overset{H}{\nearrow\!\!\!\searrow} \mathcal{S}_p(L_1) * \ldots * \mathcal{S}_p(L_n)$, we also have

$$\mathcal{S}_p(L_1 \times \ldots \times L_n)^H \nearrow\!\!\!\searrow (\mathcal{S}_p(L_1) * \ldots * \mathcal{S}_p(L_n))^H = \mathcal{S}_p(L_1)^{H_1} * \ldots * \mathcal{S}_p(L_n)^{H_n}.$$

Now we compute the Euler characteristic of $\mathcal{S}_p(G)^H$. If $\mathcal{S}_p(G)$ is acyclic, by Lemma 4.2.1 $\tilde{\chi}(\mathcal{S}_p(G)^H) = 0 \mod 2$. On the other hand, by the join decomposition,

$$\tilde{\chi}(\mathcal{S}_p(G)^H) = \pm \prod_{i=1}^{n} \tilde{\chi}(\mathcal{S}_p(L_i)^{H_i}) = \pm 1$$

since $\tilde{\chi}(\mathcal{S}_p(L_i)^{X_i}) = 1$ for each i by [AS93, Theorem 5.3], which is $1 \mod 2$.

4.3 $\mathbb{Z}$-acyclic 2-complexes and Quillen's conjecture

In the previous sections we have summarized the known results on Quillen's conjecture, together with the proof of Aschbacher-Smith.

In this section, we work with the following version of the conjecture: if $O_p(G) = 1$ then $\tilde{H}_*(\mathcal{S}_p(G), \mathbb{Z}) \neq 0$. In particular, we take homology with integer coefficients and by acyclic we mean $\mathbb{Z}$-acyclic.

We yield new cases of this version of the conjecture when relate it with the study of groups acting on acyclic 2-complexes. Our results depends on the classification of Oliver and Segev [OS02] of fixed points free actions of finite groups on acyclic 2-complexes, which depends on the CFSG.

The results of this section correspond to a work in collaboration with Ivan Sadofschi Costa and Antonio Viruel [PSV19].

We prove the following theorem.

Theorem 4.3.1. *If X is a $\mathbb{Z}$-acyclic 2-dimensional G-invariant subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$, then $O_p(G) \neq 1$.*

From Theorem 4.3.1 we immediately deduce:

Corollary 4.3.2. *Let G be a finite group. Suppose that $\mathcal{K}(\mathcal{S}_p(G))$ admits a 2-dimensional and G-invariant subcomplex homotopy equivalent to itself. Then Quillen's conjecture holds for G.*

If G has p-rank 3 then $\mathcal{K}(\mathcal{A}_p(G))$ is a G-invariant 2-dimensional homotopy equivalent subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$. Hence, Quillen's conjecture holds for groups of p-rank 3.

Corollary 4.3.3. *Quillen's conjecture holds for groups of p-rank at most 3.*

By Corollary 4.3.2, Quillen's conjecture also holds when $\mathcal{B}_p(G)$, $\mathfrak{i}(\mathcal{S}_p(G))$ or $\mathfrak{i}(\mathcal{A}_p(G))$ have height 2.

Corollary 4.3.4. *Let G be a finite group such that $\mathcal{B}_p(G)$ has height 2. If $\widetilde{H}_*(\mathcal{S}_p(G),\mathbb{Z}) = 0$ then $O_p(G) \neq 1$.*

Corollary 4.3.5. *Let G be a finite group such that either $\mathfrak{i}(\mathcal{S}_p(G))$ or $\mathfrak{i}(\mathcal{A}_p(G))$ has height 2. If $\widetilde{H}_*(\mathcal{S}_p(G),\mathbb{Z}) = 0$ then $O_p(G) \neq 1$.*

In the next section we give a series of examples satisfying the hypotheses of Corollary 4.3.2 but which are not contained in the theorems of [AS93] neither in Theorem 4.1.1.

In order to prove Theorem 4.3.1, we need to review first some of the results of [OS02]. By a G-complex we mean a G-CW complex.

Definition 4.3.6 ([OS02])**.** A G-complex X is *essential* if there is no normal subgroup $1 \neq N \lhd G$ such that for each $H \subseteq G$, the inclusion $X^{HN} \to X^H$ induces an isomorphism on integral homology.

The main theorems of [OS02] are the following.

Theorem 4.3.7 ([OS02, Theorem A])**.** *For any finite group G, there is an essential fixed point free 2-dimensional (finite) acyclic G-complex if and only if G is isomorphic to one of the simple groups*

1. *$L_2(2^k)$ for $k \geq 2$,*
2. *$L_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or*
3. *$\mathrm{Sz}(2^k)$ for odd $k \geq 3$.*

Furthermore, the isotropy subgroups of any such G-complex are all solvable.

Theorem 4.3.8 ([OS02, Theorem B])**.** *Let G be a finite group, and let X be a 2-dimensional acyclic G-complex. Let N be the subgroup generated by all normal subgroups $N' \lhd G$ such that $X^{N'} \neq \varnothing$. Then X^N is acyclic; X is essential if and only if $N = 1$; and the action of G/N on X^N is essential.*

Denote by $\mathcal{S}(G)$ the set of subgroups of G.

Definition 4.3.9 ([OS02])**.** A *family* of subgroups of G is any subset $\mathcal{F} \subseteq \mathcal{S}(G)$ closed under conjugation. A nonempty family is said to be *separating* if it has the following three properties: (a) $G \notin \mathcal{F}$; (b) if $H' \subseteq H$ and $H \in \mathcal{F}$ then $H' \in \mathcal{F}$; (c) for any $H \lhd K \subseteq G$ with K/H solvable, $K \in \mathcal{F}$ if $H \in \mathcal{F}$.

For a family $\mathcal{F}$ of subgroups of G, a $(G,\mathcal{F})$*-complex* is a G-complex all of whose isotropy subgroups lie in $\mathcal{F}$. A $(G,\mathcal{F})$-complex is H*-universal* if the fixed point set of each $H \in \mathcal{F}$ is acyclic.

Lemma 4.3.10 ([OS02, Lemma 1.2]). *Let X be any 2-dimensional acyclic G-complex without fixed points. Let $\mathcal{F}$ be the set of subgroups $H \subseteq G$ such that $X^H \neq \varnothing$. Then $\mathcal{F}$ is a separating family of subgroups of G, and X is an H-universal $(G,\mathcal{F})$-complex.*

If G is not solvable, the separating family of solvable subgroups of G is denoted by $\mathcal{SLV}$.

Proposition 4.3.11 ([OS02, Proposition 6.4]). *Assume that L is one of the simple groups $L_2(q)$ or $\mathrm{Sz}(q)$, where $q = p^k$ and p is prime ($p = 2$ in the second case). Let $G \leq \mathrm{Aut}(L)$ be any subgroup containing L, and let $\mathcal{F}$ be a separating family for G. Then there is a 2-dimensional acyclic $(G,\mathcal{F})$-complex if and only if $G = L$, $\mathcal{F} = \mathcal{SLV}$, and q is a power of 2 or $q \equiv \pm 3 \pmod 8$.*

Definition 4.3.12 ([OS02, Definition 2.1]). For any family $\mathcal{F}$ of subgroups of G define

$$i_{\mathcal{F}}(H) = \frac{1}{[N_G(H):H]}(1-\chi(\mathcal{K}(\mathcal{F}_{>H}))).$$

Lemma 4.3.13 ([OS02, Lemma 2.3]). *Fix a separating family $\mathcal{F}$, a finite H-universal $(G,\mathcal{F})$-complex X, and a subgroup $H \leq G$. For each n, let $c_n(H)$ denote the number of orbits of n-cells of type G/H in X. Then $i_{\mathcal{F}}(H) = \sum_{n\geq 0}(-1)^n c_n(H)$.*

Proposition 4.3.14 ([OS02, Tables 2,3,4]). *Let G be one of the simple groups $L_2(2^k)$ for $k \geq 2$, $L_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. Then $i_{\mathcal{SLV}}(1) = 1$.*

Using these results we prove the following.

Theorem 4.3.15. *Every acyclic 2-dimensional G-complex has an orbit with normal stabilizer.*

Proof. If $X^G \neq \varnothing$ we are done. Otherwise, G acts fixed point freely on X. Consider the subgroup N generated by the subgroups $N' \triangleleft G$ such that $X^{N'} \neq \varnothing$. Clearly N is normal in G. By Theorem 4.3.8 $Y = X^N$ is acyclic (in particular it is nonempty) and the action of G/N on Y is essential and fixed point free. By Lemma 4.3.10 $\mathcal{F} = \{H \leq G/N : Y^H \neq \varnothing\}$ is a separating family and Y is an H-universal $(G/N,\mathcal{F})$-complex. Thus, Theorem 4.3.7 asserts that G/N must be one of the groups $\mathrm{PSL}_2(2^k)$ for $k \geq 2$, $\mathrm{PSL}_2(q)$ for $q \equiv \pm 3 \pmod 8$ and $q \geq 5$, or $\mathrm{Sz}(2^k)$ for odd $k \geq 3$. In any case, by Proposition 4.3.11 we must have $\mathcal{F} = \mathcal{SLV}$. By Proposition 4.3.14, $i_{\mathcal{SLV}}(1) = 1$. Finally by Lemma 4.3.13, Y must have at least one free G/N-orbit. Therefore X has a G-orbit of type G/N and we are done. □

Now we can prove Theorem 4.3.1.

Proof of Theorem 4.3.1. By Theorem 4.3.15 there is a simplex $(A_0 < \ldots < A_j)$ of X with stabilizer $N \triangleleft G$. Since $A_0 \triangleleft N$, we see that $O_p(N)$ is nontrivial. On the other hand, $N \triangleleft G$ and $O_p(N)\,\mathrm{char}\,N$ implies that $O_p(N) \triangleleft G$. Therefore $O_p(N) \leq O_p(G)$ and $O_p(G)$ is nontrivial. □

Remark 4.3.16. A possible approach to study Quillen's conjecture is to find an acyclic 2-dimensional G-invariant subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$. If Quillen's conjecture were true, then this would be possible. Therefore, by Theorem 4.3.1, Quillen's conjecture can be restated in the following way: if $\mathcal{A}_p(G)$ is acyclic, then there exists a G-invariant 2-dimensional acyclic subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$.

4.4 Examples of the 2-dimensional case

In this section we apply Theorem 4.3.1 and Corollary 4.3.2 to establish Quillen's conjecture for some groups not included in the hypotheses of the theorems of [AS93].

The presence of simple components of G isomorphic to $L_2(2^3)$ or $U_3(2^3)$ (in the $p=3$ case) and $\mathrm{Sz}(2^5)$ (in the $p=5$ case) is an obstruction to extending [AS93, Main Theorem] (see also Theorem 4.1.2) to $p=3$ and $p=5$. The case $p=2$ is not considered in [AS93] and would require a much more detailed analysis. As we have seen in Section 4.2, the first steps in the proof of Theorem 4.1.2 is the reduction to the case $O_{p'}(G)=1$ by means of [AS93, Proposition 1.6] (see Proposition 4.2.8). To do this, [AS93, Theorems 2.3 and 2.4] (see Theorems 4.2.6 and 4.2.7) are needed and they make a strong use of the hypothesis $p>5$. Concretely, it is not possible to apply [AS93, Theorem 2.3] if a component of $C_G(O_{p'}(G))$ is isomorphic to $L_2(2^3)$, $U_3(2^3)$ (if $p=3$) or $\mathrm{Sz}(2^5)$ (if $p=5$).

Before presenting the examples for $p=3$ and $p=5$, we give some motivation. Most of the groups G in these examples satisfy the following conditions. First, $O_{p'}(G)\neq 1$ and $C_G(O_{p'}(G))$ contains a component isomorphic to $U_3(2^3)$ if $p=3$ and to $\mathrm{Sz}(2^5)$ if $p=5$. In this way, we cannot find nontrivial homology for $\mathcal{A}_p(G)$ in the same way it is done in the proof of [AS93, Proposition 1.6] since we are not able to invoke [AS93, Theorems 2.3 and 2.4] (see the proof in Section 4.2).

Secondly, by [AS93, Lemma 0.12] (see also Lemma 4.2.3) there is an inclusion

$$\tilde{H}_*(\mathcal{A}_p(G/O_{p'}(G));\mathbb{Q})\subseteq \tilde{H}_*(\mathcal{A}_p(G);\mathbb{Q}).$$

We ask for $O_p(G/O_{p'}(G))\neq 1$, so that $\tilde{H}_*(\mathcal{A}_p(G/O_{p'}(G)))=0$. Finally, we require $O_p(G)=1$.

The groups presented in Examples 4.4.5 and 4.4.7 have p-rank 4 and are constructed in the following way. We take a direct product of a group N, consisting of one or more copies of a particular simple p'-group, by a group K consisting of one or more copies of $L=U_3(2^3)$ if $p=3$ or $L=\mathrm{Sz}(2^5)$ if $p=5$. Then we take two cyclic p-groups A and B and we let them act on the direct product $N\times K$ as follows. We take a faithful action of $A\times B$ on N, and we choose a representation $A\times B\to \mathrm{Aut}(K)$ such that $O_p(K:(A\times B))\cong O_p(C_A(K))\neq 1$. The group $G=(N\times K):(A\times B)$ satisfies the conditions $O_p(G)=1$, $O_{p'}(G)=N\neq 1$, $C_G(N)=K$ and $O_p(G/N)=O_p(K:(A\times B))\neq 1$. Moreover, since the p-rank of L is at most 2, we can construct G to have p-rank 4 by adjusting the number of copies of L in K.

For these groups we show that $\mathcal{K}(\mathcal{S}_p(G))$ has a 2-dimensional G-invariant subcomplex homotopy equivalent to itself, and thus Corollary 4.3.2 applies.

In Example 4.4.6 we consider $p=5$ and construct a group of 5-rank 3 in a similar way to that of Example 4.4.7.

On the other hand, in Example 4.4.4 we take $p=3$ and consider a group G with 3-rank 3 in the hypotheses of the third step of the proof of [AS93, Main Theorem] for which their [AS93, Theorem 5.3] does not apply (see also the third step of the proof of Theorem 4.1.2 in Section 4.2).

In Examples 4.4.9 and 4.4.10 we describe two groups of 2-rank 4 such that $\mathcal{K}(\mathcal{S}_2(G))$ admits a 2-dimensional G-invariant homotopy equivalent subcomplex.

For the claims on the structure of the automorphisms groups of the finite groups of Lie type we refer to [GL83] (see also Appendix A.1).

The following lemma provides an easy way to look for the p-rank of a semidirect product.

Lemma 4.4.1. *Let $1\to N\to G\to K\to 1$ be an extension of finite groups. Then*

$$m_p(G)=\max_{A\in\mathcal{S}} m_p(C_N(A))+m_p(A),$$

where $\mathcal{S}$ is the set of elementary abelian p-subgroups $1\leq A\leq G$ such that $A\cap N=1$. In particular we have $m_p(G)\leq m_p(N)+m_p(K)$.

Proof. If $A\in\mathcal{S}$ we have $C_N(A)\times A\cong C_N(A)A$ and hence $m_p(C_N(A))+m_p(A)\leq m_p(C_N(A)A)\leq m_p(G)$. Taking maximum over $A\in\mathcal{S}$ gives the lower bound for $m_p(G)$. We now prove the other inequality. Let E be an elementary abelian p-subgroup of G and write $E=(E\cap N)A$ for some complement A of $E\cap N$ in E. Then $m_p(E\cap N)\leq m_p(C_N(A))$ and $A\in\mathcal{S}$. Now $m_p(E)=m_p(E\cap N)+m_p(A)\leq m_p(C_N(A))+m_p(A)$, giving the upper bound for $m_p(G)$. For the last claim note that $C_N(A)\leq N$ and $m_p(A)\leq m_p(K)$ by the isomorphism theorems. $\square$

We will use the following lemma to obtain proper subcomplexes of $\mathcal{K}(\mathcal{A}_p(G))$ without changing the homotopy type.

Lemma 4.4.2. *Let G be a finite group and let $H\leq G$. In addition, suppose that $O_p(C_H(E))\neq 1$ for each $E\in\mathcal{A}_p(G)$ with $E\cap H=1$. Then $\mathcal{A}_p(G)\simeq\mathcal{A}_p(H)$.*

Proof. Consider the subposet $\mathcal{N}=\{E\in\mathcal{A}_p(G) : E\cap H\neq 1\}$. By Remark 3.1.12, $\mathcal{A}_p(H)\simeq\mathcal{N}$.

Let $\mathcal{S}=\{E\in\mathcal{A}_p(G) : E\cap H=1\}$ be the complement of $\mathcal{N}$ in $\mathcal{A}_p(G)$. Take a linear extension $\{E_1,\dots,E_r\}$ of $\mathcal{S}$ such that $E_i\leq E_j$ implies $i\leq j$. Fix $E\in\mathcal{S}$ and consider $\mathcal{A}_p(G)_{>E}\cap\mathcal{N}=\{A\in\mathcal{N} : A>E\}$. By Lemma 3.1.13 $\mathcal{A}_p(G)_{>E}\cap\mathcal{N}\simeq\mathcal{A}_p(C_H(E))$, which is a homotopically trivial finite space.

Let $X^i=\mathcal{A}_p(G)-\{E_{i+1},\dots,E_r\}$. We show that $X^i\hookrightarrow X^{i+1}$ is a weak equivalence for each $0\leq i\leq r-1$.

Note that $X^{i+1} = X^i \cup \{E_i\}$. Moreover, $X^{i+1}_{>E_i} = \mathcal{A}_p(G)_{>E_i} \cap \mathcal{N}$, which is homotopically trivial by hypothesis. Therefore $X^i = X^{i+1} - \{E_i\} \hookrightarrow X^{i+1}$ is a weak homotopy equivalence (see for example [Bar11a, Proposition 6.2.2]). In consequence,

$$\mathcal{A}_p(G) = X^r \underset{w}{\approx} X^0 = \mathcal{A}_p(G) - \mathcal{S} = \mathcal{N} \simeq \mathcal{A}_p(H).$$

□

Remark 4.4.3. In the hipotheses of the above lemma, it can be shown that if $H \triangleleft G$ then $\mathcal{K}(\mathcal{A}_p(G)) \simeq_G \mathcal{K}(\mathcal{A}_p(H))$.

Example 4.4.4. Let $p = 3$ and let $L = L_2(2^3) \times L_2(2^3) \times L_2(2^3)$. Let A be a cyclic group of order 3 acting on L by permuting the copies of $L_2(2^3)$. Take $G = LA$. Since $m_3(L_2(2^3)) = 1$ and $C_L(A) \cong L_2(2^3)$, we see that $m_3(G) = 3$. By Corollary 4.3.3, G satisfies Quillen's conjecture.

Example 4.4.5. Let $p = 3$, $N = \mathrm{Sz}(2^3) \times \mathrm{Sz}(2^3) \times \mathrm{Sz}(2^3)$ and $U = U_3(2^3)$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be cyclic groups of order 3. We are going to construct a semidirect product $G = (N \times U) : (A \times B)$. To do this we need to define a map $A \times B \to \mathrm{Aut}(N \times U) = \mathrm{Aut}(N) \times \mathrm{Aut}(U)$.

Choose a field automorphism $\phi \in \mathrm{Aut}(U_3(2^3))$ of order 3. By the properties of the p-group actions, there exists an inner automorphism $x \in \mathrm{Inn}(U_3(2^3))$ of order 3 commuting with ϕ. Then $A \times B \to \mathrm{Aut}(U_3(2^3))$ is given by $a \mapsto x$ and $b \mapsto \phi$. Choose a field automorphism $\psi \in \mathrm{Aut}(\mathrm{Sz}(2^3))$ of order 3. Let A act on each coordinate of N as ψ and let B act on N by permuting its coordinates. This gives rise to a well defined map $A \times B \to \mathrm{Aut}(N)$.

The 3-rank of G is $m_3(G) = m_3(U_3(2^3)AB)$. We can take an elementary abelian subgroup $E \leq C_U(\phi)$ of order 9 containing x since $C_U(\phi) \cong \mathrm{PGU}_3(2) \cong ((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$ by [GL83, (9-1), (9-3)] (cf. [GLS99, Chapter 4, Lemma 3.10]) and $\mathcal{A}_3(\mathrm{PGU}_3(2))$ is connected of height 1. Then EAB is an elementary abelian subgroup of order 3^4. Hence, $m_3(UAB) \geq 4$. Since $m_3(U_3(2^3)) = 2$ and $m_3(AB) = 2$, by Lemma 4.4.1 we have $m_3(G) = 4$.

By Corollary 4.3.2, to show that Quillen's conjecture holds for G and $p = 3$ it is enough to find a 2-dimensional G-invariant subcomplex of $\mathcal{K}(\mathcal{S}_3(G))$ homotopy equivalent to this latter one.

Let $H = (N \times U)A$. Then $H \triangleleft G$ and $m_3(H) = 3$. Therefore, $\mathcal{K}(\mathcal{A}_3(H))$ is a 2-dimensional G-invariant subcomplex of $\mathcal{K}(\mathcal{A}_3(G))$. Now the plan is to use Lemma 4.4.2 to show that $\mathcal{A}_3(H) \underset{w}{\approx} \mathcal{A}_3(G)$. Let $E \in \mathcal{A}_3(G)$ be such that $E \cap H = 1$. Then $E \cong EH/H \leq B \cong C_3$ and hence, E is cyclic generated by some element $e \in E$. Write $e = nua^ib^j$ with $n \in N$, $u \in U$ and $i, j \in \{0, 1, 2\}$. Note that $j \neq 0$ since $E \cap H = 1$. If $v \in U$, then

$$v^e = v^{nua^ib^j} = (v^{ua^i})^{b^j}.$$

Since $j \neq 0$ and e induces an automorphism of U of order 3 in $\mathrm{Inn}(U)\phi^j$, by [GL83, (7-2)] and the definition of field automorphism, e is $\mathrm{Inndiag}(U)$-conjugate to ϕ^j and acts as a field automorphism on U. In particular, $C_U(E) = C_U(e) \cong C_U(\phi^j) = C_U(\phi)$. Observe also that $O_3(C_U(E)) \cong O_3(C_U(\phi)) \cong C_3 \times C_3 \neq 1$. Since $C_U(E) \triangleleft C_H(E)$ and $O_3(C_U(E)) \neq 1$, we conclude that $O_3(C_H(E)) \neq 1$. By Lemma 4.4.2, $\mathcal{A}_3(G) \underset{w}{\approx} \mathcal{A}_3(H)$, which is 2-dimensional and G-invariant. In conclusion, the subcomplex $\mathcal{K}(\mathcal{A}_3(H))$ satisfies the hypothesis of Corollary 4.3.2 and therefore, Quillen's conjecture holds for G.

Finally note that $O_3(G) = 1$, $O_{3'}(G) = N$, $C_G(O_{3'}(G)) = U_3(2^3)$ and $O_3(G/O_{3'}(G)) = O_3(U_3(2^3)AB) = \langle ax^{-1} \rangle \cong C_3$.

Example 4.4.6. Let $p = 5$. Let r be a prime number such that $r \equiv 2$ or $3 \mod 5$ and let $q = r^{5^n}$ with $n \geq 2$. Let N be one of the simple groups $L_2(q)$, $G_2(q)$, ${}^3D_4(q^3)$ or ${}^2G_2(3^{5^n})$ and let $A = \langle a \rangle$ be a cyclic group of order 5^n. Note that $5 \nmid |N|$. Let a act on N as a field automorphism of order 5^n. Choose a field automorphism $\phi \in \mathrm{Aut}(\mathrm{Sz}(2^5))$ of order 5 and let A act on $\mathrm{Sz}(2^5) \times \mathrm{Sz}(2^5)$ as $\phi \times \phi$. Now consider the semidirect product $G = (N \times \mathrm{Sz}(2^5) \times \mathrm{Sz}(2^5)) : A$ defined by this action.

Since the Sylow 5-subgroups of $\mathrm{Sz}(2^5)$ are cyclic of order 25, by Lemma 4.4.1 we have that $m_5(G) = 3$. By Corollary 4.3.3, Quillen's conjecture holds for G.

Finally, our group has the following properties: $O_5(G) = 1$, $O_{5'}(G) = N$, $C_G(O_{5'}(G)) = \mathrm{Sz}(2^5)^2$ and $O_5(G/O_{5'}(G)) = C_A(\mathrm{Sz}(2^5)^2) = \langle a^5 \rangle \neq 1$.

Example 4.4.7. Let $p = 5$ and let $N = L^5$, where L is one of the simple $5'$-groups of the previous example. Let $A = \langle a \rangle \cong C_{5^n}$ and $B = \langle b \rangle \cong C_5$. Let $G = (N \times \mathrm{Sz}(2^5)^2) : (A \times B)$, where a acts on each copy of L as a field automorphism of order 5^n and trivially on $\mathrm{Sz}(2^5)^2$, and b permutes the copies of L and acts as a field automorphism of order 5 on each copy of $\mathrm{Sz}(2^5)$.

To compute the 5-rank of G we use Lemma 4.4.1:

$$\begin{aligned} m_5(G) &= m_5(\mathrm{Sz}(2^5)^2 : (A \times B)) \\ &= m_5(A \times (\mathrm{Sz}(2^5)^2 B)) \\ &= m_5(A) + m_5(\mathrm{Sz}(2^5)^2 B) \\ &= 1 + 3 \\ &= 4. \end{aligned}$$

Now the aim is to apply Corollary 4.3.2 on G by finding a 2-dimensional G-invariant homotopy equivalent subcomplex X of $\mathcal{K}(\mathcal{S}_5(G))$ to proceed as in Example 4.4.5.

Let $H = (N \times \mathrm{Sz}(2^5)^2)A \cong NA \times \mathrm{Sz}(2^5)^2$. Note that $H \trianglelefteq G$, $m_5(H) = 3$ and that $\mathcal{K}(\mathcal{A}_5(H))$ is 2-dimensional and G-invariant subcomplex of $\mathcal{K}(\mathcal{A}_5(G))$. We will show that $\mathcal{A}_5(H) \underset{w}{\approx} \mathcal{A}_5(G)$ by applying Lemma 4.4.2.

Let $E \in \mathcal{A}_5(G)$ be such that $E \cap H = 1$. Then E is cyclic generated by an element e of order 5 and $e = lsa^ib^j$ with $l \in N$, $s \in \mathrm{Sz}(2^5)^2$, $0 \leq i \leq 5^n - 1$ and $j \in \{1,2,3,4\}$. Thus E acts by field automorphisms on each copy of the Suzuki group and e is $\mathrm{Inndiag}(\mathrm{Sz}(2^5))$-conjugate to the field automorphism induced by b on $\mathrm{Sz}(2^5)$ (see [GL83, (7-2)] and Example 4.4.5). Hence, $C_H(E) \cong C_{NA}(E) \times C_{\mathrm{Sz}(2^5)^2}(E)$. Observe that $C_{\mathrm{Sz}(2^5)^2}(E) \trianglelefteq C_H(E)$ and $C_{\mathrm{Sz}(2^5)^2}(E) \cong C_{\mathrm{Sz}(2^5)}(E)^2 \cong (C_5 : C_4)^2$ has a nontrivial normal 5-subgroup. Therefore $\mathcal{A}_5(G) \underset{w}{\approx} \mathcal{A}_5(H)$ by Lemma 4.4.2 and Quillen's conjecture holds for G by Corollary 4.3.2 applied to the subcomplex $\mathcal{K}(\mathcal{A}_5(H))$.

Note that $O_{5'}(G) = N$ and $C_G(O_{5'}(G)) = \mathrm{Sz}(2^5)^2$. On the other hand, $O_5(G) = 1$ and $O_5(G/O_{5'}(G)) \cong A \neq 1$.

We conclude with two examples of groups satisfying Quillen's conjecture for $p = 2$.

Proposition 4.4.8. *Let L_1 and L_2 be two finite groups in which their distinct Sylow p-subgroups intersect trivially. Let $L = L_1 \times L_2$ and take G an extension of L such that $|G : L| = p$. Then $\mathfrak{i}(\mathcal{S}_p(G))$ and $\mathcal{B}_p(G)$ have height at most 2. If in addition the Sylow p-subgroups of L_1 and L_2 have abelian Ω_1, then $\mathfrak{i}(\mathcal{A}_p(G))$ has height at most 2.*

Proof. The elements of $\mathfrak{i}(\mathcal{S}_p(L))$ are of the form $S_1 \times S_2$, $1 \times S_2$ or $S_1 \times 1$, where $S_i \leq L_i$ are Sylow p-subgroups. Hence, $\mathfrak{i}(\mathcal{S}_p(L))$ is 1-dimensional.

Now suppose that $Q_0 < Q_1 < \ldots < Q_n$ is a chain in $\mathfrak{i}(\mathcal{S}_p(G))$. Then

$$Q_0 \cap L \leq Q_1 \cap L \leq \ldots \leq Q_n \cap L$$

is a chain in $\mathfrak{i}(\mathcal{S}_p(L))$. We claim that there is at most one index i such that $Q_i \cap L = Q_{i+1} \cap L$. To see this note that

$$|Q_j : Q_j \cap L| = \begin{cases} 1 & \text{if } Q_j \subseteq L \\ p & \text{if } Q_j \not\subseteq L \end{cases}.$$

We have $|Q_{i+1} : Q_i| \cdot |Q_i : Q_i \cap L| = |Q_{i+1} : Q_{i+1} \cap L| \cdot |Q_{i+1} \cap L : Q_i \cap L|$. Then if $Q_i \cap L = Q_{i+1} \cap L$, since $|Q_{i+1} : Q_i| \geq p$ we must have $|Q_i : Q_i \cap L| = 1$ and $|Q_{i+1} : Q_{i+1} \cap L| = p$. Then $i = \max\{j : Q_j \subseteq L\}$.

From this we conclude that $h(\mathfrak{i}(\mathcal{S}_p(G))) \leq 1 + h(\mathfrak{i}(\mathcal{S}_p(L))) = 2$. By Lemma 1.3.10 $\mathcal{B}_p(G)$ is a subposet of $\mathfrak{i}(\mathcal{S}_p(G))$. Then $\mathcal{B}_p(G)$ has height at most 2. The same proof can be easily adapted to prove that, if the Sylow p-subgroups of L_1 and L_2 have abelian Ω_1, $\mathfrak{i}(\mathcal{A}_p(G))$ has height at most 2. □

In the following examples we use the fact that two distinct Sylow 2-subgroups of $\mathbb{A}_5$ or $U_3(2^2)$ intersect trivially, and that $\Omega_1(S)$ is abelian for S a Sylow 2-subgroup of either $\mathbb{A}_5$ or $U_3(2^2)$.

Example 4.4.9. Let G be the splitting group extension $(\mathbb{A}_5 \times \mathbb{A}_5) : C_2$ where the generator of C_2 acts on each coordinate as conjugation by the transposition $(1\,2)$. Then by Lemma 4.4.1, G has 2-rank 4. By Proposition 4.4.8, $\mathfrak{i}(\mathcal{A}_2(G))$, $\mathfrak{i}(\mathcal{S}_2(G))$ and $\mathcal{B}_2(G)$ have height at most 2 and then Quillen's conjecture holds for G since Corollaries 4.3.4 and 4.3.5 apply.

Example 4.4.10. Let $G = (U_3(2^2) \times \mathbb{A}_5) : C_2$ be the semidirect product constructed in the following way. Let $L = U_3(2^2) \times \mathbb{A}_5$. Then

$$\mathrm{Out}(L) \cong \mathrm{Aut}(U_3(2^2))/\mathrm{Inn}(U_3(2^2)) \times \mathrm{Aut}(\mathbb{A}_5)/\mathrm{Inn}(\mathbb{A}_5) \cong C_4 \times C_2.$$

Take $t \in \mathrm{Out}(L)$ to be the involution which acts nontrivially on both factors. Therefore $G = L\langle t\rangle$. By Lemma 4.4.1, G has 2-rank 4 and just as before, Quillen's conjecture holds for G.

4.5 The reduction $O_{p'}(G) = 1$ for Quillen's conjecture

In the previous section we have studied some examples of finite groups G which do not satisfy the hypotheses of the theorems of [AS93] since $p \leq 5$ and $O_{p'}(G) \neq 1$. Those examples were constructed evading the methods of reduction of [AS93] (see also Section 4.2). Nevertheless, in each case we reduced the Quillen poset (preserving its weak homotopy type) to an invariant subposet of height 2 and hence they satisfy Quillen's conjecture by Corollary 4.3.2.

In this section we show that the methods we have used to reduce the posets of the previous examples can be generalized. Concretely, we show that we can reduce the study of Quillen's conjecture to finite groups G with $O_{p'}(G) = 1$. We work with rational homology and therefore with the strong Quillen's conjecture.

Theorem 4.5.1. *Let G be a finite group such that $O_p(G) = 1$, $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) = 0$ and its proper subgroups satisfy the strong Quillen's conjecture. Then $O_{p'}(G) = 1$. In particular, a minimal counterexample G to the strong Quillen's conjecture has $O_{p'}(G) = 1$.*

The proof follows the homology propagation ideas of Lemma 4.2.5 of Aschbacher-Smith paper [AS93]. We prove a generalization of this lemma in Lemma 4.5.10. Below we quote the definitions and results of [AS93] that we will need.

If X is a finite poset, we denote by $\tilde{Z}_n(X)$ the set of n-cycles of the reduced chain complex $\tilde{C}_*(X)$ with rational coefficients.

Definition 4.5.2. Let X be a finite poset. A chain $a \in X'$ is *full* if for every $x \in X$ such that $\{x\} \cup a$ is a chain we have that $x \in a$ or $x \geq \max a$. A chain containing a is called *a-initial chain* if it has the form $(x_0 < x_1 < \ldots < x_m < y_0 < \ldots < y_s)$, where $(x_0 < x_1 < \ldots < x_m) = a$.

Definition 4.5.3. Let G be a finite group with $(QD)_p$ and let $m = m_p(G) - 1$. Take a nontrivial cycle $\alpha \in \tilde{H}_m(\mathcal{A}_p(G)) = \tilde{Z}_m(\mathcal{A}_p(G))$. If the chain $a = (A_0 < A_1 < \ldots < A_m)$ is a sum of the cycle α, we write $a \in \alpha$ and say that A_m *exhibits* $(QD)_p$ for G. Note that a is a full chain.

For the following definitions and propositions, we fix a finite group G and subgroups $H \leq G$ and $K \leq C_G(H)$ such that $H \cap K$ is a p'-subgroup. Note that $[H,K] = 1$, $H \cap K \leq Z(H) \cap Z(K)$ and $\mathcal{A}_p(HK) \underset{w}{\approx} \mathcal{A}_p(H/H\cap K) * \mathcal{A}_p(K/H\cap K)$ (see Lemma 4.2.2).

Definition 4.5.4. Let $a = (A_0 < \ldots < A_m)$ be a chain of $\mathcal{A}_p(H)$ and $b = (B_0 < \ldots < B_n)$ be a chain of $\mathcal{A}_p(K)$. Then

$$a * b := (A_0 < \ldots < A_m < B_0A_m < \ldots < B_nA_m)$$

is a chain in $\mathcal{A}_p(HK)$.

Suppose that $c = (0,1,2\ldots,m+n+1)$. A permutation σ of the index set $\{0,1,2,\ldots,m+n+1\}$ such that $\sigma(i) < \sigma(j)$ if $i < j \leq m$ or $m+1 \leq i < j$ is called a *shuffle*. Let $\sigma(c) := (\sigma(0),\sigma(1),\ldots,\sigma(m+n+1))$.

With the above notation, let $C_j = A_j$ if $j \leq m$ or $B_{j-(m+1)}$ if $j \geq m+1$. Define $(a \times b)_\sigma$ to be the chain whose i-th element is $C_{\sigma(0)}C_{\sigma(1)}\ldots C_{\sigma(i)}$.

Definition 4.5.5 ([AS93, Definition 0.21])**.** With the above notation, the shuffle product of a and b is

$$a \times b = \sum_{\sigma \text{ shuffle}} (-1)^\sigma (a \times b)_\sigma \in \tilde{C}_{m+n+1}(\mathcal{A}_p(HK)).$$

We extend this product by linearity to all chains of $\tilde{C}_*(\mathcal{A}_p(H))$ and $\tilde{C}_*(\mathcal{A}_p(K))$.

Proposition 4.5.6 ([AS93, Corollary 0.23])**.** *If $\alpha \in \tilde{Z}_m(\mathcal{A}_p(H))$ and $\beta \in \tilde{Z}_n(\mathcal{A}_p(K))$ then $\alpha \times \beta \in \tilde{Z}_{m+n+1}(\mathcal{A}_p(HK))$.*

Remark 4.5.7. Let X be a finite poset and $a \in X'$. Denote by $\tilde{C}_*(X)_a$ the subgroup of a-initial chains and by $\tilde{C}_*(X)_{\neg a}$ the subgroup of non-a-initial chains. Clearly we have a decomposition

$$\tilde{C}_*(X) = \tilde{C}_*(X)_a \bigoplus \tilde{C}_*(X)_{\neg a}.$$

Moreover, if ∂ denotes the border map of the chain complex,

$$\partial(\tilde{C}_*(X)_{\neg a}) \subseteq \tilde{C}_*(X)_{\neg a}.$$

If $\gamma \in \tilde{C}_*(X)$ then $\gamma = \gamma_a + \gamma_{\neg a}$, where γ_a corresponds to the a-initial part of γ, and

$$\partial\gamma = \partial(\gamma_a) + \partial(\gamma_{\neg a}) = (\partial(\gamma_a))_a + (\partial(\gamma_a))_{\neg a} + \partial(\gamma_{\neg a}).$$

Lemma 4.5.8 (cf. [AS93, Lemma 0.24])**.** *If a is a full chain then $(\partial\gamma)_a = (\partial\gamma_a)_a$.*

Lemma 4.5.9 (cf. [AS93, Lemma 0.25])**.** *Let $X \subseteq \mathcal{A}_p(G)$ be such that if $B \cap K \neq 1$ and $B \in \mathcal{A}_p(G)$ then $B \in X$. Let a be a full chain of $X \cap \mathcal{A}_p(H)$ and b be a chain of $\mathcal{A}_p(K)$. The following holds:*

1. $(a \times b)_a = (a \times b)_{\sigma=\mathrm{id}} = a * b$.

2. $(\partial(a \times b))_a = (-1)^{m+1}(a * \partial b)$*, where m is the large of the chain a.*

Now we prove a variation of Lemma 4.2.5 (see also [AS93, Lemma 0.27]) which will allow us to extend some of the results of [AS93].

Lemma 4.5.10 (Homology Propagation)**.** *Let G be a finite group. Let $H \leq G$, $K \leq C_G(H)$ and $X \subseteq \mathcal{A}_p(G)$ be such that:*

(i) If $B \in \mathcal{A}_p(G)$ and $B \cap K \neq 1$, then $B \in X$;

(ii) $H \cap K$ is a p'-group;

(iii) There exists $a \in \mathcal{A}_p(H)' \cap X'$ such that $a \in \alpha \in \tilde{C}_*(\mathcal{A}_p(H)) \cap \tilde{C}_*(X)$ and α is a cycle but not a boundary in $\tilde{C}_*(\mathcal{A}_p(H))$;

(iv) In addition, for such a, if $B \cup a \in X'$ then either $B \in a$ or $B = (\max a)C_B(H)$ and $1 \neq C_B(H) \leq K$;

(v) $\tilde{H}_*(\mathcal{A}_p(K)) \neq 0$.

Then $\tilde{H}_*(X) \neq 0$.

Proof. We essentially follow the original proof of Lemma 4.2.5 (see [AS93, Lemma 0.27]), adapted to these hypotheses.

By (v), there exists a cycle $\beta \in \tilde{C}_*(\mathcal{A}_p(K))$ which is not a boundary in $\tilde{C}_*(\mathcal{A}_p(K))$. Take a chain a and a cycle α as in the hypothesis (iii). Then $\alpha \times \beta$ is a cycle by Proposition 4.5.6 and it belongs to $\tilde{C}_*(X)$ by hypotheses (i) and (iii). Suppose that $\alpha \times \beta = \partial\gamma$ with $\gamma \in \tilde{C}_*(X)$. Write $\beta = \sum_i q_i(B_0^i < \ldots < B_t^i)$ and $\gamma = \sum_{j\in J} p_j(C_0^j < \ldots < C_{t+s+2}^j)$, where $s+1 = |a|$. Then we can take a-initial part on both sides of the equality $\alpha \times \beta = \partial\gamma$. Note that no intermediate group can be added to a due to hypothesis (iv). Let $A = \max a$. By hypotheses (i) and (ii), and Lemma 4.5.9,

$$(\alpha \times \beta)_a = q \sum_i q_i \, a \cup (AB_0^j < \ldots < AB_t^j)$$

(where $0 \neq q \in \mathbb{Q}$ is the coefficient of a in α), and it is equal to

$$\begin{aligned}(\partial\gamma)_a &= \sum_{j\in J'} p_j \sum_{k=s+1}^{t+s+2} (-1)^k \, a \cup (C_{s+1}^j < \ldots < \hat{C}_k^j < \ldots < C_{t+s+2}) \\ &= \sum_{j\in J'} p_j (-1)^{s+1} \sum_{k=0}^{t+1} a \cup (C_{s+1}^j < \ldots < \hat{C}_{k+s+1}^j < \ldots < C_{t+s+2}).\end{aligned}$$

Here, $J' = \{j \in J : a \subseteq (C_0^j < \ldots < C_{t+s+2}^j)\}$. By hypothesis (iv), $D_k^j := C_{k+s+1}^j = AE_k^j$, where $E_k^j = C_{C_{k+s+1}^j}(H) \neq 1$. Let $\tilde{\beta} = \sum_i q_i(AB_0^i < \ldots < AB_t^i)$ and $\tilde{\gamma} = \sum_{j\in J'} p_j(-1)^{s+1}(D_0^j < \ldots < D_{t+1}^j)$. Note that $q\tilde{\beta} = \partial\tilde{\gamma}$. Let $\mathcal{N} = \{E \in \mathcal{A}_p(G) : E \cap K \neq 1\}$ and consider the retraction

$r:\mathcal{N}\to\mathcal{A}_p(K)$ given by $r(E)=E\cap K$. Note that r is a homotopy equivalence and that $\mathcal{N}\subseteq X$ by hypothesis (i). Therefore,

$$q\beta = r_*(q\tilde{\beta}) = r_*(\partial(\tilde{\gamma})) = \partial(r_*(\tilde{\gamma}))$$

and $r_*(\tilde{\gamma})\in\tilde{C}_*(\mathcal{A}_p(K))$. Since q is invertible, we have a contradiction. □

Remark 4.5.11. The proof would work with integer coefficients if we could choose the chain a and the cycle $\alpha\in\tilde{Z}_*(\mathcal{A}_p(H))$ such that a has coefficient 1 in α.

If coefficients are taken in $\mathbb{Z}$, then, in the proof of Lemma 4.5.10, $q\in\mathbb{Z}$ implies that $q\beta=0$ in the homology of $\mathcal{A}_p(K)$. That is, β is a torsion element of $\tilde{H}_*(\mathcal{A}_p(K),\mathbb{Z})$. The proof would also work for integer coefficients if we could suppose that β is not a torsion element of $\tilde{H}_*(\mathcal{A}_p(K))$, or that its order is prime to the coefficient of a in α.

Now we are in conditions to prove Theorem 4.5.1.

Proof of Theorem 4.5.1. Let G be as in the hypotheses of the theorem. Then $G=\Omega_1(G)$ and $O_p(G)=1$. By Lemma 4.2.2, we may assume that $Z(G)=1$. Let $L=O_{p'}(G)$ and suppose that $L\neq 1$. If every $A\in\mathcal{A}_p(G)$ acts non-faithfully on L, then, by considering the order p subgroups, $L\leq C_G(\Omega_1(G))=Z(G)=1$, a contradiction. Therefore, some $A\in\mathcal{A}_p(G)$ acts faithfully on L.

Let $\mathcal{P}=\{A\in\mathcal{A}_p(G): A \text{ acts faithfully on } L\}$. Then $\mathcal{P}\neq\varnothing$. Let $\mathcal{P}=\{A_1,\dots,A_r\}$ be a linear extension of $\mathcal{P}$ such that $A_i<A_j$ implies $i<j$. Let $i=\max\{k: O_p(C_G(LA_k))=1\}$, with $i=0$ if this set is empty, and let $X_k=\mathcal{A}_p(G)-\{A_k,\dots,A_r\}$.

We prove that $X_{i+1}\subseteq\mathcal{A}_p(G)$ is a weak equivalence by showing that $X_j\subseteq X_{j+1}$ is a weak equivalence for each $j>i$. Put $X:=X_{j+1}=X_j\cup\{A\}$ with $A:=A_j$. If $B\in X_{>A}$ then $A<B$ and B does not act faithfully on L. Hence, $C_B(L)\neq 1$. Let $\mathcal{N}=\{E\in\mathcal{A}_p(G): C_E(L)\neq 1\}$. By Remark 3.1.12, $\mathcal{N}\simeq\mathcal{A}_p(C_G(L))$. Then $X_{>A}=\mathcal{A}_p(G)_{>A}\cap\mathcal{N}$. By Lemma 3.1.13 it is homotopy equivalent to $\mathcal{A}_p(C_{C_G(L)}(A))=\mathcal{A}_p(C_G(LA))\underset{w}{\approx}*$, since $O_p(C_G(LA))\neq 1$. Therefore $X-\{A\}\hookrightarrow X$ is a weak equivalence.

By induction we conclude that $X_{i+1}\subseteq\mathcal{A}_p(G)$ is a weak equivalence. Let $X=X_{i+1}$. Note that $\mathcal{A}_p(LC_G(L))\subseteq X$. If $i=0$, then $\mathcal{A}_p(G)\underset{w}{\approx}X=\mathcal{N}\simeq\mathcal{A}_p(C_G(L))$ But $C_G(L)\trianglelefteq G$ implies $O_p(C_G(L))=1$, and since L is not central in G, $C_G(L)<G$. By the inductive hypothesis, $0\neq\tilde{H}_*(\mathcal{A}_p(C_G(L)),\mathbb{Q})=\tilde{H}_*(\mathcal{A}_p(G),\mathbb{Q})$, a contradiction. Therefore, $i>0$ and we let $A=A_i$.

Now we check the hypotheses of Lemma 4.5.10 with $H=LA$ and $K=C_G(LA)$.

(i) The elements of $\mathcal{A}_p(G)-X$ acts faithfully on L, so they intersect trivially to K;

(ii) $H\cap K\leq Z(L)$ is a p'-group;

(iii) Let $a\in\mathcal{A}_p(LA)'$ be a chain exhibiting $(QD)_p$ in some cycle $\alpha\in\tilde{C}_*(\mathcal{A}_p(LA))$. Then $\alpha\in\tilde{C}_*(X)$ since $\mathcal{A}_p(LA)\subseteq X$.

(iv) It is clear since a is a full chain.

(v) It holds since $O_p(K) = 1$ and $K < G$.

Therefore, $\tilde{H}_*(\mathcal{A}_p(G)) \cong \tilde{H}_*(X) \neq 0$.

□

Remark 4.5.12. As we have noted in Remark 4.5.11, the above reduction can be carried out with integer coefficients if we can choose the cycle $\alpha \in \tilde{Z}_{m_p(A)-1}(\mathcal{A}_p(LA))$ with at least one of its terms with coefficient equals to 1.

It would also work with integer coefficients if the nontrivial cycle β chosen in the homology of $\mathcal{A}_p(K)$ has order prime to the coefficient of some chain $a \in \alpha$.

We relate this result with Corollary 4.3.2, which is stated in terms of integral homology.

Corollary 4.5.13. *If the proper subgroups of G satisfy the strong Quillen's conjecture but $O_p(G) = 1$ and $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) = 0$, then $\mathcal{K}(\mathcal{S}_p(G))$ has no 2-dimensional G-invariant homotopy equivalent subcomplex. In particular, $m_p(G) \geq 4$.*

Proof. By Theorem 4.5.1, $O_{p'}(G) = 1$. Since the strong Quillen's conjecture holds for almost simple groups, G is not almost simple.

Suppose that the statement is false and take K such subcomplex. If K has free abelian integral homology then K is $\mathbb{Z}$-acyclic and by Theorem 4.3.1 $O_p(G) \neq 1$, a contradiction. Now we show that $H_*(K, \mathbb{Z})$ is free abelian.

By a dimension argument, $H_2(K, \mathbb{Z})$ and $H_0(K, \mathbb{Z})$ are free abelian groups. It remains to show that $H_1(K, \mathbb{Z})$ is free. By Theorem 3.4.1, since $O_{p'}(G) = 1$ and G is not almost simple, $\pi_1(\mathcal{A}_p(G))$ is a free group. Therefore, $H_1(K, \mathbb{Z})$ is a free abelian group. □

We can extend Corollary 4.3.3 of the p-rank 3 case of the conjecture to the strong version.

Corollary 4.5.14. *The strong Quillen's conjecture holds for groups of p-rank at most* 3.

4.6 The p-rank 4 case of the stronger conjecture

In this section we reduce the study of the strong Quillen's conjecture to groups with components of p-rank at least 2, and prove that it holds for groups of p-rank at most 4. The ideas behind them roughly follow those in the proofs of Lemma 4.4.2 and Theorem 4.5.1, with the use of Lemma 4.5.10. We begin with some general remarks.

Remark 4.6.1. Let $L \leq G$ and let $E \in \mathcal{A}_p(N_G(L))$. Then $E \cap (LC_G(L)) = 1$ if and only if E acts by outer automorphisms on L.

Suppose $E \cap (LC_G(L)) = 1$ and that $x \in E$ acts as inner automorphism on L. Then there exists $y \in L$ such that $z = y^{-1}x$ acts trivially on L. Therefore $z \in C_G(L)$ and $x = yz \in LC_G(L)$. Since $E \cap (LC_G(L)) = 1$, we conclude that $x = 1$.

The reciprocal is immediate.

Remark 4.6.2. Let G be a finite group with $C_G(F^*(G)) = 1$ and $F^*(G) = E(G)$. Note that $Z(E(G)) = 1$. Let L be a component of G.

If $B \leq G$ is such that $B \cap L \neq 1$, then $B \leq N_G(L)$. This holds because if $b \in B$ then $L^b \cap L \geq B \cap L$ is nontrivial, and it forces to $L^b = L$.

On the other hand, if $N = O_p(C_G(L))$ and $K \in \mathcal{C}(G)$ is a component of G, then either $K \in \mathcal{C}(C_G(L))$ or $K = L$. In both cases, $[N,K] = 1$, so $1 = [N, E(G)] = [N, F^*(G)]$ since $E(G) = F^*(G)$. Therefore $N \leq C_G(E(G)) = C_G(F^*(G)) = 1$. In conclusion, $O_p(C_G(L)) = 1$.

The following theorem deals with groups with some component of p-rank 1. In particular, it deals with the excluded cases $L_2(2^3)$ with $p = 3$ and $\mathrm{Sz}(2^5)$ with $p = 5$ in[AS93]. Hence, this theorem really represents an extension of the works of Aschbacher-Smith.

Theorem 4.6.3. *Let $L \leq G$ be a component such that $L/Z(L)$ has p-rank 1. If the strong Quillen's conjecture holds for proper subgroups of G then it holds for G.*

Proof. Suppose otherwise. By Theorem 4.5.1, $O_p(G) = 1 = O_{p'}(G)$ and L is a simple group of p-rank 1. By Remark 4.6.2, $O_p(C_G(L)) = 1$. Let $\mathcal{N} = \{E \in \mathcal{A}_p(N_G(L)) : E \cap (LC_G(L)) \neq 1\}$. We split the proof in two cases.

Case 1: $\mathcal{A}_p(N_G(L)) = \mathcal{N}$. In this case, there is no outer automorphism of order p of L inside G, and $\Omega_1(N_G(L)) \cong L \times \Omega_1(C_G(L))$. Let $A \in \mathcal{A}_p(L)$. If $B \in \mathcal{A}_p(G)_{>A}$ then $B = AC_B(L)$, so $\mathcal{A}_p(G)_{>A} \subseteq A \times C_G(L)$. Moreover, since L has p-rank 1, A is a connected component of $\mathcal{A}_p(L)$ and it exhibits $(QD)_p$ for L. The hypotheses of the Homology Propagation Lemma 4.2.5 are verified with $H = L$ and $K = C_G(L)$.

Case 2: $\mathcal{A}_p(N_G(L)) \neq \mathcal{N}$. In this case, every $E \in \mathcal{A}_p(N_G(L)) - \mathcal{N}$ acts by outer automorphisms on L by Remark 4.6.1, and has order p since $m_p(L) = 1$ (see Table A.4). By Lemma 4.6.4, we can suppose that L is not isomorphic to $L_2(2^3)$ ($p = 3$) nor to $\mathrm{Sz}(2^5)$ ($p = 5$). Therefore $m_p(LE) = 2$ and LE has $(QD)_p$ (i.e. it is connected, see Table A.4). Moreover, every $A \in \mathcal{A}_p(LE)$ of order p^2 equals $\Omega_1(S)$ for some $S \in \mathrm{Syl}_p(LE)$ by [GLS99, Chapter 4, Lemma 5.1(b)]. Hence, every $A \in \mathcal{A}_p(LE)$ of order p^2 exhibits $(QD)_p$ for LE, and two of them are LE-conjugates. Pick such a subgroup A and note that $LA = LE$, so $C_G(LA) = C_G(LE)$. Let $K := C_G(LE)$. If $B \in \mathcal{A}_p(G)_{>A}$ then $B \cap L \neq 1$ and $B = AC_B(L) \leq AC_G(LA) = AK$, with $C_B(L) \neq 1$. It shows that $B \in \mathcal{A}_p(G)_{>A} \mapsto r(B) = C_B(L) \in \mathcal{A}_p(K)$ is a retraction with inverse $C \mapsto AC$. Therefore, $\mathcal{A}_p(G)_{>A} \simeq \mathcal{A}_p(K)$. If $O_p(K) = 1$ then we are done by the Homology Propagation Lemma 4.2.5 with $H = LA$.

If $O_p(K) \neq 1$, then we can extract all these A and obtain a weak homotopy equivalence $\mathcal{A}_p(G) - \{A \in \mathcal{A}_p(LE) : |A| = p^2\} \hookrightarrow \mathcal{A}_p(G)$. Suppose the same conclusion holds with any choice of $E \in \mathcal{A}_p(N_G(L)) - \mathcal{N}$, that is, $O_p(C_G(LE)) \neq 1$. Let $\mathcal{S} = \{A \in \mathcal{A}_p(N_G(L)) : |A| = p^2$ acts faithfully on $L\}$. Then $X := \mathcal{A}_p(G) - \mathcal{S} \hookrightarrow \mathcal{A}_p(G)$ is a weak homotopy equivalence. Let $A \in \mathcal{A}_p(L)$. If $B \in X_{>A}$ then $B \cap L \geq A \neq 1$ implies that $B \in \mathcal{A}_p(N_G(L))$, and since B cannot act faithfully on L, we have that $B = AC_B(L) \leq AC_G(L)$. As before, $X_{>A} \simeq \mathcal{A}_p(C_G(L))$ via the retraction $B \mapsto C_B(L)$. Let $\alpha = (A) \in C_0(\mathcal{A}_p(L))$. Since $X_{>A} \subseteq A \times C_G(L)$, the hypotheses of Lemma 4.5.10 are verified with $H = L$, $K = C_G(L)$ and $a = (A) = \alpha$, so $\tilde{H}_*(\mathcal{A}_p(G)) \cong \tilde{H}_*(X)$ is nontrivial. □

The following lemma deals with some excluded cases by Aschbacher-Smith in [AS93].

Lemma 4.6.4. *Let G be a finite group such that its proper subgroups satisfy the strong Quillen's conjecture. If L is a component of G such that $L/Z(L)$ is isomorphic to $L_2(2^3)$ (and $p = 3$) or $\mathrm{Sz}(2^5)$ (and $p = 5$), then G satisfy the strong Quillen's conjecture.*

Proof. Suppose otherwise. Therefore $O_p(G) = 1 = O_{p'}(G)$. Let L be such component. Then L is a simple group and $L \cong L_2(2^3)$ or $\mathrm{Sz}(2^5)$ with $p = 3$ or $p = 5$ respectively. Note that $\mathrm{Aut}(L) \cong L \rtimes C_p$ where C_p acts on L by field automorphisms. If $x \in \mathrm{Aut}(L)$ is a non-inner automorphism of order p of L, then x acts on L by field automorphisms and $C_L(x) \cong \mathbb{S}_3 \cong C_3 \rtimes C_2$ if $p = 3$ and $L \cong L_2(2^3)$, or $C_L(x) \cong C_5 \rtimes C_4$ if $p = 5$ and $L \cong \mathrm{Sz}(2^5)$. That is, $\Omega_1(C_L(x)) \cong C_p$.

We proceed similarly as in the above theorem. Let $\mathcal{N} = \{E \in \mathcal{A}_p(N_G(L)) : E \cap (LC_G(L)) \neq 1\}$.

Case 1: $\mathcal{A}_p(N_G(L)) = \mathcal{N}$. This case follows exactly as in the previous theorem.

Case 2: $\mathcal{A}_p(N_G(L)) \neq \mathcal{N}$, so every $E \in \mathcal{A}_p(N_G(L)) - \mathcal{N}$ acts by outer field automorphisms on L, $|E| = p$, $LE \cong \mathrm{Aut}(L)$ and $N_G(L) = LEC_G(L)$.

Fix $E \in \mathcal{A}_p(N_G(L)) - \mathcal{N}$. We prove that $\mathcal{A}_p(G)_{>E}$ is contractible. Let $C = \Omega_1(C_L(E)) \in \mathcal{A}_p(L)$ and fix a generator $c \in C$. Suppose that $b, b' \in C_G(E)$ are such that c^b and $c^{b'}$ belong to a component $L_1 \in \mathcal{C}(G)$. Then $1 \neq c^b \in L_1 \cap L^b$, so $L_1 = L^b$. Similarly, $L_1 = L^{b'}$. Hence,

$$b'b^{-1} \in N_G(L) \cap C_G(E) = C_{N_G(L)}(E) = C_L(E)EC_G(LE).$$

On the other hand, $[C, C_L(E)EC_G(LE)] = 1$, so $[C, b'b^{-1}] = 1$ and therefore $c^b = c^{b'}$.

Consider the set $\mathcal{I} = \{c^b : b \in C_G(E)\}$. It is not hard to show that the elements of $\mathcal{I}$ commute pairwise. Let $\hat{c} = \prod_{c' \in \mathcal{I}} c'$. Then $\hat{c} \in E(G)$ and it is a nontrivial element of order p (the projection onto L is c). Let $\hat{C} = \langle \hat{c} \rangle$ $(\neq 1)$. Note that $[\hat{C}, E] = 1$ and $\hat{C} \in \mathcal{A}_p(LC_G(L))$. If $B \in \mathcal{A}_p(G)_{>E}$ and $b \in B$, then b permutes the elements of $\mathcal{I}$. In particular, $\hat{c}^b = \hat{c}$ and hence $[\hat{C}, B] = 1$. Therefore $\mathcal{A}_p(G)_{>E}$ is conically contractible via the homotopy $B \leq \hat{C}B \geq \hat{C}E$, so E is an up weak point of $\mathcal{A}_p(G)$.

The structure of $\mathcal{A}_p(\mathrm{Aut}(L))$ can be easily described: its connected components have the form $\mathcal{A}_p(E)$, where E is elementary abelian of order p^2 generated by an order p element of

L and some outer field automorphism of L. This is because $\mathrm{Aut}(L) \cong \mathrm{Ree}(3)$ or $\mathrm{Aut}(\mathrm{Sz}(2^5))$ and the Sylow p-subgroups of these groups intersect trivially by Theorem A.1.3. In particular, for every $C \in \mathcal{A}_p(L)$ there exists a unique $E_C \in \mathcal{A}_p(\mathrm{Aut}(L))$ of order p^2 such that $C \leq E_C$, and $C = \Omega_1(C_L(x))$ for every $x \in E_C - C$.

If $C \in \mathcal{A}_p(L)$, then $|C| = p$ and $\mathcal{A}_p(G)_{>C} \simeq \mathcal{A}_p(E_C C_G(L))$, which is contractible since $1 \neq E_C \leq Z(E_C C_G(LC))$. In conclusion, the subgroups of the set $\mathcal{S} := \mathcal{A}_p(L) \cup \{E \in \mathcal{A}_p(N_G(L)) : E$ acts by field automorphisms on $L\}$ have order p and are up weak points. Thus, $X := \mathcal{A}_p(G) - \mathcal{S} \hookrightarrow \mathcal{A}_p(G)$ is a weak homotopy equivalence.

Now, note that $X \cap \mathcal{A}_p(N_G(L)) = \mathcal{A}_p(N_G(L)) - \mathcal{S}$ and that it contains $\mathcal{A}_p(C_G(L))$. Let $\mathcal{S}' = \{F \in \mathcal{A}_p(N_G(L)) : |F| = p^2$ and acts faithfully on $L\}$. Note that $\mathcal{S}' \subseteq X$. Clearly $X_{>F} = \mathcal{A}_p(G)_{>F} \simeq \mathcal{A}_p(C_G(LF))$ and the retraction $r : X_{>F} \to \mathcal{A}_p(C_G(LF))$ defined by $r(B) = C_B(L)$ is a homotopy equivalence.

If $O_p(C_G(LF)) \neq 1$ for all $F \in \mathcal{S}'$, consider $\mathcal{S}'' = \{E \in \mathcal{A}_p(G) - \mathcal{A}_p(L)$: E acts faithfully on $L\}$, and let $Y = \mathcal{A}_p(G) - \mathcal{S}''$. Then $Y \hookrightarrow \mathcal{A}_p(G)$ is a weak homotopy equivalence (we can extract first the points of $\mathcal{S}''$ of order p as up weak points, and then those of order p^2). Note that $\mathcal{A}_p(L) \subseteq Y$ and take $C \in \mathcal{A}_p(L)$. If $B \in Y_{>C}$, then $B \cap L \geq C \neq 1$ implies that $B \leq N_G(L)$ and $B \not\leq L$, and since we have extracted those who act faithfully on L, $B = CC_B(L)$. Hence, $Y_{>C} \subseteq C \times C_G(L)$ and by Lemma 4.5.10 applied with the subposet Y, $H = \mathcal{A}_p(L)$, $K = C_G(L)$ and $a = (C) = \alpha$, $\tilde{H}_*(\mathcal{A}_p(G)) \cong \tilde{H}_*(Y)$ is nontrivial.

Now suppose that for some $F \in \mathcal{A}_p(N_G(L))$ of order p^2 and acting faithfully on L we have that $O_p(C_G(LF)) = 1$. Let $\alpha = (F) \in C_0(\mathcal{A}_p(LF))$. Then α is not a boundary in $\tilde{C}_*(\mathcal{A}_p(LF))$ and $F \in X$. Since $O_p(C_G(LF)) = 1$, by induction we can take a nontrivial cycle $0 \neq \beta \in \tilde{H}_*(\mathcal{A}_p(C_G(LF)))$. The hypotheses of Lemma 4.5.10 are clearly satisfied with $H = LF$, $K = C_G(LF)$ and $\alpha = a = (F)$, and therefore $\tilde{H}_*(\mathcal{A}_p(G)) \cong \tilde{H}_*(X) \neq 0$. □

As a corollary, one can show that [AS93, Main Theorem] extends to $p = 5$. The obstruction to extending this theorem to $p = 3$ relies on [AS93, Theorem 5.3], which is proved for $p > 3$ and it is strongly used in the third step of the proof of [AS93, Main Theorem] (see also the discussion in Section 4.2).

Corollary 4.6.5. *The conclusions of the Main Theorem of [AS93] hold for $p = 5$.*

Now we continue with some preliminaries remarks before proving the p-rank 4 case of the strong conjecture.

Remark 4.6.6. Suppose that $H \leq G$ and $m_p(H) = m_p(G) =: r$. Then we have an inclusion in the top dimensional homology group $\tilde{H}_{r-1}(\mathcal{A}_p(H)) \subseteq \tilde{H}_{r-1}(\mathcal{A}_p(G))$. In particular, if H has $(QD)_p$ then so does G.

If $L = L_1 \times \ldots \times L_n$ is a direct product and each L_i has $(QD)_p$ then L has $(QD)_p$. It follows from the weak equivalence $\mathcal{A}_p(L) \underset{w}{\approx} \mathcal{A}_p(L_1) * \ldots * \mathcal{A}_p(L_n)$ and the homology decomposition of a join.

Remark 4.6.7. Let G be such that $O_p(G) = 1 = O_{p'}(G)$ and $G = \Omega_1(G)$. Suppose that L is a normal component of G. Let $H = L \times C_G(L)$ and let $x \in G$ of order p. If x acts by an inner automorphism on L, then $x = yz$ where $y \in L$ and $z \in C_G(L)$ (see Remark 4.6.1). Therefore, if every element of order p of G acts by inner automorphisms on L then $G = \Omega_1(G) = L \times C_G(L)$. In particular, $\mathcal{A}_p(G) \underset{w}{\approx} \mathcal{A}_p(L) * \mathcal{A}_p(C_G(L))$ by Proposition 3.1.16. Finally, if $\mathcal{A}_p(C_G(L))$ is not $\mathbb{Q}$-acyclic then neither is $\mathcal{A}_p(G)$ since $\mathcal{A}_p(L)$ is not $\mathbb{Q}$-acyclic (by the almost simple case of the conjecture).

Theorem 4.6.8. *The strong Quillen's conjecture holds for groups of p-rank at most* 4.

Proof. Let G be a minimal counterexample to the statement. Then $G = \Omega_1(G)$, $m_p(G) = 4$, $O_p(G) = 1 = O_{p'}(G)$ and $\tilde{H}_*(\mathcal{A}_p(G), \mathbb{Q}) = 0$ by Theorem 4.5.1 and Corollary 4.5.14. These hypotheses imply that $F^*(G) = L_1 \times \ldots \times L_n$ is the direct product of simple groups of order divisible by p. By the almost simple case of the conjecture, $n \geq 2$, and by Lemma 4.4.1 $n \leq 4$. By Theorem 4.6.3, we can suppose that G does not have components of p-rank 1, so $m_p(L_i) \geq 2$ for all i and it forces to $n = 2$, with $m_p(L_1) = 2 = m_p(L_2)$.

By Remark 4.6.7, if G has a normal component L_i then some order p element of G acts by outer automorphisms on L_i and in particular $p \mid |\mathrm{Out}(L_i)|$.

If both $\mathcal{A}_p(L_1)$ and $\mathcal{A}_p(L_2)$ are connected, then L_1 and L_2 have $(QD)_p$, and so $F^*(G)$ has $(QD)_p$, leading to a contradiction by Remark 4.6.6. In consequence we may assume that $\mathcal{A}_p(L_1)$ is disconnected, i.e. L_1 has a strongly p-embedded subgroup. By Table A.4, L_1 is isomorphic to one of the following groups:

1. $L_2(2^2) = \mathbb{A}_5$ or $U_3(2^2)$ with $p = 2$, or
2. $L_3(2^2)$ with $p = 3$.

We are using the fact that if p is odd then L_1 is normal in G and $p \mid |\mathrm{Out}(L_1)|$. We deal with each case separately.

1. $p = 2$ and $L_1 \cong \mathbb{A}_5$ or $U_3(2^2)$. Suppose that some involution $x \in G$ permutes L_1 with L_2 (i.e. they are not normal in G). Let $X = \langle x \rangle$. Note that $F^*(G) \leq N_G(L_1)$ and $G = N_G(L_1)X$. Since $\pi_1(\mathcal{A}_2(F^*(G)X))$ is a nontrivial free group by Theorem 3.4.8, $F^*(G) < N_G(L_1)$ (which is a subgroup of $\mathrm{Aut}(L_1) \times \mathrm{Aut}(L_2)$). We can suppose that $N_G(L_1) = \Omega_1(N_G(L_1))$ and then, $N_G(L_1)/F^*(G)$ is a nontrivial subgroup of $C_2 \times C_2$.

   Since $m_2(\mathrm{Aut}(U_3(2^2))) = 3$ and $\mathbb{S}_5 \times \mathbb{S}_5$ has $(QD)_2$ and 2-rank 4, $N_G(L_1) = F^*(G)\langle \phi \rangle$ where ϕ is an involution acting by an outer automorphism on L_1 and L_2. If $L_1 \cong U_3(2^2)$, then $m_2(C_{L_1}(\phi)) = 2$ and $m_2(L_1\langle \phi \rangle) = 3 = m_2(L_2\langle \phi \rangle)$. This leads to $m_2(N_G(L_1)) = 5$, a contradiction. In consequence, $G \cong ((\mathbb{A}_5 \times \mathbb{A}_5) : \langle \phi \rangle) : X$, where ϕ acts on each copy of $\mathbb{A}_5$ as an outer involution and X permutes the copies of $\mathbb{A}_5$.

A similar proof to the one of Examples 4.4.9 and 4.4.10 shows that $\mathfrak{i}(\mathcal{A}_p(G))$ has height 2, leading to a contradiction when combined with Corollary 4.5.13 in this context. It can also be tested with GAP [GAP18, FPSC19].

Therefore, we can suppose that $L_1 \trianglelefteq G$.

Consider $H = L_1 \times C_G(L_1)$, and $\mathcal{N} = \{E \in \mathcal{A}_2(G) : E \cap H \neq 1\}$. The complement $\mathcal{S} := \mathcal{A}_2(G) - \mathcal{N}$ consists of subgroups of order 2 since otherwise $\Omega_1(\mathrm{Aut}(L_1)) \leq G$ (see Remark 4.6.7) and hence $G = \Omega_1(\mathrm{Aut}(L_1)) \times G_2$ for some $G_2 \leq \mathrm{Aut}(L_2)$. Their links are $\mathcal{A}_2(G)_{>E} \underset{w}{\approx} \mathcal{A}_2(C_{L_1}(E)) * \mathcal{A}_2(C_G(L_1E))$, for $E \in \mathcal{S}$. If all these links are homotopically trivial then we are done. Assume they are not, so in particular for some $E \in \mathcal{S}$ we have that $O_p(C_G(L_1E)) = 1$.

The centralizers of outer involutions of $L_1 \cong \mathbb{A}_5$ and $U_3(2^2)$ are $\mathbb{S}_3$ and $\mathbb{A}_5$ respectively, which have disconnected poset of 2-subgroups. Note that $m_2(\mathbb{S}_3) = 1$.

Since $\mathbb{S}_5$ has $(QD)_2$, $2 = m_2(\mathbb{A}_5) = m_2(\mathbb{S}_5)$ and $\mathbb{A}_5$ does not have $(QD)_2$, for every involution $x \in \mathbb{S}_5 - \mathbb{A}_5$ there exists an involution $y \in \mathbb{A}_5$ such that $\langle x, y\rangle$ exhibits $(QD)_2$ in $\mathbb{S}_5$.

If $L_1 \cong \mathbb{A}_5$ then every $E \in \mathcal{A}_2(G) - \mathcal{N}$ acts by outer automorphisms on L_1 and therefore, for some $A \in \mathcal{A}_2(L_1)$, AE exhibits $(QD)_2$ for $L_1E \cong \mathbb{S}_5$. Fix $E \in \mathcal{S}$ with $O_p(C_G(L_1E)) = 1$ and take $A \in \mathcal{A}_2(L_1)$ with $|A| = p$ and AE exhibiting $(QD)_2$ for L_1E. Then $L_1E = L_1AE$ has $(QD)_2$ exhibited by AE and $O_p(C_G(L_1AE)) = O_p(C_G(L_1E)) = 1$. The hypotheses of the Homology Propagation Lemma 4.2.5 can be checked and hence $\tilde{H}_*(\mathcal{A}_2(G), \mathbb{Q}) \neq 0$.

In consequence, $L_1 \cong U_3(2^2)$ and $L_2 \not\cong \mathbb{A}_5$. Moreover, there exists some involution $x \in G$ acting by outer automorphisms on both L_1 and L_2 by Remark 4.6.7. Since $C_{L_1}(x) \cong C_{U_3(2^2)}(x) \cong \mathbb{A}_5$ has 2-rank 2, it must be that $C_{L_2}(x)$ has 2-rank 1 and this forces to $L_2 \cong L_2(q)$ with q odd and x inducing diagonal automorphisms on L_2 (see [GLS98, Theorem 4.10.5]).

Suppose there is an involution $\phi \in C_G(L_1)$ acting by outer field automorphism on $L_2 = L_2(r^a)$, with r an odd prime. Then $C_{L_2}(\phi) \cong L_2(r^{a/2})$ has 2-rank 2, and hence $L_1 \times (L_2\langle \phi\rangle) \leq H$ has 2-rank at least 5, a contradiction. In consequence $C_G(L_1)$ contains no involution acting as field automorphism on L_2. By the above reasoning, an outer involution of both L_1 and L_2 must act by diagonal automorphisms on L_2. This shows that G contains no field automorphisms of L_2 and in particular, $G \leq \mathrm{Aut}(L_1) \times \mathrm{Inndiag}(L_2)$.

Take $A \in \mathcal{A}_2(L_2)$ exhibiting $(QD)_2$ for L_2. Since $L_2 \trianglelefteq G$, $O_p(C_G(L_2A)) = O_p(C_G(L_2)) = 1$ and if $B \in \mathcal{A}_p(G)_{>A}$ then $B/C_B(L_2) \leq \mathrm{Inndiag}(L_2)$ which has 2-rank 2. Hence $C_B(L_2) \neq 1$ and $B = AC_B(L_2)$. By the Homology Propagation Lemma 4.2.5 applied to $H = L_2$ and $K = C_G(L_2)$, $\tilde{H}_*(\mathcal{A}_2(G), \mathbb{Q}) \neq 0$.

2. Suppose $p = 3$ and $L_1 \cong L_3(2^2)$.

Note that $m_3(L_2) = 2$, $\mathrm{Out}(L_3(2^2)) \cong D_{12} = C_3 : (C_2 \times C_2)$ and $\mathrm{Inndiag}(L_3(2^2)) \cong L_3(2^2) : C_3$, so without loss of generality $G \leq \mathrm{Inndiag}(L_3(2^2)) \times \mathrm{Aut}(L_2)$. We also may assume that G is not the direct product of almost simple groups and that some element $x \in G - L_1$ of order 3 acts by diagonal automorphisms on $L_1 \cong L_3(2^2)$ and by outer automorphisms on L_2. Let $C = \langle x \rangle$. Observe that $(L_1 \times L_2)C$ contains every non-inner diagonal automorphism of L_1, and every such automorphism acts non trivially on L_2.

Since $\mathcal{A}_3(L_3(2^2))$ is disconnected but $\mathcal{A}_3(L_3(2^2)C)$ is a connected (not simply connected) poset of height 1, after changing C for other non-inner diagonal automorphism of $L_1 \cong L_3(2^2)$ (which is in $(L_1 \times L_2)C$), for some $A \in \mathcal{A}_3(C_{L_1}(C))$, AC exhibits $(QD)_3$ for L_1C (this holds since $C_{L_3(2^2)}(C) \cong \mathbb{A}_5$ or $C_7 : C_3$ by direct computation). Let $H = L_1 \times C_G(L_1)$ and $\mathcal{N} = \{E \in \mathcal{A}_3(G) : E \cap H \neq 1\}$. Then $\mathcal{S} := \mathcal{A}_3(G) - \mathcal{N}$ consists of minimal elements acting by non-inner diagonal automorphisms on $L_1 \cong L_3(2^2)$.

Recall that $\mathcal{A}_3(G)_{>E} \simeq \mathcal{A}_3(C_{L_1}(E) \times C_G(L_1E)) \underset{w}{\approx} \mathcal{A}_3(C_{L_3(2^2)}(E)) * \mathcal{A}_3(C_G(L_1E))$ and $C_G(L_1E) = C_G(L_1C)$, for $E \in \mathcal{S}$. If $O_p(C_G(L_1C)) \neq 1$ then $\mathcal{A}_3(G) \underset{w}{\approx} \mathcal{N} \simeq \mathcal{A}_3(H)$ and we are done. Otherwise, $C \in \mathcal{S}$, $O_p(C_G(L_1C)) = 1$ and for some $A \in \mathcal{A}_3(C_{L_1}(C))$, AC exhibits $(QD)_3$ for L_1C. By the Homology Propagation Lemma 4.2.5 with $H = L_1(AC)$ and $K = C_G(L_1(AC)) = C_G(L_1C)$, $\tilde{H}_*(\mathcal{A}_3(G), \mathbb{Q}) \neq 0$.

This concludes the proof of the p-rank 4 case. $\square$

Appendix

A.1 Finite Simple groups

By the classification of the finite simple groups (CFSG for short), a finite simple groups belongs to one of the following families:

1. Cyclic groups C_p of order p prime (the abelian simple groups),
2. Alternating groups $\mathbb{A}_n$ with $n \geq 5$,
3. Finite simple groups of Lie Type,
4. The 26 Sporadic groups.

Recall that a finite group G has a strongly p-embedded subgroup if there exists $M < G$ such that $|M|_p = |G|_p$ and $M \cap M^g$ is a p'-group for all $g \in G - M$. By Quillen's result (see Proposition 3.1.1), G has a strongly p-embedded subgroup if and only if $\mathcal{A}_p(G)$ is disconnected. The following theorem classifies the groups with this property.

Theorem A.1.1 ([Asc93, (6.1)]). *The finite group G has a strongly p-embedded subgroup (i.e. $\mathcal{A}_p(G)$ is disconnected) if and only if either $O_p(G) = 1$ and $m_p(G) = 1$, or $\Omega_1(G)/O_{p'}(\Omega_1(G))$ is one of the following groups:*

1. *Simple of Lie type of Lie rank 1 and characteristic p,*
2. *$\mathbb{A}_{2p}$ with $p \geq 5$,*
3. *$\mathrm{Ree}(3)$, $L_3(2^2)$ or M_{11} with $p = 3$,*
4. *$\mathrm{Aut}(\mathrm{Sz}(2^5))$, ${}^2F_4(2)'$, McL, or Fi_{22} with $p = 5$,*
5. *J_4 with $p = 11$.*

Remark A.1.2. The simple groups of Lie type and Lie rank 1 are the groups $L_2(q)$, $U_3(q)$, $\mathrm{Sz}(q)$ and ${}^2G_2(q)$. In characteristic 2, these are $L_2(2^n)$, $U_3(2^n)$ and $\mathrm{Sz}(2^n)$ and they are the unique simple groups with a strongly 2-embedded subgroup. There are no simple groups of 2-rank 1 (see [GLS98, Theorem 4.10.5(a)]).

From the list it can be deduced that if L is a simple group with a strongly p-embedded subgroup then its Sylow p-subgroups intersect trivially. It is deduced from [GLS98, Theorem 7.6.2]. See also [Sei82, Theorem 7].

Theorem A.1.3 ([GLS98, Theorem 7.6.2]). *If G has p-rank 1 or it is one of the almost simple groups listed in Theorem A.1.1, then the Sylow p-subgroups of G intersect trivially.*

A.1.1 Finite simple groups of Lie type

The family of simple groups of Lie type is the biggest family of simple groups, and it is frequently subdivided in the following subfamilies:

Classical: $L_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$,$U_n(q)$, ${}^2D_n(q)$;
Exceptional: $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, $G_2(q)$;
(Nonclassical) Twisted: $\mathrm{Sz}(2^m)$, $\mathrm{Ree}(3^m)$, ${}^3D_4(q)$, ${}^2F_4(2^m)$, ${}^2E_6(q)$.

Here, $q = p^f$, where p is a prime number, and, for example $L_n(q)$ means that it is the Projective Special Linear group defined over the finite field of q elements $\mathbb{F}_q$.

We will call the groups $L_n(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$ the *untwisted classical groups* or *untwisted Chevalley groups*. The groups $U_n(q)$, ${}^2D_n(q)$, ${}^2E_6(q)$, ${}^3D_4(q)$ are *Steinberg variations* (which are twisted). The *exceptional groups* are $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, $G_2(q)$ (and they are not twisted). The *Suzuki-Ree groups* are the groups $\mathrm{Sz}(q)$, ${}^2F_4(q)$ and $\mathrm{Ree}(q)$ (they are twisted).

In Table A.1 we give the names and orders of the different finite simple groups of Lie type. Denote by (a,b) the greatest common divisor between the integers a and b.

The following groups appearing in Table A.1 are not simple. See [GL83, (3-1)].

- $L_2(2)$ and $L_2(3)$ are solvable.
- $U_3(2)$ is solvable.
- $\mathrm{Sz}(2)$ is solvable.
- $B_2(2)$ is not simple, but $B_2(2)'$ is.
- $G_2(2)$ is not simple, but $G_2(2)'$ is.
- ${}^2F_4(2)$ is not simple, but ${}^2F_4(2)'$ is and it is called the *Tits group*. Its outer automorphism group is C_2.
- $\mathrm{Ree}(3)$ is not simple, but $\mathrm{Ree}(3)'$ is.

Group	Order	Other names
Untwisted classical groups of Lie type		
$L_n(q), n \geq 2$	$\dfrac{q^{\frac{(n-1)n}{2}}}{(n,q-1)} \prod_{i=1}^{n-1}(q^{i+1}-1)$	$\mathrm{PSL}_n(q)$, $A_{n-1}(q)$
$B_n(q), n \geq 2$	$\dfrac{q^{n^2}}{(2,q-1)} \prod_{i=1}^{n}(q^{2i}-1)$	$O_{2n+1}(q)$ $\Omega_{2n+1}(q)(q \text{ odd})$
$C_n(q), n \geq 3$	$\dfrac{q^{n^2}}{(2,q-1)} \prod_{i=1}^{n}(q^{2i}-1)$	$\mathrm{PSp}_{2n}(q)$
$D_n(q), n \geq 4$	$\dfrac{q^{n(n-1)}(q^n-1)}{(4,q^n-1)} \prod_{i=1}^{n-1}(q^{2i}-1)$	$O^+_{2n}(q)$, $\mathrm{P}\Omega^+_{2n}(q)$
Steinberg variations		
$U_n(q), n \geq 3$	$\dfrac{q^{\frac{n(n-1)}{2}}}{(n,q+1)} \prod_{i=1}^{n-1}(q^{i+1}-(-1)^{i+1})$	$\mathrm{PSU}_n(q)$ ${}^2A_{n-1}(q^2)$
${}^2D_n(q), n \geq 4$	$\dfrac{q^{n(n-1)(q^n+1)}}{(4,q^n+1)} \prod_{i=1}^{n-1}(q^{2i}-1)$	${}^2D_n(q^2)$ $O^-_{2n}(q)$ $\mathrm{P}\Omega^-_{2n}(q)$
${}^2E_6(q)$	$\frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{(3,q+1)}$	${}^2E_6(q^2)$
${}^3D_4(q)$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	${}^3D_4(q^3)$
Exceptional groups of Lie type		
$E_6(q)$	$\frac{q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)}{(3,q-1)}$	
$E_7(q)$	$\frac{q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)}{(2,q-1)}$	
$E_8(q)$	$q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)$ $(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	
$F_4(q)$	$q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	
$G_2(q)$	$q^6(q^6-1)(q^2-1)$	
Suzuki-Ree groups		
$\mathrm{Sz}(2^{2n+1}), n \geq 1$	$q^2(q^2+1)(q-1), q = 2^{2n+1}$	${}^2B_2(2^{2n+1})$
${}^2F_4(2^{2n+1}), n \geq 0$	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1), q = 2^{2n+1}$	
$\mathrm{Ree}(3^{2n+1}), n \geq 0$	$q^3(q^3+1)(q-1), q = 3^{2n+1}$	${}^2G_2(3^{2n+1})$

Table A.1: The order of the finite simple groups of Lie type and their names.

There exists some isomorphisms among the simple groups of the different families, and they are given in the theorem below. See [GL83, (3-2) & (3-3)].

Theorem A.1.4. *We have the following isomorphisms between the different families of simple groups:*

$$L_2(2) \cong \mathbb{S}_3; \quad L_2(3) \cong \mathbb{A}_4; \quad L_2(2^2) \cong L_2(5) \cong \mathbb{A}_5$$

$$L_2(3^2) \cong B_2(2)' \cong \mathbb{A}_6; \quad B_2(2) \cong \mathbb{S}_6; \quad L_4(2) \cong \mathbb{A}_8;$$

$$L_2(7) \cong L_3(2); \quad L_2(2^3) \cong \mathrm{Ree}(3)'; \quad B_n(2^m) \cong C_n(2^m)$$

$$B_2(3) \cong U_4(2); \quad U_3(3) \cong G_2(2)'$$

$$B_2(q^n) \cong C_2(q^n); \quad D_3(q^n) \cong L_4(q^n); \quad {}^2D_3(q^n) \cong U_4(q^n); \quad {}^2D_2(q^n) \cong L_2(q^{2n})$$

In Table A.3, we briefly describe the outer automorphisms group of the finite simple groups of Lie type. We adopt the convention of [GL83, Section 7] for the name of the different automorphisms of a group of Lie type, which in fact follows [Ste68]. We refer to [GL83] for more details on the treatment of the automorphisms of the Lie type groups.

Let L be a finite simple group of Lie type in defining field $\mathbb{F}_q$. An automorphism of L is *inner* if it lies in $\mathrm{Inn}(L) \cong L/Z(L)$. An *inner-diagonal automorphisms* of L is an automorphism of L which is a product of an inner and a diagonal automorphism (in the sense of [Ste68]). The group of inner-diagonal automorphisms of L is denoted by $\mathrm{Inndiag}(L)$. We have that $\mathrm{Inndiag}(L) \trianglelefteq \mathrm{Aut}(L)$ and $\mathrm{Outdiag}(L) = \mathrm{Inndiag}(L)/\mathrm{Inn}(L)$. The elements of $\mathrm{Inndiag}(L) - \mathrm{Inn}(L)$ are called *diagonal automorphisms*. There is a subgroup $\Phi_L \leq \mathrm{Aut}(L)$ which essentially is the group of automorphisms of the defining field. That is, $\Phi_L \cong \mathrm{Aut}(\mathbb{F}_q)$ except if L is a Steinberg variation, in which case $\Phi_L \cong \mathrm{Aut}(\mathbb{F}_{q^2})$ if $L = U_n(q)$, ${}^2D_n(q)$ or ${}^2E_6(q)$, and $\Phi_L \cong \mathrm{Aut}(\mathbb{F}_{q^3})$ if $L = {}^3D_4(q)$. Note that Φ_L is always a cyclic group. There is a subgroup $\Gamma_L \leq \mathrm{Aut}(L)$ consisting of graph automorphisms, isomorphic to the group of symmetries of the Dynkin diagram of L. There could be many choices for Φ_L and Γ_L. We fix one of them. Then $[\Phi_L, \Gamma_L] = 1$ and $\Phi_L\Gamma_L$ is a subgroup of $\mathrm{Aut}(L)$. The elements of Φ_L and its conjugates are called *field automorphisms*. The elements of $\Phi_L\Gamma_L - \Phi_L$ generating a (cyclic) group disjoint from Γ_L, together with their $\mathrm{Aut}(L)$-conjugates, are called *graph-field automorphisms*. The elements of $\Gamma_L \mathrm{Inndiag}(L) - \mathrm{Inndiag}(L)$ are called *graph automorphisms*, except if $L = B_2(q)$, $F_4(q)$ or $G_2(q)$ and $\Gamma_L \neq 1$, in which case all elements of $\Phi_L\Gamma_L - \Phi_L$ are called graph-field automorphisms. If L is a Steinberg variation $U_n(q)$, ${}^2D_n(q)$, ${}^2E_6(q)$ or ${}^3D_4(q)$, then Φ_L is cyclic of order $2f$, $2f$, $2f$ and $3f$ respectively and the elements of order not divisible by 2, 2, 2 and 3 resp. (and their conjugates) are called field automorphisms. The remaining elements of Φ_L are called graph automorphisms.

We have that $\mathrm{Aut}(L)$ is a split-extension $\mathrm{Inndiag}(L) : (\Phi_L \times \Gamma_L)$ and its outer automorphisms group is $\mathrm{Outdiag}(L) : (\Phi_L \times \Gamma_L)$. These three groups are cyclic except for $D_n(q)$ with q odd, where $\mathrm{Outdiag}(D_n(q)) = C_2 \times C_2$, and if $n = 4$ then $\Gamma_L = \mathbb{S}_3$.

Every automorphism of L can be written as $i.d.f.g$, where i is an inner automorphism, d is a diagonal automorphism, f is a field automorphism and g is a graph automorphism.

Put $q = p^f$ with p prime. We adopt the convention that n denotes the cyclic group of order n, and n^m is the direct product C_n^m.

Group	Out structure
Untwisted classical groups of Lie type	
$L_n(q), n \geq 2$	$\begin{cases} (2,q-1):(f.1) & n=2 \\ (n,q-1):(f\times 2) & n>2 \end{cases}$
$B_n(q), n \geq 2$	$\begin{cases} (2,q-1):(f.1) & q \text{ odd or } n>2 \\ (2,q-1):(f\times 2) & q \text{ even, } n=2 \end{cases}$
$C_n(q), n \geq 3$	$(2,q-1):(f.1)$
$D_n(q), n \geq 4$	$\begin{cases} (2,q-1)^2:(f\times \mathbb{S}_3) & n=4 \\ (2,q-1)^2:(f\times 2) & n>4 \text{ even} \\ (4,q^n-1):(f\times 2) & n \text{ odd} \end{cases}$
Steinberg variations	
$U_n(q), n \geq 3$	$(n,q+1):(2f.1)$
${}^2D_n(q), n \geq 4$	$(4,q^n+1):(2f.1)$
${}^2E_6(q)$	$(3,q+1):(2f.1)$
${}^3D_4(q)$	$1.3f.1$
Exceptional groups of Lie type	
$E_6(q)$	$(3,q-1):(f\times 2)$
$E_7(q)$	$(2,q-1):(f.1)$
$E_8(q)$	$1.f.1$
$F_4(q)$	$\begin{cases} 1.f.1 & q \text{ odd} \\ 1.(f\times 2) & q \text{ even} \end{cases}$
$G_2(q)$	$\begin{cases} 1.f.1 & p\neq 3 \\ 1.(f\times 2) & p=3 \end{cases}$
Suzuki-Ree groups	
$\mathrm{Sz}(2^{2n+1}), n \geq 1$	$1.(2n+1).1$
${}^2F_4(2^{2n+1}), n \geq 0$	$\begin{cases} 1.(2n+1).1 & n\neq 0 \\ 1.2.1 & n=0 \end{cases}$
$\mathrm{Ree}(3^{2n+1}), n \geq 0$	$1.(2n+1).1$

Table A.3: Outer automorphisms structure of simple groups of Lie type.

The following table summarize the p-ranks and outer automorphisms structures for the groups of the list of Theorem A.1.1. In each case $q = p^a$ is the order of the definition field.

Group G	Order	Out(G)	$m_p(G)$	$m_p(\mathrm{Out}(G))$
p-rank 1 groups				
G		cyclic p-Sylow	1	≤ 1
Lie rank 1 in characteristic p				
$L_2(p^a)$	$\frac{q(q^2-1)}{(2,q-1)}$	$(2,q-1) \rtimes C_a$	a	$= m_p(C_a) \leq 1$
$U_3(p^a)$	$\frac{q^3(q^2-1)(q^3+1)}{(3,q+1)}$	$(3,q+1) \rtimes C_{2a}$	$\begin{cases} a & p=2 \\ 2a & p \neq 2 \end{cases}$	$= m_p(C_{2a}) \leq 1$
Sz(2^a), a odd	$q^2(q^2+1)(q-1)$	C_a	a	0
Ree(3^a), a odd	$q^3(q^3+1)(q-1)$	C_a	$2a$	$= m_p(C_a) \leq 1$
Alternating groups, $p \geq 5$				
$\mathbb{A}_{2p}$	$\frac{(2p)!}{2}$	C_2	2	0
$p = 3$ exceptions				
Ree(3)	$2^3.3^3.7$	1	2	0
$L_3(2^2)$	$2^6.3^2.5.7$	D_{12}	2	1
M_{11}	$2^4.3^2.5.11$	1	2	0
$p = 5$ exceptions				
Aut(Sz(2^5))	$2^{10}.5^3.31.41$	1	2	0
${}^2F_4(2)'$	$2^{11}.3^3.5^2.13$	C_2	2	0
McL	$2^7.3^6.5^3.7.11$	C_2	2	0
Fi$_{22}$	$2^{17}.3^9.5^2.7.11.13$	C_2	2	0
$p = 7$ exceptions				
J_4	...	1	1	0

Table A.4: Structure of groups with a strongly p-embedded subgroup and p'-free core.

A.1.2 Sporadic groups

We list the order of the sporadic groups. The outer automorphisms group of a sporadic group is either trivial or the cyclic group C_2. Most of them are named after the mathematicians who discovered them.

Group	Order	Out	Other names
Mathieu groups			
M_{11}	$2^4.3^2.5.11$	1	
M_{12}	$2^6.3^3.5.11$	2	
M_{22}	$2^7.3^2.5.7.11$	2	
M_{23}	$2^7.3^2.5.7.11.23$	1	
M_{24}	$2^{10}.3^3.5.7.11$	1	
Janko groups			
J_1	$2^3.3.5.7.11.19$	1	
J_2	$2^7.3^3.5^2.7$	2	
J_3	$2^7.3^5.5.17.19$	2	
J_4	$2^{21}.3^3.5.7.11^3.23.29.31.37.43$	1	
Conway groups			
Co_1	$2^{21}.3^9.5^4.7^2.11.13.23$	1	.1
Co_2	$2^{18}.3^6.5^3.7.11.23$	1	.2
Co_3	$2^{10}.3^7.5^3.7.11.23$	1	.3
Fischer groups			
Fi_{22}	$2^{17}.3^9.5^2.7.11.13$	2	$M(22)$
Fi_{23}	$2^{18}.3^{13}.5^2.7.11.13.17.23$	1	$M(23)$
Fi'_{24}	$2^{21}.3^{16}.5^2.7^3.11.13.17.23.29$	2	$M(24)'$
HS	$2^9.3^2.5^3.7.11$	2	
McL	$2^7.3^6.5^3.7.11$	2	Mc
He	$2^{10}.3^3.5^2.7^3.17$	2	F_7
Ru	$2^{14}.3^3.5^3.7.13.29$	1	
Suz	$2^{13}.3^7.5^2.11.13$	2	
O'N	$2^9.3^4.5.7^3.11.19.31$	2	ON
HN	$2^{14}.3^6.5^6.7.11.19$	2	F_5
Ly	$2^8.3^7.5^6.7.11.31.37.67$	2	
Th	$2^{15}.3^{10}.5^3.7^2.13.19.31$	1	F_3
B	$2^{41}.3^{13}.5^6.7^2.11.13.17.19.23.31.47$	1	F_2
M	$2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71$	1	F_1

Table A.5: Order and Out structure of Sporadic groups

We list the p-ranks of the Sporadic Groups in Table A.6. An empty cell means that the p-rank is at most 1.

L	$m_2(L)$	$m_3(L)$	$m_5(L)$	$m_7(L)$	$m_{11}(L)$	$m_{13}(L)$
M_{11}	2	2				
M_{12}	3	2				
M_{22}	4	2				
M_{23}	4	2				
M_{24}	6	2				
J_1	3					
J_2	4	2	2			
J_3	4	3				
J_4	11	2			2	
Co_1	11	6	3	2		
Co_2	10	4	2			
Co_3	4	5	2			
Fi_{22}	10	5	2			
Fi_{23}	11	6	2			
Fi'_{24}	11	7	2	2		
HS	4	2	2			
McL	4	4	2			
He	6	2	2	2		
Ru	6	2	2			
Suz	6	5	2			
O'N	3	4		2		
HN	6	4	3			
Ly	4	5	3			
Th	5	5	2	2		
B	12 to 18	6	3	2		
M	13 to 22	8	4	3	2	2

Table A.6: The p-ranks of the sporadic groups.

A.2 GAP codes

In this section we present some of the programming codes in **GAP** that we have used to compute the examples. We use the package [FPSC19], whose code is available in the **github** repository.

A.2.1 Computing the core of a poset

Using the package [FPSC19], we can compute the core of a poset loaded in **GAP** in the following way.

```
gap> LoadPackage("posets");;
gap> G:=AlternatingGroup(5);;
gap> p:=2;;
gap> P:=QuillenPoset(G,p);
<finite poset of size 20>
gap> Core(P);
<finite poset of size 5>
```

If $G = \mathbb{S}_3 \wr C_2$ with $p = 2$, by Example 1.3.4, $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ are not homotopy equivalent. We have computed their cores with the following program in **GAP**. This group G has id $(72,40)$ in the library of SMALLGROUPS of **GAP**.

```
gap> G:=SmallGroup(72,40);
<pc group of size 72 with 5 generators>
gap> p:=2;
2
gap> StructureDescription(G);
"(S3 x S3) : C2"
gap> ApG:=QuillenPoset(G,2);
<finite poset of size 39>
gap> SpG:=BrownPoset(G,2);
<finite poset of size 57>
gap> Core(ApG);
<finite poset of size 39>
gap> Core(SpG);
<finite poset of size 21>
```

The following code computes the core of the p-subgroup posets of the group of Example 1.3.17, which corresponds to the counterexample to Stong's question. It has id $(576,8654)$ in the library of SMALLGROUPS of **GAP**.

```
gap> G:=SmallGroup(576,8654);
<pc group of size 576 with 8 generators>
gap> ApG:=QuillenPoset(G,2);
<finite poset of size 321>
gap> Core(ApG);
<finite poset of size 100>
gap> RpG:=RobinsonPoset(G,2);
<finite poset of size 48431>
gap> Core(RpG);
```

```
<finite poset of size 2065>
gap> SdApG:=FacePoset(OrderComplex(ApG));
<finite poset of size 3287>
gap> Core(SdApG);
<finite poset of size 631>
```

We can also compute the orbit poset of the subdivided p-subgroup posets. In the following example we compute the core of $\mathcal{A}_p(G)'/G$ when G is the group of Example 1.3.17 with $p = 2$.

```
gap> G:=SmallGroup(576,8654);
<pc group of size 576 with 8 generators>
gap> OrbitSdApG:=OrbitSubdivisionPosetOfElementaryAbelianpSubgroups(G,2);
<finite poset of size 9>
gap> Core(OrbitSdApG);
<finite poset of size 1>
```

In the following example we compute the core of $\mathcal{S}_p(G)'/G$ when $G = \mathrm{PSL}_2(7)$ and $p = 2$.

```
gap> G:=PSL(2,7);
Group([ (3,7,5)(4,8,6), (1,2,6)(3,4,8) ])
gap> OrbitSdSpG:=OrbitSubdivisionPosetOfpSubgroups(G,2);
<finite poset of size 19>
gap> Core(OrbitSdSpG);
<finite poset of size 13>
gap> OrbitSdApG:=OrbitSubdivisionPosetOfElementaryAbelianpSubgroups(G,2);
<finite poset of size 5>
gap> Core(OrbitSdApG);
<finite poset of size 1>
```

A.2.2 Computing the fundamental group

The following program computes the fundamental group of a p-subgroup poset.

```
gap> G:=AlternatingGroup(9);
Alt( [ 1 .. 9 ] )
gap> BpG:=BoucPoset(G,3);
<finite poset of size 2324>
gap> pi1:=FundamentalGroup(BpG);
<fp group of size infinity with 2997 generators>
gap> NonFreePart(pi1);
Free group of rank 2997.
<pc group of size 1 with 0 generators>
```

Here, the function `NonFreePart` takes a finitely presented group $Q = \langle X | R \rangle$ and returns the finitely presented group $\langle X' | R \rangle$ where X' consists of those elements of X which appear in some relation of R. When the set of relations is empty, it returns the trivial group and informs that the group is free of rank $|X|$. Otherwise, it prints the rank of the free part, i.e. $|X|-|X'|$, and returns the finitely presented group $\langle X'|R\rangle$.

```
NonFreePart := function (Q)
    local a, r, v, F, code, involved_generators, rels,
    rels1, rels_coded, relators_coded;

    rels:=RelatorsOfFpGroup(Q);;
    rels1:=List(rels,LetterRepAssocWord);;
    involved_generators:=Set(List(Set(Concatenation(rels1)),AbsInt));;
    code:=function(x)
        if x > 0 then
            return PositionSorted(involved_generators,x);
        else
            return - PositionSorted(involved_generators,-x);
        fi;
    end;;

    if Size(involved_generators) > 0 then
        Print("The free part has rank ",
        Size(GeneratorsOfGroup(Q)) - Size(involved_generators), ".\n");
        F:=FreeGroup(Size(involved_generators));
        a:=GeneratorsOfGroup(F)[1];
        relators_coded:=List(rels1, r-> List(r,code));
        rels_coded:=List(relators_coded,
        v-> AssocWordByLetterRep(FamilyObj(a),v));
        return F/rels_coded;
    else
        Print("Free group of rank ",Size(GeneratorsOfGroup(Q)), ".\n");
        return TrivialGroup();
    fi;
end;;
```

With this program we have computed the fundamental group of $\mathcal{A}_p(G)$ with $G=\mathbb{A}_{10}$ and $p=3$.

```
gap> G:=AlternatingGroup(10);
Alt( [ 1 .. 9 ] )
gap> BpG:=BoucPoset(G,3);
<finite poset of size 24620>
gap> pi1:=FundamentalGroup(BpG);
<fp group of size infinity with 25242 generators>
gap> Q:=NonFreePart(pi1);
The free part has rank 25200.
<fp group of size infinity with 42 generators>
```

The group Q is non-free since its abelianization is fre abelian of rank 42 and it has nontrivial commuting relations.

```
gap> A:=AbelianInvariants(Q);;
gap> Size(A);
42
gap> Unique(A);
[ 0 ]
gap> Size(RelatorsOfFpGroup(Q));
861
gap> RelatorsOfFpGroup(Q){[1..2]};
[ f41^-1*f36^-1*f41*f36, f38^-1*f34^-1*f38*f34 ]
```

The code shows that the unique abelian invariant of Q is 0, that is, the infinite cyclic group $\mathbb{Z}$, and that there are 42 copies of it. This means that the abelianization of Q is $\mathbb{Z}^{42}$.

List of Symbols

Groups and actions

Let G, H and N be groups and let X be a G-set.

$\mathbb{A}_n$	the alternating group on n letters
C_n	the cyclic group of order n with multiplicative notation
D_n	the dihedral group of order n
F_n	the free group of rank n
$L_n(q)$	equals $\mathrm{PSL}_n(q)$
$\mathrm{PSL}_n(q)$	the projective special linear group over $\mathbb{F}_q$
$\mathrm{PSU}_n(q)$	the projective special unitary group over $\mathbb{F}_q$
$\mathbb{S}_n$	the symmetric group on n letters
$\mathrm{Sz}(q)$	for $q = 2^{2n+1}$ denotes the Suzuki group over $\mathbb{F}_q$
$U_n(q)$	the projective special unitary group over $\mathbb{F}_q$
$\mathbb{F}_q$	the finite field of order q
$\mathbb{Z}_n$	the cyclic group of order n with additive notation
$\lvert G\rvert$	the order of G
$\lvert g\rvert$	the order of $g \in G$
$\lvert G\rvert_\pi$	the π-part of the order of G, with π a set of primes
$\mathrm{Aut}(G)$	the automorphisms group of G
$\mathrm{Inn}(G)$	the inner automorphisms group of G
$\mathrm{Out}(G)$	the outer automorphisms group of G
$H \leq G$	a subgroup of G
$H < G$	a proper subgroup of G
$N \trianglelefteq G$	a normal subgroup of G
$N \triangleleft G$	a proper normal subgroup of G
$N \operatorname{char} G$	a characteristic subgroup of G
$N_G(H)$	the normalizer of H in G
$C_G(H)$	the centralizer of H in G
$[G,H]$	commutator of G and H

$[G:H]$	index of H in G
$N \times H$	direct product N by H
$N \rtimes H$	split extension of N by H
$N:H$	split extension of N by H
NH	inner split extension of N by H
$N.H$	non-split extension of N by H
G/H	set of right-cosets or quotient group if $H \trianglelefteq G$
$\langle S \rangle$	is the subgroup generated by $S \subset G$
$\mathcal{C}g$	the conjugacy class of g
h^g	$= g^{-1}hg$ for $g,h \in G$
H^g	$= g^{-1}Hg$ for $H \leq G$
$[g,h]$	the commutator $ghg^{-1}h^{-1}$
G'	the derived subgroup of G
$F(G)$	the Fitting subgroup of G
$F^*(G)$	the generalized Fitting subgroup of G
$L(G)$	the layer of G
$\Phi(G)$	the Frattini subgroup of G
$Z(G)$	the center of G
$O_p(G)$	the largest normal p-subgroup of G
$O_{p'}(G)$	the largest normal p'-subgroup of G
$O^p(G)$	the smallest normal subgroup of G with $G/O^p(G)$ a p-group
$O^{p'}(G)$	the smallest normal subgroup of G with $G/O^{p'}(G)$ a p'-group
$\Omega_1(G)$	for a fixed prime p, the subgroup of G generated by elements of order p
$\mathrm{Syl}_p(G)$	set of Sylow p-subgroups of G
n_p	equals $\lvert \mathrm{Syl}_p(G) \rvert$
$m_p(G)$	the p-rank of G
$r_p(G)$	equals $\log_p(\lvert G \rvert_p)$
$A * B$	the free product of A and B
$G \curvearrowright X$	a group action of G on X (at right)
x^g	the element $g \in G$ acting on $x \in X$
$\mathcal{O}_x$	the orbit of $x \in X$
G_x	the stabilizer (or isotropy group) of $x \in X$
$\mathrm{Fix}_H(Y)$	the fixed point set of $H \subseteq G$ on $Y \subseteq X$
Y^G	the set $\{y^g : y \in Y, g \in G\}$ if $Y \subseteq X$
CFSG	Classification of the finite simple groups

Topological spaces

Let X and Y be two topological spaces.

$\mathbb{D}^n$	the unit disk in $\mathbb{R}^n$
$\mathbb{S}^n$	the unit sphere in $\mathbb{R}^{n+1}$
$X \simeq Y$	means X and Y are homotopy equivalent
$X \underset{w}{\approx} Y$	means there is a space Z together with weak equivalences $Z \to X$ and $Z \to Y$
$X \underset{\approx}{\to} Y$	weak equivalence from X to Y
X^n	the n-skeleton of a CW-complex X

Posets and simplicial complexes

Let X, Y be G-posets and K, L be G-simplicial complexes.

$\lvert K \rvert$	the geometric realization of K
$\mathcal{K}(X)$	the order complex of X
$\mathcal{X}(K)$	the face poset of K
$X^{(n)}$	the n-th subdvidision or derived poset of X
$K^{(n)}$	the n-th subdvidision of K
$\mathrm{Lk}(v, K)$	the link of a vertex v in K
$\mathrm{St}(v, K)$	the open star of a vertex v in K
$X_{>x}$	$= \{y \in X : y > x\}$
$X_{\geq x}$	$= \{y \in X : y \geq x\}$
F_x^Y	equals $X_{\geq x} \cap Y$ if $Y \subseteq X$
$\hat{F}_x^Y$	equals $X_{>x} \cap Y$ if $Y \subseteq X$
$X_{<x}$	$= \{y \in X : y < x\}$
$X_{\leq x}$	$= \{y \in X : y \leq x\}$
U_x^Y	equals $X_{\leq x} \cap Y$ if $Y \subseteq X$
$\hat{U}_x^Y$	equals $X_{<x} \cap Y$ if $Y \subseteq X$
f/y	$= \{x \in X : f(x) \leq y\}$ is the fiber of a map $f : X \to Y$ under y
f/y	$= \{x \in X : f(x) \geq y\}$ is the opposite fiber of a map $f : X \to Y$ under y
μ_X	McCord's map from $\lvert \mathcal{K}(X) \rvert$ to X
X^{op}	the poset X with the opposite order
$x \prec y$	the element $x \in X$ is covered by $y \in X$
$h(x)$	the height of $x \in X$
$h(X)$	the height of X
$\mathrm{Max}(X)$	the maximal elements of X
$\mathrm{Min}(X)$	the minimal elements of X
$X \searrow\searrow Y$	strong collapse from X to $Y \subseteq X$

$Y \nearrow\!\!\!\nearrow X$	strong expansion from $Y \subseteq X$ to X
$X \searrow\!\!\!\searrow^{G} Y$	strong G-equivariant collapse from X to $Y \subseteq X$
$X \searrow^{e} Y$	elementary simple collapse from X to $Y \subseteq X$
$Y \,{}^{e}\!\!\nearrow X$	elementary simple expansion from $Y \subseteq X$ to X
$X \searrow Y$	simple collapse from X to $Y \subseteq X$
$Y \nearrow X$	simple expansion from $Y \subseteq X$ to X
$X \diagup\!\!\searrow Y$	X and Y are simple homotopy equivalent
$X \searrow^{Ge} Y$	elementary simple G-equivariant collapse from X to $Y \subseteq X$
$X \searrow^{G} Y$	simple G-equivariant collapse from X to $Y \subseteq X$
$X \diagup\!\!\searrow^{G} Y$	X and Y are simple G-equivariant homotopy equivalent

Families of posets and simplicial complexes

$\mathcal{A}_p(G)$	the Quillen poset of nontrivial elementary abelian p-subgroups of G
$\mathcal{B}_p(G)$	the Bouc poset of nontrivial radical p-subgroups of G
$K_p(G)$	the commuting complex of G at p
$\mathcal{S}_p(G)$	the Brown poset of nontrivial p-subgroups of G
$\mathcal{R}_p(G)$	the Robinson subcomplex of $\mathcal{K}(\mathcal{S}_p(G))$ of chains $(P_0 < \ldots < P_n)$ such that $P_i \trianglelefteq P_n$ for all i
$X_p(G)$	the poset of $P \in \mathcal{S}_p(G)$ such that $P \trianglelefteq S$ if $S \in \mathrm{Syl}_p(G)$ and $P \leq S$

Bibliography

[Ale37] Pavel Sergeevich Alexandroff, *Diskrete räume*, Mat. Sb. (N.S.) **2** (1937), 501–518.

[AB79] J. Alperin and Michel Broué, *Local methods in block theory*, Ann. of Math. (2) **110** (1979), no. 1, 143–157.

[Asc00] M. Aschbacher, *Finite group theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 10, Cambridge University Press, Cambridge, 2000.

[Asc93] Michael Aschbacher, *Simple connectivity of p-group complexes*, Israel J. Math. **82** (1993), no. 1-3, 1–43.

[AKO11] Michael Aschbacher, Radha Kessar, and Bob Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011.

[AK90] Michael Aschbacher and Peter B. Kleidman, *On a conjecture of Quillen and a lemma of Robinson*, Arch. Math. (Basel) **55** (1990), no. 3, 209–217.

[AS92] Michael Aschbacher and Yoav Segev, *The uniqueness of groups of Lyons type*, J. Amer. Math. Soc. **5** (1992), no. 1, 75–98.

[AS93] Michael Aschbacher and Stephen D. Smith, *On Quillen's conjecture for the p-groups complex*, Ann. of Math. (2) **137** (1993), no. 3, 473–529.

[Bar11a] Jonathan A. Barmak, *Algebraic topology of finite topological spaces and applications*, Lecture Notes in Mathematics, vol. 2032, Springer, Heidelberg, 2011.

[Bar11b] Jonathan Ariel Barmak, *On Quillen's Theorem A for posets*, J. Combin. Theory Ser. A **118** (2011), no. 8, 2445–2453.

[BM08a] Jonathan Ariel Barmak and Elías Gabriel Minian, *One-point reductions of finite spaces, h-regular CW-complexes and collapsibility*, Algebr. Geom. Topol. **8** (2008), no. 3, 1763–1780.

[BM08b] Jonathan Ariel Barmak and Elías Gabriel Minian, *Simple homotopy types and finite spaces*, Adv. Math. **218** (2008), no. 1, 87–104.

[BM12a] Jonathan Ariel Barmak and Elías Gabriel Minian, *G-colorings of posets, coverings and presentations of the fundamental group*, arXiv e-prints (2012), arXiv:1212.6442.

[BM12b] Jonathan Ariel Barmak and Elías Gabriel Minian, *Strong homotopy types, nerves and collapses*, Discrete Comput. Geom. **47** (2012), no. 2, 301–328.

[Bjo03] Anders Bjorner, *Nerves, fibers and homotopy groups*, J. Combin. Theory Ser. A **102** (2003), no. 1, 88–93.

[Bou84] Serge Bouc, *Homologie de certains ensembles ordonnés*, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 2, 49–52.

[Bre67] Glen E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Mathematics, vol. 34, Springer-Verlag, Berlin-New York, 1967.

[Bre72] Glen E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York-London, 1972, Pure and Applied Mathematics, Vol. 46.

[BLO03] Carles Broto, Ran Levi, and Bob Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856.

[Bro75] Kenneth S. Brown, *Euler characteristics of groups: the p-fractional part*, Invent. Math. **29** (1975), no. 1, 1–5.

[Bro94] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original.

[Bro06] Ronald Brown, *Topology and groupoids*, BookSurge, LLC, Charleston, SC, 2006, Third edition of ıt Elements of modern topology[McGraw-Hill, New York, 1968; MR0227979],With 1 CD-ROM (Windows, Macintosh and UNIX).

[Bux99] Kai-Uwe Bux, *Orbit spaces of subgroup complexes, Morse theory, and a new proof of a conjecture of Webb*, Topology Proc. **24** (1999), no. Spring, 39–51.

[CD92] Carles Casacuberta and Warren Dicks, *On finite groups acting on acyclic complexes of dimension two*, Publicacions Matemàtiques (1992), 463–466.

[Cra11] David A. Craven, *The theory of fusion systems*, Cambridge Studies in Advanced Mathematics, vol. 131, Cambridge University Press, Cambridge, 2011, An algebraic approach.

[Das95] Kaustuv Mukul Das, *Simple connectivity of the Quillen complex of* $\mathrm{GL}_n(q)$, J. Algebra **178** (1995), no. 1, 239–263.

[Das98] Kaustuv Mukul Das, *Some results about the Quillen complex of* $\mathrm{Sp}_{2n}(q)$, J. Algebra **209** (1998), no. 2, 427–445.

[Das00] Kaustuv Mukul Das, *The Quillen complex of groups of symplectic type: the characteristic* 2 *case*, J. Algebra **223** (2000), no. 2, 556–561.

[DR16] Antonio Díaz Ramos, *On quillen's conjecture for p-solvable groups*, Journal of Algebra **513** (2016), 246–264.

[FPSC19] Ximena L. Fernández, Kevin Iván Piterman, and Iván Sadofschi Costa, Posets - *Posets and Finite Spaces, Version 1.0.0*, GAP package, https://github.com/isadofschi/posets, 2019.

[GAP18] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.9.3*, 2018.

[GL83] Daniel Gorenstein and Richard Lyons, *The local structure of finite groups of characteristic* 2 *type*, Mem. Amer. Math. Soc. **42** (1983), no. 276, vii+731.

[GLS96] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The Classification of the Finite Simple Groups. Number 2. Part I. Chapter G*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1996, General group theory.

[GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The Classification of the Finite Simple Groups. Number 3. Part I. Chapter A: Almost simple $\mathcal{K}$-groups*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998.

[GLS99] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The Classification of the Finite Simple Groups, Number 4. Part II, Chapters 1-4: Uniqueness theorems*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1999, General group theory.

[Gro16] Jesper Grodal, *Endotrivial modules for finite groups via homotopy theory*, arXiv e-prints (2016), arXiv:1608.00499.

[HI88] Trevor Hawkes and I. M. Isaacs, *On the poset of p-subgroups of a p-solvable group*, J. London Math. Soc. (2) **38** (1988), no. 1, 77–86.

[Isa08] I. Martin Isaacs, *Finite group theory*, Graduate Studies in Mathematics, vol. 92, American Mathematical Society, Providence, RI, 2008.

[KR89] Reinhard Knörr and Geoffrey R. Robinson, *Some remarks on a conjecture of Alperin*, J. London Math. Soc. (2) **39** (1989), no. 1, 48–60.

[Kot97] Sonja Kotlica, *Verification of Dade's conjecture for Janko group J_3*, J. Algebra **187** (1997), no. 2, 579–619.

[Kso03] Rached Ksontini, *Simple connectivity of the Quillen complex of the symmetric group*, J. Combin. Theory Ser. A **103** (2003), no. 2, 257–279.

[Kso04] Rached Ksontini, *The fundamental group of the Quillen complex of the symmetric group*, J. Algebra **282** (2004), no. 1, 33–57.

[Lib08] Assaf Libman, *Webb's conjecture for fusion systems*, Israel J. Math. **167** (2008), 141–154.

[Lin09] Markus Linckelmann, *The orbit space of a fusion system is contractible*, Proc. Lond. Math. Soc. (3) **98** (2009), no. 1, 191–216.

[McC66] Michael C. McCord, *Singular homology groups and homotopy groups of finite topological spaces*, Duke Math. J. **33** (1966), 465–474.

[MP18] Elías Gabriel Minian and Kevin Iván Piterman, *The homotopy types of the posets of p-subgroups of a finite group*, Adv. Math. **328** (2018), 1217–1233.

[MP19] Elías Gabriel Minian and Kevin Iván Piterman, *The fundamental group of the p-subgroup complex*, arXiv e-prints (2019), arXiv:1903.03549.

[OS02] Bob Oliver and Yoav Segev, *Fixed point free actions on $\mathbf{Z}$-acyclic 2-complexes*, Acta Math. **189** (2002), no. 2, 203–285.

[Pit16] Kevin Iván Piterman, *El tipo homotópico de los posets de p-subgrupos*, Tesis de Licenciatura, Departamento de Matemática, FCEyN, UBA. Available at http://cms.dm.uba.ar/academico/carreras/licenciatura/tesis/2016/Kevin_Piterman.pdf, 2016.

[Pit19] Kevin Iván Piterman, *A stronger reformulation of Webb's conjecture in terms of finite topological spaces*, J. Algebra **527** (2019), 280–305.

[PSV19] Kevin Iván Piterman, Iván Sadofschi Costa, and Antonio Viruel, *Acyclic 2-dimensional complexes and Quillen's conjecture*, Accepted in Publicacions Matemàtiques (2019).

[Pui06] Lluis Puig, *Frobenius categories*, J. Algebra **303** (2006), no. 1, 309–357.

[PW00] Jürgen Pulkus and Volkmar Welker, *On the homotopy type of the p-subgroup complex for finite solvable groups*, J. Austral. Math. Soc. Ser. A **69** (2000), no. 2, 212–228.

[Qui71] Daniel Quillen, *The spectrum of an equivariant cohomology ring. I, II*, Ann. of Math. (2) **94** (1971), 549–572; ibid. (2) 94 (1971), 573–602.

[Qui78] Daniel Quillen, *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Adv. in Math. **28** (1978), no. 2, 101–128.

[Rob88] Geoffrey R. Robinson, *Some remarks on permutation modules*, J. Algebra **118** (1988), no. 1, 46–62.

[Seg94] Yoav Segev, *Simply connected coset complexes for rank* 1 *groups of Lie type*, Math. Z. **217** (1994), no. 2, 199–214.

[Sei82] Gary M. Seitz, *Generation of finite groups of Lie type*, Trans. Amer. Math. Soc. **271** (1982), no. 2, 351–407.

[Sha04] John Shareshian, *Hypergraph matching complexes and Quillen complexes of symmetric groups*, J. Combin. Theory Ser. A **106** (2004), no. 2, 299–314.

[Smi11] Stephen D. Smith, *Subgroup complexes*, Mathematical Surveys and Monographs, vol. 179, American Mathematical Society, Providence, RI, 2011.

[Ste68] Robert Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968, Notes prepared by John Faulkner and Robert Wilson.

[Sto66] R. E. Stong, *Finite topological spaces*, Trans. Amer. Math. Soc. **123** (1966), 325–340.

[Sto84] R. E. Stong, *Group actions on finite spaces*, Discrete Math. **49** (1984), no. 1, 95–100.

[Sym98] Peter Symonds, *The orbit space of the p-subgroup complex is contractible*, Comment. Math. Helv. **73** (1998), no. 3, 400–405.

[TW91] J. Thévenaz and P. J. Webb, *Homotopy equivalence of posets with a group action*, J. Combin. Theory Ser. A **56** (1991), no. 2, 173–181.

[Thé92] Jacques Thévenaz, *On a conjecture of Webb*, Arch. Math. (Basel) **58** (1992), no. 2, 105–109.

[tD87] Tammo tom Dieck, *Transformation groups*, De Gruyter Studies in Mathematics, vol. 8, Walter de Gruyter & Co., Berlin, 1987.

[UY02] Katsuhiro Uno and Satoshi Yoshiara, *Dade's conjecture for the simple O'Nan group*, J. Algebra **249** (2002), no. 1, 147–185.

[Wal69] John H. Walter, *The characterization of finite groups with abelian Sylow* 2*-subgroups*, Ann. of Math. (2) **89** (1969), 405–514.

[Web87] P. J. Webb, *Subgroup complexes*, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), Proc. Sympos. Pure Math., vol. 47, Amer. Math. Soc., Providence, RI, 1987, pp. 349–365.