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Variedades combinatorias no homogéneas y dualidad de Alexander

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Resumen

En esta Tesis introducimos la teoría de NH-variedades, una extensión de la teoría clásica de variedades combinatorias al contexto no homogéneo. Las NH-variedades poseen una estructura local que consiste en versiones simpliciales de espacios euclídeos de distintas dimensiones, lo que les confiere propiedades muy parecidas a las de las variedades usuales. Nuestro trabajo permite extender los resultados principales de la teoría clásica de variedades a una clase mucho más amplia de espacios; entre estos resultados, el teorema de expansiones regulares de Alexander y la existencia de entornos regulares. A lo largo de esta Tesis exhibimos muchos ejemplos de espacios que forman parte de esta teoría pero no están incluidos en la teoría clásica. Introducimos también la noción de *shelling* no homogéneo y caracterizamos todas las NH-variedades shellables en el sentido de Björner y Wachs. La teoría de NH-variedades puede aplicarse al estudio de variedades clásicas y lo exhibimos en el caso concreto de la factorización de operaciones simpliciales entre variedades combinatorias (*starrings*, *shellings* y *bistellar moves*). En particular, se muestra que dos variedades son PL-homeomorfas si y sólo si pueden relacionarse por medio de NH-factorizaciones involucrando una sucesión de NH-variedades.

En la segunda parte del trabajo analizamos la relación entre la teoría clásica y la no homogénea en el contexto de la dualidad de Alexander combinatoria. Estudiamos el dual de Alexander de las bolas y esferas combinatorias y mostramos que los doble duales de estos complejos son NH-bolas y NH-esferas, las versiones no homogéneas de las bolas y esferas clásicas. Además, definimos la noción de NH-bola y NH-esfera *minimal*, bolas y esferas no puras que satisfacen una condición de minimalidad en la cantidad de símplices maximales. Las NH-bolas y NH-esferas minimales caracterizan completamente la clase del simplex y del borde del simplex en la relación de equivalencia generada por tomar dual de Alexander. Uno de los resultados principales de este trabajo es la generalización al contexto no homogéneo de los resultados de Dong y Santos-Sturmfels sobre el tipo homotópico del dual de Alexander de las bolas y esferas combinatorias: el dual de Alexander de una NH-bola es un espacio contráctil y el dual de Alexander de una NH-esfera es homotópicamente equivalente a una esfera. Nuestra generalización muestra que el tipo homotópico del dual de Alexander es preservado para una clase mucho más amplia de espacios que los contemplados en los resultados originales de Dong y Santos-Sturmfels. Por ejemplo, incluye todas las NH-bolas y NH-esferas exhibidas (explícita o implícitamente) en esta Tesis.

Palabras clave. Complejos simpliciales, variedades combinatorias, NH-variedades, dualidad de Alexander, shellabilidad no homogénea, dual de Alexander.

Non-homogeneous combinatorial manifolds and Alexander duality

Abstract

In this Thesis we introduce the theory of NH-manifolds, an extension of the classical theory of combinatorial manifolds to the non-homogeneous setting. NH-manifolds have a local structure consisting of simplicial versions of euclidean spaces of different dimensions, giving them features very similar to those of polyhedral manifolds. In our work we extend the main results from the classical theory of manifolds to a larger class of spaces; among these results, the Alexander's theorem on regular expansions and the existence of regular neighbourhoods. Throughout this Thesis we present many examples of spaces which are part of this theory but are not included in the classical theory. We also introduce the notion of *non pure shelling* and characterize all shellable NH-manifolds in the sense of Björner and Wachs. The theory of NH-manifolds can be applied to the study of classical manifolds. This is exhibited in the concrete case of factorization of simplicial moves between combinatorial manifolds (*starrings*, *shellings* and *bistellar moves*). In particular, it is shown that two manifolds are PL-homeomorphic if and only if they are related through NH-factorizations involving a sequence of NH-manifolds.

In the second part of this work we analyze the relation between the classical and the non-pure theory within the context of combinatorial Alexander duality. We study the Alexander dual of combinatorial balls and spheres and show that the *double duals* of these classes of complexes are NH-balls and NH-spheres, the non-homogeneous versions of classical balls and spheres. We also define the notion of *minimal* NH-ball and NH-sphere, a subfamily of non-pure manifolds satisfying a minimality condition on the number of maximal simplices. Minimal NH-balls and NH-spheres completely characterize the class of the simplex and the boundary of the simplex in the equivalence relation generated by taking Alexander dual. One of the main results of this Dissertation is a generalization to the non-homogeneous setting of the results of Dong and Santos-Sturmfels on the homotopy type of the Alexander dual of combinatorial balls and spheres: the Alexander dual of an NH-ball is a contractible space and the Alexander dual of an NH-sphere is homotopically equivalent to a sphere. Our generalization shows that the homotopy type of the Alexander dual is preserved for a far larger class of spaces than the original results by Dong and Santos-Sturmfels. For example, all NH-balls and NH-spheres exhibited (explicitly or implicitly) in this Thesis are included.

Key words. Simplicial complexes, combinatorial manifolds, NH-manifolds, Alexander duality, non-pure shellability, Alexander duals.

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“Two roads diverged in a wood, and I...
I took the one less traveled by,
and that has made all the difference.”*
John Keating

*From “The Road Not Taken” by Robert Frost.

Introducción

La teoría de variedades es un tema central de la Topología y la Geometría, y el uso de triangulaciones nos permite atacar problemas de naturaleza topológica y geométrica por medio de herramientas combinatorias (o simpliciales). El estudio de las triangulaciones fue un tópico muy importante a comienzos del 1900 y algunas conjeturas fundamentales como la *Hauptvermutung*, que preguntaba si dos triangulaciones del mismo espacio eran combinatoriamente equivalentes, y la *Triangulation Conjecture*, preguntando si toda variedad topológica compacta podía ser triangulada, ocuparon la mente de topólogos y geómetras a lo largo de todo el siglo. Ambas conjeturas resultaron falsas, pero estos problemas, junto con la Conjetura de Poincaré (ahora un Teorema), inspiraron a topólogos algebraicos y diferenciales.

Una de las más potentes teorías de triangulación es la de *variedades combinatorias (o poliedrales)*, que constituyen las versiones simpliciales de las variedades topológicas. Una *variedad* PL (de dimensión d) es un poliedro en el que cada punto posee un entorno que es homeomorfo lineal a trozos (PL-homeomorfo) a una d-bola de $\mathbb{R}^d$. Una variedad combinatoria es una triangulación (fija) de una variedad PL. Es sabido desde los años cuarenta que las variedades diferenciables pueden triangularse por variedades combinatorias, por lo que un especial interés fue puesto en el estudio de estos complejos, principalmente impulsado por la búsqueda de la solución de algunos problemas abiertos, como la Conjetura (generalizada) de Poincaré. La teoría de variedades combinatorias ha sido desarrollada por más de noventa años y resulta central en el estudio de muchos problemas clásicos, jugando un papel principal en el desarrollo de la Topología Algebraica.

Una *d-bola combinatoria* es un complejo que posee una subdivisión en común con un d-simplex Δ^d y una *d-esfera combinatoria* es un complejo que posee una subdivisión en común con el borde $\partial\Delta^{d+1}$ de un $(d+1)$-simplex. Una *d-variedad combinatoria (o poliedral)* es un complejo cuyos vértices poseen un entorno simplicial (*star*) que es una d-bola combinatoria. Más convenientemente, es un complejo M tal que el *link* $lk(v, M) = \{\sigma \in M \,|\, v * \sigma \in M\}$ de cada vértice $v \in M$ es una $(d-1)$-bola combinatoria o una $(d-1)$-esfera combinatoria (aquí $*$ denota el *join* de los dos símplices; es decir, el simplex generado por los vértices de ambos símplices). Con esta última definición, uno puede distinguir entre vértices en el interior y vértices en el borde de la variedad. Puede probarse que las d-bolas y d-esferas combinatorias son un tipo especial de d-variedades combinatorias. Las variedades poliedrales tienen muchas propiedades estructurales, como la regularidad de los símplices (el star de todo simplex es una d-bola combinatoria), la invariancia por subdivisiones y la fuerte conexión (en el caso conexo). De especial interés para esta teoría son los conceptos de *colapsos* y *expansiones* simpliciales, introducidos por J.H.C. Whitehead en los años treinta con el objetivo de estudiar las deformaciones continuas de espacios por medio de métodos simpliciales. En este contexto, Whitehead desarrolló la teoría de entornos regulares, que permitió usar herramientas de variedades combinatorias en el estudio de complejos generales. Entre los más importantes resultados de la teoría de variedades poliedrales mencionamos el Teorema de Alexander sobre expansiones regulares (Teorema 1.3.2), el Teorema de Newman sobre el complemento de una d-bola en una d-esfera (Teorema 1.2.6) y un teorema de Whitehead que afirma que las d-variedades combinatorias colapsables son d-bolas combinatorias (Teorema 1.3.1). Este último resultado fue un gran avance hacia la caracterización de espacios a través de sus propiedades homotópicas (el interrogante central en la Conjetura de Poincaré).

Las d-variedades combinatorias son complejos homogéneos (o puros) de dimensión d; esto es, todos sus símplices maximales tienen la misma dimensión d. Los espacios más familiares están triangulados por este tipo de complejos, como las curvas y superficies (singulares), las variedades homológicas y los polítopos. Una propiedad distintiva de los complejos homogéneos de dimensión d es que poseen una noción natural de *borde*, que resulta un complejo homogéneo de dimensión $d-1$. En particular, los símplices del borde de una variedad poliedral son precisamente aquellos cuyos links son bolas combinatorias, y el borde de una d-variedad combinatoria es una $(d-1)$-variedad combinatoria sin borde, como es de esperarse.

El primer propósito de esta Tesis es desarrollar la teoría de *variedades combinatorias no-homogéneas*, que nosotros llamaremos *NH-variedades*. Esta es una clase de complejos no necesariamente puros con una configuración local regular, con buenas propiedades estructurales y satisfaciendo muchos de los resultados (generalizados) fuertes de la teoría de variedades PL, como las versiones no homogéneas del Teorema de Alexander sobre expansiones regulares y la existencia de entornos regulares. En particular, nuestro trabajo permite extender los resultados principales de la teoría clásica de variedades a una clase mucho más amplia de espacios. A lo largo de esta Tesis exhibimos muchos ejemplos de espacios (en muchos casos de manera gráfica para bajas dimensiones) que forman parte de esta teoría pero no están incluidos en la teoría clásica. El estudio de la NH-variedades fue en parte inspirado por el trabajo de Björner y Wachs [7] sobre shellabilidad en complejos no puros y las variedades no puras surgen naturalmente en muchos contextos de la teoría clásica, mayormente evidenciado en la factorización de movimientos simpliciales entre variedades (*starrings* y *welds*, *shellings* y *stellar exchanges*). Sin embargo, es dentro del contexto de la dualidad de Alexander que esta teoría adquiere una relevancia fundamental. Mostramos en la segunda parte de este trabajo que las NH-bolas y NH-esferas, las versiones no puras de las bolas y esferas combinatorias, aparecen como *duales de Alexander* (representantes simpliciales del complemento del espacio) de bolas y esferas clásicas. Por un lado, definimos las NH-bolas y NH-esferas *minimales*, una familia de complejos que satisface una condición de minimalidad en el número de símplices maximales, y probamos que caracterizan completamente las clases de Δ^d y $\partial\Delta^{d+1}$ en la relación de equivalencia generada por $K \sim K^*$, donde K^* denota el dual de Alexander de K. Por otro lado, mostramos que las NH-esferas (resp. NH-bolas) son los *doble duales* (el espacio complementario en esferas de mayor dimensión del complemento) de las esferas (resp. bolas) combinatorias clásicas. Más aún, como consecuencia de la estructura regular de las NH-variedades, y como uno de los resultados principales de este trabajo, probamos una generalización al contexto no homogéneo de un teorema de Dong y Santos-Sturmfels [22, 49] sobre el tipo homotópico del dual de Alexander de bolas y esferas simpliciales: el dual de Alexander de una NH-bola es un espacio contráctil y el dual de Alexander de una NH-esfera es homotópicamente equivalente a una esfera. Esto no es válido en general para complejos arbitrarios. Nuestra generalización muestra que el tipo homotópico del dual de Alexander es preservado para una clase mucho más amplia de espacios que los contemplados en los resultados originales de Dong y Santos-Sturmfels. Por ejemplo, incluye todas las NH-bolas y NH-esferas exhibidas (explícita o implícitamente) en esta Tesis.

Casi todos los resultados en este trabajo son nuevos. Con la excepción de los contenidos del Capítulo 1 y §3.1, ambos con definiciones preliminares y resultados previos, el resto de esta Tesis es trabajo original. Algunos de los resultados aquí presentados aparecen en nuestros artículos [15, 16, 17].

La primera parte de este trabajo está dedicado a proveer las definiciones fundacionales de la teoría de variedades no homogéneas y a establecer los resultados básicos fundamentales de esta clase de triangulaciones. Las NH-variedades son complejos con una estructura local que consiste en versiones simpliciales de espacios euclídeos de variadas dimensiones. Esto les confiere propiedades muy similares a la de las variedades combinatorias pero sin forzar la homogeneidad. Un tipo especial de variedades no homogéneas son las NH-bolas y NH-esferas, las versiones no necesariamente puras de las bolas y esferas combinatorias respectivamente, las cuales juegan un rol fundamental en la teoría. Las NH-variedades, NH-bolas y NH-esferas se definen en conjunto por medio de un argumento inductivo.

Definición. Una *NH-variedad* (resp. *NH-bola*, *NH-esfera*) de dimensión 0 es una colección finita de vértices (resp. un vértice, dos vértices). Una NH-esfera dimensión -1 es, por convención, $\{\emptyset\}$. Para $d \geq 1$, definimos por inducción

- Una *NH-variedad de dimensión d* es un complejo M de dimensión d tal que $lk(v, M)$ es una NH-bola de dimensión $0 \leq k \leq d-1$ o una NH-esfera de dimensión $-1 \leq k \leq d-1$ para todo vértice $v \in M$.

- Una *NH-bola de dimensión d* es una NH-variedad B de dimensión d colapsable; esto es, existe una subdivisión de B que colapsa simplicialmente a un vértice.

- Una *NH-esfera de dimensión d y dimensión homotópica k* es una NH-variedad S de dimensión d tal que existe una NH-bola $B \subset S$ de dimensión d y una k-bola combinatoria $L \subset S$, ambos subcomplejos generados por símplices maximales de S, tales que $B + L = S$ y $B \cap L = \partial L$. Llamamos $S = B + L$ una *descomposición* de S.

La idea detrás de la definición de variedad no pura es introducir una mínima (y necesaria) modificación en la estructura local de las variedades PL para alcanzar la no homogeneidad. No es difícil ver que esta teoría se genera (inductivamente) a partir del "primer" ejemplo no trivial de esfera no pura: la unión de una 1-bola combinatoria y un vértice aislado (ver Figura 1). Por otro lado, la noción de NH-bola y NH-esfera está inspirada en los Teoremas de Whitehead y Newman (Teoremas 1.2.6 and 1.3.1).

Bajo estas propiedades definitorias, las NH-variedades (resp. NH-bolas, NH-esferas) homogéneas son precisamente las variedades (resp. bolas, esferas) combinatorias (Teorema 2.1.2). La invariancia por subdivisión de todas estas clases (Teorema 2.1.6) y la regularidad de los símplices en una variedad no pura (Proposition 2.1.3) puede establecerse luego de unas pocas observaciones técnicas sobre starrings elementales. Más aún, el resultado esperado acerca del join simplicial entre bolas y esferas no homogéneas se satisface (Teorema 2.1.9). Finalmente, como una característica propia de las variedades PL, puede mostrarse que las NH-variedades poseen las propiedades (generalizadas) de las

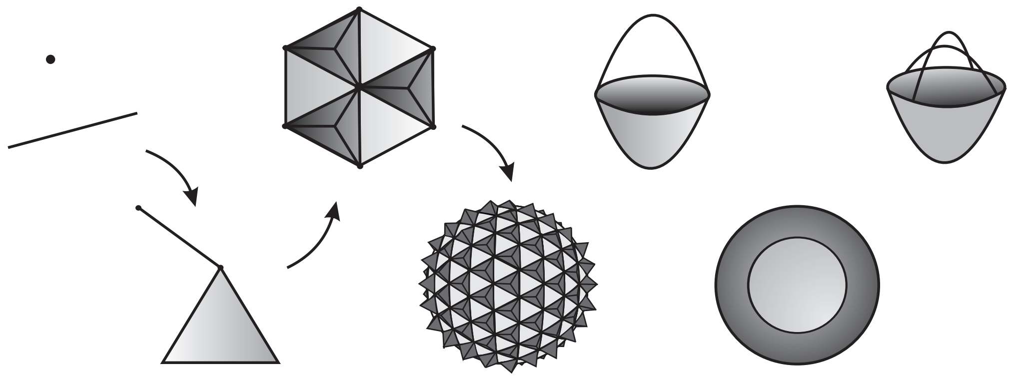


Figure 1: Ejemplos de NH-variedades, NH-bolas y NH-esferas.

pseudo variedades: regularidad en codimensión 1 y fuerte conexión (Lema 2.1.10). Esto en particular conduce a la noción de *pseudo variedad no homogénea*.

Siguiendo la misma caracterización de los símplices del borde de las variedades combinatorias, nos apoyamos en la estructura de las NH-variedades para definir una noción de borde en el contexto no puro. El *pseudo borde* $\tilde{\partial} M$ de una NH-variedad M es la colección de símplices cuyos links son NH-bolas. El pseudo borde no es generalmente un complejo. El *borde* ∂M es el complejo generado por $\tilde{\partial} M$ (el complejo obtenido al agregar las caras de los símplices en el pseudo borde). Además de proveer una noción de borde para una clase de espacios no homogéneos, estos conceptos juegan un rol esencial en la generalización de algunos teoremas clásicos de variedades con borde. Con esta definición, puede mostrarse que las NH-variedades poseen *spines* (es decir, colapsan a un complejo de dimensión menor).

El concepto de bola y esfera no homogénea puede generalizarse al de NH-bouquet. Un *NH-bouquet* de índice 0 es una NH-bola e, inductivamente, un NH-bouquet G de índice k es una NH-variedad tal que existe un NH-bouquet $S \subset G$ de índice $k-1$ y una bola combinatoria $L \subset G$, ambos subcomplejos generados por símplices maximales de G, tales que $S + L = G$ y $S \cap L = \partial L$. En particular, los NH-bouquets de índice 1 son las NH-esferas. La razón por la cual uno puede proceder con esta generalización es que, contrario al caso homogéneo, las esferas no puras sí poseen borde. La importancia de los NH-bouquets reside en el hecho que caracterizan a las NH-variedades shellables (Teorema 2.3.4). Como mencionamos anteriormente, el estudio de la shellabilidad en el contexto no homogéneo, introducido por Björner and Wachs [7] en los años noventa, fue en parte lo que motivó el desarrollo de la teoría de NH-variedades.

Uno de los resultados fundacionales de la teoría de NH-variedades es la versión no homogénea del Teorema de Alexander sobre expansiones regulares. Una expansión regular (clásica) en una d-variedad combinatoria M consiste en la unión de una d-bola combinatoria B ($M \to M+B$) de manera tal que $M \cap B = \partial B \cap \partial M$ es una $(d-1)$-bola combinatoria. El Teorema de Alexander afirma que el resultado de una expansión regular es nuevamente una d-variedad combinatoria; más aún, es isomorfa lineal a trozos (PL-isomorfa) a la variedad original (ver Teorema 1.3.2). La teoría no pura admite una manera mucho más general de expandir una NH-variedad (que la simple restricción al contacto sobre una bola combinatoria), que nosotros llamamos *expansión regular no homogénea*

Teorema 2.4.3. *Sea M una NH-variedad y B^r una r-bola combinatoria. Supongamos que $M \cap B^r \subseteq \partial B^r$ es una NH-bola o una NH-esfera generada por caras propias maximales* *de símplices maximales de M o B^r y que $(M \cap B^r)^\circ \subseteq \tilde{\partial} M$. Entonces $M + B^r$ es una NH-variedad. Más aún, si M es un NH-bouquet de índice k y $M \cap B^r \neq \emptyset$ para $r \neq 0$, entonces $M + B^r$ es un NH-bouquet of índice k (si $M \cap B^r$ es una NH-bola) o $k+1$ (si $M \cap B^r$ es una NH-esfera).*

Las implicaciones de este resultado son muchas y el Teorema 2.4.3 aparece virtualmente en la demostración de los principales teoremas de la teoría. Como una primera aplicación, definimos la noción de *shelling no puro*, que consiste en una expansión regular no homogénea que involucra un único simplex. Con esta definición, la shellabilidad no pura de las NH-variedades es equivalente a la existencia de una sucesión de shellings (no puros) elementales que terminan en un simplex (Corolario 2.4.5), al igual que sucede para variedades clásicas.

Una expansión regular no homogénea produce una nueva NH-variedad que podría no ser PL-isomorfa a la original. Esta discrepancia con el Teorema de Alexander clásico reside en la generalidad de la expansión (permitiendo que $M \cap B$ sea más general) y en el hecho que la equivalencia por subdivisiones es un requerimiento muy fuerte en el contexto no homogéneo. Es por esto que una versión más precisa (y menos general) del Teorema de Alexander para NH-variedades se obtiene al restringir $M \cap B$ al contacto regular sobre una bola combinatoria e introduciendo una nueva relación de equivalencia entre NH-variedades llamada *NH-equivalencia*: $M \simeq_{NH} M'$ si M y M' están relacionados por una sucesión de shellings no homogéneos (directos e inversos) e isomorfismos simpliciales (Teorema 2.4.7). Esta equivalencia está inspirada en un reconocido teorema de Pachner [44], quien probó que dos variedades combinatorias con borde son PL-isomorfas si y sólo si están relacionadas por shellings directos e inversos clásicos (ver Teorema 1.4.1).

La teoría de NH-variedades tiene un alcance potencialmente amplio. Como un ejemplo de sus implicaciones, establecemos versiones equivalentes para variedades no homogéneas de dos elementos clásicos de la teoría de variedades: los entornos regulares y las ecuaciones de Dehn-Sommerville. Por un lado, puede mostrarse que cualquier subcomplejo K de una NH-variedad M posee un entorno regular; esto es, una subvariedad no homogénea $N \subset M$ tal que $K \subset N$ y N colapsa a K (Teorema 2.5.4). Por otro lado, las NH-variedades satisfacen una versión general de las ecuaciones de Dehn-Sommerville que involucran el pseudo borde de la variedad. En particular, satisfacen la versión módulo 2 de las ecuaciones clásicas de Dehn-Sommerville (Teorema 2.5.6). Todas esta generalizaciones evidencian que la estructura de NH-variedades es lo suficientemente rígida para permitir el desarrollo de tales teorías en este contexto más general.

Como una aplicación de la teoría de NH-variedades, mostramos que las variedades no puras aparecen naturalmente en factorizaciones de movimientos simpliciales entre variedades. Puede mostrarse que *starrings* y *welds* elementales (los movimientos de Alexander) y *shellings* y *stellar exchanges* (una generalización de *bistellar flip*) pueden factorizarse en pasos intermedios que involucran NH-variedades. Estas factorizaciones están basadas en las nociones de *conings* y *NH-factorizaciones*. Los *Conings* son movimientos que localmente expanden una región de la variedad tomando un cono sobre un subcomplejo propio de la misma. Con esta noción, *starrings* y *welds* pueden ser factorizados usando *conings* y *shellings* no puros, donde cada complejo involucrado en la factorización es una NH-variedad (Teorema 2.6.5). Por otro lado, las NH-factorizaciones proveen una manera de factorizar movimientos estelares, capturando la deformación implícita que tiene lugar en la transformación producida por este movimiento. Una NH-factorización relaciona dos variedades por una NH-variedad intermedia, por lo que las variedades poliedrales están implícitamente relacionadas a través de variedades no homogéneas (Corolario 2.6.9).

Además, las NH-factorizaciones realizadas en el borde de la variedad M describen naturalmente a los shellings en M en términos de *bistellar moves* sobre ∂M (Teorema 2.6.11).

La segunda parte de esta Tesis está dedicada a la aplicación de la teoría de variedades no homogéneas al estudio de los duales de Alexander de bolas y esferas. En esta área proporcionamos soluciones a preguntas que sólo pueden ser contestadas por la teoría de NH-variedades. La dualidad de Alexander es un resultado clásico que relaciona la homología de una subespacio A de la esfera d-dimensional S^d con la cohomología de su complemento $S^d - A$. Cuando el espacio A es triangulable, la dualidad de Alexander admite una formulación puramente combinatoria que involucra un representante simplicial (homotópico) A^* de $S^d - A$ llamado el *dual de Alexander* de A: para un conjunto base de vértices V que contiene los vértices de la triangulación de A se tiene $H_i(A^*) = H^{n-i-3}(A)$, donde n es el cardinal de V y los grupos de homología y cohomología son reducidos.

La relación entre las propiedades topológicas de un complejo y su dual no va en general más allá que las proporcionadas por la dualidad de Alexander. Por ejemplo, el dual de Alexander de una variedad PL no tiene estructura especial alguna. Por otro lado, el dual puede comportarse mal respecto del tipo homotópico: hay ejemplos de espacios contráctiles cuyo dual de Alexander no es contráctil y de esferas homotópicas cuyo dual de Alexander no tiene el tipo homotópico de una esfera (aún cuando en ambos casos tienen su homología respectiva). Sin embargo, en relación con esta situación, en 2002 Dong [22] usó la teoría de proyecciones de polítopos para probar que el dual de Alexander de esferas simpliciales es homotópicamente equivalente a una esfera y, un año más tarde, Santos y Sturmfels [49] mostraron que el dual de Alexander de una bola simplicial es un espacio contráctil, poniendo en evidencia que para espacios regulares los tipos homotópicos se preservan bajo dualidad.

Las NH-bolas y NH-esferas aparecen naturalmente como la noción dual a las bolas y esferas clásicas dentro del contexto de la dualidad de Alexander combinatoria. Esto se ve evidenciado a través de los tres resultados previamente mencionados: por un lado, las bolas y esferas no homogéneas son los *doble duales* de Alexander de las bolas y esferas combinatorias; por otro lado, las NH-bolas y NH-esferas *minimales* caracterizan completamente la clase de Δ^d y $\partial\Delta^d$ en la relación de equivalencia generada por $K \sim K^*$; finalmente, la estructura local de las NH-bolas y NH-esferas permiten extender los resultados de Dong y Santos-Sturmfels a una familia mucho más amplia de poliedros: el dual de Alexander de una NH-bola es un espacio contráctil y el dual de Alexander de una NH-esfera es homotópicamente equivalente a una esfera.

A continuación, precisamos estos resultados. Para un conjunto base de vértices V que contiene a los vértices V_K del complejo K, el *dual de Alexander* de K (relativo a V) es el complejo $K^{*_V} = \{\sigma \in \Delta(V) \,|\, \Delta(V - V_\sigma) \notin K\}$, donde $\Delta(\mathsf{X})$ denota el simplex generado por los vértices en el conjunto de vértices X. La dualidad queda evidenciada en el hecho que $(K^{*_V})^{*_V} = K$. Si $\tau = \Delta(V - V_K)$ escribimos $K^\tau := K^{*_V}$, manteniendo la notación K^* para $K^{*_{V_K}}$. Como primer paso, estudiamos la relación entre el dual de Alexander relativo a diferentes conjuntos base de vértices y proveemos un fórmula que es de esencial relevancia para este trabajo (ver Lema 3.2.1): $K^\tau = \partial\tau * \Delta_K + \tau * K^*$.

Los resultados de Dong y Santos-Sturmfels muestran que un complejo con una configuración estable permite trasladar propiedades topológicas al complejo dual. Sin embargo, los argumentos en las demostraciones originales de estos teoremas no recaen en la estructura local de las bolas y las esferas, sino en la convexidad. En este trabajo proporcionamos una demostración original completamente alternativa de los teoremas de Dong y Santos-Sturmfels inspirada en la estructura local de las variedades, en contraste con los tratamientos anteriores (Teorema 3.3.4). Este enfoque está basado en la relación elemental entre *links* y *deletion* de vértices en el dual de Alexander de un complejo y en propiedades básicas de las variedades combinatorias. También proporcionamos dos nuevas demostraciones del resultado original de Dong para esferas politopales: una de ellas, aplicando los teoremas clásicos de Ewald y Shephard [24] y Pachner [42] de la teoría de polítopos (Teoremas 3.3.6 y 3.3.7); la otra, usando la teoría de complejos *vertex-decomposables*.

Los resultados principales de la segunda parte de esta Tesis son las siguientes generalizaciones de los teoremas de Dong and Santos-Sturmfels sobre el dual de Alexander de las bolas y esferas simpliciales.

Teorema 4.3.1. *El dual de Alexander de una NH-bola es un espacio contráctil.*

Teorema 4.3.8. *El dual de Alexander de una NH-esfera es homotópicamente equivalente a una esfera.*

Al igual que nuestra demostración alternativa de los resultados originales de Dong y Santos-Sturmfels, nuestro enfoque está basado en la naturaleza local de las NH-variedades.

Para conjuntos de vértices $V' \supset V \supset V_K$, el complejo $(K^{*_V})^{*_{V'}}$ se llama un *doble dual* de K. Geométricamente, representa el complemento de un complejo visto como subespacio de esferas de distintas dimensiones; esto es, al espacio $S^{d'} - (S^d - A)$ para $A \subset S^d \subsetneq S^{d'}$. Los doble duales no son en general similares al complejo original. Sin embargo, comparten muchas de sus propiedades, como la shellabilidad. Para bolas y esferas no homogéneas, probamos la siguiente propiedad distintiva.

Teorema 4.1.3. *K es una NH-bola (resp. NH-esfera) si y sólo si $(K^{*_V})^{*_{V'}}$ es una NH-bola (resp. NH-esfera).*

Esto es, las bolas y esferas no homogéneas son clases cerradas bajo doble dualidad. En particular se tiene el siguiente

Corolario 4.1.4. *Las NH-bolas son los doble duales de las bolas combinatorias. Las NH-esferas son los doble duales de las esferas combinatorias.*

Una conexión mucho más fuerte entre bolas y esferas puras y no puras está presente al considerar ejemplos con mínima cantidad de vértices. Un complejo simplicial K de dimensión d es *vertex-minimal* si es un d-simplex o tiene $d+2$ vértices. Las d-bolas vertex-minimales son exactamente los *starrings* elementales de un d-simplex, mientras que la única d-esfera vertex-minimal es $\partial\Delta^{d+1}$. La versión no homogénea de estos espacios son las NH-bolas y NH-esferas *minimales*, las cuales satisfacen una condición de minimalidad en la cantidad de símplices maximales (una propiedad que es estrictamente más fuerte que ser vertex-minimal en el contexto no puro). Una NH-esfera S es *minimal* is el número de símplices maximales es $\dim_h(S)+2$, donde $\dim_h(S)$ es la dimensión homotópica de S. Una NH-bola es *minimal* si es parte de una descomposición $S = B + L$ de una NH-esfera minimal S. Tanto las NH-bolas minimales como las NH-esferas minimales son complejos vertex-minimales (Proposiciones 4.2.3 y 4.2.8). La característica más extraordinaria de estos espacios es que su familia es cerrada bajo la acción de tomar dual de Alexander (Teoremas 4.2.6 y 4.2.11). Esta propiedad puede utilizarse para mostrar que esta subclase de variedades no homogéneas caracteriza completamente las clases de Δ^d y $\partial\Delta^{d+1}$ en la relación de equivalencia generada por $K \sim K^*$. Esto queda contemplado en el siguiente

Teorema 4.2.1. *Para un complejo K, sea $K^{*(0)} := K$ y $K^{*(m)} := (K^{*(m-1)})^*$.*

(i) Existe $m \in \mathbb{N}_0$ tal que $K^{*(m)} = \partial\Delta^d$ si y sólo si K es una NH-esfera minimal.

(ii) Existe $m \in \mathbb{N}_0$ tal que $K^{*(m)} = \Delta^d$ si y sólo si K es una NH-bola minimal.

El Teorema 4.2.1 (i) (resp. (ii)) caracteriza todos los complejos cuya sucesión de *duales de Alexander iterados* $(K^{*(m)})_{m\in\mathbb{N}}$ converge a una esfera (resp. bola) vertex-minimal. El Teorema 4.2.1 es usado para calcular que la cantidad de NH-bolas y NH-esferas minimales de dimensión d es 2^d en cada caso; siendo la cantidad de NH-esferas minimales de dimensión homotópica k igual a $\binom{d}{k}$ (Proposición 4.2.16).

La Tesis está organizada como sigue. En el Capítulo 1 introducimos las definiciones básicas de la teoría de complejos simpliciales y damos un rápido repaso de la teoría de variedades combinatorias (incluyendo una sección sobre colapsos y expansiones simpliciales y una sección sobre movimientos simpliciales entre variedades). También damos un breve repaso de homología simplicial.

El Capítulo 2 está dedicado a asentar las bases de la teoría de NH-variedades. En las primeras dos secciones introducimos las definiciones y propiedades básicas de esta clase de complejos. En §2.3 introducimos los NH-bouquets y caracterizamos las NH-variedades shellables y §2.4 esta dedicada a probar el teorema de expansiones regulares no homogéneas, uno de los resultados principales de la primera parte de este trabajo. En §2.5 desarrollamos versiones para NH-variedades de la teoría de entornos regulares y de las ecuaciones de Denh-Sommerville y en §2.6 estudiamos como los movimientos simpliciales clásicos entre variedades pueden factorizarse por medio de pasos intermedios que involucran NH-variedades.

En el Capítulo 3 recordamos el Teorema de Dualidad de Alexander y esbozamos la demostración combinatoria que aparece en [4, 6]. En §3.2 estudiamos la relación entre los duales de Alexander de un complejo relativos a diferentes conjuntos base de vértices y en §3.3 damos la demostración alternativa de los resultados de Dong y Santos-Sturmfels que se basa en la estructura local de las variedades combinatorias. En §3.3.2 y §3.3.3 presentamos dos nuevas demostraciones del resultado de Dong para esferas politopales basado en elementos clásicos de la teoría de polítopos.

El Capítulo 4 contiene los resultados más fuertes de este trabajo. En §4.1 establecemos que las clases de NH-bolas y NH-esferas son cerradas bajo doble duales y probamos que las bolas y esferas no homogéneas son naturalmente los doble duales de las bolas y esferas clásicas. En §4.2 introducimos la teoría de NH-bolas y NH-esferas minimales y probamos que esta subclase de NH-variedades son cerradas bajo la acción de tomar dual de Alexander. Esto conduce a la caracterización de la clase del simplex y del borde del simplex en la relación de equivalencia generada por tomar dual de Alexander. Finalmente, §4.3 está dedicado a probar la generalización de los resultados de Dong y Santos-Sturmfels al contexto no homogéneo.

Introduction

The theory of manifolds is a central subject in Topology and Geometry, and the use of triangulations allows us to attack problems of topological and geometric nature by means of combinatorial (or simplicial) tools. The study of triangulations was a very important topic at the beginnings of 1900 and some fundamental conjectures such as the *Hauptvermutung*, asking if two triangulations of a same space are combinatorially equivalent, and the *Triangulation Conjecture*, asking whether any compact topological manifold can be triangulated, occupied the mind of topologists and geometers throughout the entire century. Both conjectures turned out to be false, but these problems together with Poincaré Conjecture (now a Theorem), inspired algebraic and differential topologists.

One of the most powerful triangulation theories is that of *combinatorial (or polyhedral) manifolds*, which are the simplicial versions of topological manifolds. A *PL-manifold* (of dimension d) is a polyhedron in which every point has a neighborhood piece-wise linear homeomorphic to a d-ball of $\mathbb{R}^d$. A combinatorial manifold is a (fixed) triangulation of a PL-manifold. Since the forties it is known that differentiable manifolds can be triangulated by combinatorial manifolds, so special interest was placed in studying these complexes, mainly impulsed by the quest to solve some open problems, like the (generalized) Poincaré Conjecture. The theory of combinatorial manifolds has been developed for over ninety years and it is central in the study of many classical problems, playing a significant role in the development of Algebraic Topology.

A *combinatorial d-ball* is a complex which has a subdivision in common with a d-simplex Δ^d and a *combinatorial d-sphere* has a subdivision in common with the boundary $\partial\Delta^{d+1}$ of a $(d+1)$-simplex. A *combinatorial* (or *polyhedral*) *d-manifold* is a complex whose vertices have a simplicial neighborhood (*star*) which is a combinatorial d-ball. More conveniently, it is a complex M such that the *link* $lk(v, M) = \{\sigma \in M \mid v * \sigma \in M\}$ of every vertex $v \in M$ is either a combinatorial $(d-1)$-ball or $(d-1)$-sphere (here $*$ denotes the *join* of the two simplices; i.e. the simplex spanned by the vertices of both simplices). With this last definition, one may distinguish between vertices in the interior and in the boundary of the manifold. It can be seen that combinatorial d-balls and d-spheres are actually a special type of combinatorial manifolds. Polyhedral manifolds have many structural properties, such as regularity of simplices (the star of every simplex is a combinatorial d-ball), invariance under subdivision and strong connectivity (in the connected case). Of special interest to this theory are the concepts of simplicial *collapse* and *expansion* introduced by J.H.C. Whitehead in the thirties in order to study continuous deformations of spaces by simplicial methods. In this setting, he developed the theory of regular neighborhoods which helped to use tools from combinatorial manifold theory to study general complexes. Among the most important results we mention Alexander's Theorem on regular expansions (Theorem 1.3.2), Newman's Theorem on the complement of a d-ball inside a d-sphere (Theorem 1.2.6) and a theorem of Whitehead's stating that a collapsible combinatorial d-manifold is actually a combinatorial d-ball (Theorem 1.3.1). This last result was a huge step towards characterizing spaces via their homotopic properties (the central interrogative in Poincaré Conjecture).

Combinatorial d-manifolds are d-homogeneous (or pure) complexes; that is, all of its maximal simplices have the same dimension d. Most familiar spaces are triangulated by such complexes, such as (singular) curves and surfaces, homology manifolds and polytopes. A distinctive property of d-homogeneous complexes is a natural notion of *boundary*, which is a homogeneous complex of dimension $d-1$. In particular, the boundary simplices in a polyhedral manifold are precisely those whose link is a combinatorial ball and the boundary complex of a combinatorial d-manifold is a combinatorial $(d-1)$-manifold without boundary, as expected.

The first purpose of this Thesis is to develop the theory of *non-homogeneous combinatorial manifolds*, which we shall call NH*-manifolds*. This is a class of non-necessarily pure complexes with a regular local configuration, possessing good structural properties and fulfilling many (generalized) strong results of PL-manifold theory, such as non-homogeneous versions of Alexander's Theorem on regular expansions and the existence of regular neighborhoods. In particular, our work extends the main results from the classical theory of manifolds to a larger class of spaces. Throughout this Thesis we present many examples of spaces (in many cases graphically for low dimensions) which are part of this theory but are not included in the classical theory. The study of non-homogeneous manifolds was in part inspired by the work of Björner and Wachs [7] on non-pure shellability and NH-manifolds arise naturally in many contexts of the classical theory, most evidently in the factorization of classical simplicial moves between manifolds (*starrings* and *welds*, *shellings* and *stellar exchanges*). However, it is within the context of Alexander duality that this theory acquires a major significance. It is shown in the second part of this work that NH-balls and NH-spheres, the non-pure versions of combinatorial balls and spheres, appear as the *Alexander dual* (a simplicial representative of the complement of the space) of classical balls and spheres. On one hand, we define *minimal* NH-balls and NH-spheres, families of non-pure manifolds which satisfy a minimality condition on the number of maximal simplices, and prove that they completely characterize the classes of Δ^d and $\partial\Delta^{d+1}$ in the equivalence relation generated by $K \sim K^*$, where K^* denotes the Alexander dual of K. On the other hand, we show that NH-spheres (resp. NH-balls) are the *double duals* (the complementary spaces in higher dimensional spheres of the complement) of classical combinatorial spheres (resp. balls). Furthermore, as a consequence of the regular structure of NH-manifolds, and as one of the main results of this Dissertation, we prove a generalization to the non-pure setting of a theorem by Dong and Santos-Sturmfels [22, 49] on the homotopy type of the Alexander dual of simplicial balls and spheres: the Alexander dual of an NH-ball is a contractible space and the Alexander dual of an NH-sphere is homotopy equivalent to a sphere. This does not hold in general for arbitrary complexes. Our generalization shows that the homotopy type of the Alexander dual is preserved for a far larger class of spaces than the original results by Dong and Santos-Sturmfels. For example, all NH-balls and NH-spheres exhibited (explicitly or implicitly) in this Thesis are included.

Almost all results in this Thesis are new. With the exception of the contents of Chapter 1 and §3.1, both with preliminary definitions and previous results, the rest of this Dissertation is original work. Some of the results presented here appear in our articles [15, 16, 17].

The first part of this work is devoted to provide the foundational definitions of the theory of non-pure manifolds and to establish the ground fundamental results of this class of triangulations. NH-manifolds are complexes with a local structure consisting in simplicial versions of euclidean spaces of varying dimensions. This endows them with properties very similar to those of combinatorial manifolds but without forcing homogeneity. A special type of non-homogeneous manifold are the NH-balls and NH-spheres, a not-necessarily-pure version of combinatorial balls and spheres respectively, which play a fundamental role in the theory. NH-manifolds, NH-balls and NH-spheres are defined together via an inductive argument.

Definition. An *NH-manifold* (resp. *NH-ball*, *NH-sphere*) of dimension 0 is any finite collection of vertices (resp. one vertex, two vertices). An NH-sphere of dimension -1 is, by convention, $\{\emptyset\}$. For $d \geq 1$, we define by induction

- An *NH-manifold of dimension d* is a complex M of dimension d such that $lk(v, M)$ is an NH-ball of dimension $0 \leq k \leq d-1$ or an NH-sphere of dimension $-1 \leq k \leq d-1$ for all vertices $v \in M$.
- An *NH-ball of dimension d* is a collapsible NH-manifold B of dimension d; that is, there is a subdivision of B which simplicially collapses to a single vertex.
- An *NH-sphere of dimension d and homotopy dimension k* is an NH-manifold S of dimension d such that there exists an NH-ball $B \subset S$ of dimension d and a combinatorial k-ball $L \subset S$, both subcomplexes generated by maximal simplices of S, such that $B + L = S$ and $B \cap L = \partial L$. We call $S = B + L$ a *decomposition* of S.

The idea behind the definition of non-pure manifold is to introduce a minimal (required) modification in the local structure of PL-manifolds to attain no-homogeneity. This theory is easily shown to span (by induction) from the "first" non-trivial non-pure sphere: the union of a combinatorial 1-ball and an isolated vertex (see Figure 2). On the other hand, the notion of NH-ball and NH-sphere is inspired in Whitehead's and Newman's Theorems (Theorems 1.2.6 and 1.3.1).

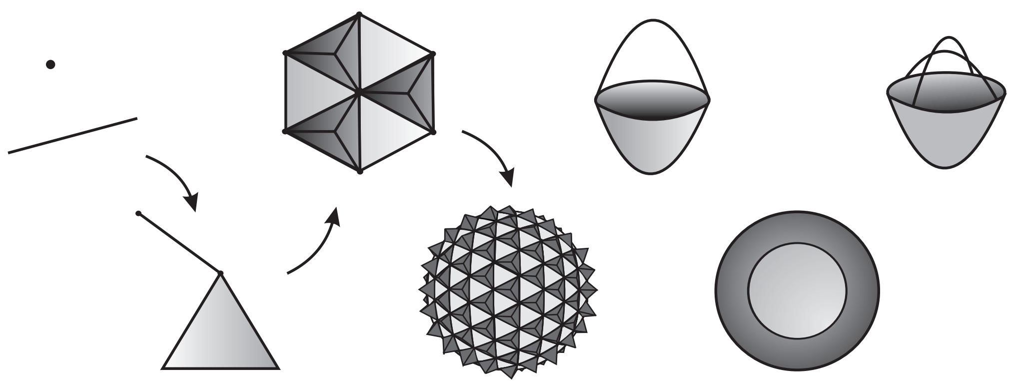


Figure 2: Examples of NH-manifolds, NH-balls and NH-spheres.

Under these defining features, homogeneous NH-manifolds (resp. NH-balls, NH-spheres) are readily seen to be combinatorial manifolds (resp. balls, spheres) (Theorem 2.1.2). The invariance under subdivision of all of these classes (Theorem 2.1.6) and the regularity of the simplices in a non-pure manifold (Proposition 2.1.3) may be settled after a few technical remarks on elementary starrings. Furthermore, the expected outcome of the simplicial join between non-pure balls and spheres is satisfied (Theorem 2.1.9). As a final property inherent to PL-manifolds, it can be seen that NH-manifolds have the (generalized) properties of pseudo manifolds: regularity in codimension 1 and strong connectivity (Lemma 2.1.10). This in particular leads to the notion of *non-homogeneous pseudo manifold*.

Following the same characterization of boundary simplices in combinatorial manifolds, we lean on the structure of NH-manifolds to define a notion of boundary in the non-pure setting. The *pseudo boundary* $\tilde{\partial} M$ of an NH-manifold M is the collection of simplices whose links are NH-balls (which is generally not a complex); the *boundary* ∂M is the complex generated by $\tilde{\partial} M$. Apart from providing a notion of boundary for a class of non-homogeneous spaces, these concepts play an essential role in the generalization of some classical theorems of manifolds with boundary. With this definition, it can be proved that NH-manifolds have *spines* (i.e. they collapse to a complex of smaller dimension).

The concept of non-pure ball and sphere may be further generalize to that of NH-bouquet. An NH-bouquet of index 0 is an NH-ball and, inductively, an NH-bouquet G of index k is an NH-manifold such that there exist an NH-bouquet $S \subset G$ of index $k-1$ and a combinatorial ball $L \subset G$, both subcomplexes generated by maximal simplices of G, such that $S + L = G$ and $S \cap L = \partial L$. Thus, NH-bouquets of index 1 are NH-spheres. The reason one may proceed with this generalization is that, as opposed to the pure situation, non-homogeneous spheres do have boundary simplices. The importance of NH-bouquets lies in the fact that they characterize shellable NH-manifolds (Theorem 2.3.4). As mentioned, the study of shellability in the non-pure context, introduced by Björner and Wachs [7] in the nineties, was in part what motivated the development of the theory of NH-manifolds.

One of the foundational results in NH-manifold theory is the non-pure version of Alexander's Theorem on regular expansions. A (classical) regular expansion on a combinatorial d-manifold M is the adding $M \to M + B$ of a combinatorial d-ball B in such a way that $M \cap B = \partial B \cap \partial M$ is a combinatorial $(d-1)$-ball. Alexander Theorem asserts that the result of a regular expansion is again a combinatorial d-manifold; moreover, it is piece-wise linear isomorphic to the original one (see Theorem 1.3.2). The non-pure theory admits a far more general way to expand an NH-manifold (than restricting to a contact over a combinatorial ball) which we call *non-homogeneous regular expansion*.

Theorem 2.4.3. *Let M be an NH-manifold and B^r a combinatorial r-ball. Suppose $M \cap B^r \subseteq \partial B^r$ is an NH-ball or an NH-sphere generated by ridges of M or B^r and that $(M \cap B^r)^\circ \subseteq \tilde{\partial} M$. Then $M + B^r$ is an NH-manifold. Moreover, if M is an NH-bouquet of index k and $M \cap B^r \neq \emptyset$ for $r \neq 0$, then $M + B^r$ is an NH-bouquet of index k (if $M \cap B^r$ is an NH-ball) or $k+1$ (if $M \cap B^r$ is an NH-sphere).*

The implications of this result are many and Theorem 2.4.3 appears virtually in the proof of the main theorems of the theory. As a first application, we define the notion of *non-pure shelling* as a non-homogeneous regular expansion involving a single simplex. With this definition, shellability of NH-manifolds is shown to be equivalent to the existence of a sequence of elementary shellings leading to a single simplex (Corollary 2.4.5), just as for classical manifolds.

A non-homogeneous regular expansion produces a new NH-manifold which may not be piece-wise linear isomorphic to the original one. This discrepancy with the classic Alexander Theorem lies in the more general type of the expansion (based on the contact complex $M \cap B$) and on the fact that piece-wise linear equivalence is a too strong requirement for the non-pure context. Thus, a more precise (and less general) version of Alexander's Theorem for NH-manifolds is attained by restricting $M \cap B$ to standard contact over a combinatorial ball and by introducing a new equivalence relation between NH-manifolds called *NH-equivalence*: $M \simeq_{NH} M'$ if M and M' are related by a sequence of non-homogeneous (direct and inverse) shellings and simplicial isomorphisms (Theorem 2.4.7). This equivalence is inspired in a celebrated theorem of Pachner [44], who proved that two combinatorial manifolds with boundary are piece-wise linear isomorphic if and only if they are related by (standard) direct and inverse shellings (see Theorem 1.4.1).

The theory of NH-manifolds has a potentially ample scope. As an example of its implications, we establish equivalent versions for non-pure manifolds of two classic elements of manifold theory: regular neighborhoods and the Dehn-Sommerville equations. On one hand, it can be shown that every subcomplex K of an NH-manifold M has a regular neighborhood; that is, a non-pure submanifold $N \subset M$ such that $K \subset N$ and N collapses to K (Theorem 2.5.4). On the other hand, NH-manifolds satisfy a general version of the Dehn-Sommerville equations which involves the pseudo boundary of the manifold. In particular, it satisfies a mod 2 version of the classic Dehn-Sommerville equations (Theorem 2.5.6). All these generalizations show that the structure of NH-manifolds is rigid enough to allow such strong theories to hold in this wider setting.

As an evidence of the interaction between the pure and non-pure theories, we show that non-homogeneous manifolds appear naturally in factorizations of simplicial moves between manifolds: *elementary starrings* and *welds* (Alexander subdivision moves) and *direct* and *inverse shellings* and *stellar exchanges* (a generalization of *bistellar flips*) can be factorized in intermediate steps involving NH-manifolds. These factorizations are based on the notions of *conings* and *NH-factorizations*. *Conings* are moves that locally expand a region of the manifold by taking a cone over a proper subcomplex. With this notion, starrings and welds may be factorized using conings and non-pure shellings where every complex involved in the factorization is an NH-manifold (Theorem 2.6.5). On the other hand, NH-factorizations are a way to factorize stellar exchanges by capturing the implicit deformation that takes place in the transformation produced by this move. An NH-factorization relates two manifolds by an intermediate NH-manifold, so polyhedral manifolds are also implicitly related through non-pure manifolds in this way (Corollary 2.6.9). Additionally, NH-factorizations performed on the boundary of a manifold M naturally describe inverse and direct shellings in M in terms of (internal) bistellar moves on ∂M (Theorem 2.6.11).

The second part of this Thesis is devoted to applying the theory of non-pure manifolds to study the Alexander dual of balls and spheres. In this area we provide solutions to questions that can only be answered by elements of NH-manifold theory. Alexander duality is a classical result relating the homology of a subspace A of the d-dimensional sphere S^d with the cohomology of its complement $S^d - A$. When the space A is triangulable, Alexander duality admits a purely combinatorial formulation involving a simplicial (homotopy) representative A^* of $S^d - A$ called the *Alexander dual* of A: for a ground set of vertices V containing the vertices of the triangulation of A we have $H_i(A^*) = H^{n-i-3}(A)$, where n is the cardinal of V and both homology and cohomology groups are reduced.

The relation between the topological properties of a complex and its dual goes in general no further than the one given by Alexander duality. For example, the Alexander dual of a PL-manifold has no special structure. On the other hand, it can even be homotopically bad-behaved: there are examples of contractible complexes whose Alexander dual is no contractible and of homotopy spheres whose Alexander dual does not have the homotopy type of a sphere (even though they have their respective homology). However, regarding this matter, in 2002 Dong [22] used the theory of projection of polytopes to prove that the Alexander duals of simplicial spheres are homotopy equivalent to spheres and, a year later, Santos and Sturmfels [49] showed that the Alexander dual of a simplicial ball is a contractible space, evidencing that for regular spaces the homotopy type is preserved under duality.

NH-balls and NH-spheres appear as natural dual notion to (classical) balls and spheres when seen in the context of combinatorial Alexander duality. This is reflected in the three results mentioned above: on one hand, non-homogeneous balls and spheres are the Alexander *double duals* of combinatorial balls and spheres; on the other hand *minimal* NH-balls and NH-spheres completely characterize the class of Δ^d and $\partial\Delta^d$ in the equivalence relation generated by $K \sim K^*$; finally, the local structure of NH-manifolds makes possible to extend the results of Dong and Santos-Sturmfels to the family of NH-balls and NH-spheres.

Let us now precise these results. For a ground set of vertices V containing the vertices V_K of the complex K, the *Alexander dual* of K (relative to V) is the complex $K^{*_V} = \{\sigma \in \Delta(V) \,|\, \Delta(V - V_\sigma) \notin K\}$, where $\Delta(\mathsf{X})$ denotes the simplex spanned by the vertices in a vertex set X. Duality is reflected in the fact that $(K^{*_V})^{*_V} = K$. If $\tau = \Delta(V - V_K)$ we write $K^\tau := K^{*_V}$, keeping the notation K^* for $K^{*_{V_K}}$. As a first step, we study the relation between the Alexander dual relative to different ground set of vertices and provide a formula which is of essential relevance to this work (see Lemma 3.2.1): $K^\tau = \partial\tau * \Delta_K + \tau * K^*$.

Dong's and Santos-Sturmfels' results show that a complex with a stable configuration permits to translate topological properties to the dual complex. However, the arguments in the original proofs of these theorems do not lie on the local structure of balls and spheres, but on convexity arguments. A completely alternative proof of Dong's and Santos-Sturmfels' original results is produced in this work inspired on the local structure of manifolds, in contrast to the previous treatments (Theorem 3.3.4). This approach is based on an elementary relation between links and deletion of vertices in the Alexander dual of a complex and basic properties of combinatorial manifolds. We also provide two new proofs of Dong's original result for polytopal spheres: one of then, applying the classical theorems of Ewald and Shephard [24] and Pachner [42] from the theory of polytopes (Theorems 3.3.6 and 3.3.7), and the other one, based on the theory of vertex-decomposable complexes.

The main results of the second part of the Thesis are the following generalizations of Dong's and Santos-Sturmfels' theorems on the Alexander dual of balls and spheres.

Theorem 4.3.1. *The Alexander dual of an NH-ball is contractible.*

Theorem 4.3.8. *The Alexander dual of an NH-sphere is homotopy equivalent to a sphere.*

As our proof of the original version of these results, our approach is based on the local nature of the NH-manifolds.

For vertex sets $V' \supset V \supset V_K$, the complex $(K^{*_V})^{*_{V'}}$ is called a *double dual* of K. It geometrically amounts to the complement of a complex when seen as subspace of spheres of different dimensions; that is, the space $S^{d'} - (S^d - A)$ for $A \subset S^d \subsetneq S^{d'}$. Double duals need not be similar to the original complex; however, they share many of its properties (shellability is an example). For non-pure balls and spheres we have the following distinctive property.

Theorem 4.1.3. *K is an NH-ball (resp. NH-sphere) if and only if $(K^{*_V})^{*_{V'}}$ is an NH-ball (resp. NH-sphere).*

Thus, non-homogeneous balls and spheres are closed classes under double duality. In particular, we have

Corollary 4.1.4. *NH-balls are the double duals of combinatorial balls. NH-spheres are the double duals of combinatorial spheres.*

A far stronger connection between pure and non-pure balls and spheres is present when considering the vertex-minimal case. A simplicial complex K of dimension d is *vertex-minimal* if it is a d-simplex or it has $d+2$ vertices. Vertex-minimal d-balls are the elementary starrings of a d-simplex, while the only vertex-minimal d-sphere is $\partial\Delta^{d+1}$. The non-pure version of these spaces are *minimal* NH-balls and NH-spheres, which satisfy a minimality condition on the number of maximal simplices (a property strictly stronger than vertex-minimality in this setting). An NH-sphere S is *minimal* if the number of maximal simplices is $\dim_h(S)+2$, where $dim_h(S)$ is the homotopy dimension of S. An NH-ball is *minimal* if it is part of a decomposition $S = B + L$ of a minimal NH-sphere S. Both minimal NH-balls and NH-spheres are shown to be vertex-minimal complexes (Propositions 4.2.3 and 4.2.8). The most remarkable feature of these spaces is that their family is closed under taking Alexander dual (Theorems 4.2.6 and 4.2.11). This property may be used to show that these subclasses of non-pure manifolds completely characterize the classes of Δ^d and $\partial\Delta^{d+1}$ in the equivalence relation generated by $K \sim K^*$. This is contemplated in the following

Theorem 4.2.1. *For a complex K, let $K^{*(0)} = K$ and $K^{*(m)} = (K^{*(m-1)})^*$.*

(i) There is an $m \in \mathbb{N}_0$ such that $K^{*(m)} = \partial\Delta^d$ if and only if K is a minimal NH-sphere.

(ii) There is an $m \in \mathbb{N}_0$ such that $K^{*(m)} = \Delta^d$ if and only if K is a minimal NH-ball.

Theorem 4.2.1 (i) (resp. (ii)) characterizes all complexes whose sequence of *iterated Alexander duals* $(K^{*(m)})_{m\in\mathbb{N}}$ converge to vertex-minimal spheres (resp. balls). Theorem 4.2.1 is used to show that the number of d-dimensional minimal NH-balls and NH-spheres is 2^d, of which the number of minimal NH-spheres with homotopy dimension k is $\binom{d}{k}$ (Proposition 4.2.16).

The Thesis is organized as follows. In Chapter 1 we introduce the basic definitions of the theory of simplicial complexes and give a quick overview of the theory of combinatorial manifolds (including a section on simplicial collapses and expansions and a section on simplicial moves between manifolds). We also give a brief review of simplicial homology theory.

Chapter 2 is devoted to laying the foundations of the theory of NH-manifolds. In the first two sections we introduce the basic definitions and properties of these classes of complexes. In §2.3 we introduce NH-bouquets and characterize shellable NH-manifolds and §2.4 is dedicated to prove the theorem on non-homogeneous regular expansions, one of the key results of the first part of this work. In §2.5 we develop versions for NH-manifolds of regular neighborhood's theory and the Dehn-Sommerville equations and in §2.6 we study how classical simplicial moves between manifolds may be factorized via intermediate steps involving NH-manifolds.

In Chapter 3 we recall the Alexander Duality Theorem and sketch the combinatorial proof which appears in [4, 6]. In §3.2 we study the relationship between the Alexander duals of a complex relative to different ground sets of vertices and in §3.3 we provide the alternative proof of Dong's and Santos-Sturmfels' results which relies on the local structure of combinatorial manifolds. In §3.3.2 and §3.3.3 we present two new proofs of Dong's result for polytopal spheres based on elements of classical polytopes theory.

Chapter 4 contains the strongest results of this work. In §4.1 we establish that the class of NH-balls and NH-spheres is closed under taking double dual and prove that non-pure balls and spheres are the natural double duals of classical balls and spheres. In §4.2 we introduce the theory of minimal NH-balls and NH-spheres and prove that these subclasses of NH-manifolds are (independently) closed under taking Alexander dual. This leads to the characterization of the class of the simplex and the boundary of the simplex in the equivalence class generated by taking Alexander dual. Finally, §4.3 is devoted to prove the generalization of Dong's and Santos-Sturmfels' results to the non-homogeneous setting.

Contents

Chapter 1

Preliminaries

In this chapter we recall the basic definitions and notations we shall be using throughout this work and give a general overview of the theory of combinatorial manifolds. For more details, we refer the reader to the classical texts [2, 25, 36, 40, 47].

§1.1 Simplicial complexes

Combinatorial methods are a fundamental tool to study many geometric and topological problems of manifolds and general spaces. The most classical approach is based on providing a decomposition of the space (triangulation) in a finite collection of "nicely-assembled" convex blocks (simplices) and use this discrete description to obtain information about the topological properties of the space. The form and structure of this decomposition is contained in a *simplicial complex*, a purely combinatorial entity.

We recall first the basic notions of the theory of simplicial complexes. An (abstract) *simplicial complex* K is a pair (V_K, S_K) where V_K is a set and S_K is a collection of subsets of V_K such that every subset in V_K with a single element belongs to S_K and every subset of a member of S_K is also a member of S_K. The set V_K is the *vertex set* of K and the set S_K the *set of simplices*.

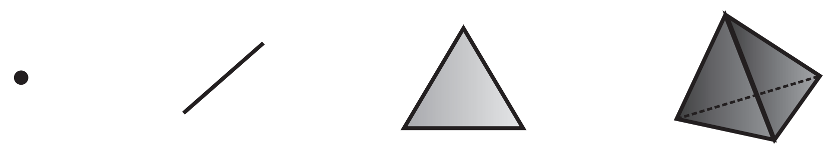
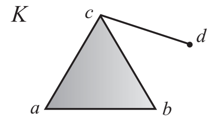


Figure 1.1: Low dimensional simplices.

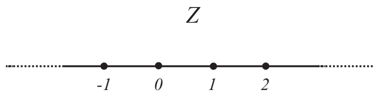
Simplicial complexes model a decomposition of topological spaces in the following way. Simplices are pictured geometrically as points, line segments, triangles, tetrahedra, and their high-dimensional analogues (see Figure 1.1); and the way two of these blocks fit together is determined by the common subset of vertices between them. We shall name this representation a *geometric simplex* (a concept that will be made rigorous in following paragraphs).

EXAMPLE 1.1.1. Let K be the simplicial simplex with vertex set $V_K = \{a, b, c, d\}$ and simplices $S_K = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}\}$. This complex models the following space.



EXAMPLE 1.1.2. A single simplex σ can be seen as a complex whose simplices are the power set of its vertex set.

EXAMPLE 1.1.3. We denote by Z the simplicial complex with $V_Z = \mathbb{Z}$ (the integers) and $S_Z = \{\{n\} \,|\, n \in \mathbb{Z}\} \cup \{\{n, n+1\} \,|\, n \in \mathbb{Z}\}$. The *geometric realization* of the simplicial complex Z is the real line.



A simplex $\sigma \in K$ with $d+1$ vertices is said to have dimension d and it is called a d-*simplex*. Note that 0-simplices are the vertices of K. The *dimension* $\dim(K)$ of K is the maximum, if it exists, of the dimension of its simplices. Also, if $\sigma \subset \tau$ then we say that σ is a *face* of τ and we write $\sigma < \tau$. Thus, two geometric simplices intersect always on a common face. When $\dim(\sigma) = \dim(\tau) - 1$ we say that σ is an *immediate* face of τ and write $\sigma \prec \tau$. A *principal* or *maximal* simplex in K is a simplex which is not a proper face of any other simplex of K and a *ridge* in K is an immediate face of a maximal simplex. By the definition of simplicial complex, principal simplexes are sufficient to describe completely any simplicial complex. Also, since every vertex in K is a simplex we shall from now on identify K with its set of simplices.

Two complexes require some comments: $\emptyset$ and $\{\emptyset\}$. On one hand, $\emptyset$ is the complex with absolutely no simplices and $\{\emptyset\}$ is the complex whose solely simplex is the empty simplex. Although these complexes have no importance from the topological point of view, they are useful from the combinatorial viewpoint. Note that $\dim(\{\emptyset\}) = -1$.

A *subcomplex* $L \subset K$ is a subcollection of simplices of K which contains all the faces of all its members. Note that any simplex $\sigma \in K$, together with all its faces, determines a subcomplex of K that is also denoted by σ. The set of all simplices of K of dimension less than or equal to k is a subcomplex called the k-skeleton of K. We denote the union of two complexes K, L as $K + L$. The *join* $\sigma * \tau$ of two non-empty non-intersecting simplices σ, τ is the simplex spanned by $V_\sigma \cup V_\tau$. By convention, $\sigma * \emptyset = \sigma$. The join of two complexes K, L is the complex $K * L = \{\sigma * \tau \,|\, \sigma \in K\,, \tau \in L\}$. In particular, $K * \{\emptyset\} = K$ and $K * \emptyset = \emptyset$.

We recall the main two constructions associated to the notion of neighborhood in the simplicial setting: the *link* of σ in K is the subcomplex $lk(\sigma, K) = \{\tau \in K : \ \tau \cap \sigma = \emptyset,\ \tau * \sigma \in K\}$ and the *star* of σ in K the subcomplex $st(\sigma, K) = \sigma * lk(\sigma, K) = \{\eta \in K \,|\, \eta < \tau \text{ and } \sigma < \tau\}$. This constructions are of key importance to investigate the local properties of complexes. Note that for a complex K and a simplex σ we have that $lk(\sigma, K) = \{\emptyset\}$ if and only if $\sigma \in K$ is a principal simplex and that $lk(\sigma, K) = \emptyset$ if and only if $\sigma \notin K$.

Remark 1.1.4. A simple computation shows that

$$lk(\sigma * \tau, K) = lk(\sigma, lk(\tau, K)) = lk(\tau, lk(\sigma, K)).$$

This basic identity will be used frequently from now on.

A *simplicial map* $f : K \to L$ between complexes is an application $V_K \to V_L$ such that $f(\sigma)$ is a simplex of L for every $\sigma \in K$. If K and L are (simplicially) *isomorphic* we write $K \equiv L$.

Convention. *In this Thesis we shall be working exclusively with finite complexes; that is, V_K is a finite set. In particular, all complexes considered are finite-dimensional.*

We next formalize the idea about how geometric simplices effectively decompose a (triangulable) space. The *geometric realization* $|K|$ of a simplicial complex K is the topological space consisting of the set of functions $\alpha : V_K \to [0,1]$ such that

- $supp(\alpha) := \{v \in V_K \,|\, \alpha(v) > 0\}$ is a simplex of K.
- $\sum_{v \in V_K} \alpha(v) = 1$,

with the metric $d(\alpha,\beta) = (\sum_{v \in V_\sigma} (\alpha(v) - \beta(v))^2)^{\frac{1}{2}}$. It is a standard result that if $\dim(K) = d$ then $|K|$ can be embedded in $\mathbb{R}^{2d+1}$. In particular, $|K|$ is a metric space with the standard (inherited) euclidean metric. For $\sigma \in K$ with $V_\sigma = \{v_0, \dots, v_k\}$, the space $|\sigma|$ is the subspace of $|K|$ formed by the functions α such that $supp(\alpha) \subset V_\sigma$. When seen inside a euclidean space, $|\sigma|$ can also be identify with the convex hull of $k+1$ points $p_0, \dots, p_k$ in general position by the natural homeomorphism sending $\alpha \in |\sigma|$ to $\sum_{i=0}^{k} \alpha(v_i) p_i$. This is the rigorous notion of *geometric simplex*.

We shall frequently identify the simplicial complex K with its geometric realization, so we shall often speak about complexes which are homeomorphic or homotopy equivalent. Recall that two maps $f, g : X \to Y$ between topological spaces are *homotopic*, written $f \simeq g$, if there is a mapping $H : X \times I \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$; i.e. if the image of f may be continuously deformed into the image of g. X and Y are said to be *homotopy equivalent* (or to *have the same homotopy type*) if there are mappings $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. This says that X can be continuously deformed into Y and we write $X \simeq Y$.

Not every topological space may be described by a simplicial complex, but the most commonly used can, and they form the class of *Polyhedra*. A *polyhedron* X is a topological space such that there exists a complex K such that X is homeomorphic to $|K|$. A choice of such a complex is called a *triangulation* of the space X. The study of triangulations of topological spaces was one of the most important topics at the beginnings of the 1900's. Many attempts were undertaken on the first half of the century to determine the existence and equivalence of triangulations of manifolds and general spaces. Two of the most important open problems at that time were the *Triangulation Conjecture*, asking whether every compact topological manifold could be triangulated, and the *Hauptvermutung*, asking if two triangulation of a same space were combinatorially equivalent; that is, if the simplicial description of the space is equivalent to the topological one. It is known since the forties that C^∞ manifolds are (not only) triangulable (but also) by a combinatorial manifold [13, 53], and a positive answer to the Hauptvermutung would traduce the problem of the study of the topology of these spaces to the combinatorial setting. However, the Hauptvermutung was proven false by Kirby and Siebenmann in the late sixties [30] and a counterexample to the Triangulation Conjecture was produced recently by Manolescu [38].

Different triangulations of the same space can be described in terms of subdivision of complexes. A complex K is a *subdivision* of a complex L if $|K| = |L|$ and every geometric simplex $|\sigma| \in |K|$ is contained in a geometric simplex of $|L|$. We shall consider three types of subdivisions in this work: arbitrary subdivisions, derived subdivisions and stellar subdivisions. A *derived subdivision* of a complex K, denoted by δK, is the complex having one vertex v_σ for each simplex $\sigma \in K$ and as simplices the sets $\{v_{\eta_1}, \ldots, v_{\eta_k}\}$ for $\eta_1 < \cdots < \eta_k$ in K. Geometrically, a derived subdivision consists in choosing a point a_σ in the interior of $|\sigma|$ and then replacing each simplex $\tau \in K$ with $a_\tau * \partial\tau$ in order of increasing dimension. A special (important) case of a derived subdivision is the *barycentric subdivision* K' of K, where each a_σ is taken to be $\hat{\sigma}$, the barycenter of σ. Of course, from the combinatorial viewpoint, there is not distinction between derived or barycentric subdivision. Finally, stellar subdivisions are based on the concept of *elementary starring* of a simplex. If $\sigma \in K$ and $a \notin K$, then $(\sigma, a)K$ is the subdivision of K obtained by locally replacing $st(\sigma, K)$ with $a * \partial\sigma * lk(\sigma, K)$. A *stellar subdivision* sK of K is a finite sequence of these elementary starrings. The operation inverse to an elementary starring is called an *elementary weld* and is denoted by $(\sigma, a)^{-1}K$. It is not hard to see that derived subdivisions are stellar.

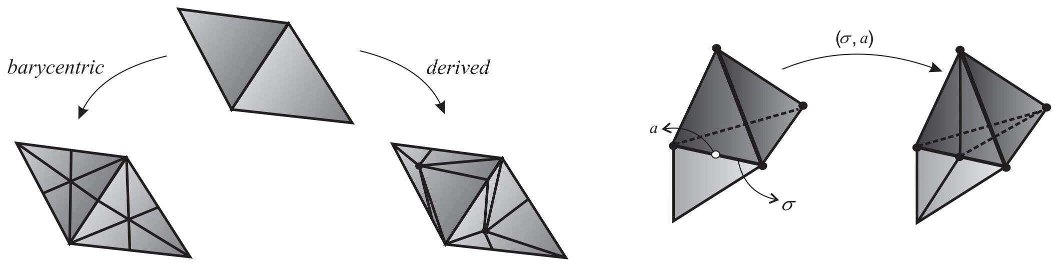


Figure 1.2: Derived and stellar subdivisions.

Lemma 1.1.5. *Let (τ, a) be an elementary starring of a complex K. Then for every $v \in V_K$ we have that $lk(v, (\tau, a)K)$ is isomorphic to some elementary starring of $lk(v, K)$ and $lk(v, (\tau))$. The same is true for welds.*

Proof. It is not hard to see that the following identity holds.

$$lk(v, (\tau, a)K) \equiv \begin{cases} lk(v, K) & v \notin st(\tau, K) \\ (\tau, a)lk(v, K) & v \in lk(\tau, K) \\ (\sigma, \hat{\sigma})lk(v, K) & v \in \tau \text{ and } \tau = v * \sigma \end{cases}$$

where the first two isomorphisms are identities and the one in the last equation consists in sending a to $\widehat{\sigma}$ and leaving all the other vertices of $lk(v, K)$ fixed.

The very same reasoning proves the result for welds. □

Definition. Two complexes K and L are *combinatorially equivalent* or *PL-isomorphic* (or *PL-homeomorphic*), denoted $K \simeq_{PL} L$, if there are subdivisions αK and βL such that $\alpha K \equiv \beta L$ (i.e. some subdivision of them are isomorphic).

It can be proved that two triangulations of the same space are always PL-isomorphic (see for example [25, Corollary I.4]). It was proven by Alexander that starrings and welds are enough to characterize this equivalence relation.

Theorem 1.1.6. *Two complex are PL-isomorphic if and only if they are related by a sequence of starrings, welds and (simplicial) isomorphisms.*

See [25, Theorem II.17] for a proof of this theorem.

§1.2 Combinatorial Manifolds

The notion of manifold is central in Mathematics. A d-manifold is an object which is locally like the euclidean space $\mathbb{R}^d$, where the local properties depend on the category one is working in (piecewise linear, topological, differentiable). Combinatorial (or polyhedral) manifolds are the simplicial versions of topological manifolds, in which the local regularity condition is defined in a purely combinatorial way. The first axiomatic description of a combinatorial theory of manifolds was presented in Heegaard and Dehn's Enzyclopädie article [26] of 1907, which is considered the beginning of combinatorial topology (see [29]).* Polyhedral manifolds were one of the main objects of study at the first half of the last century and they had a major impact in the development of algebraic topology of the last one hundred years. The standard references for this subject are [25, 28, 36, 47].

Combinatorial manifolds are a special case of homogeneous complex, a notion that is central in this Thesis.

Definition. A complex is said to be *homogeneous* or *pure* of dimension d if all of its principal simplices have dimension d. We also call it *d-homogeneous*.

Most familiar spaces, such as (triangulable) topological manifolds, are triangulated by homogeneous complexes. A distinctive property of this family of complexes is a natural notion of *boundary*. The boundary ∂K of a d-homogeneous complex K is the subcomplex generated by the mod 2 union of its $(d-1)$-simplices. We note that some authors prefer to define the boundary as the subcomplex generated by the union of the $(d-1)$-simplices lying in only one d-simplex. For our purposes, either of these definitions is suitable since they both coincide in the manifold case. The following properties of an homogeneous complex are easy to establish.

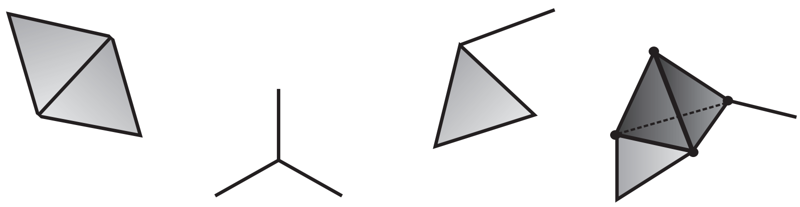


Figure 1.3: The two leftmost complexes are homogeneous; the two rightmost complexes are non-homogeneous.

Lemma 1.2.1. *Let K be a d-homogeneous complex. Then*

(1) $lk(\sigma, K)$ is homogeneous for every $\sigma \in K$.

(2) ∂K is $(d-1)$-homogeneous.

(3) $\partial(K * L) = \partial K * L + K * \partial L$.

(4) $\partial(\partial K) = \emptyset$.

(5) If αK is an arbitrary subdivision then αK is d-homogeneous, $\alpha(\partial K)$ is $(d-1)$-homogeneous and $\alpha(\partial K) = \partial(\alpha K)$.

*Curiously, in this article is introduced the word "homotopy" for the first time, although with a different meaning.

Note the resemblance between (3) of the previous lemma and the derivation rule of the product. This can be understood from the equivalence between d-homogeneous complexes and $(d+1)$-forms provided by the algebraic approach to the theory of simplicial complexes introduced by Alexander [2]. For a d-homogeneous complex K with vertex set V_K we consider the ring of polynomials $\mathbb{Z}_2[V_K]$ and associate each d-simplex $\{x_0, \dots, x_d\} \in K$ to the monomial $x_0 \dots x_d$ and the complex K to the $(d+1)$-form $f = \sum x_{i_0} \dots x_{i_d}$ for every d-simplex $\{x_{i_0}, \dots, x_{i_d}\} \in K$. In this way, the complex K can be described algebraically from its generating simplices. It is straightforward to see that ∂K coincides precisely with the complex associated to the d-form $\partial f := \sum_j \frac{\partial f}{\partial x_j} \in \mathbb{Z}_2[V_K]$. As we shall see in the next chapter, this can be used as a point of departure to define the notion of boundary in the non-homogeneous setting.

Remark 1.2.2. Regarding (5) of Lemma 1.2.1, note that the boundary also behaves well with respect to welds. For let s represent an elementary starring and consider $s^{-1}K = L$. On one hand $\partial(s^{-1}K) = \partial L$. On the other hand, $K = sL$, so $\partial K = \partial(sL) = s(\partial L) = s(\partial(s^{-1}K))$. Therefore, $\partial K = s(\partial(s^{-1}K))$, from where $s^{-1}(\partial K) = \partial(s^{-1}K)$.

Notation. We shall denote by Δ^d a generic d-simplex and by $\partial \Delta^d$ its boundary.

Definition. A *combinatorial d-ball* is a complex which is PL-homeomorphic to Δ^d. A *combinatorial d-sphere* is a complex PL-homeomorphic to $\partial \Delta^{d+1}$. By convention, $\{\emptyset\} = \partial \Delta^0$ is considered a sphere of dimension -1. A *combinatorial d-manifold* is a complex M such that for every $v \in V_M$, $lk(v, M)$ is a combinatorial $(d-1)$-ball or $(d-1)$-sphere.

It is easy to verify that combinatorial d-manifolds are homogeneous complexes of dimension d: if σ is a simplex in a d-manifold M and $v \in V_\sigma$ then $lk(v, M)$ is a combinatorial $(d-1)$-ball or sphere, and hence $(d-1)$-homogeneous by an inductive argument. Therefore, $lk(v, \sigma)$ is a face of a $(d-1)$-simplex $\tau \in lk(v, M)$ and hence σ is a face of the d-simplex $v * \tau$.

The following is a relevant characterization of the joins of combinatorial balls and spheres (see [14, Lemma 2.1.6] for a proof).

Proposition 1.2.3. *Let B^p stand for a combinatorial p-ball and S^q for a combinatorial q-sphere. Then*

(1) $B^p * B^q$ is a combinatorial $(p+q+1)$-ball.

(2) $B^p * S^q$ is a combinatorial $(p+q+1)$-ball.

(3) $S^p * S^q$ is a combinatorial $(p+q+1)$-sphere.

A simplex in a combinatorial manifold is said to be *regular* if its link is either a combinatorial ball or a combinatorial sphere. Note that a combinatorial d-manifold is a complex all of whose vertices are regular. The two main basic features of combinatorial manifolds are contained in the following proposition. We include its proof to show how the properties of these complexes can be traced back to their local structure.

Proposition 1.2.4. *Let M be a combinatorial d-manifold. Then*

(1) If $N \simeq_{PL} M$ then N is also a combinatorial d-manifold (combinatorial d-manifolds are closed under PL-homeomorphisms).

(2) Every simplex in M is regular.

This proposition confirms the intuitive idea that combinatorial d-balls and d-spheres are combinatorial d-manifolds (since Δ^d and $\partial\Delta^{d+1}$ trivially are). This proof is essentially the same as the one given in [25].

Proof of Proposition 1.2.4. We use induction to prove both statements simultaneously. Let $(1)_d$ and $(2)_d$ be the assertions that (1) and (2), respectively, hold for every combinatorial k-manifold with $k \leq d$. We shall prove that $(1)_{d-1}$ implies $(2)_d$ and $(2)_d$ implies $(1)_d$. The base cases $(1)_0$ and $(2)_0$ are easy to check.

Assume $(1)_{d-1}$ so, by the previous comment, combinatorial k-balls and spheres with $k \leq d-1$ are combinatorial manifolds. Let $\tau \in M$ be a k-simplex. We shall show that τ is regular. Since $k \geq 1$ then we can write $\tau = a * \sigma$ where $a \in M$ is a vertex. By Remark 1.1.4, $lk(\tau, M) = lk(\sigma, lk(a, M))$. Since $lk(a, M)$ is a combinatorial $(d-1)$-manifold then the inductive hypothesis implies that $lk(\tau, M)$ is a combinatorial ball or sphere. Thus, $(2)_d$ is settled.

Assume now $(2)_d$. By Theorem 1.1.6 it suffice to prove the cases $N = (\sigma, a)M$ and $N = (\sigma, a)^{-1}M$. Suppose first $N = (\sigma, a)M$ and let $v \in V_N$. We shall show that v is regular. If $v \neq a$ then $lk(v, N)$ is isomorphic to some elementary starring of $lk(v, M)$ by Lemma 1.1.5. In particular, $lk(v, N)$ is a combinatorial ball or sphere. Finally, $lk(a, N) = \partial\sigma * lk(\sigma, M)$. Since by $(2)_d$ $lk(\sigma, M)$ is a combinatorial ball or sphere then so is $\partial\sigma * lk(\sigma, M)$ by Proposition 1.2.3.

The case $N = (\sigma, a)^{-1}M$ follows the same lines. $\square$

Note that the boundary of a combinatorial d-manifold M is precisely the set of simplices σ for which $lk(\sigma, M)$ is a combinatorial ball. This is consistent with the intuition that the points in the boundary of (topological) manifolds are exactly the ones whose neighborhoods are homeomorphic to $\mathbb{R}^d_+ := \mathbb{R}^d \cap \{x_d \geq 0\}$ (see Figure 1.4). It is not hard to show that the boundary of a combinatorial d-manifold is a combinatorial $(d-1)$-manifold without boundary (see for example [14, Proposition 2.1.15]). The set of simplices $\sigma \in M$ for which $lk(\sigma, M)$ is a combinatorial sphere is called the *interior* of M and is denoted M°. This is not a subcomplex.

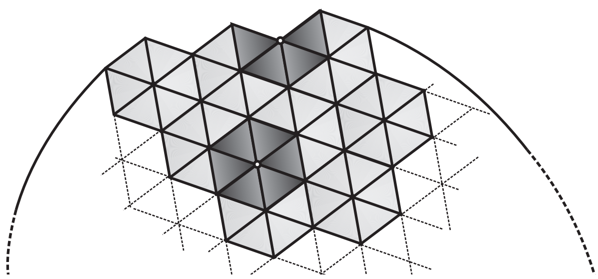


Figure 1.4: Boundary and internal links.

Let us mention that, equivalently, a combinatorial d-manifold is a complex such that each of its vertices has a star which is a combinatorial d-ball. Colloquially, this says that the simplicial notion of neighborhood of a vertex is a simplicial notion of ball. When seen in this way, the definition of combinatorial manifold is a direct translation to simplicial setting of that of topological manifold. The advantage of considering the definition involving links is twofold: on one hand, one can distinguish between internal and boundary simplices; on the other hand, it allows one to use inductive arguments based on the smaller dimension of the link.

Some global properties of combinatorial manifolds can be stated in terms of more general type of complex called *pseudo manifolds*.

Definition. A *weak d-pseudo manifold* is a d-homogeneous complex K satisfying that every $(d-1)$-simplex is a face of at most two d-simplices. A *d-pseudo manifold* is a weak d-pseudo manifold K which is *strongly connected*, i.e. given two d-simplices σ, σ', there exists a sequence of d-simplices $\sigma = \sigma_0, \ldots, \sigma_k = \sigma'$ such that $\sigma_i \cap \sigma_{i+1}$ is $(d-1)$-dimensional for all $i = 0, \ldots, k-1$.

If every $(d-1)$-simplex is in exactly two d-simplices we say that it is a weak d-pseudo manifold *without boundary*. Note that the defining property of a weak pseudo manifold P states that $lk(\sigma, P)$ is either Δ^0 or $\partial\Delta^1$ for every ridge in P. It is not hard to show that any connected combinatorial d-manifold is a d-pseudo manifold (see [14, Corollary 2.1.13 and Proposition 2.1.16] for a proof).

For these complexes we have the following properties. We shall borrow some notations and definitions from homology theory (see §1.5 below).

Lemma 1.2.5. *Let P be a weak d-pseudo manifold without boundary and let $\sigma \in P$. Then*

(1) $lk(\sigma, P)$ is a weak $(d - \dim(\sigma) - 1)$-pseudo manifold without boundary.

(2) $H_d(P; \mathbb{Z}_2) \neq 0$.

Sketch of proof. On one hand, $lk(\sigma, P)$ is homogeneous by Lemma 1.2.1 (1) and if η is a ridge of $lk(\sigma, P)$ then $\eta * \sigma$ is a ridge of P. Therefore, $\partial\Delta^1 = lk(\eta * \sigma, P) = lk(\eta, lk(\sigma, P))$.

On the other hand, the d-chain which is the formal sum of all the d-simplices of P is a generating d-cycle. □

We end this section stating a classical result of Newman [41] that will be a central interest in this work (see also [25, 28, 36]). The *closure* $\overline{S}$ of a set of simplices S (not necessarily a complex) coincides with the complex generated by S; that is, the complex obtained by adding all the faces of the simplices in S.

Theorem 1.2.6 (Newman's Theorem)**.** *If S is a combinatorial d-sphere containing a combinatorial d-ball B, then the closure $\overline{S - B}$ is a combinatorial d-ball.*

§1.3 Collapses and expansions

One of the main drivers of the study of combinatorial manifold theory was the quest for a solution of Poincaré Conjecture (1908): "Every simply connected closed 3-manifold is homeomorphic to the 3-sphere". Major contributions to this area are due to J.H.C. Whitehead, who introduced the concepts of simplicial *collapse* and *expansion* in order to discretize the notion of a continuous deformation and study the homotopy type of polyhedra by simplicial methods. The tools he developed, such as regular neighborhoods and simple homotopy theory, were of key importance in the development of combinatorial topology of the twentieth century. Whitehead research led eventually to the development of the theory of CW-complexes and his work settled the bases of the development of K-theory.

The basic notion of Whitehead's combinatorial deformation theory is that of *simplicial collapse*. A simplex τ of a complex K is said to be *collapsible* in K if it has a free face σ, i.e. a proper face which is not a face of any other simplex of K. Note that, in particular,

τ must be maximal simplex and σ must be a ridge. In this situation, the operation which transforms K into $K - \{\tau, \sigma\}$ is called an *elementary (simplicial) collapse*, and it is usually denoted by $K \searrow^e K - \{\tau, \sigma\}$. The inverse operation is called an *elementary (simplicial) expansion*. If there is a sequence $K \searrow^e K_1 \searrow^e \cdots \searrow^e L$ we say that K *collapses* to L (or equivalently, L *expands* to K) and write $K \searrow L$ (or $L \nearrow K$). It is not hard to see that if $K \searrow L$ then L is a strong deformation retract of K. In particular, $K \simeq L$ (homotopy equivalent). Two complexes are *simply homotopy equivalent* if they are related by a sequence of collapses and expansions. Hence, simply homotopy equivalent spaces have the same homotopy type. The converse does not hold. The obstruction, found by Whitehead himself, lies in the so called *Whitehead group*, which depends on the fundamental group of the complex.

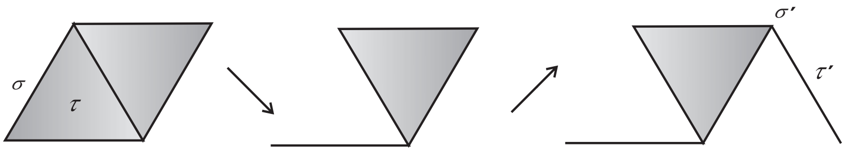


Figure 1.5: Collapsing and expanding.

A stronger notion than simple homotopy is *collapsibility*. A complex K is said to be *collapsible* if it has a subdivision which collapses to a single vertex. Thus, in this case, no expansions are involved in the deformation (see Figure 1.6). The theory of collapsible complexes is very rich and useful. The following celebrated theorem of Whitehead, reflecting the nice behaviour of polyhedral manifolds within the context of simplicial deformation theory, is a key result in the theory of PL-manifolds (see [25, Corollary III.17] for a proof).

Theorem 1.3.1 (Whitehead's Theorem)**.** *A collapsible combinatorial d-manifold is a combinatorial d-ball.*

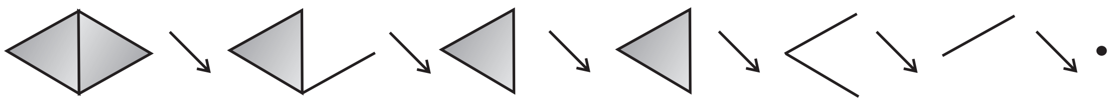


Figure 1.6: Collapsable complex.

There is a more general type of collapse called *geometrical collapse*. If $K = K_0 + B^d$, where B^d is a combinatorial d-ball and $B^d \cap K_0 = B^{d-1}$ is a combinatorial $(d-1)$-ball contained in the boundary of B^d, then the move $K \to K_0$ it called an *elementary geometrical collapse*. A finite sequence of elementary geometrical collapses (resp. expansions) is a geometrical collapse (resp. expansion). It can be shown that any geometrical collapse $K \to K_0$ can be turned into a simplicial collapse $\alpha K \searrow \alpha K_0$ for some convenient subdivision α (that can be actually taken to be stellar; see [25, Theorem III.6]). Though, both deformation theories are intrinsically equivalent. One of the most important results about discrete deformation of manifolds is the following theorem by Alexander [2] (see also [25, 36]).

Theorem 1.3.2 (Alexander's Theorem)**.** *Let M be a combinatorial d-manifold and let $M \to M + B^d$ be a geometrical expansion such that $M \cap B \subset \partial M$. Then, $M + B \simeq_{PL} M$.*

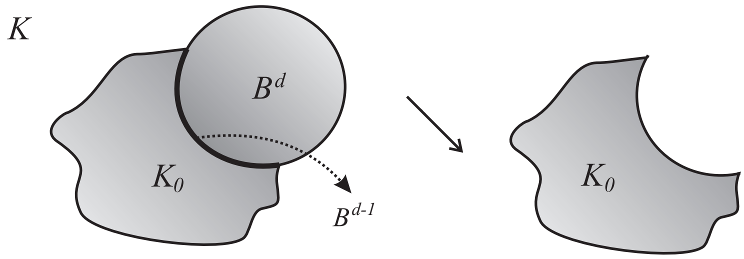


Figure 1.7: A geometric collapse.

Theorem 1.3.2 states in particular that under a good intersection situation the result of an expansion on a manifold is still a manifold. This is why a geometrical expansion $M \to N = M + B$ such that $M \cap B \subset \partial M$ is called a *regular expansion*. Note that in a sequence of regular expansions on a d-manifold the dimension of all the balls being expanded must be d.

A concrete example of the reaches of Whitehead's simplicial deformation theory is a conjecture stated by Zeeman in his 1963's paper "On the dunce hat" [54]: "If K is a contractible 2-complex then $K \times I$ is geometrically collapsible". This seemingly innocent statement of low-dimensional topology has as a corollary the (original) Poincaré Conjecture. Zeeman himself presented a ten-line-proof of it based on his conjecture. Although Zeeman's conjecture is still an open problem, this is another evidence of the power of combinatorial study of spaces. A very nice and complete account of the work around Zeeman's conjecture can be seen in [27, §11].

§1.4 Simplicial moves on manifolds

A number of structure-preserving moves which transform a manifold into another one are of central interest in the theory of combinatorial manifolds: shellings and bistellar moves (or, more generally, stellar exchanges). They were first studied by Newman [41] (see also [36, 47, 52]) in an attempt to shift the attention of invariance from the subdivision paradigm originally developed by Dehn and Heegaard, which Newman believed had many technical disadvantages. Newman constructed a combinatorial manifold theory around the concept of transforming one space into another by means of boundary operations, consisting in addition and deletion of maximal simplices in a regular fashion (*shellings*), and internal operations, which affected inner parts of the manifold (*bistellar moves*). Although he introduced a consistent self-contained theory, he did not prove that his approach was equivalent to that of Heegaard and Dehn's. However, his theory was finally ratified in the 90's when Pachner [42] showed that subdivision and Newman's moves provided the same invariance theory. That is, two combinatorial manifolds are PL-isomorphic if and only if one can obtain one from the other by a sequence of elementary shellings, inverse shellings and bistellar moves (see Theorem 1.4.1 below). More strongly, for manifolds with boundary shelling moves are sufficient (see [44]). After Pachner's work, these moves were renamed *Pachner moves*.

Definition. Let M be a combinatorial d-manifold with boundary. Suppose there is a d-simplex $\rho = \sigma * \tau \in M$ with $\dim \sigma, \dim \tau \geq 0$ such that $\sigma \in \overset{\circ}{M}$ and $\partial\sigma * \tau \subset \partial M$. Then the move $M \xrightarrow{sh} M_1 = \overline{M - \rho}$ is called an *elementary shelling*. The opposite move is called an *inverse shelling*.

It is straightforward to see that these moves are special cases of regular collapses and expansions (where the collapsing or expanding ball is a single simplex) and, therefore, the result of any of these moves produces a new d-manifold. Actually, a "regular collapse (resp. expansion) involving a single d-simplex" is an equivalent definition of elementary shelling (resp. inverse shelling).

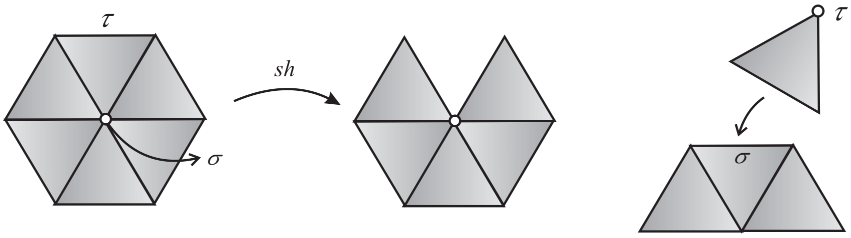


Figure 1.8: Examples of direct and inverse shellings.

A combinatorial d-manifold which can be transformed into a single d-simplex by a sequence of elementary shellings is said to be *shellable*. Since $M \stackrel{sh}{\to} M'$ implies $M \searrow M'$ then, in particular, shellable combinatorial d-manifolds are collapsible and, hence, combinatorial d-balls. The definition of *shellability* can also be extended to combinatorial d-spheres by declaring S to be shellable if for some d-simplex σ, $\overline{S-\sigma}$ is a shellable d-ball.

There is an alternative (constructive) definition of shellability which is also valid for arbitrary d-homogeneous complexes.

Definition. A d-homogeneous complex K is *shellable* if there exists a linear order $F_1, \ldots, F_t$ of all the d-simplices of K such that $F_k \cap (F_1 + \cdots + F_{k-1})$ is $(d-1)$-homogeneous for all $2 \leq k \leq t$.

Shellability expressed in this form became a classical notion ever since the (incomplete) proof of Schläfli of the high-dimensional Euler-Poincaré formula. In 1852, Schläfli [50]† extended the famous formula $V - E + F = 2$, relating the number of vertices, edges and faces of polygons, to d-dimensional polytopes assuming that the boundary of a convex polytope admitted such a decomposition order. It was not until the work of Bruggesser and Mani [12] of 1971 that the shellability of the boundary of any convex polytope was settled (and hence the complete proof of Euler-Poincaré formula).

In geometric topology, shellability is usually studied for simplicial balls. It is a classical subject in triangulation theory to look for examples of non-shellable balls and spheres. It is known that every ball of dimension less than 3 is shellable. Examples of non-shellable 3-balls abound in the bibliography. The first example was presented by Furch in 1924 and many others have been provided ever since (see [55] for a survey of non-shellable 3-balls). A general way for constructing non-shellable spheres for every $d \geq 3$ was presented by Lickorish in [35].

As mentioned above, the simplicial moves in the interior of a manifold considered by Newman are called *bistellar moves*. We shall actually define a more general notion (see [36]).

Definition. Let K be a simplicial complex of dimension d. Suppose there is a simplex $\sigma \in K$ such that $lk(\sigma, K) = \partial\tau * L$ for some $\tau \notin K$ and some $L \subset K$. Then the move that

†The work of Schäfli was actually not published until 1901, after Schäfli's death

replaces $\sigma * \partial\tau * L$ with $\partial\sigma * \tau * L$ is called an *stellar exchange* and we denote it $\kappa(\sigma, \tau)$. When $L = \{\emptyset\}$ we call it a *bistellar move* or *bistellar flip* and denote it $\chi(\sigma, \tau)$.

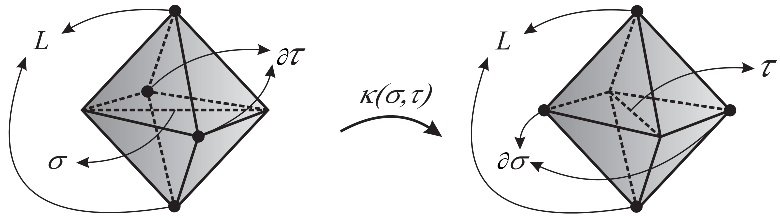


Figure 1.9: An stellar exchange.

Geometrically, a bistellar move simply exchanges the combinatorial ball $\sigma * \partial\tau$ with its complementary ball in the sphere $\partial(\sigma * \tau)$ (both balls share the same boundary). Note that a bistellar move $\chi(\sigma, \tau)$ can only be performed on interior simplices since $lk(\sigma, K) = \partial\tau$ in this case. This is the reason why bistellar moves are considered the interior moves of manifolds. Note also that $\chi(\tau, \sigma)\chi(\sigma, \tau)K = K$, so the inverse of a bistellar move is also a bistellar move. In general, bistellar moves are insufficient to determine PL-equivalence of complexes. However, the more general stellar exchanges produce an equivalent theory of equivalence. This follows from the fact that $(\sigma, a) = \kappa(\sigma, a)$ for any starring and $(\tau, b)^{-1} = \kappa(\tau, b)$ for any weld (see Theorem 1.1.6).

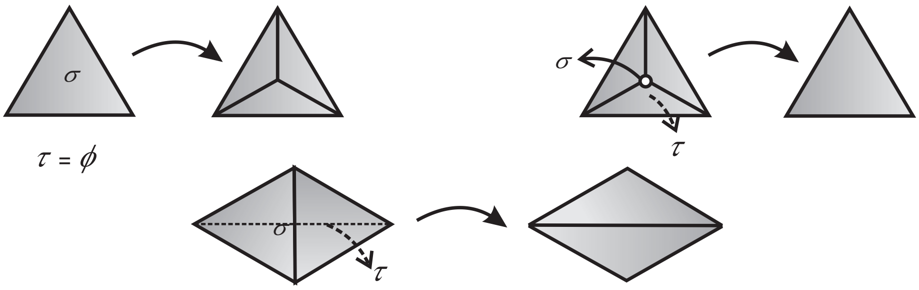


Figure 1.10: All possible bistellar moves in dimension 2.

As commented before, shellings and bistellar moves provide the same equivalence between complexes as PL-invariance. We state next this result for future reference.

Theorem 1.4.1 (Pachner)**.** *Two combinatorial d-manifolds are PL-isomorphic if and only if one can get from one to the other by a sequence of inverse and direct shellings and bistellar moves. Moreover, if both manifolds have non-empty boundary, only inverse and direct shellings are necessary; and if both manifolds are boundaryless, only bistellar moves are necessary.*

§1.5 Homology and cohomology

We recall the basic construction of the simplicial homology and cohomology groups of triangulable spaces (we refer the reader to [40] for a complete exposition). Let K be a complex of dimension d and G an abelian group (we shall work exclusively with $G = \mathbb{Z}$ or $G = \mathbb{Z}_2$). For every $\sigma \in K$ choose an orientation of the vertices of σ and define two orientations of V_σ to be equivalent if they differ by an even permutation. The simplex σ together with a chosen orientation class is called an *oriented simplex* and we write $\sigma = [v_0, \dots, v_p]$ if $V_\sigma = \{v_0, \dots, v_p\}$ for that given orientation. Assume for the rest of the section that all simplexes considered are oriented. Let $C_p(K;G) = \{\sum g_\sigma \sigma \mid \dim(\sigma) = p\,, g_\sigma \in G\}$ be the abelian group of formal G-linear combinations of p-simplices of K $(-1 \le p \le d)$ where we identify $-\sigma$ with the simplex σ with the opposite orientation. An element of $C_p(K;G)$ is called a *p-chain* (with coefficients in G). For two oriented simplices $\sigma, \tau \in K$ such that $\sigma \prec \tau = [v_0, \dots, v_k]$ we must have that $\sigma = \pm[v_0, \dots, \hat{v}_i, \dots, v_k]$ for some v_i; define the *incidence number* between σ and τ by

$$\langle \sigma, \tau \rangle = \begin{cases} 1 & \text{if } \sigma = [v_0, \dots, \hat{v}_i, \dots, v_k] \\ -1 & \text{if } \sigma = -[v_0, \dots, \hat{v}_i, \dots, v_k] \end{cases}$$

We have the *boundary map* $\partial_p : C_p(K;G) \to C_{p-1}(K;G)$ defined in the basis of $C_p(K;G)$ by

$$\partial_p(\sigma) = \sum_{\eta \prec \sigma} \langle \eta, \sigma \rangle \eta.$$

We shall frequently let the context imply the index p in ∂_p and write simply ∂. The boundary map satisfies $\partial^2 = 0$ so $(C_*(K;G), \partial)$ forms a chain complex called the *simplicial chain complex of* K. The *pth (reduced)*[‡] *homology group* of K (with coefficients in the group G) is the abelian group

$$H_p(K;G) = \frac{Ker(\partial_p)}{Im(\partial_{p+1})}.$$

The elements of $Ker(\partial_p)$ are the *p-cycles* of K (boundaryless p-chains). A p-cycle c is called *generating cycle* if the class $[c] \in H_p(K;G)$ is not zero. A complex K is *acyclic* if all the homology groups vanish.

Cohomology is the dual notion to homology. Let $C^p(K;G)$ denote the set of G-linear maps $T : C_p(K;G) \to G$. A basis for $C^p(K;G)$ are the functionals σ^*, for each p-simplex $\sigma \in K$, taking the value 1 in σ and 0 in every other p-simplex of K. The *coboundary* map $\delta_p : C^p(K;G) \to C^{p+1}(K;G)$ is defined by

$$\delta(\sigma^*) = \sum_{\eta \succ \sigma} \langle \eta, \sigma \rangle \eta^*.$$

It can be readily seen that $\delta_p T(\sigma) = T(\partial_{p+1}\sigma)$. It follows that $\delta^2 = 0$ and the *pth (reduced) cohomology group* of K (with coefficients in the group G) is

$$H^p(K;G) = \frac{Ker(\delta_p)}{Im(\delta_{p-1})}.$$

Given a subcomplex $L \subset K$ we have that $C_p(L;G)$ is a subgroup of $C_p(K;G)$ and we can form the abelian groups $C_p(K,L;G) := C_p(K;G)/C_p(L;G)$. It is easily seen that the restriction $\partial_p : C_p(K;G)/C_p(L;G) \to C_{p-1}(K;G)/C_{p-1}(L;G)$ is well defined and $\partial^2 = 0$. Then the group $H_p(K,L;G) := H_p(C_*(K,L;G))$ is called the *pth relative homology group*. In this case there is a long exact sequence of homology groups

$$\cdots \to H_p(L;G) \to H_p(K;G) \to H_p(K,L;G) \to H_{p-1}(L;G) \to H_{p-1}(K;G) \to \cdots$$

[‡]Since the empty simplex belongs to every (non-empty) complex we have that $C_{-1}(K;G) = G$ and $\partial_{-1} : C_0(K;G) \to G$ is trivially surjective.

The other long exact sequence we shall be using is the Mayer-Vietoris sequence. Suppose $K = A + B$ is a decomposition of the complex K in the union of two subcomplexes $A, B \subset K$. Then, the *(homological) Mayer-Vietoris sequence* is the following long exact sequence

$$\cdots \to H_p(A \cap B; G) \to H_p(A; G) \oplus H_p(B; G) \to H_p(K; G) \to H_{p-1}(A \cap B; G) \to \cdots$$

Convention. *When $G = \mathbb{Z}$ we shall omit the reference to G and write simply $H_p(K)$ for $H_p(K; \mathbb{Z})$.*

Resumen en castellano del Capítulo 1

En este capítulo introducimos las definiciones y notaciones básicas que utilizaremos a lo largo de nuestro trabajo y damos un repaso general de la teoría de variedades combinatorias. El tratamiento de los temas es superficial y referimos al lector a los textos clásicos [2, 25, 36, 40, 47] para una exposición más detallada.

Los métodos combinatorios constituyen una herramienta fundamental para estudiar problemas geométricos y topológicos de las variedades y espacios en general. El enfoque clásico consiste en descomponer el espacio (triangulación) en una colección finita de bloques convexos (símplices) y usar esta descripción discreta para extraer propiedades topológicas del espacio. La información acerca de esta descomposición está contenida en un *complejo simplicial*, un objeto puramente combinatorio.

Un complejo simplicial es un par (V_K, S_K) donde S_K es una colección de subconjuntos del conjunto de vértices V_K tal que cada conjunto unipuntual de V_K pertenece a S_K y cada subconjunto de un miembro de S_K es también un miembro de S_K. Los símplices pueden verse geométricamente como puntos, segmentos, triángulos, tetraedros, y sus análogos en dimensiones mayores; y la manera como se pegan dichos bloques queda determinada por el subconjunto de vértices que tienen en común. Siguiendo esta idea geométrica, un simplex con $d+1$ vértices tiene dimensión d (y se lo llama *d-simplex*) y la dimensión de un complejo es la dimensión del simplex de mayor dimensión en él. Si $\sigma \subset \tau$ decimos que σ es una *cara* de τ (notado $\sigma < \tau$) y si $\dim(\sigma) = \dim(\tau) - 1$ decimos que es una cara *inmediata* de τ (notado $\sigma \prec \tau$). Un simplex es *maximal* si no es cara de ningún otro simplex del complejo y un *ridge* es una cara maximal de un simplex maximal.

El *join* de dos símplices σ, τ (el simplex generado por los vértices de ambos) es denotado por $\sigma * \tau$. El join de dos complejos es $K * L = \{\sigma * \tau \,|\, \sigma \in K\,, \tau \in L\}$. La unión de dos complejos es notada $K + L$. El *link* de σ en K es el subcomplejo $lk(\sigma, K) = \{\tau \in K : \tau \cap \sigma = \emptyset,\ \tau * \sigma \in K\}$ y el *star* de σ en K el subcomplejo $st(\sigma, K) = \sigma * lk(\sigma, K) = \{\eta \in K \,|\, \eta < \tau \text{ and } \sigma < \tau\}$. En esta Tesis trabajamos exclusivamente con complejos finitos; esto es, V_K es finito.

La realización geométrica $|K|$ de un complejo simplicial K es el espacio topológico de las funciones $\alpha : V_K \to [0, 1]$ tales que

- $supp(\alpha) := \{v \in V_K \,|\, \alpha(v) > 0\}$ es un simplex de K.
- $\sum_{v \in V_K} \alpha(v) = 1$,

con la métrica $d(\alpha, \beta) = (\sum_{v \in V_\sigma} (\alpha(v) - \beta(v))^2)^{\frac{1}{2}}$. Para $\sigma \in K$ con $V_\sigma = \{v_0, \dots, v_k\}$, el espacio $|\sigma|$ es el subespacio de $|K|$ de las funciones α tales que $supp(\alpha) \subset V_\sigma$. Frecuentemente identificamos el complejo simplicial K con su realización geométrica y escribimos $K \simeq L$ cuando $|K|$ es homotópicamente equivalente a $|L|$.

Un complejo K es una *subdivisión* de un complejo L si $|K| = |L|$ y para cada $\sigma \in K$ existe un $\tau \in L$ tal que $|\sigma| \subset |\tau|$. Consideramos tres tipos de subdivisiones en este trabajo: subdivisiones arbitrarias, subdivisiones derivadas y subdivisiones estelares, basadas estas últimas en las nociones de *starrings* y *welds* (ver página 34).

Dos complejos K y L son *combinatoriamente equivalentes* o *PL-isomorfos* (o *PL-homeomorfos*), denotado $K \simeq_{PL} L$, si existen subdivisiones αK y βL tales que $\alpha K \equiv \beta L$. Fue probado por Alexander que dos complejos son PL-isomorfos si y sólo si están relacionados por una sucesión de starrings, welds e isomorfismos simpliciales.

En §1.2 damos un breve repaso a la teoría de variedades combinatorias. La noción de variedad es central en Matemática. Una variedad es un objeto que localmente es como el espacio euclídeo $\mathbb{R}^d$. Las variedades combinatorias (o poliedrales) son las versiones simpliciales de las variedades topológicas, en donde la condición local de regularidad es definida de una manera puramente combinatoria. Las referencias estándar para este tema son [25, 28, 36, 47].

Las variedades combinatorias son un tipo especial de complejo homogéneo.

Definición. Un complejo se dice *homogéneo* o *puro* de dimensión d si todos sus símplices maximales tienen dimensión d.

La mayoría de los espacios más familiares, como las variedades topológicas (triangulables), están trianguladas por complejos homogéneos. Una propiedad característica de esta familia de complejos es que poseen una noción natural de borde: el *borde* ∂K de un complejo d-homogéneo K es el subcomplejo generado por la unión módulo 2 de los $(d-1)$-símplices. El Lema 1.2.1 enuncia algunas propiedades básicas del borde de un complejo.

Definición. Una *d-bola combinatoria* es un complejo PL-homeomorfo a un d-simplex Δ^d. Una *d-esfera combinatoria* es un complejo PL-homeomorfo al borde de un $(d+1)$-simplex $\partial\Delta^{d+1}$. Por convención, $\{\emptyset\} = \partial\Delta^0$ es considerado una esfera de dimensión -1. Una *d-variedad combinatoria* es un complejo M tal que para todo $v \in V_M$, $lk(v, M)$ es una $(d-1)$-bola o $(d-1)$-esfera combinatoria.

Es fácil ver que las variedades combinatorias son complejos homogéneos. A continuación probamos dos propiedades características de las variedades combinatorias.

Proposición 1.2.4. *Sea M una d-variedad combinatoria. Entonces,*

(1) Si $N \simeq_{PL} M$ entonces N también es una d-variedad combinatoria.

(2) El link de todo simplex en M es una bola o esfera combinatoria.

Esta proposición confirma la idea intuitiva que las bolas y esferas combinatorias son variedades combinatorias. Es fácil ver que el borde de una variedad combinatoria coincide precisamente con el conjunto de símplices cuyo link es una bola combinatoria y que el borde ∂M de una d-variedad combinatoria M es una $(d-1)$-variedad combinatoria sin borde.

A continuación, mencionamos que las variedades combinatorias conexas son pseudo variedades.

Definición. Una *d-pseudo variedad* es un complejo d-homogéneo K que satisface que todo $(d-1)$-simplex es cara de a lo sumo dos d-símplices y que dados dos d-símplices σ, σ', existe una sucesión de d-símplices $\sigma = \sigma_0, \dots, \sigma_k = \sigma'$ tal que $\sigma_i \cap \sigma_{i+1}$ es $(d-1)$-dimensional para todo $i = 0, \dots, k-1$.

Cerramos la sección enunciando el Teorema de Newman, que resulta de gran utilidad para la teoría de variedades combinatorias.

Teorema 1.2.6. *Si S es una d-esfera combinatoria que contiene una d-bola combinatoria B, entonces el complemento $\overline{S-B}$ es una d-bola combinatoria.*

En §1.3 recordamos las nociones de colapsos y expansiones de Whitehead. Un simplex maximal τ de un complejo K se dice *colapsable* en K si tiene una cara libre σ. En esta situación, la operación que transforma K en $K - \{\tau, \sigma\}$ es llamado un *colapso simplicial elemental*, y denotado $K \searrow^{e} K - \{\tau, \sigma\}$. La operación inversa es una *expansión simplicial elemental*. Si se tiene una sucesión $K \searrow^{e} K_1 \searrow^{e} \cdots \searrow^{e} L$ se dice que K *colapsa* a L (o L se *expande* a K) y se nota $K \searrow L$ o $L \nearrow K$ respectivamente. La Figura 1.5 muestra ejemplos de estos movimientos. Un complejo se dice *colapsable* si colapsa a un vértice. Uno de los resultado más fuertes de esta teoría de deformación discreta es el siguiente teorema de Whitehead.

Teorema 1.3.1. *Una d-variedad combinatoria colapsable es una d-bola combinatoria.*

Un tipo más general de expansión es la *expansión geométrica*. Si $K = K_0 + B^d$, donde B^d es una d-bola combinatoria y $B^d \cap K_0 = B^{d-1}$ es una $(d-1)$-bola combinatoria contenida en el borde de B^d, entonces el movimiento $K \to K_0$ se llama *colapso geométrico elemental* (y el movimiento inverso *expansión geométrica elemental*). Una sucesión de tales colapsos (resp. expansiones) forman un *colapso geométrico* (resp. *expansión geométrica*). Otro de los resultados fundacionales de esta teoría es el siguiente teorema de Alexander sobre expansiones geométricas regulares.

Teorema 1.3.2. *Sea M una d-variedad combinatoria y sea $M \to M + B^d$ una expansión geométrica tal que $M \cap B \subset \partial M$. Entonces, $M + B \simeq_{PL} M$.*

En §1.4 recordamos dos movimientos simpliciales fundamentales para la teoría de variedades combinatorias: *shellings* y *movimientos biestelares* (*bistellar moves*). Estas operaciones preservan la estructura de las variedades y son conocidas hoy en día como *movimientos de Pachner*.

Definición. Sea M una d-variedad combinatoria con borde. Supongamos que existe un d-simplex $\rho = \sigma * \tau \in M$ con $\dim \sigma, \dim \tau \geq 0$ tal que $\sigma \in \overset{\circ}{M}$ y $\partial\sigma * \tau \subset \partial M$. Entonces $M \xrightarrow{sh} M_1 = \overline{M - \rho}$ se llama *shelling elemental*. El movimiento opuesto es un *shelling inverso*.

Estos movimientos son casos especiales de expansiones y colapsos regulares (que involucran un único simplex). La figura 1.8 muestra ejemplos de shellings inversos y directos.

Una variedad combinatoria que pueden llevarse a un único simplex por medio de shellings se dice *shellable*. Es fácil ver que las únicas variedades shellables son necesariamente bolas combinatorias. Existe una definición alternativa de shellabilidad para complejos d-homogéneos que requiere la existencia de un orden $F_1, \dots, F_t$ de todos los d-símplices del complejo de manera que $F_k \cap (F_1 + \cdots + F_{k-1})$ sea $(d-1)$-homogéneo para todo $2 \leq k \leq t$.

Resumen en castellano del Capítulo 1

Definición. Sea K un complejo simplicial de dimensión d. Supongamos que existe un simplex $\sigma \in K$ tal que $lk(\sigma, K) = \partial\tau * L$ para cierto $\tau \notin K$ y cierto $L \subset K$. Entonces el movimiento que reemplaza $\sigma * \partial\tau * L$ con $\partial\sigma * \tau * L$ se llama un *intercambio estelar* y se nota $\kappa(\sigma, \tau)$. Cuando $L = \{\emptyset\}$ se llama *movimiento biestelar* (*bistellar move*) y es denotado $\chi(\sigma, \tau)$.

Geométricamente, un movimiento biestelar intercambia $\sigma * \partial\tau$ con su bola complementaria en la esfera $\partial(\sigma * \tau)$. La Figura 1.10 muestra ejemplos de movimientos biestelares.

El siguiente teorema de Pachner es central en el desarrollo de nuestra teoría.

Teorema 1.4.1. *Dos d-variedades combinatorias son PL-isomorfas si y sólo si se puede ir de una a la otra por una sucesión de shellings directos e inversos y movimientos biestelares. Para variedades con borde, solo los shellings son necesarios.*

Finalmente, la última sección 1.5 se encarga de reunir las definiciones básicas de la teoría de homología simplicial.

Chapter 2

Non-homogeneous Combinatorial Manifolds

In this chapter we present the main definitions of the theory of NH-manifolds, establish their basic properties and prove the key results. Motivated in part by the notion of non-pure shellability due to Björner and Wachs [7], we develop a generalization of the concept of polyhedral manifold to the non-homogeneous setting and prove that strong structural features are still present in these more general complexes. The core of the theory is introduced in the first two sections and one of the most important results of this chapter is the generalization of Alexander's Theorem on regular expansions given in §2.4. This theory is new and all results presented here are original.

§2.1 Definitions and basic properties

In this section we provide the definition of NH-manifold and prove their basic properties. Non-homogeneous combinatorial manifolds are simplicial complexes whose underlying spaces are locally like Euclidean spaces of varying dimensions assembled in a regular manner. Figure 2.1 shows some example of this local structure. The non-homogeneous version of balls and spheres, called NH-balls and NH-spheres, are a fundamental type of NH-manifolds defined to fit with two characterizing properties of classical balls and spheres. NH-manifolds, NH-balls and NH-spheres are defined together using an inductive argument.

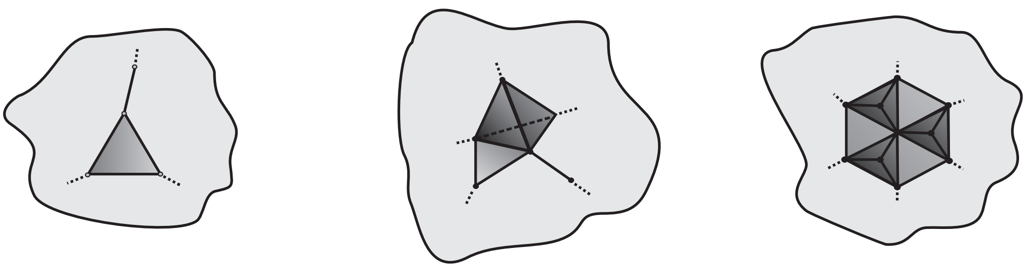


Figure 2.1: Local structure of NH-manifolds.

Definition. Let K be a complex. A subcomplex $L \subset K$ is said to be *top generated* in K if it is generated by principal simplices of K; i.e. every maximal simplex of L is also maximal in K.

Definition. An *NH-manifold* (resp. *NH-ball*, *NH-sphere*) of dimension 0 is a manifold (resp. ball, sphere) of dimension 0. An NH-sphere of dimension -1 is, by convention, $\{\emptyset\}$. For $d \geq 1$, we define by induction

- An *NH-manifold* of dimension d is a complex M of dimension d such that $lk(v, M)$ is an NH-ball of dimension $0 \leq k \leq d-1$ or an NH-sphere of dimension $-1 \leq k \leq d-1$ for all $v \in V_M$.
- An *NH-ball* of dimension d is a collapsible NH-manifold of dimension d.
- An *NH-sphere* of dimension d and *homotopy dimension* k is an NH-manifold S of dimension d such that there exist a top generated NH-ball $B \subset S$ of dimension d and a top generated combinatorial k-ball $L \subset S$ such that $B + L = S$ and $B \cap L = \partial L$. We say that $S = B+L$ is a *decomposition* of S and we denote $\dim_h(S)$ the homotopy dimension of S.

The definition of NH-ball is motivated by Whitehead's theorem and the definition of NH-sphere by that of Newman's (Theorems 1.3.1 and 1.2.6). Note that an NH-ball of dimension 1 is simply a combinatorial 1-ball. An NH-sphere of dimension 1 is either a 1-sphere (if the homotopy dimension is 1) or the disjoint union of a point and a combinatorial 1-ball (if the homotopy dimension is 0). In general, an NH-sphere of homotopy dimension 0 consists of a disjoint union of a point and an NH-ball. These are the only NH-spheres which are not connected.

Figure 2.2 shows various examples of NH-manifolds, NH-balls and NH-spheres. Note that NH-manifolds are in some way the smallest class of polyhedra with non-homogeneous members containing all combinatorial manifolds. This follows from the fact that this theory is basically "spanned" from 1-dimensional NH-spheres of homotopy dimension 0, which have the minimal distortion one may introduce to achieve non-homogeneity (see Figure 2.2).

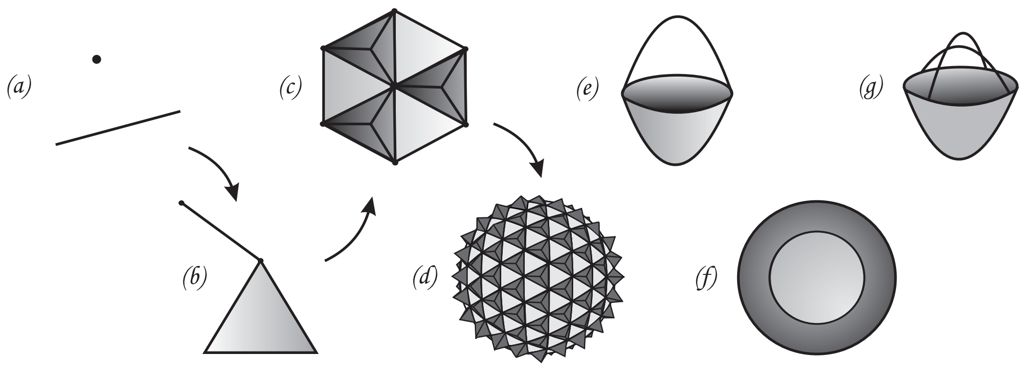


Figure 2.2: Examples of NH-manifolds. (a), (d) and (e) are NH-spheres of dimension 1, 3 and 2 and homotopy dimension 0, 2 and 1 respectively. (b) is an NH-ball of dimension 2 and (c), (f) are NH-balls of dimension 3. (g) is an NH-manifold which is neither an NH-ball nor an NH-sphere. The sequence (a)-(d) evidences how NH-manifolds are inductively defined.

Remark 2.1.1. Note that the decomposition of an NH-sphere need not be unique. However the homotopy dimension of an NH-sphere S is well defined since its geometric realization is a homotopy $\dim_h(S)$-sphere.

The rest of the section is devoted to establish the basic properties of NH-manifolds. We shall prove that non-pure manifolds possess the same (generalized) features as combinatorial manifolds. We begin by showing that this theory is effectively an extension to the non-homogeneous context of the classical polyhedral theory.

Theorem 2.1.2. *A complex K is a homogeneous NH-manifold (resp. NH-ball, NH-sphere) of dimension d if and only if it is a combinatorial d-manifold (resp. d-ball, d-sphere).*

Proof. We may assume $d \geq 1$. It is straightforward that the result holds for NH-manifolds of dimension d provided that it holds for NH-balls and NH-spheres of dimension less than d. It then remains to prove that the result holds for NH-balls and NH-spheres of dimension d if it holds for NH-manifolds of dimension d.

For NH-balls the result follows at once by Whitehead's Theorem 1.3.1. Suppose now that $S = B + L$ is a homogeneous NH-sphere of dimension d. It follows that B and L are combinatorial d-balls. Take $\sigma \in \partial L$ a maximal simplex. Since $lk(\sigma, S) = lk(\sigma, B) + \{v\}$ for some vertex $v \in L$ and S is an d-pseudo manifold then $lk(\sigma, B)$ is also a single vertex. It follows that $\sigma \in \partial B$. Since both ∂L and ∂B are combinatorial $(d-1)$-spheres, this implies that $\partial L = \partial B$. This proves that S is a combinatorial d-sphere. Conversely, any d-simplex of a combinatorial d-sphere can play the role of L in its decomposition as an NH-sphere. The result then follows from Newman's Theorem 1.2.6. □

We next show that NH-manifolds fulfill the basic properties of combinatorial manifolds contained in Proposition 1.2.4.

Proposition 2.1.3. *Let M be an NH-manifold of dimension d and let $\sigma \in M$ be a k-simplex. Then $lk(\sigma, M)$ is an NH-ball or an NH-sphere of dimension less than $d - k$.*

Proof. We proceed by induction on k. We may assume $k \geq 1$. In this case, we may write $\sigma = v * \eta$ with $0 \leq \dim(\eta) \leq k - 1$. Since by inductive hypothesis $lk(\eta, M)$ is an NH-manifold then $lk(\sigma, M) = lk(v, lk(\eta, M))$ is an NH-ball or NH-sphere. This completes the proof. □

As in the classic theory, the property stated in the preceding proposition will be called *regularity*. In order to show that the class of NH-manifolds is closed under PL-homeomorphisms, we need some preliminary results. The following lemma, which is needed in this form at this point, will be generalized in Theorem 2.1.9.

Lemma 2.1.4. *Let K be an NH-ball or an NH-sphere and let σ be a simplex disjoint from K. Then,*

(1) $\sigma * K$ is an NH-ball.

(2) $\partial\sigma * K$ is an NH-ball (if K is an NH-ball) or an NH-sphere (if K is an NH-sphere).

Proof. For the first part of the lemma, we proceed by double induction. Suppose that $\dim \sigma = 0$, i.e. $\sigma = v$ is a vertex, and that the result holds for NH-balls and NH-spheres K of dimension less than d. Note that $v * K \searrow 0$, so we only need to verify that $v * K$ is an NH-manifold. Take $w \in V_K$. Since $lk(w, v * K) = v * lk(w, K)$, by induction applied to $lk(w, K)$, it follows that $lk(w, v * K)$ is an NH-ball. On the other hand, $lk(v, v * K) = K$, which is an NH-ball or an NH-sphere by hypothesis. This shows that $v * K$ is an NH-manifold and proves the case $\dim \sigma = 0$. If now $\dim \sigma \geq 1$ then write $\sigma = v * \eta$ for some $v \in \sigma$. Since $\sigma * K = v * (\eta * K)$ then the result follows by induction applied to v and η.

For the second part of the lemma, suppose that $\dim \sigma = k \geq 1$ and let K be an NH-ball or an NH-sphere of dimension d. It is easy to see that the result is valid if $d = 0$. Suppose then that $d \geq 1$ and that the result holds for $t < d$. For any vertex $v \in \partial\sigma * K$, we have

$$lk(v, \partial\sigma * K) = \begin{cases} \partial\sigma * lk(v, K) & v \notin \partial\sigma \\ lk(v, \partial\sigma) * K & v \in \partial\sigma \end{cases}$$

In the first case, it follows by induction on d that $lk(v, \partial\sigma * K)$ is an NH-ball or NH-sphere. In the second case, we may use induction on k since $lk(v, \partial\sigma) = \partial lk(v, \sigma)$. This proves that $\partial\sigma * K$ is an NH-manifold. Now, if K is an NH-ball then $\partial\sigma * K \searrow 0$ and $\partial\sigma * K$ is again an NH-ball. If K is an NH-sphere write $K = B + L$ with B an NH-ball, L a combinatorial ball and $B \cap L = \partial L$. Since $\partial(\partial\sigma * L) = \partial\sigma * \partial L = \partial\sigma * B \cap \partial\sigma * L$, then $\partial\sigma * K = \partial\sigma * B + \partial\sigma * L$ is a valid decomposition of an NH-sphere by the previous case. This concludes the proof. □

Remark 2.1.5. In particular from Lemma 2.1.4 we deduce that M is an NH-manifold if and only if $st(v, M)$ is an NH-ball for all $v \in V_M$.

Theorem 2.1.6. *The classes of NH-manifolds, NH-balls and NH-spheres are closed under PL-homeomorphisms.*

Proof. It suffices to prove that K is an NH-manifold (resp. NH-ball, NH-sphere) if and only if any starring $(\tau, a)K$ is an NH-manifold (resp. NH-ball, NH-sphere). We suppose first that the result is valid for NH-manifolds of dimension d and prove that it is valid for NH-balls and NH-spheres of the same dimension. If $(\tau, a)K$ is an NH-ball of dimension d then K is also an NH-ball since it is an NH-manifold with $\alpha((\tau, a)K) \searrow 0$ for some subdivision α. On the other hand, if K is an NH-manifold of dimension d with $\alpha K \searrow 0$, by [25, Theorem I.2] we can find a stellar subdivision δ and an arbitrary subdivision β such that $\beta((\tau, a)K) = \delta(\alpha K)$. Since stellar subdivisions preserve collapses, $(\tau, a)K$ is collapsible and hence an NH-ball. Now, if K is an NH-sphere of dimension d with decomposition $B + L$ then the result holds by the previous case and the following identities.

$$(\tau, a)K = \begin{cases} (\tau, a)B + L, \text{ with } (\tau, a)B \cap L = \partial L & a \in B - L \\ \\ B + (\tau, a)L, \text{ with } B \cap (\tau, a)L = \partial L & a \in L - B \\ \\ (\tau, a)B + (\tau, a)L, \text{ with } (\tau, a)B \cap (\tau, a)L = (\tau, a)\partial L & a \in B \cap L = \partial L \end{cases}$$

In the last equation we used that $(\tau, a)\partial L = \partial(\tau, a)L$. The converse follows by replacing (τ, a) with $(\tau, a)^{-1}$ (see Remark 1.2.2).

We assume now that the result is valid for NH-balls and NH-spheres of dimension d and prove that it is valid for NH-manifolds of dimension $d + 1$. Suppose K is an NH-manifold of dimension $d + 1$ and let $v \in (\tau, a)K$. If $v \neq a$ then $lk(v, (\tau, a)K)$ is PL-homeomorphic to an elementary starring of $lk(v, K)$ by Lemma 1.1.5. The inductive hypothesis on $lk(v, K)$ shows that $lk(v, (\tau, a)K)$ is also an NH-ball or NH-sphere. On the other hand, $lk(a, (\tau, a)K) = \partial\tau * lk(\tau, K)$, which is an NH-ball or an NH-sphere by Lemma 2.1.4. Once again, the converse follows by replacing (τ, a) with $(\tau, a)^{-1}$ (the case $v = a$ not being possible). □

From Theorem 2.1.6 it follows that the simplex σ in Lemma 2.1.4 may be replaced by a general combinatorial ball and that $\partial\sigma$ by a general combinatorial sphere. We next prove the strongest statement that the join between NH-balls and/or NH-spheres produces again NH-balls and NH-spheres. We need two technical results.

Proposition 2.1.7. *Let K be an d-dimensional complex and let B be a combinatorial r-ball. Suppose $K+B$ is an NH-manifold such that*

(1) $K\cap B\subset\partial B$ is homogeneous of dimension $r-1$ and

(2) $lk(\sigma,K)$ is collapsible for all $\sigma\in K\cap B$

Then, K is an NH-manifold.

Proof. We show first that $K,B\subset K+B$ are top generated. Clearly, B is top generated since it intersects K in dimension $r-1$. On the other hand, a principal simplex in K which is not principal in $K+B$ must lie in $K\cap B$. Then, by hypothesis, it has a collapsible link in K. But this contradicts the fact that it is principal in K. Therefore $K,B\subset K+B$ are top generated and, in particular, $r\leq d$.

We prove the result by induction on r (the case $r=0$ being trivial). Let $r\geq 1$ and $v\in K$. If $v\notin B$ then $lk(v,K)=lk(v,K+B)$, which is an NH-ball or NH-sphere by hypothesis. Suppose now that $v\in K\cap B$ (and hence $lk(v,B)$ is a ball). If $r=1$, then $lk(v,K+B)=lk(v,K)+*$. It follows that $lk(v,K)$ is an NH-ball. Suppose $r\geq 2$ (and hence $d\geq 2$). We will see that the pair $lk(v,K),lk(v,B)$ also satisfies the conditions of the theorem. Note that $lk(v,K)+lk(v,B)=lk(v,K+B)$ is an NH-manifold by hypothesis and $lk(v,K)\cap lk(v,B)=lk(v,K\cap B)\subset\partial lk(v,B)$ is homogeneous of dimension $r-2$. On the other hand, if $\eta\in lk(v,K)\cap lk(v,B)$ then $v*\eta\in K\cap B$, so $lk(\eta,lk(v,K))=lk(v*\eta,K)$ is collapsible. By induction, it follows that $lk(v,K)$ is an NH-manifold, and, since it is also collapsible, it is an NH-ball. This shows that K is an NH-manifold. □

Lemma 2.1.8. *Suppose $S_1=G_1+L_1$ and $S_2=G_2+L_2$ are two disjoint NH-spheres. Then, $G_1*S_2+L_1*G_2$ is collapsible.*

Proof. Since G_1 and G_2 are collapsible, there exist subdivisions ϵ_1,ϵ_2 such that $\epsilon_1G_1\searrow 0$ and $\epsilon_2G_2\searrow 0$. We can extend these subdivisions to S_1 and S_2 and then suppose without loss of generality that $G_1\searrow 0$ and $G_2\searrow 0$. Note that

$$G_1*S_2\cap L_1*G_2=\partial L_1*G_2.$$

We will show that some subdivision of L_1*G_2 collapses to (the induced subdivision of) ∂L_1*G_2. Let α be an arbitrary subdivision of L_1 and δ a derived subdivision of Δ^r such that $\alpha L_1=\delta\Delta^r$. Then, $\alpha(L_1*G_2)=\delta(\Delta^r*G_2)$. Since $G_2\searrow 0$, then $\Delta^r*G_2\searrow\partial\Delta^r*G_2$ ([25, Corollary III.4]). Therefore

$$\alpha(L_1*G_2)=\delta(\Delta^r*G_2)\searrow\delta(\partial\Delta^r*G_2)=\alpha(\partial L_1*G_2).$$

We extend α to $(G_1*S_2+L_1*G_2)$ and then

$$\alpha(G_1*S_2+L_1*G_2)=\alpha(G_1*S_2)+\alpha(L_1*G_2)\searrow\alpha(G_1*S_2)+\alpha(\partial L_1*G_2)=\alpha(G_1*S_2).$$

By [25, Theorem III.6] there is a stellar subdivision s such that $s\alpha G_1\searrow 0$ and therefore

$$s\alpha(G_1*S_2+L_1*G_2)\searrow s\alpha(G_1*S_2)=s\alpha G_1*s\alpha S_2\searrow 0.\quad\square$$

Theorem 2.1.9. *Let B_1, B_2 be NH-balls and S_1, S_2 be NH-spheres. Then,*

(1) $B_1 * B_2$ and $B_1 * S_2$ are NH-balls.

(2) $S_1 * S_2$ is an NH-sphere.

Proof. Let K_1 represent B_1 or S_1 and let K_2 represent B_2 or S_2. We must show that $K_1 * K_2$ is an NH-ball or an NH-sphere. We proceed by induction on $s = \dim K_1 + \dim K_2$. If $s = 0, 1$ the result follows from Lemma 2.1.4. Let $s \geq 2$. We show first that $K_1 * K_2$ is an NH-manifold. Let $v \in K_1 * K_2$ be a vertex. Then,

$$lk(v, K_1 * K_2) = \begin{cases} lk(v, K_1) * K_2 & v \in K_1 \\ K_1 * lk(v, K_2) & v \in K_2 \end{cases}$$

Since $\dim lk(v, K_1) + \dim K_2 = \dim K_1 + \dim lk(v, K_2) = s-1$, then by induction, $lk(v, K_1 * K_2)$ is an NH-ball or an NH-sphere. It follows that $K_1 * K_2$ is an NH-manifold. Now, if $K_1 = B_1$ or $K_2 = B_2$, then $K_1 * K_2 \searrow 0$ and $K_1 * K_2$ is an NH-ball. We finally prove that $S_1 * S_2$ is an NH-sphere. Decompose $S_1 = G_1 + L_1$ and $S_2 = G_2 + L_2$ and note that $S_1 * S_2 = (G_1 * S_2 + L_1 * G_2) + L_1 * L_2$ and $(G_1 * S_2 + L_1 * G_2) \cap (L_1 * L_2) = \partial(L_1 * L_2)$. It then suffices to show that $(G_1 * S_2 + L_1 * G_2)$ is an NH-ball. Since by Lemma 2.1.8 it is collapsible we only need to check that it is an NH-manifold. In order to prove this, we apply Proposition 2.1.7 to the complex $G_1 * S_2 + L_1 * G_2$ and the combinatorial ball $L_1 * L_2$. The only non-trivial fact is that $lk(\sigma, G_1 * S_2 + L_1 * G_2)$ is collapsible for $\sigma \in \partial(L_1 * L_2)$. To see this, let $\eta \in \partial(L_1 * L_2) = \partial L_1 * L_2 + L_1 * \partial L_2$ and write $\eta = l_1 * l_2$ with $l_1 \in L_1$, $l_2 \in L_2$. Then,

$$lk(\eta, G_1 * S_2 + L_1 * G_2) = lk(l_1, G_1) * lk(l_2, S_2) + lk(l_1, L_1) * lk(l_2, G_2).$$

Now, if $l_1 \in L_1 - \partial L_1$ then $lk(l_1 * l_2, G_1 * S_2) = \emptyset$ and $lk(\eta, G_1 * S_2 + L_1 * G_2) = lk(l_1, L_1) * lk(l_2, G_2) \searrow 0$. By a similar argument, the same holds if $l_2 \in L_2 - \partial L_2$. If $l_1 \in \partial L_1$ and $l_2 \in \partial L_2$ then $lk(l_1, S_1) = lk(l_1, G_1) + lk(l_1, L_1)$ and $lk(l_2, S_2) = lk(l_2, G_2) + lk(l_2, L_2)$ are NH-spheres by Lemma 2.2.8. It follows by Lemma 2.1.8 that $lk(\eta, G_1 * S_2 + L_1 * G_2)$ is also collapsible. By Proposition 2.1.7, we conclude that $G_1 * S_2 + L_1 * G_2$ is an NH-manifold. □

As we pointed out earlier, the concept of pseudo manifold is intimately related to that of combinatorial manifold. We next introduce the non-pure counterpart of pseudo manifolds and prove that connected NH-manifolds are non-pure pseudo manifolds. We first need a definition.

Definition. Two principal simplices $\sigma, \tau \in M$ are said to be *adjacent* if the intersection $\tau \cap \sigma$ is an immediate face of σ or τ.

Definition. An *NH-pseudo manifold* of dimension d is a complex of dimension d such that (1) for each ridge $\sigma \in M$, $lk(\sigma, M)$ is either a point or an NH-sphere of homotopy dimension 0; and (2) given any two principal simplices $\sigma, \tau \in M$, there exists a sequence $\sigma = \eta_1, \ldots, \eta_s = \tau$ of principal simplices of M such that η_i is adjacent to η_{i+1} for every $1 \leq i \leq s-1$.

Lemma 2.1.10. *A connected NH-manifold is an NH-pseudo manifold.*

The proof of Lemma 2.1.10 will follow from the next result.

Lemma 2.1.11. *If K is a connected complex such that $st(v, K)$ is an NH-pseudo manifold for all $v \in V_K$ then K is an NH-pseudo manifold.*

Proof. We will show that K satisfies properties (1) and (2) of Lemma 2.1.10. Let $\sigma \in K$ be a ridge and let $v \in \sigma$ be any vertex. Then σ is also a ridge in $st(v, K)$ and $lk(\sigma, K) = lk(\sigma, st(v, K))$. Therefore K satisfies property (1).

Let $\nu, \tau \in K$ be maximal simplices and let $v \in \nu,\ w \in \tau$. Take an edge path from v to w. We will prove that K satisfies property (2) by induction on the length r of the edge path. If $r = 0$, then $v = w$. In this case, $\nu, \tau \in st(v, K)$ and the results follows by hypothesis. Suppose now that $e_1, \ldots, e_r$ is an edge path from v to w of length $r \geq 1$. Take maximal simplices η_i such that $e_i < \eta_i$. Note that $\eta_1 \cap \eta_2$ contains the vertex $e_1 \cap e_2$. By hypothesis, $st(e_1 \cap e_2, K)$ satisfies property (2) and therefore we can join η_1 with η_2 by a sequence of adjacent maximal simplices. Now the result follows by induction. □

Proof of Lemma 2.1.10. We proceed by induction on the dimension d of M. By Lemma 2.1.11, it suffices to prove that $st(v, M)$ is an NH-pseudo manifold for every vertex v. The case $d = 0$ is trivial. Suppose that $d \geq 1$ and that the result is valid for $k \leq d - 1$. Now, if $lk(v, M)$ is an NH-ball or a connected NH-sphere then, by induction, it is an NH-pseudo manifold. It follows that $st(v, M)$ is also an NH-pseudo manifold since it is a cone of an NH-pseudo manifold. In the other case, $lk(v, M)$ is an NH-sphere of homotopy dimension 0 of the form $B + \{w\}$, for some NH-ball B and vertex w. Since $v * B$ is an NH-pseudo manifold, it follows that $st(v, M)$ is also an NH-pseudo manifold. □

We close this section characterizing NH-spheres with maximal homotopy dimension. As one may conjecture, these are combinatorial spheres.

Proposition 2.1.12. *Let M be a connected NH-manifold of dimension d such that $H_d(M; \mathbb{Z}_2) \neq 0$. Then, M is a combinatorial d-manifold (without boundary). In particular, if S is an NH-sphere with $\dim_h(S) = \dim(S)$ then S is a combinatorial sphere.*

Proof. By Theorem 2.1.6 it suffices to prove that M is homogeneous. Let c be a generating d-cycle of $H_d(M; \mathbb{Z}_2)$ and let $K \subset M$ be the subcomplex generated by the d-simplices appearing in c with nonzero coefficients. We shall show that $M = K$. Note that since $K \subset M$ is top generated and M is an NH-pseudo manifold then K is a weak pseudo manifold without boundary (since c is a cycle). If $M \neq K$, let $\eta \in M - K$ be a principal simplex adjacent to K and set $\rho = \eta \cap K$. Since by dimensional considerations $\rho \prec \eta$ then $lk(\rho, M) = lk(\rho, M - \eta) + lk(\rho, \eta)$ is an NH-sphere of homotopy dimension 0. But $lk(\rho, K) \subset lk(\rho, M - \eta)$ is a weak pseudo manifold without boundary and hence $H_{\dim(lk(\rho,K))}(lk(\rho, K); \mathbb{Z}_2) \neq 0$. This contradicts the fact that $lk(\rho, M - \eta)$ is an NH-ball since a generating cycle in $lk(\rho, K)$ is also generating in $lk(\rho, M - \eta)$. Note also that $\partial M = \partial K = \emptyset$. □

§2.2 Boundary and pseudo boundary

The concept of boundary is central in manifold theory. As we saw earlier, it is classically defined for complexes that are homogeneous. The boundary of a general (not necessarily homogeneous) complex is not defined, probably because of lack of an intuitive geometrical notion of this concept in this setting. In this section we shall provide a definition of boundary and "boundary-like" subset of simplices (called pseudo boundary) based on the structural properties of these complexes. These concepts are fundamental for the generalization of classical theorems to the non-homogeneous theory.

We first digress briefly on a possible definition of boundary for general complexes taking into account the equivalence between d-homogeneous complexes and $(d+1)$-forms provided by the algebraic approach to simplicial complexes studied by Alexander [2] (see paragraph after Lemma 1.2.1). There is no restriction in describing a *general* complex L from its maximal simplices by a (non-necessarily-homogeneous) polynomial $g \in \mathbb{Z}_2[V_L]$ whose monomials are variable-wise square free. Under this identification the boundary of L may be considered as the complex associated to $\partial g := \sum_j \frac{\partial g}{\partial x_j}$. Note that every monomial appearing with non-zero coefficients comes necessarily from a ridge of L. Caution must be taken since the monomials appearing in ∂g with non-zero coefficients do not longer represent principal simplices of ∂L as some higher degree polynomials may contain some lower degree ones. For example, letting $h = x_1x_2x_3 + x_3x_4$ we have $\partial h = x_2x_3 + x_1x_3 + x_1x_2 + x_3 + x_4$, where x_3 is not a principal face of ∂h. Of course, this does not change the spanned complex ∂L. But if we want to preserve bijectivity between the geometrical an algebraic representations, we must *normalize* ∂h so to discard every monomial x^α such that $x^\alpha | x^\beta$.

We shall define the boundary of an NH-manifold by use of the characterization of the boundary of combinatorial manifolds. As it turns out, this coincides with the algebraic notion just introduced.

Definition. Let M be an NH-manifold. The *pseudo boundary* of M is the set of simplices $\tilde{\partial} M$ whose links are NH-balls. The *boundary* of M is the subcomplex ∂M spanned by $\tilde{\partial} M$. In other words, ∂M is the closure $\overline{\tilde{\partial} M}$.

It is not hard to see that ∂M is generated by ridges of M. We note that the pseudo boundary of M is not in general a complex as the examples in Figure 2.3 show. It is also clear that $\tilde{\partial} M = \partial M$ for any combinatorial manifold M. It is interesting to see that this is the only case where this happens. We prove this in Proposition 2.2.3.

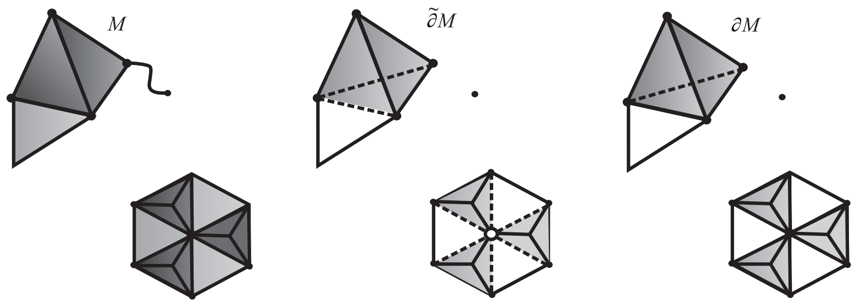


Figure 2.3: Boundary and pseudo boundary of NH-manifolds.

Remark 2.2.1. It is easy to see that $\partial(\alpha M) = \alpha \partial M$ for any subdivision α. It follows from collapsibility that a non-trivial NH-ball has non-empty boundary.

Lemma 2.2.2. *Let M be an NH-manifold and let $\sigma \in M$. If σ is a face of two principal simplices of different dimensions then $\sigma \in \partial M$.*

Proof. Let $\tau_1 = \sigma * \eta_1$ and $\tau_2 = \sigma * \eta_2$ be principal simplices such that $\dim \tau_1 \neq \dim \tau_2$. By Lemma 2.1.10 we may assume that τ_1 and τ_2 are adjacent. Let $\rho = \tau_1 \cap \tau_2$ and suppose

$\rho \prec \tau_1$. Then, $lk(\rho, M)$ is an NH-sphere of homotopy dimension 0 with decomposition $lk(\rho, M - \tau_1) + lk(\rho, \tau_1)$. Since $\dim lk(\rho, M - \tau_1) \geq 1$ then $\tilde{\partial} lk(\rho, M - \tau_1) = \tilde{\partial} lk(\rho, M)$ is non-empty by the previous remark. For any simplex ν in $\tilde{\partial} lk(\rho, M)$, $\nu * \rho \in \tilde{\partial} M$. Thus $\sigma \in \partial M$. □

Proposition 2.2.3. *If M is a connected NH-manifold such that $\tilde{\partial} M = \partial M$ then M is a combinatorial manifold. In particular, NH-manifolds without boundary (or pseudo boundary) are combinatorial manifolds.*

Proof. If M is non-homogeneous, by Lemma 2.1.10 there exist two adjacent principal simplices τ_1, τ_2 of different dimensions. By Lemma 2.2.2, $\rho = \tau_1 \cap \tau_2 \in \partial M - \tilde{\partial} M$. □

The following result will be used in the next sections. It is the non-homogeneous version of the well-known fact that any d-homogeneous subcomplex of a d-combinatorial manifold with non-empty boundary has also a non-empty boundary.

Lemma 2.2.4. *Let M be a connected NH-manifold with non-empty boundary and let $L \subseteq M$ be a top generated NH-submanifold. Then, $\partial L \neq \emptyset$.*

Proof. We may assume $L \neq M$. We proceed by induction on $d = \dim M$. Since the 1-dimensional case is clear we let $d \geq 2$. Take adjacent principal simplices $\sigma \in L$ and $\tau \in M - L$ and let $\rho = \sigma \cap \tau$. If $\dim \sigma = \dim \tau$ then $lk(\rho, M) = \partial \Delta^1$ and therefore, $\rho \in \partial L$. If $\dim \sigma \neq \dim \tau$ then $lk(\rho, M) = B + \{v\}$ is a non-homogeneous NH-sphere of homotopy dimension 0. We analyze both cases: $\rho \prec \sigma$ and $\rho \prec \tau$. If $\rho \prec \sigma$ then $lk(\rho, L)$ is either a 0-ball, which implies $\rho \in \tilde{\partial} L$, or a non-homogeneous NH-sphere of homotopy dimension 0. In this case, $\tilde{\partial} lk(\rho, L) \neq \emptyset$ by Proposition 2.2.3. If $\rho \prec \tau$ then $\tilde{\partial} lk(\rho, L) \neq \emptyset$ by induction applied to $lk(\rho, L) \subset B$. In any case, if $\eta \in \tilde{\partial} lk(\rho, L)$ then $\eta * \rho \in \tilde{\partial} L$. □

Corollary 2.2.5. *If M is a connected NH-manifold of dimension $d \geq 1$ containing a top generated combinatorial manifold L without boundary then $M = L$.*

Note that if $S = B + \{v\}$ is a non-homogeneous NH-sphere of homotopy dimension 0 and M is a non-trivial top generated combinatorial d-manifold contained in S, then $M \subseteq B$. This implies that $\partial M \neq \emptyset$ by Corollary 2.2.5. For future reference, we state this fact in the following

Corollary 2.2.6. *A non-homogeneous NH-sphere of homotopy dimension 0 cannot contain a non-trivial top generated combinatorial manifold without boundary.*

In contrast to the classical situation, the boundary of an NH-manifold M is not in general an NH-manifold, so the boundary of ∂M is not defined. However, similarly as in the homogeneous setting, boundary of NH-manifolds preserve the property of not possessing free faces. Also, since the links of simplices in NH-manifolds are also NH-manifolds, it makes sense to study their boundary, and we show that the standard relation holds.

Proposition 2.2.7. *Let M be an NH-manifold and let $\sigma \in M$. Then,*

(1) $lk(\sigma, \partial M) = \partial lk(\sigma, M)$.

(2) ∂M has no collapsible simplices.

Proof. For (1) we prove the double inclusion. Let $\eta \in \partial lk(\sigma, M)$. Then, there exists a simplex $\tau \in \tilde{\partial} lk(\sigma, M)$ with $\eta < \tau$. Since $lk(\sigma * \tau, M) = lk(\tau, lk(\sigma, M))$ then $\sigma * \tau \in \tilde{\partial} M$. Hence, $\sigma * \eta \in \partial M$ as we wanted. Conversely, if $\eta \in lk(\sigma, \partial M)$ then there is simplex $\tau > \eta$ such that $\sigma * \tau \in \tilde{\partial} M$. By the same equality as before, $\tau \in \tilde{\partial} lk(\sigma, M)$ and, therefore, $\eta \in \partial lk(\sigma, M)$.

For (2), let ρ be a ridge in ∂M. By (1), it suffices to show that the boundary of any NH-ball or NH-sphere cannot be a singleton. But if B is an NH-ball with $\partial B = \{v\}$ then $\tilde{\partial} B = \{v\}$ and $lk(v, B)$ is an NH-ball. If $lk(v, B)$ is trivial then Lemma 2.2.2 implies that $B - v$ is homogeneous of dimension 1, from where it follows that B is a 1-ball. If $lk(v, B)$ is not trivial then there is a $\rho \in \partial lk(v, B)$ by Remark 2.2.1 and $v * \rho \in \partial B$. In any case, we reach a contradiction. Suppose now that $S = B + L$ is an NH-sphere with $\partial S = \{v\}$. If $\dim L = 0$ then $\partial S = \partial B$ and a contradiction arises from the previous case. If $\dim L \geq 1$ then S is connected and there is a ridge $\eta \in \partial S - \tilde{\partial} S$ by Lemma 2.2.2. This is a contradiction since $v \in \tilde{\partial} S$. □

We note that (2) of the last proposition reformulates in some way the fact that the boundary of a combinatorial manifold has no boundary.

We can readily show that the notion of boundary defined in terms of the structure of NH-manifolds coincides with the algebraic approach given in page 56. Let $f \in \mathbb{Z}_2[V_M]$ be the polynomial associated to M. Let $\sigma \in \partial M$ be principal (i.e. $\sigma \in \tilde{\partial} M$) and write $V_\sigma = \{x_1, \ldots, x_t\}$. Since $\partial lk(\sigma, M) = lk(\sigma, \partial M) = \{\emptyset\}$ then $lk(\sigma, M) = v$. Therefore, σ appears only as a face of $\tau = v * \sigma$, and hence $x_1 \ldots x_t$ has a non-zero coefficient in ∂f. Conversely, if $x_1 \ldots x_t$ appears in the normalized form of ∂f with non-zero coefficient then σ is a ridge and face of an odd number of principal t-simplices of M. Since M is an NH-pseudo manifold then σ is face of only one t-simplex $\tau = v * \sigma$. Since ∂f is normalized then $lk(\sigma, M) = v$ and hence $\sigma \in \tilde{\partial} M$.

A simplex $\sigma \in M$ will be called *internal* if $lk(\sigma, M)$ is an NH-sphere, i.e. if $\sigma \notin \tilde{\partial} M$. We denote by $\overset{\circ}{M}$ the relative interior of M, which is the set of its internal simplices.

Lemma 2.2.8. *Let S be an NH-sphere with decomposition $B + L$. Then, every $\sigma \in L$ is internal in S. Furthermore, if $\sigma \in \partial L$ then $lk(\sigma, S) = lk(\sigma, B) + lk(\sigma, L)$ is a valid decomposition. In particular, $\tilde{\partial} S = \tilde{\partial} B - L$.*

Proof. This is a particular case of Lemma 2.3.3. □

A *spine* of a combinatorial d-manifold M is a subcomplex K such that $M \searrow K$ and $\dim(K) \leq d - 1$. Spines play an important role in the study of manifolds. It is a standard result that a polyhedral manifold with boundary always has spines. The following result, which is a corollary of Proposition 2.1.12, show that NH-manifolds with boundary also have them.

Theorem 2.2.9 (Existence of spines for NH-manifolds)**.** *Every connected NH-manifold M with non-empty boundary has a spine (i.e. it collapses to a subcomplex of smaller dimension).*

Proof. Let d be the dimension of M and let Y^d be the d-homogeneous subcomplex of M (i.e. the subcomplex of M generated by the d-simplices). Start collapsing the d-simplices of Y^d and suppose we get stuck before depleting all the d-simplices. Then, there is a boundaryless d-pseudo manifold $L \subset Y^d \subset M$ and hence $0 \neq H_d(L; \mathbb{Z}_2) \subset H_d(M; \mathbb{Z}_2)$. By Proposition 2.1.12, M is a combinatorial manifold without boundary, which is a contradiction. □

We finish the section proving that NH-manifolds admit a regular decomposition in pure pieces. More concretely, NH-manifolds can be constructed from pseudo manifolds (of different dimensions) *in a unique way* which interconnect through the *anomaly complex*, which we next introduce.

Definition. Let M be an NH-manifold. The *anomaly complex* of M is the subcomplex

$$\mathsf{A}(M) = \{\sigma \in M : lk(\sigma, M) \text{ is not homogeneous}\}.$$

The fact that $\mathsf{A}(M)$ is a simplicial complex follows from Remark 1.1.4. The anomaly complex gathers in some way the information about the non-pure parts of an NH-manifold. Figure 2.4 shows examples of anomaly complexes.

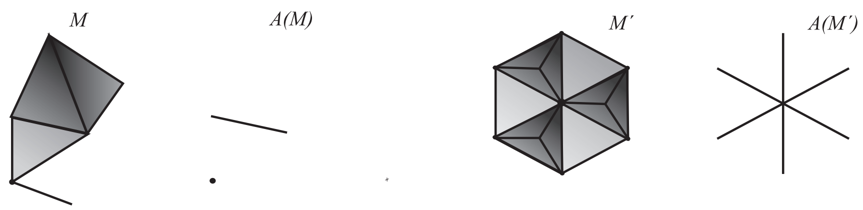


Figure 2.4: Anomaly complex.

Proposition 2.2.10. *For any NH-manifold M, $\partial M = \tilde{\partial} M + \mathsf{A}(M)$.*

Proof. If $\sigma \in \mathsf{A}(M)$ then σ is face of two principal simplices of M of different dimensions. Therefore $\sigma \in \partial M$ by Lemma 2.2.2. For the other inclusion, let $\sigma \in \partial M - \tilde{\partial} M$. Then $lk(\sigma, M)$ is an NH-sphere and $\sigma < \tau$ with $\tau \in \tilde{\partial} M$. Write $\tau = \sigma * \eta$, so $lk(\tau, M) = lk(\eta, lk(\sigma, M))$. If $\sigma \notin \mathsf{A}(M)$ then $lk(\sigma, M)$ is a combinatorial sphere and so is $lk(\tau, M)$, contradicting the fact that $\tau \in \tilde{\partial} M$. □

Proposition 2.2.11. *Let M be an NH-manifold. There exists a* unique *collection $P_1, \ldots, P_t$ of top generated pseudo manifolds in M, each of which is (inclusion-wise) maximal, and whose union equals all M. Also, $\sigma \in \mathsf{A}(M)$ if, and only if, $\sigma \in P_i \cap P_j$ for some $i \neq j$.*

Proof. Start by noting that for any principal simplex $\sigma \in M$ there exist an unique top generated pseudo manifold which contains σ and is (inclusion-wise) maximal in M. Indeed, suppose $\sigma \in P \cap P'$, where P and P' are two such pseudo manifolds. Since σ is principal and P, P' are top generated then $\dim P = \dim P'$. The complex $P'' = P + P'$ is a top generated pseudo manifold in M since M is an NH-pseudo manifold and any ridge in P'' is a ridge in M. Since P and P' are maximal then $P = P'$ and the claim is proven. To form the collection $P_1, \ldots, P_t$, take a principal simplex $\sigma \in M$ and consider the *only* maximal pseudo manifold P_1 in M containing σ. Take next any principal simplex $\sigma' \notin P_1$ and let P_2 be the only maximal pseudo manifold in M containing σ'. This process eventually ends because of finiteness.

For the second statement, suppose $\sigma \in P_i \cap P_j$ for some $i \neq j$. Since the pseudo manifolds are top generated, there are principal simplices $\tau \in P_i$ and $\tau' \in P_j$ with $\sigma \in \tau, \tau'$. Write $\tau = \sigma * \eta$ and $\tau' = \sigma * \eta'$. If $\dim P_i \neq \dim P_j$ then $\eta, \eta' \in lk(\sigma, M)$ are principal of different dimensions, and hence $\sigma \in \mathsf{A}(M)$. If $\dim P_i = \dim P_j$ and $lk(\sigma, M)$ is homogeneous then we find a sequence of principal simplices $\eta = E_0, E_1, \ldots, E_s = \eta'$, all of the same dimensions. The sequence $\tau = \sigma * E_0, \sigma * E_1, \ldots, \sigma * E_s = \tau'$ shows that $P_i + P_j$ is a pseudo manifold, and hence $P_i = P_j$, contradicting that $i \neq j$.

Finally, if $\sigma \in \mathsf{A}(M)$ belongs to only one P_i then $lk(\sigma, M) = lk(\sigma, P_i)$, which is a homogeneous complex. Therefore, there exists a P_j containing σ with $\dim P_i \neq \dim P_j$. □

Let us stress that *any* complex K admits a maximal decomposition as an union of pseudo manifolds (proceeding in the very same way as in the first part of the last proof). However, for a general complex, the decomposition may not be unique (see Example 2.2.12). Also, one would hope that every NH-manifold could be decomposed in a *unique* way as an union of maximal combinatorial manifolds, instead of pseudo manifolds. This is in general not possible as Example 2.2.13 shows.

Example 2.2.12. A decomposition in maximal pseudo manifolds for the complex in Figure 2.5 (a) consists in a pseudo manifold P_1 with two triangles and a pseudo manifold P_2 with one. There are three ways to choose these complexes.

Example 2.2.13. A decomposition in maximal *manifolds* for the NH-manifold in Figure 2.5 (b) has necessarily three complexes: a tetrahedron and two 2-dimensional manifolds. This is because the two triangles containing the vertex v cannot belong to the same manifold. There is therefore to ways to choose the 2-dimensional manifolds (one of which will consist in a single triangle).

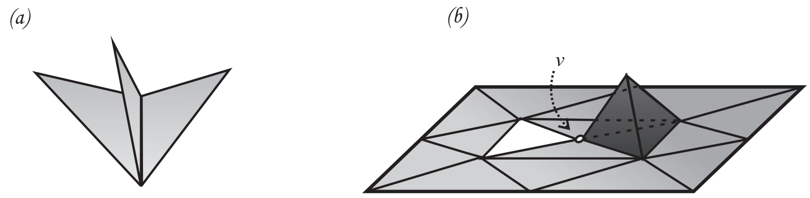


Figure 2.5: Examples (a) and (b).

§2.3 NH-bouquets and shellability

Shellability in the non-pure setting was introduced by Björner and Wachs [7] in the nineties with the motivation to analyze examples coming from the theory of subspace arrangements.

In this section we study and characterize shellable NH-manifolds. These are a special family of manifolds called NH-bouquets, which actually further generalize NH-balls and NH-spheres.

2.3.1 NH-bouquets

Similarly as in the homogeneous setting, an NH-sphere is obtained by "gluing" a combinatorial ball to an NH-ball along its entire boundary. In the homogeneous case one can no longer glue another ball to a sphere for it would produce a complex which is not a manifold (not even a pseudo manifold). The existence of boundary in non-homogeneous NH-spheres allows us to glue balls and obtain again an NH-manifold. This is the idea behind the notion of NH-bouquet. This concept arises naturally when studying shellability of non-homogeneous manifolds as we shall see in subsequent sections.

Definition. We define an NH-*bouquet* G of dimension d and *index* k by induction on k.

- If $k = 0$ then G is an NH-ball of dimension d.
- If $k \geq 1$ then G is an NH-manifold of dimension d such that there exist a top generated NH-bouquet S of dimension d and index $k-1$ and a top generated combinatorial ball L, such that $G = S + L$ and $S \cap L = \partial L$.

Clearly an NH-bouquet of index 1 is an NH-sphere. It is easy to see that for every $d \geq 0$ and every $k \geq 0$ there exists an NH-bouquet G of dimension d and index k. Figure 2.6 shows some examples of underlying spaces of NH-bouquets of low dimensions. Note that following the inductive construction of an NH-bouquet G of index k we may define, just as for NH-spheres, a *decomposition* $G = B + L_1 + \cdots + L_k$ consisting of top generated subcomplexes of G such that B is an NH-ball, L_i is a combinatorial ball for each $i = 1, \ldots, k$ and $(B + \cdots + L_i) \cap L_{i+1} = \partial L_{i+1}$. Of course, this decomposition is not unique; although the index k in any decomposition is uniquely determined, as we will shortly see.

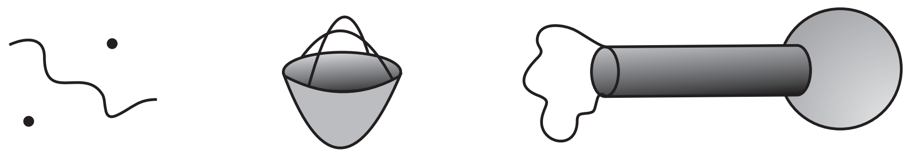


Figure 2.6: NH-bouquets of index 2. Note that the rightmost example is obtained from a solid cylinder by attaching a 1-disk to one side and a 2-disk to the other.

NH-bouquets can also be seen as generalizations of combinatorial balls and spheres since, leaving aside homotopy properties, the construction is the same for every index k. Actually, it is not hard to see that a homogeneous NH-bouquet of dimension $d \geq 1$ is a combinatorial d-ball or d-sphere. This follows at once from Theorem 2.1.2 and Corollary 2.2.5. Similarly as in Theorem 2.1.6, it can be proved that the class of NH-bouquets is closed under PL-homeomorphisms.

We show next that the index of an NH-bouquets is well defined and characterize the homotopy type of NH-bouquets.

Lemma 2.3.1. *If $G = B + L_1 + \cdots + L_k$ is a decomposition of an NH-bouquet of index $k \geq 2$, then $L_i \cap L_j = \partial L_i \cap \partial L_j$ for all $1 \leq j < i \leq k$.*

Proof. $L_i \cap L_j \subseteq \partial L_i$ by definition. Suppose that $L_i \cap L_j \nsubseteq \partial L_j$. Then there exists a simplex $\sigma \in L_i \cap L_j$ such that $lk(\sigma, L_j)$ is a sphere. By Corollaries 2.2.5 and 2.2.6, $lk(\sigma, L_j) = lk(\sigma, G)$. In particular, $lk(\sigma, L_i) \subseteq lk(\sigma, L_j)$. But if $\nu \in lk(\sigma, L_i)$ is principal, then $\sigma * \nu$ is a maximal simplex in G which is contained in $L_i \cap L_j \subseteq \partial L_i$, a contradiction. □

Proposition 2.3.2. *If $G = B + L_1 + \cdots + L_k$ is a decomposition of an NH-bouquet, then $\partial L_i \subseteq B$ for every $i = 1, \ldots, k$. In particular, an NH-bouquet of index k is homotopy equivalent to a bouquet of spheres of dimensions $\dim L_i$, for $1 \leq i \leq k$.*

Proof. $\partial L_1 \subseteq B$ by definition. For $i \geq 2$ the result follows immediately by induction and Lemma 2.3.1.

For the second statement, note that, since $\partial L_i \subseteq B$ for every i, G is homotopy equivalent to a CW-complex obtained by attaching cells of dimensions $\dim L_i$ to a point. □

The following result extends Lemma 2.2.8 and will be frequently used later.

Lemma 2.3.3. *Let $G = B + L_1 + \cdots + L_k$ be a decomposition of an NH-bouquet. Then every simplex in each L_i is internal in G. Furthermore, if $\sigma \in \partial L_i$ then $lk(\sigma, G)$ is an NH-sphere with decomposition $lk(\sigma, B) + lk(\sigma, L_i)$. In particular, $\tilde{\partial} G = \tilde{\partial} B - \cup_i L_i$.*

Proof. It is clear that every simplex internal in L_i is internal in G. Given $\sigma \in \partial L_i$, by Proposition 2.3.2 $lk(\sigma, G) = lk(\sigma, B) + lk(\sigma, L_i)$. Also $lk(\sigma, L_i) \cap lk(\sigma, B) = \partial lk(\sigma, L_i)$. □

2.3.2 Shellable NH-manifolds

The notion of (non-pure) shellability is a straight generalization of the classical constructive definition. As it is evident, a definition by means of elementary shellings is not possible without assuming some structural properties of the complexes. In the next section we shall provide an equivalent definition based on a newly non-pure version of shelling.

Definition. A finite (non-necessarily homogeneous) simplicial complex is shellable if there is a linear order $F_1, \ldots, F_t$ of its maximal simplices such that $F_k \cap (F_1 + \cdots + F_{k-1})$ is $(\dim F_k - 1)$-homogeneous for all $2 \leq k \leq t$.

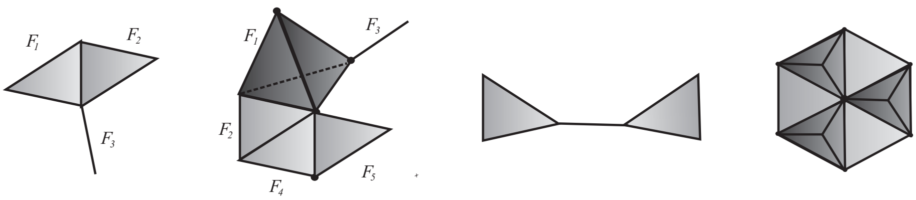


Figure 2.7: Shellable and non-shellable complexes. The two leftmost complexes are non-pure shellable with the given shelling order (which is not unique). The two rightmost complexes are not shellable.

To characterize shellable NH-manifolds we introduce first some definitions. A simplex F_k is said to be a *spanning simplex* if $F_k \cap (F_1 + \cdots + F_{k-1}) = \partial F_k$. It is not hard to see that the spanning simplices may be moved to any later position in the shelling order (see for example [33]). It is known that a shellable complex is homotopy equivalent to a wedge of spheres, which are indexed by the spanning simplices (see [33, Theorem 12.3]). In particular, shellable NH-balls cannot have spanning simplices and shellable NH-spheres have exactly one spanning simplex. In general, a shellable NH-bouquet of index k must have exactly k spanning simplices.

Theorem 2.3.4. *Let M be a shellable NH-manifold. Then, for every shelling order $F_1, \ldots, F_t$ of M and every $0 \leq l \leq t$, $\mathcal{F}_l(M) = F_1 + \cdots + F_l$ is an NH-manifold. Moreover, $\mathcal{F}_l(M)$ is an NH-bouquet of index $\sharp\{F_j \in \mathcal{T} \,|\, j \leq l\}$, where $\mathcal{T}$ is the set of spanning simplices. In particular, M is an NH-bouquet of index $\sharp\mathcal{T}$.*

Proof. We proceed by induction on $n = \dim M$. Suppose $n \geq 1$ and fix a shelling order $F_1, \ldots, F_t$. Let $1 \leq l \leq t$ and let $v \in M$ be a vertex. Since $lk(v, M)$ is a shellable NH-ball or NH-sphere with shelling order $lk(v, F_1), \ldots, lk(v, F_t)$ (some of them possibly empty), then by induction $\mathcal{F}_j(lk(v, M))$ is an NH-bouquet of index at most 1 for all $1 \leq j \leq l$. Since $lk(v, \mathcal{F}_l(M)) = \mathcal{F}_l(lk(v, M))$ then $\mathcal{F}_l(M)$ is an NH-manifold. To see that $\mathcal{F}_l(M)$ is actually an NH-bouquet, reorder $F_1, \ldots, F_l$ so that the spanning simplices are placed at the end of the order. If F_{p+1} is the first spanning simplex in the order, then $\mathcal{F}_p(M)$ is a collapsible NH-manifold (see [33, Theorem 12.3]) and hence an NH-ball. Then, $\mathcal{F}_l(M) = \mathcal{F}_p(M) + F_{p+1} + \cdots + F_l$ is an NH-bouquet of index $\sharp\{F_j \in \mathcal{T} | j \leq l\}$ by definition. □

§2.4 Regular expansions

Recall that a regular expansion in a d-combinatorial manifold M is a geometrical expansion $M \to M + B^d$ such that $M \cap B^d \subset \partial M$. Alexander's Theorem asserts that this move produces a new combinatorial d-manifold (see Theorem 1.3.2). In this section we provide two versions of a generalization of this theorem to the non-pure context. The first of these generalizations shows that a more general type of "expansion" can be made and still retain the structure of NH-manifold. The second generalization involves classical regular expansions but extracts a more strong consequence: the new NH-manifold is equivalent (in a generalized piece-wise linear way) to the original one.

2.4.1 Non-homogeneous regular expansions

The present section is devoted entirely to prove one of the central theorems of NH-manifold theory: a first generalization of Alexander's theorem on regular expansions. As it will evident in following chapters, this result will be of fundamental need in virtually every application of non-homogeneous manifolds. We need two preliminary results.

Lemma 2.4.1. *Let B be a combinatorial d-ball and let $L \subset \partial B$ be a combinatorial $(d-1)$-ball. Then, there exists a stellar subdivision s such that $sB \searrow sL$.*

Proof. By [25, Lemma III.8] there exists a derived subdivision δ and a subdivision α such that $\delta B = \alpha\Delta^d$ and $\delta L = \alpha\Delta^{d-1}$, where Δ^{d-1} is an $(d-1)$-face of Δ^d. Now, by [25, Lemma III.7] there exists a stellar subdivision $\tilde{s}$ such that $\tilde{s}\alpha\Delta^d \searrow \tilde{s}\alpha\Delta^{d-1}$ and therefore $\tilde{s}\delta B \searrow \tilde{s}\delta L$. □

Corollary 2.4.2. *Let B be a combinatorial d-ball and let $K \subset \partial B$ be a collapsible complex. Then, there exists a stellar subdivision s such that $sB \searrow sK$.*

Proof. Subdivide B barycentrically twice and consider a regular neighborhood N of K'' in $\partial B''$ (see [25, Corollary III.17] and §2.5.1). Since K'' is collapsible, then N is an $(d-1)$-ball. Since $N \subset \partial B''$, by the previous lemma, there is a stellar subdivision $\tilde{s}$ such that $\tilde{s}B'' \searrow \tilde{s}N$. We conclude that $\tilde{s}B'' \searrow \tilde{s}N \searrow \tilde{s}K''$. □

Theorem 2.4.3. *Let M be an NH-manifold and B^r a combinatorial r-ball. Suppose $M \cap B^r \subseteq \partial B^r$ is an NH-ball or an NH-sphere generated by ridges of M or B^r and that $(M \cap B^r)^\circ \subseteq \tilde{\partial} M$. Then $M + B^r$ is an NH-manifold. Moreover, if M is an NH-bouquet of index k and $M \cap B^r \neq \emptyset$ for $r \neq 0$, then $M + B^r$ is an NH-bouquet of index k (if $M \cap B^r$ is an NH-ball) or $k+1$ (if $M \cap B^r$ is an NH-sphere).*

Proof. We note first that $M, B^r \subset M + B^r$ are top generated. Since $M \cap B^r \subseteq \partial B^r$ then B^r is top generated. On the other hand, if σ is a principal simplex in M which is not principal in $M + B^r$ then σ must be in $M \cap B^r$. Since $\sigma \notin \tilde{\partial} M$ then $\sigma \notin (M \cap B^r)^\circ$. Hence, σ is not principal in $M \cap B^r$, which contradicts the maximality of σ in M.

We shall prove the result by induction on r. The case $M \cap B^r = \emptyset$ is clear, so let $r \geq 1$ and assume $M \cap B^r \neq \emptyset$. We need to prove that every vertex in $M + B^r$ is regular. It is clear that the vertices in $(M - B^r) + (B^r - M)$ are regular since B^r and M are NH-manifolds. Consider then a vertex $v \in M \cap B^r$. We claim that the pair $lk(v, M), lk(v, B^r)$ fulfills the hypotheses of the theorem. Note that $lk(v, M)$ is an NH-ball or NH-sphere, $lk(v, B^r)$ is a combinatorial ball, since $v \in M \cap B^r \subseteq \partial B^r$, and $lk(v, M \cap B^r)$ is an NH-ball or NH-sphere contained in $\partial lk(v, B^r)$. Note also that the inclusion $(M \cap B^r)^\circ \subseteq \tilde{\partial} M$ implies that $lk(v, M \cap B^r)^\circ \subseteq \tilde{\partial} lk(v, M)$. We now check that $lk(v, M \cap B^r)$ is generated by ridges of $lk(v, M)$ or $lk(v, B^r)$. This is easily seen if $lk(v, M \cap B^r) \neq \emptyset$. For the case $lk(v, M \cap B^r) = \emptyset$ we need to show that there is a principal 0-simplex in $lk(v, M)$ or $lk(v, B^r)$. Now, $lk(v, M \cap B^r) = \emptyset$ implies that v is principal in $M \cap B^r$, so $v \in (M \cap B^r)^\circ \subseteq \tilde{\partial} M$ and $lk(v, M)$ is an NH-ball (and hence, collapsible). And since $v \in M \cap B^r \subseteq \partial B^r$ then $lk(v, B^r)$ is a ball. Now, if v is a ridge in B^r then $r = 1$ and, hence, $lk(v, B^1) = *$. If, on the other hand, v is a ridge of M then there exists a principal 1-simplex σ with $v \prec \sigma$. Since σ is principal in M, $* = lk(v, \sigma)$ is principal in $lk(v, M)$. Since $lk(v, M)$ is collapsible, then $lk(v, M) = *$.

Therefore, by induction, $lk(v, M + B^r)$ is an NH-manifold. Now, if $lk(v, M \cap B^r) \neq \emptyset$, then $lk(v, M + B^r)$ is an NH-ball or an NH-sphere if $lk(v, M)$ is an NH-ball and it is an NH-sphere if $lk(v, M)$ is an NH-sphere. If $lk(v, M \cap B^r) = \emptyset$, we showed above that $lk(v, M) = *$ and $lk(v, B^r)$ is a ball or $lk(v, B^r) = *$ and $lk(v, M)$ is an NH-ball. In either case, $lk(v, M + B^r)$ is an NH-sphere of homotopy dimension 0. This proves that $M + B^r$ is an NH-manifold.

We prove now the second part of the statement. We proceed by induction on the index k. Suppose first that $k = 0$, i.e. M is an NH-ball. Let α be a subdivision such that $\alpha M \searrow 0$, and extend α to all $M + B^r$. If $M \cap B^r$ is an NH-ball we can apply Corollary 2.4.2 to $\alpha(M \cap B^r) \subset \alpha \partial B^r$ and find a stellar subdivision s such that $s\alpha B^r \searrow s\alpha(M \cap B^r)$. This implies that $s\alpha(M + B^r) \searrow s\alpha M \searrow 0$ and therefore $M + B^r$ is an NH-ball. If $M \cap B^r$ is an NH-sphere S with decomposition $S = G + L$, take any maximal simplex $\tau \in L$ with an immediate face σ in ∂L and consider the starring $(\tau, \hat{\tau})S$ of S (see Figure 2.8). Let $\rho = \hat{\tau} * \sigma \in (\tau, \hat{\tau})S$. We claim that $(\tau, \hat{\tau})S - \{\rho\}$ is an NH-ball. On one hand, it is clear that $((\tau, \hat{\tau})S - \{\rho\}) \cap \rho = \partial \rho$. On the other hand, $(\tau, \hat{\tau})L - \{\rho, \sigma\}$ is a combinatorial ball because it is PL-homeomorphic to L. Since G is an NH-ball, $(\tau, \hat{\tau})L - \{\rho, \sigma\}$ is a combinatorial ball and $G \cap ((\tau, \hat{\tau})L - \{\rho, \sigma\}) = \partial L - \{\sigma\}$, which is a combinatorial ball by Newman's Theorem, it follows that $(\tau, \hat{\tau})S - \{\rho\}$ is an NH-ball, as claimed. Now, since $\tau \in L \subset M \cap B^r$ is principal then it must be a ridge of M or of B^r. We analyze both cases. Suppose τ is a ridge of B^r and let $\tau \prec \eta \in B^r$. Write $\eta = w * \tau$ (see Figure 2.8). Note that the starring $(\tau, \hat{\tau})S$ performed earlier also subdivides η and the simplex ρ lies in the boundary of $(\tau, \hat{\tau})\eta$. Consider the simplex $\nu = w * \rho$, which is one of the principal simplices in which η has been subdivided. Now make the starring $(\nu, \hat{\nu})$ in $(\tau, \hat{\tau})\eta$ (see Figure 2.8). By removing the simplex $\hat{\nu} * \rho$ from $(\nu, \hat{\nu})(\tau, \hat{\tau})B^r$, we obtain a complex which is PL-homeomorphic to B^r. Then

$$(\nu, \hat{\nu})(\tau, \hat{\tau})B^r - \{\hat{\nu} * \rho\}$$

is a combinatorial ball and it intersects M in $(\tau, \hat{\tau})S - \{\rho\}$, which is an NH-ball. It follows that

$$(\nu, \hat{\nu})(\tau, \hat{\tau})(M + B^r) - \{\hat{\nu} * \rho\} = (\tau, \hat{\tau})M + (\nu, \hat{\nu})(\tau, \hat{\tau})B^r - \{\hat{\nu} * \rho\}$$

is again an NH-ball. If we now plug the simplex $\hat{\nu} * \rho$, $(\nu, \hat{\nu})(\tau, \hat{\tau})(M + B^r)$ is an NH-sphere by definition. This completes the case where τ is a ridge of B^r. The case that τ is a ridge of M is analogous.

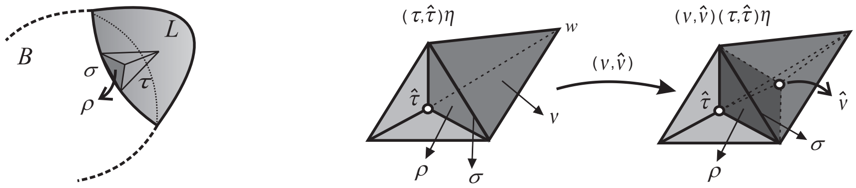


Figure 2.8: The starrings of Theorem 2.4.3.

Suppose now that M is an NH-bouquet of index $k \geq 1$. Write $M = G + L$ with G an NH-bouquet of index $k-1$ and L a combinatorial ball glued to G along its entire boundary. If $r = 0$ we obtain an NH-bouquet. Suppose then that $M \cap B^r \neq \emptyset$. We claim that $B^r \cap L \subseteq \partial L$. Suppose $(L - \partial L) \cap B^r \neq \emptyset$ and let $\eta \in (L - \partial L) \cap B^r$. Now, $lk(\eta, M) = lk(\eta, L)$ is a combinatorial sphere and Corollaries 2.2.5 and 2.2.6 imply that $lk(\eta, B^r) \subset lk(\eta, M)$. But if $\tau \in B^r$ is a principal simplex containing η then $lk(\eta, \tau) \in lk(\eta, M)$ and $\tau \in M \cap B^r \subseteq \partial B^r$, contradicting the maximality of τ in B^r. This proves that $B^r \cap L \subseteq \partial L$ and, therefore $M \cap B^r = G \cap B^r$. Also, $(G \cap B^r)^\circ \subseteq \tilde{\partial} M = \tilde{\partial} G - L \subset \tilde{\partial} G$. By induction, $G + B^r$ is an NH-bouquet of index $k-1$ (if $G \cap B^r = M \cap B^r$ is an NH-ball) or k (if $G \cap B^r = M \cap B^r$ is an NH-sphere). In either case, $M + B^r = G + L + B^r = (G + B^r) + L$ with $(G + B^r) \cap L = G \cap L + B^r \cap L = \partial L$. Thus, $M + B^r$ is an NH-bouquet of index k or $k+1$. This completes the proof. □

Theorem 2.4.3 gives (very general) necessary conditions for the "adding" of a combinatorial ball to preserve the structure of NH-manifold. In light of this result, we introduce some definitions.

Definition. Assume the hypotheses of Theorem 2.4.3. If $M \cap B^d$ is an NH-ball then the move $M \to M + B$ is called a *non-homogeneous regular expansion.*

Note that in the conditions of the last definition, $M \to M + B$ is not a geometrical expansion, but a more general "move" between NH-manifolds. Also, it is easy to see that the condition $(M \cap B)^\circ \subset \tilde{\partial} M$ corresponds to $M \cap B \subset \partial M$ in the homogeneous case. This shows, in particular, that classical regular expansions are a special case of non-homogeneous regular expansions (when $M \cap B^d$ is a combinatorial $(d-1)$-ball).

We can now characterize shellings in terms of geometrical expansions. Recall that an inverse shelling in a combinatorial d-manifold M corresponds to a (classical) regular expansion $M \to M + \sigma$ involving a single d-simplex σ. An elementary shelling is the inverse move. With this in mind we can extend the notion of (elementary and inverse) shelling to non-homogeneous manifolds.

Definition. Let M be an NH-manifold. An *inverse shelling* is a geometrical expansion $M \to M + \sigma$ that is a non-homogeneous regular expansion. Here σ is a single simplex. An *elementary shelling* is the inverse move.

Thus, an inverse shelling is a move $M \to M + \sigma$ where $M \cap \sigma \subset \partial \sigma$ is a combinatorial $(d-1)$-ball and $M \cap \sigma \subset \tilde{\partial} M$. We now show that this concept is consistent with the shellability definition of Björner and Wachs by characterizing shellable NH-balls in terms of elementary shellings. We first need a result.

Proposition 2.4.4. *Let $M \to M + B$ be a geometrical expansion in an NH-manifold M. If $M + B$ is an NH-manifold and $M, B \subset M + B$ are top generated then $(M \cap B)^\circ \subset \tilde{\partial} M$ (i.e. $M \to M + B$ is a regular expansion).*

Proof. Take $\rho \in (M \cap B)^\circ$. Since $lk(\rho, M \cap B)$ is a sphere contained in the sphere $\partial lk(\rho, B)$, then $lk(\rho, M \cap B) = \partial lk(\rho, B)$. Suppose $\rho \notin \tilde{\partial} M$. Then $lk(\rho, M + B) = lk(\rho, M) + lk(\rho, B)$ is an NH-bouquet of index 2 since $lk(\rho, M), lk(\rho, B) \subset lk(\rho, M + B)$ are top generated by hypothesis. This contradicts the fact that $M + B$ is an NH-manifold. □

Corollary 2.4.5. *An NH-ball B is shellable if and only if B can be transformed into a single maximal simplex by a sequence of elementary shellings.*

Proof. It follows at once from Proposition 2.4.4 and Theorem 2.3.4. □

2.4.2 NH-equivalences and a second generalization of Alexander's Theorem

In this section we present a equivalence relation between NH-manifolds using the concepts of elementary and inverse shelling in the non-pure context. As it may be already evident, combinatorial equivalence is a very strong requirement for the non-homogeneous case. Complexes very much alike such as the ones shown in Figure 2.9 are not PL-isomorphic, but are however related by a single inverse (non-pure) shelling. In the spirit of Pachner's Theorem 1.4.1, we introduce the following equivalence relation among NH-manifolds.

Definition. Let M and M' be two NH-manifolds. We say that M is *NH-isomorphic* (or NH-equivalent) to M', and write $M \simeq_{NH} M'$, if M' can be obtained from M by a finite sequence of elementary shellings, inverse shellings and PL-isomorphisms.

Clearly $M \simeq_{NH} M'$ implies $M \nearrow\searrow M'$ (i.e. they are simply equivalent). On the other hand, $M \simeq_{PL} M'$ trivially implies $M \simeq_{NH} M'$. It is easy to see that these implications are strict.

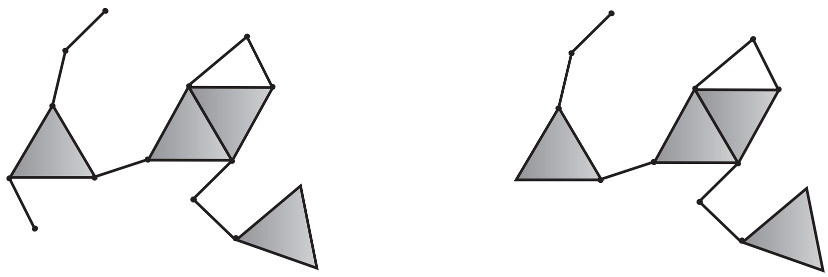


Figure 2.9: Example of non-PL-isomorphic NH-manifolds.

Note immediately that in the homogeneous context $M \simeq_{NH} M'$ if, and only if, $M \simeq_{PL} M'$ by Pachner's Theorem. So, NH-equivalence is a relation that it is camouflaged in the homogeneous context and only detaches from PL-equivalence in the non-pure scenario.

We next prove the second version of the Alexander's Theorem for NH-manifolds involving the notion of NH-equivalence. We first need a preliminary result.

Proposition 2.4.6. *Let M be an NH-manifold of dimension d, let $v \notin M$ and let B be a combinatorial $(r-1)$-ball such that $B^\circ \subset \partial M$. Then $M + v * B \simeq_{NH} M$.*

Proof. We may subdivide B by α so αB is shellable (see [12, Proposition 1]). Extend α to all M arbitrarily and to $v * B$ as $v * \alpha B$. We may assume that this is given and obviate the subdivision α. Since B is shellable let $F_1, \ldots, F_t$ be a shelling order of B. We will show that

$$M = M_0 \overset{v*F_1}{\longrightarrow} M_1 \overset{v*F_2}{\longrightarrow} M_2 \longrightarrow \cdots \overset{v*F_t}{\longrightarrow} M_t = M + v * B$$

is a sequence of inverse shellings, where $M_i \overset{v*F_{i+1}}{\longrightarrow} M_{i+1}$ represents the move $M_i \to M_{i+1} = M_i + v * F_{i+1}$. We prove the claim for the general step $M_i \overset{v*F_{i+1}}{\longrightarrow} M_{i+1}$. On one hand,

$$M_i = M + (v * F_1 + \cdots + v * F_i)$$

from where

$$\begin{aligned} M_i \cap (v * F_{i+1}) &= (M \cap (v * F_{i+1})) + [(v * F_1 + \cdots + v * F_i) \cap (v * F_{i+1})] \\ &= F_{i+1} + v * [(F_1 + \cdots + F_i) \cap F_{i+1}]. \end{aligned}$$

which is clearly homogeneous of dimension $\dim F_{i+1}$. Moreover, $M_i \cap (v * F_{i+1}) \neq \partial(v * F_{i+1})$ because $\partial(v * F_{i+1}) = F_{i+1} + (v * \dot{F}_{i+1})$ and $(F_1 + \cdots + F_i) \cap F_{i+1} \neq \partial F_{i+1}$ since B has no spanning simplices. It remains to prove that $(M_i \cap (v * F_{i+1}))^\circ \subset \partial M_i$. By Theorem 2.4.3 it suffice to show that M_{i+1} is an NH-manifold. We prove this by induction on i, the induction starting with $M_0 = M$. Assume then that M_i is an NH-manifold. Let $u \in V_{M_{i+1}}$. If $u \notin M_i \cap (v * F_{i+1})$ then $lk(u, M_{i+1}) = lk(u, M_i)$, which is an NH-ball or NH-sphere. Assume $u \in M_i \cap (v * F_{i+1})$. Now, v is regular because $lk(v, M_{i+1}) = \cup_{j \leq i+1} F_j$, which is a combinatorial ball since $F_1, \ldots, F_t$ is a shelling of B. Suppose $u \neq v$. In this case, since $u \in v * F_{i+1}$ then $u \in F_{i+1} \subset B \subset M$. Now,

$$lk(u, M_{i+1}) = lk(u, M + (v * F_1 + \cdots + v * F_{i+1})) = lk(u, M) + lk(u, v * F_1 + \cdots + v * F_{i+1}),$$

where $lk(u, M)$ is an NH-ball or an NH-sphere and $lk(u, v * F_1 + \cdots + v * F_{i+1}) = v * lk(u, F_1 + \cdots + F_{i+1})$ is a combinatorial ball. Furthermore,

$$lk(u, M) \cap lk(u, v * (F_1 + \cdots + F_{i+1})) = lk(u, M \cap v * (F_1 + \cdots + F_{i+1})) = lk(u, F_1 + \cdots + F_{i+1}),$$

where this last complex is either a combinatorial ball or sphere. We claim we are in the conditions of Theorem 2.4.3 for $lk(u, M)$ and $v * lk(u, F_1 + \cdots + F_{i+1})$. If $\eta \in lk(u, \cup_{j \leq i+1} F_j)^\circ$ then $lk(\eta, lk(u, F_1 + \cdots + F_{i+1}) = lk(u * \eta, F_1 + \cdots + F_{i+1})$ is a sphere. Then, $u * \eta \in (F_1 + \cdots + F_{i+1})^\circ \subset B^\circ \subset \partial M$. We conclude that $lk(u * \eta, M) = lk(\eta, lk(u, M))$ is an NH-ball, and hence $\eta \in \partial lk(u, M)$. This proves the claim. We next analyze the two possibilities.

- $u \in (F_1 + \cdots + F_{i+1})^\circ$. In this case, $lk(u, F_1 + \cdots + F_{i+1}) = \partial lk(u, F_1 + \cdots + F_{i+1})$ and $u \in B^\circ \subset \partial M$, so $lk(u, M)$ is an NH-ball. By Theorem 2.4.3, $lk(u, M_{i+1})$ must be an NH-sphere.

- $u \notin (F_1 + \cdots + F_{i+1})^\circ$. In this case, $lk(u, F_1 + \cdots + F_{i+1})$ is a $(r-1)$-ball and $lk(u, M_{i+1})$ is an NH-ball or an NH-sphere depending on whether $lk(v, M)$ is an NH-ball or an NH-sphere respectively.

This proves that M_{i+1} is an NH-manifold and concludes the proof of the proposition. □

For the proof of the next result we shall make use of the classical result that any combinatorial ball B may be *starred*: it can be taken to $v * \partial B$ by a sequence of starrings and welds of interior simplices (see [25, Theorem II.11]). In particular, ∂B is not altered.

Theorem 2.4.7. *Let M be an NH-manifold of dimension d and let B be a combinatorial r-ball. Suppose $M \cap B \subset \partial B$ is a combinatorial $(r-1)$-ball and that $(M \cap B)^\circ \subset \tilde{\partial} M$. Then $M + B \simeq_{NH} M$.*

Proof. Let $C := \partial B - (M \cap B)$. C is a combinatorial ball by Newman's Theorem and $\partial C = \partial(M \cap B)$. Star $C \simeq_{PL} w * \partial C = w * \partial(M \cap B)$ and star $B \simeq_{PL} v * \partial B$. We have

$$B \simeq_{PL} v * \partial B = v * ((M \cap B) + C) = v * (M \cap B) + v * C \simeq_{PL} v * (M \cap B) + v * w * \partial(M \cap B).$$

Extending the starrings and welds of $B \simeq_{PL} v * (M \cap B) + v * w * \partial(M \cap B)$ to $M + B$ we obtain

$$M + B \simeq_{PL} M + v * (M \cap B) + v * w * \partial(M \cap B).$$

Now, since $M \cap v * (M \cap B) = M \cap B$ is a combinatorial ball fulfilling the requirements of Proposition 2.4.6 we have $M + w * (M \cap B) \simeq_{NH} M$. On the other hand,

$$(M + v * (M \cap B)) \cap v * w * \partial(M \cap B = v * \partial(M \cap B),$$

which is again a combinatorial ball. To use Proposition 2.4.6 again we must check that $(v * \partial(M \cap B))^\circ \subset \partial(M + v * (M \cap B))$. Now, since $\partial(v * \partial(M \cap B)) = \partial(M \cap B)$ then $(v * \partial(M \cap B))^\circ = v * \partial(M \cap B) - \partial(M \cap B)$. So, if $\eta \in \partial(v * (M \cap B))^\circ$ then $\eta = v * \eta'$ with $\eta' \in \partial(M \cap B)$. Therefore

$$lk(\eta, M + v * (M \cap B)) = lk(\eta, v * (M \cap B)) = lk(\eta', M \cap B)$$

which is a combinatorial ball since $\eta' \in \partial(M \cap B)$. This implies $\eta \in \partial(M + v * (M \cap B))$ and we conclude by Proposition 2.4.6 that

$$(M + v * (M \cap B)) \simeq_{NH} (M + v * (M \cap B)) + v * w * \partial(M \cap B). \quad \square$$

Note that inverse and direct (pure) shellings do not provide a completely satisfactory way to move through NH-manifolds as they do for manifolds (see Theorem 1.4.1). For example, Figure 2.10 (a) exhibits a very simple NH-ball that cannot be taken to a single simplex by inverse and direct pure shellings. On the other hand, Figure 2.10 (b) shows a complex that cannot be reached from a single simplex by allowing intersections in combinatorial s-balls of dimensions $0 \le s \le r-1$ (a close analysis shows that at some point a contact over an NH-ball must take place). The more general notion of non-homogeneous regular expansion allows us to introduce a more flexible notion of shelling.

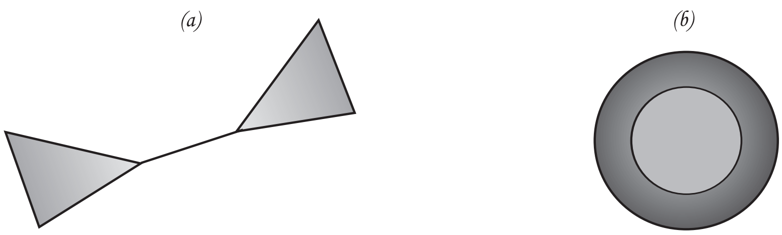


Figure 2.10

Definition. An *inverse NH-shelling* is a non-homogeneous regular expansion involving a single simplex. The inverse move is an *elementary NH-shelling.*

Thus, NH-shellings are a far more general way to move through NH-manifolds. It can be seen that the complexes in Figures (a) and (b) are related to a single simplex by NH-shellings. The same is true for simplices of different dimensions. In particular, all simplices belong to the same equivalence class.

Conjecture 2.4.8. *Every NH-ball can be taken to a simplex by a sequence of (direct and inverse) NH-shellings.*

§2.5 Further properties of NH-manifolds

In this section we prove the existence of *regular neighborhoods* of subcomplexes of non-pure manifolds and prove their existence and we show that NH-manifolds satisfy a generalized version of the Dehn-Sommerville equations involving the concept of pseudo boundary.

2.5.1 Regular neighbourhoods in NH-manifolds

The theory of regular neighborhoods of Whitehead is one of the most important tools in PL-manifold theory. In this section we prove the existence of regular neighborhoods in NH-manifolds.

A *regular neighborhood* of a subcomplex K of a combinatorial d-manifold M is a subcomplex $U \subset M$ such that

(1) U is a combinatorial d-manifold;

(2) $U \searrow K$.

Thus, regular neighborhoods are "manifold models" of subcomplexes of manifolds. We next state Whitehead's famous result on the existence and uniqueness of regular neighborhoods in combinatorial manifolds. We first need some definitions.

Definition. Let K be a complex and $L \subset K$ a subcomplex. The *simplicial neighborhood* of L in K is the subcomplex generated by the principal simplices in K intersecting L. We denote it $N(L, K)$.

We note that a simplicial neighborhood of L is the smallest subcomplex of K for which its geometric realization is a topological neighborhood of $|L|$. Note also that it can be define for any topological subspace $X \subset |K|$ in the same manner.

Definition. If $L \subset K$ then $S_L K$ denotes the subdivision of K obtained by starring the simplices of $K - L$ in order of decreasing dimension. That is, if $\sigma_1, \ldots, \sigma_t$ are the simplices of $K - L$ ordered such that $\dim(\sigma_i) \geq \dim(\sigma_j)$ if $i < j$, then $S_L K$ consists of the simplices $\tau * \Delta(\hat{\sigma}_{i_1}, \ldots, \hat{\sigma}_{i_r})$ where $\tau \in L$ and $\tau < \sigma_{i_1} < \cdots < \sigma_{i_r}$. We denote $S_L^2 K = S_L(S_L K)$.

Theorem 2.5.1 (Whitehead's Regular Neighborhood Theorem)**.** *Suppose K is a subcomplex of a combinatorial d-manifold M.*

(1) $N(K, S_K^2 M)$ is a regular neighborhood of K.

(2) Any two regular neighborhoods of K are PL-isomorphic.

Let us mention that Whitehead's Theorem 1.3.1 is a corollary of Theorem 2.5.1. To see this, suppose M is a d-manifold such that $M \searrow v$ for some $v \in V_M$. By definition, M is a regular neighborhood of v. Since $st(v, M)$ is also a regular neighborhood of v then $M \simeq_{PL} st(v, M)$ by Theorem 2.5.1.

As commented above, our purpose is to prove the existence of regular neighborhoods in NH-manifolds. We shall define two notions of this concept.

Definition. Let K be a subcomplex of an NH-manifold M.

(1) A *weak regular neighborhood* of K in M is an NH-manifold $N \subset M$ such that $N \searrow K$.

(2) A *strong regular neighborhood* of K in M is an NH-manifold $N \subset M$ such that $N \searrow K$ and $N \simeq_{PL} N(S^2_K, M)$.

The additional condition in the definition of the strong notion of regular neighborhood forces the neighborhood to "have the shape" of the complex K. This is the spirit in the homogeneous case. One may think that we want the neighborhood to be close to K (see Figure 2.11). Weak and strong neighborhoods are the standard regular neighborhoods if M is homogeneous; note that the condition $N \simeq_{PL} N(S^2_K, M)$ is a theorem in the pure case.

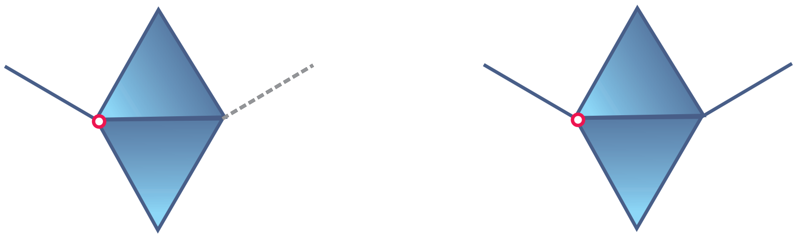


Figure 2.11: Weak and strong regular neighborhoods.

The main result of this section is that every subcomplex of an NH-manifold has a strong regular neighborhood. We shall be needing some preliminary definitions and results. We will follow the treatment of [25].

Definition. A *collapsible neighborhood* N of K is a complex containing L such that

(i) K is *full* in N; that is, if $\sigma \in N$ is such that $V_\sigma \subset V_K$ then $\sigma \in K$.

(ii) Any $\sigma \in N$ is a face of a simplex intersecting K.

(iii) For every $\sigma \in N$ not intersecting K, $lk(\sigma, N) \cap K$ is collapsible.

Theorem 2.5.2 ([25, Theorem III.5])**.** *If N is a collapsible neighborhood of K, then $N \searrow K$.*

It is in general not true that the simplicial neighborhood of L in K is a collapsible neighborhood. For example, if $B = (\Delta^d, \hat{\Delta}^d)\Delta^d$ then $N(\partial B, B) = B$ but B does not collapse to ∂B. In this case, a does not fulfill (iii) in the definition of collapsible neighborhood. The following result gives us a way to construct, up to subdivision, collapsible neighborhoods of subcomplexes.

Proposition 2.5.3 ([25, Corollary III.11])**.** *If L is a subcomplex of K, then $N := N(L, S^2_L K)$ is a collapsible neighborhood of L. Also, if $\sigma \in N - L$ then $L \cap lk(\sigma, N)$ is a single simplex.*

The main result of the section is the following

Theorem 2.5.4. *Let M be an NH-manifold and let $K \subset M$. Then, $N(K, S^2_K M)$ is a strong regular neighborhood of K.*

Proof. The proof follows the same arguments as the proof of [25, Theorem II.15n] with minor modifications. Suppose first that the following two conditions are fulfilled.

(a) K is full in M.

(b) If $\sigma \in M - K$ then $K \cap lk(\sigma, M)$ is a simplex (possibly empty).

Under these conditions, we shall prove in four steps that $N := N(K, M)$ is an NH-manifold which collapses to K.

Step 1. *N is a collapsible neighborhood of K in M.* We must check that the three conditions of the definition of collapsible neighborhood in page 70 are fulfilled. Note that (ii) follows straightforward from the definition of simplicial neighborhood and (i) follows from (a) since $K \subset M$. For (iii), let $\sigma \in N$ not intersecting K. Then, $\sigma \in N - K \subset M - K$ and $K \cap lk(\sigma, M)$ is a simplex by (b). And easy computation shows that $K \cap lk(\sigma, N) = K \cap lk(\sigma, M)$, so $K \cap lk(\sigma, N)$ is a simplex and (iii) is fulfilled.

Step 2. *If v is a vertex not in K then $lk(v, N) = N(K \cap lk(v, M), lk(v, M))$.* Let $\sigma \in lk(v, N)$ be a maximal simplex. On one hand, $\sigma \in lk(v, N) \subset lk(v, M)$. On the other hand, $|v * \sigma| \cap |K| \neq \emptyset$ by the maximality of σ and the definition of N. Since by hypothesis $v \notin K$ then $|\sigma| \cap |K| \neq \emptyset$. Therefore, $|\sigma \cap (|K \cap lk(v, M)|)| \neq \emptyset$ and, hence, $\sigma \in N(K \cap lk(v, M), lk(v, M))$.

Let now $\sigma \in N(K \cap lk(v, M), lk(v, M))$ be a maximal simplex. By maximality, $|\sigma| \cap |K \cap lk(v, M)| \neq \emptyset$. Now $\sigma \in N$ since $|\sigma| \cap |K| \neq \emptyset$ and $\sigma \in lk(v, M)$. Therefore, $v * \sigma \in N$ and $\sigma \in lk(v, N)$.

Step 3. *If v is a vertex not in K then $K \cap lk(v, M)$ fulfills the conditions (a) and (b) as a subcomplex of $lk(v, M)$.* To prove (a), let $\sigma \in lk(v, M)$ such that $w \in K \cap lk(v, M)$ for all $w \in V_\sigma$. Since K is full in M then the condition (a) for K in M implies $\sigma \in K$. Therefore, $\sigma \in K \cap lk(v, M)$, which proves that $K \cap lk(v, M)$ if full in $lk(v, M)$.

To prove (b), let $\sigma \in lk(v, M) - (K \cap lk(v, M))$. Then, in particular, $v * \sigma \in M - K$. By the condition (b) for K in M it follows that $K \cap lk(v * \sigma, M)$ is a simplex. But $K \cap lk(v * \sigma, M) = K \cap lk(\sigma, lk(v, M))$ by Remark 1.1.4. Since trivially $lk(\sigma, lk(v, M)) \subset lk(v, M)$ then

$$K \cap lk(\sigma, lk(v, M)) = (K \cap lk(v, M)) \cap (lk(\sigma, lk(v, M))),$$

and the claim is proved.

Step 4. *N is an NH-manifold.* We proceed by induction on $\dim(M) = d$. We may assume $d \geq 1$. Let $v \in N$ be a vertex. We shall see that $lk(v, N)$ is an NH-ball or NH-sphere. If $v \in K$ then $lk(v, N) = lk(v, M)$, which is already an NH-ball or NH-sphere. If $v \notin K$ then, by step 2, $lk(v, N) = N(K \cap lk(v, M), lk(v, M))$. Since $\dim(lk(v, M)) < \dim(M)$ and $K \cap lk(v, M)$ fulfills conditions (a) and (b) in $lk(v, M))$ by step 3, inductive hypothesis applies to show that $lk(v, N) = N(K \cap lk(v, M), lk(v, M))$

is an NH-manifold collapsing to $K \cap lk(v, M)$. Now, since $v \notin K$ then $K \cap lk(v, M)$ is a simplex by condition (b) of K in M. In particular, $K \cap lk(v, M) \searrow 0$. Therefore, $lk(v, N) = N(K \cap lk(v, M), lk(v, M)) \searrow K \cap lk(v, M) \searrow 0$. We conclude that $lk(v, N)$ is an NH-ball as wanted.

We finally show that the pair $(K, S^2_K M)$ satisfies the conditions (a) and (b). By Proposition 2.5.3, $N(K, S^2_K M)$ is a collapsible neighborhood of K in $S^2_K M$. In particular, K is full in $S^2_L M$. Moreover, the same proposition shows that if $\sigma \in N(K, S^2_K M) - K$ then $K \cap lk(\sigma, S^2_K M)$ is a simplex. Let $\sigma \in S^2_K M - K$. If $K \cap lk(\sigma, S^2_K M) \neq \emptyset$ then $\sigma \in N(K, S^2_K M)$, which is a simplex by Proposition 2.5.3 again. This completes the proof. $\square$

The very same ideas of [25, Remark p. 81] go through to give the following

Corollary 2.5.5. *If K is a subcomplex of an NH-manifold M then $N(K'', M'')$ is a strong regular neighborhood of K'' in M''.*

2.5.2 Dehn-Sommerville equations

A major combinatorial property of combinatorial manifolds is the verification of the Dehn-Sommerville equations. For any complex K of dimension d define the f-vector $f(K) = (f_0, \ldots, f_d)$ of K where f_i equals the number of i-simplices in K. The Dehn-Sommerville equations for a d-dimensional combinatorial manifold M relate the f-vector of M with the f-vector of its boundary:

$$(1 - (-1)^{d-k}) f_k(M) + \sum_{i=k+1}^{d} (-1)^{d-k-1} \binom{k+1}{i+1} f_i(M) = f_k(\partial M) \tag{2.1}$$

In particular, when M is boundaryless this turns into the classical Dehn-Sommerville equations. As proved by Klee in [31], this equations further hold for Eulerian manifolds, which are simplicial complexes whose simplices have links with the same euler characteristic as the sphere of the corresponding dimension.

The $d \times d$ matrix defined by the coefficients of the left is called the *d-th Dehn-Sommerville Matrix* and is given explicitly as:

$$D(d) = \begin{cases} \begin{pmatrix} 0 & \binom{2}{1} & -\binom{3}{1} & \binom{4}{1} & -\binom{5}{1} & \ldots & \binom{d}{1} & -\binom{d+1}{1} \\ 0 & 2 & -\binom{3}{2} & \binom{4}{2} & -\binom{5}{2} & \ldots & \binom{d}{2} & -\binom{d+1}{2} \\ 0 & 0 & 0 & \binom{4}{3} & -\binom{5}{3} & \ldots & \binom{d}{3} & -\binom{d+1}{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 2 & -\binom{d+1}{d} \end{pmatrix} & \text{for } d \text{ even} \\ \\ \begin{pmatrix} 2 & -\binom{2}{1} & \binom{3}{1} & -\binom{4}{1} & \binom{5}{1} & \ldots & \binom{d}{1} & -\binom{d+1}{1} \\ 0 & 0 & \binom{3}{2} & -\binom{4}{2} & \binom{5}{2} & \ldots & \binom{d}{2} & -\binom{d+1}{2} \\ 0 & 0 & 2 & -\binom{4}{3} & \binom{5}{3} & \ldots & \binom{d}{3} & -\binom{d+1}{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 2 & -\binom{d+1}{d} \end{pmatrix} & \text{for } d \text{ odd} \end{cases}$$

The Dehn-Sommerville equations (2.1) can then be written:

$$D(d)f(M) - f(\partial M) = 0.$$

Chen and Yan [19] extended these equations to general polyhedra by sorting the simplices of the complexes based on the Euler characteristic of their links. For a simplicial complex K, define $K_j := \{\sigma \in K \mid \chi(lk(\sigma, K)) = 1 - (-1)^{\dim \sigma} + (-1)^{\dim \sigma} j\}$. Then, the following equation holds.

$$D(d)f(K) = \sum_{j \neq 1-(-1)^d} (1 - (-1)^d + (-1)^d j) f(K_j). \tag{2.2}$$

For an NH-manifold M, we must have $M_j = \emptyset$ for $j \neq 0, 1, 2$ since $0 \leq 1 - (-1)^{\dim \sigma} + (-1)^{\dim \sigma} j \leq 2$ only for those values of j because $0 \leq \chi(lk(\sigma, M)) \leq 2$ if $\sigma \in M$. Note immediately that $M_1 = \tilde{\partial} M$. We introduce a name for M_0 and M_2:

Definition. Let M be an NH-manifold of dimension d. We define the *deviation set* of M by

$$B(M) = \begin{cases} M_2 = \left\{\sigma \in M \mid \chi(lk(\sigma, M)) = 1 + (-1)^{\dim(\sigma)}\right\} & \text{if } d \text{ is even} \\ M_0 = \left\{\sigma \in M \mid \chi(lk(\sigma, M)) = 1 - (-1)^{\dim(\sigma)}\right\} & \text{if } d \text{ is odd} \end{cases}$$

Note that the deviation set can be interpreted as the collection of simplices whose link does not have the "correct" Euler characteristic in relation with the dimension of the complex (if it were Eulerian). For example, a 1-simplex in a combinatorial 2-manifold has Euler characteristic 2, so a 1-simplex of an NH-manifold of dimension 2 is in the deviation set if its link has Euler characteristic 0. Applying (2.2) to an NH-manifold we have the following

Theorem 2.5.6 (Dehn-Sommerville for *NH*-manifolds)**.** *For an NH-manifold M of dimension d we have*

$$D(d)f(M) - f(\tilde{\partial} M) = 2f(B(M)). \tag{2.3}$$

In particular,

$$D(d)f(M) - f(\tilde{\partial} M) = 0 \pmod 2.$$

§2.6 Non-pure factorizations and Pachner moves

Shellings and bistellar flips are simplicial moves, not based upon subdivision, that preserve the PL class of manifolds. The question whether the converse holds (i.e. if any two PL-isomorphic polyhedral manifolds are always related by a sequence of shellings and bistellar moves) was answered in the positive by U. Pachner [43] in 1990, thus completing the work of Newman [41] (see §1.4). Furthermore, Pachner himself [44] proved that only shelling were required for manifolds with boundary (see Theorem 1.4.1).

In this section we study how simplicial transformations on combinatorial manifolds can be "decomposed" in a natural way so all complexes involved in the factorization are NH-manifolds. More concretely, we show that both elementary starrings (and welds) and stellar exchanges are the result of combining some local expansions of top generated subcomplexes with moves in NH-manifolds. Also, by applying one of this factorizations to boundary simplices, we naturally rediscover the relation between shellings and bistellar moves, the basic *Pachner moves*, in that bistellar flips are the internal versions of elementary shellings.

2.6.1 Factorization of starrings and welds: Conings

Here we shall show how elementary starrings in combinatorial manifolds are factorized via NH-manifolds. An elementary starring $(\sigma, a)M$ introduces the new vertex a in the triangulation by replacing the interior of $st(\sigma, M)$ with a cone (with apex a) over its boundary. That is, it exchanges the subcomplex $st(\sigma, M)$ of $\partial(a * st(\sigma, M))$ with its complementary ball $a * \partial st(\sigma, M)$. We then may think an elementary starring as the result of taking a cone over the subcomplex $st(\sigma, M)$ and then removing the simplices $a * \eta$ for each η in the interior of $st(\sigma, M)$. This is the idea behind the notion of *coning*.

Definition. Let M be a combinatorial d-manifold, $L \subset M$ a subcomplex and $v \notin V_M$. Then, the move $M \to C_M(L) := M + v * L$ is called the *coning* of L.

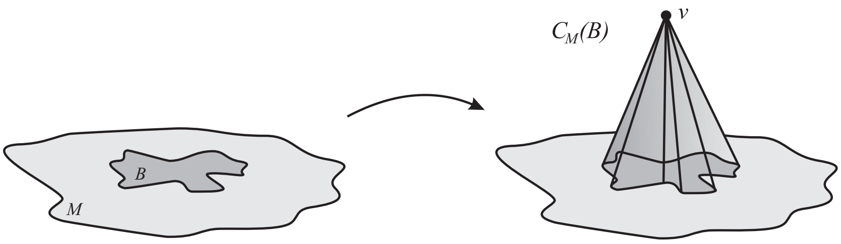


Figure 2.12: A coning of a combinatorial ball B.

Note that the inverse move $C_M(L) \to M$ is simply the deletion $M - v = \{\sigma \in M \mid v \notin \sigma\}$. We will be exclusively interested in conings of combinatorial d-balls, the reasons being the following

Proposition 2.6.1. *Let M be a combinatorial d-manifold and $B \subset M$ a combinatorial d-ball. Then $C_M(B)$ is an NH-manifold.*

Proof. We prove the proposition by induction on d. Let $v \in C_M(B) = M + b * B$. If $v \notin B$, then $lk(v, C_M(B)) = lk(v, M)$. If $v \in \mathring{B}$ then $lk(v, C_M(B)) = b * lk(v, M)$, which is a combinatorial d-ball. If $v \in \partial B$ then $lk(v, C_M(B)) = lk(v, M) + b * lk(v, B)$ is an NH-manifold by out inductive hypothesis. Since $lk(v, B)$ is collapsible then $lk(v, C_M(B)) \searrow lk(v, M)$, so $lk(v, C_M(B))$ is an NH-ball if $v \in \partial M$. If $v \notin \partial M$ then $lk(v, B)$ is strictly contained in $lk(v, M)$. It follows that there is an d-simplex $\eta \in M - B$ containing v. By Newman's Theorem, $lk(v, M) - lk(v, \eta)$ is an $(d-1)$-ball. It follows that $lk(v, C_M(B))$ is an NH-sphere with decomposition

$$(lk(v, M - \eta) + b * lk(v, B)) + lk(v, \eta)$$

since $lk(v, M - \eta) + b * lk(v, B)$ is an NH-ball by the previous case and

$$(lk(v, M - \eta) + b * lk(v, B)) \cap lk(v, \eta) = (lk(v, M) - lk(v, \eta)) \cap lk(v, \eta) = \partial lk(v, \eta). \qquad \square$$

Remark 2.6.2. Note that, since B is collapsible, then $C_M(B) \searrow M$.

Since $C_M(B)$ is an NH-manifold we may proceed with direct and inverse (non-pure) shellings. The main result of this section is that starrings and welds between combinatorial d-manifolds can be factorized through conings and non-pure shellings (and they inverses) where all the complexes involved in the process are NH-manifolds (Theorem 2.6.5). To settle this theorem, we shall need some preliminary results. We first introduce a slight improvement of Proposition 2.4.4.

Proposition 2.6.3. *Let M and M' be NH-manifolds related by a sequence of geometrical expansions*

$$M = M_0 \overset{B_1}{\to} M_1 \overset{B_2}{\to} M_2 \overset{B_3}{\to} \cdots \overset{B_t}{\to} M_t = M'$$

where $M_{k+1} = M_k + B_{k+1}$ and $M_k, B_{k+1} \subset M_{k+1}$ are top generated. Then, M_k is an NH-manifold for all $k = 0, \dots, t$. In particular, $(M_k \cap B_{k+1})^\circ \subset \tilde{\partial} M_k$ for all $k = 0, \dots, t$.

Proof. We proceed by induction on $d = \dim(M) = \dim(M')$. We may assume $d \geq 1$. Fix $0 \leq k \leq t$ and take $v \in M_k$. We shall prove $lk(v, M_k)$ is an NH-ball or NH-sphere. Let j be the minimal index such that $v \in M_j$. Then, $lk(v, M_j) = lk(v, M)$ if $j = 0$ and $lk(v, M_j) = lk(v, B_j)$ if $j \geq 1$. If $v \in B_j^\circ$ it follows from Corollaries 2.2.5 or 2.2.6 that $lk(v, B_j) = lk(v, M')$, and hence $lk(v, M_s) = lk(v, B_j)$ for all $j \leq s \leq t$. The claim then follows from the fact that M' is an NH-manifold. Suppose $v \in \partial B_j$. We claim that $v \in \partial B_s$ for all $j \leq s \leq t$. Otherwise, let $s > j$ be such that $v \in B_s^\circ$. Then $lk(v, B_j) \subset lk(v, M_j) \subset lk(v, M_s) = lk(v, B_s)$, where the last equality holds by same reasoning as before. This implies that B_j has a principal simplex $\rho \in B_s$. Hence, $\rho \in M_j \cap B_s \subset M_{s-1} \cap B_s \subset \partial B_s$, which contradicts the top generation of B_j in M_s. The claim is thus proved. Consider now the sequence of geometrical expansions

$$N_j \xrightarrow{L_{j+1}} N_{j+1} \xrightarrow{L_{j+2}} N_{j+2} \xrightarrow{L_{j+3}} \cdots \xrightarrow{L_t} N_t = N'$$

where $N_i = lk(v, M_i)$ and $L_i = lk(v, B_i)$ (where we omit the moves where $lk(v, B_i) = \emptyset$). By inductive hypothesis the N_i's are NH-manifolds for every $j \leq i \leq t$. Since N_j and N' are NH-bouquets of index less than 2 then so are the N_i's for $j \leq i \leq t$ (see Theorem 2.4.3). This proves that $lk(v, M_k)$ is an NH-ball or NH-sphere for all k and concludes the proof. $\square$

Proposition 2.6.4. *Let M be a combinatorial d-manifold and let $\sigma \in M$ be a simplex such that $lk(\sigma, M)$ is shellable. Then, M and $(\sigma, a)M$ are related by a sequence of local conings, elementary shellings and their inverses. In the process, every complex involved is an NH-manifold.*

Proof. Set $A = st(\sigma, M)$ and $B = st(a, (\sigma, a)M)$. Since $\partial\sigma$ is the boundary of no simplex in $(\sigma, a)M + v * B$ we can consider the new complex $\tilde{M}$ obtained from $C_{(\sigma,a)M}(B)$ by adding the simplex σ together with $\{\sigma * \rho \,|\, \rho \in a * lk(\sigma, M)\}$. We claim that $\tilde{M}$ is an NH-manifold. Note that $\tilde{M} = C_{(\sigma,a)M}(B) + a * \sigma * lk(\sigma, M)$ and $C_{(\sigma,a)M}(B) \cap a * \sigma * lk(\sigma, M) = a * \partial\sigma * lk(\sigma, M)$, which is a combinatorial d-ball in $\partial(a * \sigma * lk(\sigma, M))$. Also, $lk(\eta, C_{(\sigma,a)M}(B))$ is a cone over v for every simplex η in the interior of $a * \partial\sigma * lk(\sigma, M)$. Hence, $\tilde{M}$ is an NH-manifold by Theorem 2.4.3. Note that $\tilde{M}$ can also be obtained from $C_M(A)$ by adding the vertex v together with the simplices $\{v * \eta \,|\, \eta \in a * \partial\sigma * lk(\sigma, M)\}$. The proposition will be established if we show that both $C_M(A)$ and $C_{(\sigma,a)M}(B)$ can be transformed to $\tilde{M}$ by a sequence of inverse shellings.

Let $F_1, \dots, F_t$ be a shelling of $A = \sigma * lk(\sigma, M) \subset \tilde{M}$ and consider the sequence

$$C_{(\sigma,a)M}(B) = M_0 \xrightarrow{a*F_1} M_1 \xrightarrow{a*F_2} \cdots \xrightarrow{a*F_t} M_t = \tilde{M}$$

where $M_{k+1} = M_k + a * F_{k+1} = C_{(\sigma,a)M}(B) + a * F_1 + \cdots + a * F_{k+1}$. It is not hard to see that $M_k \cap (a * F_{k+1}) = a * \partial\sigma * lk(\sigma, F_{k+1}) + a * (F_{k+1} \cap (F_1 + \cdots + F_k))$, and since $F_{k+1} \notin M_k \cap (a * F_{k+1})$ this intersection is a combinatorial d-ball (being a proper d-homogeneous complex in $\partial(a * F_{k+1})$). It follows that $M_k \to M_{k+1}$ is a geometric expansion for all $k = 0, \dots, t-1$ and, therefore, a sequence of inverse shellings by Proposition 2.6.3.

Analogously, let $E_1, \dots, E_s$ be a shelling order of $B = a * \partial\sigma * lk(\sigma, M) \subset C_M(A)$ (note that B is shellable by [42, Lemma 4.12]) and consider the sequence:

$$C_M(A) = N_0 \xrightarrow{v*E_1} N_1 \xrightarrow{v*E_2} \cdots \xrightarrow{v*E_s} N_s = \tilde{M}$$

where $N_{k+1} = N_k + v * E_{k+1} = C_M(A) + v * E_1 + \cdots + v * E_{k+1}$. Again, it is easy to see that $N_k \cap (v * E_{k+1}) = E_{k+1} + v * (E_{k+1} \cap (E_1 + \cdots + E_k))$, which is a combinatorial d-ball in $\partial(v * E_{k+1})$ since $v * lk(a, E_{k+1}) \notin N_k \cap (v * E_{k+1})$. Again by Proposition 2.6.3, $N_k \to N_{k+1}$ are inverse shellings for all $k = 0, \dots, s-1$. This concludes the proof. □

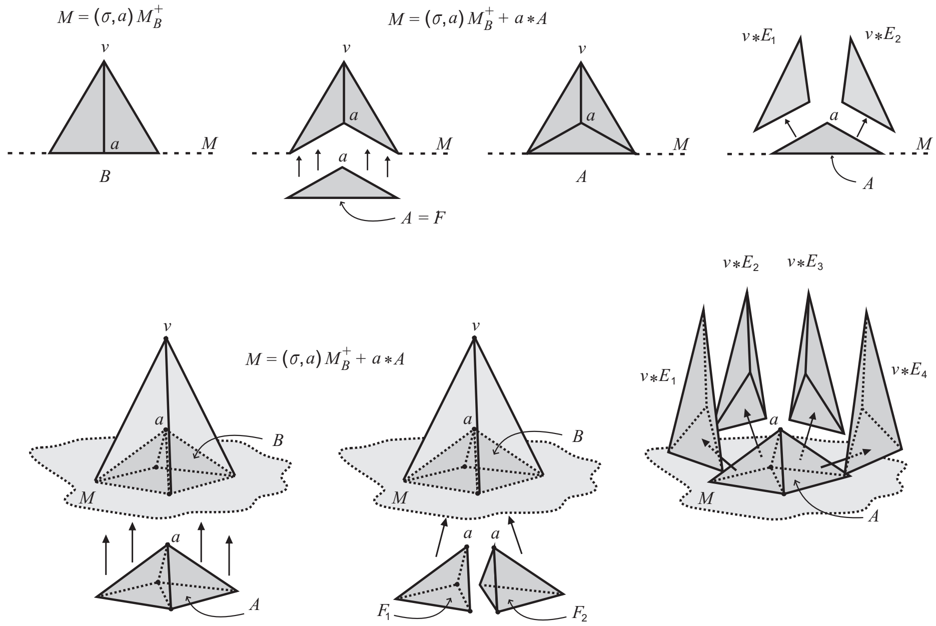


Figure 2.13: The shellings in the proof of Proposition 2.6.4.

Theorem 2.6.5. *Let M be a combinatorial d-manifold. Any starring or weld in M can be factorized by a sequence of conings, elementary shellings and their inverses. In the process, any complex involved is an NH-manifold.*

Proof. The proof is essentially the same as [18, Proposition 4] with a minor generalization.* We include the main ideas next for the sake of completeness. It suffices to prove the theorem for a starring (τ, a) of M. By [42, Lemma 4.13] there is a decomposition $lk(\sigma, M) = \partial P * L'$ where P is a polytope such that $\partial P = \partial\Delta_1 * \cdots * \partial\Delta_p$, for some simplices Δ_i, and P is maximal with this property. Now, L' is a combinatorial m-ball or m-sphere (by iteration of [36, Lemma 4.6]) and, hence, there is a sequence of elementary starrings and welds $s_1, \dots, s_r$ taking L' to Δ^m or $\partial\Delta^{m+1}$ respectively. If $m \leq 2$ or $r \leq 1$ the result follows from Proposition 2.6.4 since shellability of L' implies that of $lk(\sigma, M)$ (by iteration of [36, Lemma 5.5]). Here we use [42, Lemma 4.12] and that every ball or sphere of dimension less than 3 is shellable. Otherwise, we proceed by induction in m and r. We first study the case $s_1 = (\tau, b)$. Since $\tau \in lk(\sigma, M)$ there is by [2, Theorem 10.2] a

*Actually, the main ideas of this proof were developed by Pachner in [42, 43].

factorization of $M \stackrel{(\sigma,a)}{\longrightarrow} (\sigma,a)M$ given by

$$M \stackrel{(\sigma*\tau,b)}{\longrightarrow} M_1 \stackrel{(\sigma,a)}{\longrightarrow} M_2 \stackrel{(a*\tau,b)}{\longleftarrow} (\sigma,a)M$$

The link of every simplex being starred or welded in each of these elementary stellar moves has a decomposition with a minor m or r. Hence, the result follows from induction in each of these moves.

We next study the case $s_1 = (\rho,c)^{-1}$. This implies that $lk(c,L') = \partial\rho * L''$ where L'' is a combinatorial ball or sphere of dimension less than m and $\rho \notin L'$. We analyze the cases $\rho \notin M$ and $\rho \in M$. Suppose first $\rho \notin M$. Since $c \in lk(\sigma,M)$ then we can apply [2, Theorem 10.2] to factorize $M \stackrel{(\sigma,a)}{\longrightarrow} (\sigma,a)M$ as follows:

$$M \stackrel{(\sigma*c,d)}{\longrightarrow} M_1 \stackrel{(\sigma,a)}{\longrightarrow} M_2 \stackrel{(a*c,d)}{\longleftarrow} (\sigma,a)M$$

where d is a vertex not in M. Now, since $\partial\rho \in lk(d,M_1)$ and $\rho \notin M_1$ then the weld $(\rho,d)^{-1}$ is admissible in M_1. Also, since $\rho \in lk(\sigma,(\rho,d)^{-1}M_1)$ then [2, Theorem 10.3] gives us

$$M_1 \stackrel{(\rho,d)}{\longleftarrow} M_3 \stackrel{(\sigma,a)}{\longrightarrow} M_4 \stackrel{(\rho,d)}{\longrightarrow} M_2,$$

where we have written $M_3 = (\rho,d)^{-1}M_1$. Therefore, we obtain the following factorization of $M \stackrel{(\sigma,a)}{\longrightarrow} (\sigma,a)M$:

$$M \stackrel{(\sigma*c,d)}{\longrightarrow} M_1 \stackrel{(\rho,d)}{\longleftarrow} M_3 \stackrel{(\sigma,a)}{\longrightarrow} M_4 \stackrel{(\rho,d)}{\longrightarrow} M_2 \stackrel{(a*c,d)}{\longleftarrow} (\sigma,a)M.$$

As the previous case, the links of the involved simplices in these moves have a maximal decomposition verifying our inductive hypothesis; hence, the result follows.

If now $\rho \in M$, let $v \in \rho$ be a vertex. Since $v \in lk(\sigma,M)$, [2, Theorem 10.2] allows us to factorize $M \stackrel{(\sigma,a)}{\longrightarrow} (\sigma,a)M$ as follows:

$$M \stackrel{(\sigma*v,e)}{\longrightarrow} M_1 \stackrel{(\sigma,a)}{\longrightarrow} M_2 \stackrel{(a*v,e)}{\longleftarrow} (\sigma,a)M$$

for some $e \notin M$. Now, on one hand the inductive argument on m applies for the stellar moves $M \stackrel{(\sigma*v,e)}{\longrightarrow} M_1$ and $M_2 \stackrel{(a*v,e)}{\longleftarrow} (\sigma,a)M$. On the other hand, the move $M_1 \stackrel{(\sigma,a)}{\to} M_2$ falls into the already proven case since $\rho \notin M_1$. The results follows now from induction. This concludes the proof. □

2.6.2 Factorization of stellar exchanges: NH-factorizations

In this section we provide a way to factorize stellar exchanges in manifolds. Similarly to the case of starrings and welds, an stellar exchange $\kappa(\sigma,\tau)$, which replaces $st(\sigma,M) = \sigma*\partial\tau*L$ with $\partial\sigma*\tau*L$, can be thought as an exchange modulo L between the complementary balls $\sigma*\partial\tau$ and $\partial\sigma*\tau$ in $\partial(\sigma*\tau)$. Just as for the case of conings, an NH-factorization realizes the ball $\sigma*\tau$ where the transformation is taking place.

Definition. Let K be a simplicial complex and let $\sigma \in K$ be a simplex such that $lk(\sigma,K) = \partial\tau * L$ with $\tau \notin K$. An *NH-factorization* is the move $K \to F(\sigma,\tau)K = K + \sigma*\tau*L$. When $L = \emptyset$, we call it a *bistellar factorization*.

Note that by definition the following diagram commutes. This justifies the term "factorization".

$$\begin{tikzcd} K \arrow[rr, "\kappa(\sigma,\tau)"] \arrow[dr, "F(\sigma,\tau)"'] & & \kappa(\sigma,\tau)K \arrow[dl, "F(\tau,\sigma)"] \\ & K+\sigma*\tau*L & \end{tikzcd} .$$

When τ is a single vertex $b \notin M$, $F(\sigma,b)M = C_M(st(\sigma,M))$. In particular, $F(\sigma,b)M \searrow M$.

Proposition 2.6.6. *Let M be a combinatorial d-manifold and let $M \longrightarrow N = F(\sigma,\tau)M$ be an NH-factorization. Then N is an NH-manifold.*

Proof. Let $N = M + \sigma * \tau * L$ with $\tau \notin M$. Since $(\tau, b)N = C_M(st(\sigma, M))$, the result follows from Theorem 2.1.6 and Proposition 2.6.1 $\square$

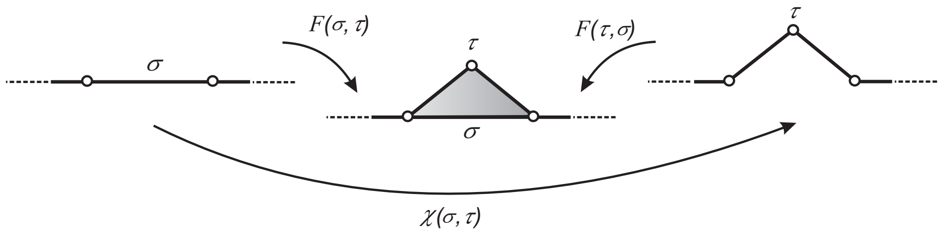


Figure 2.14: An NH-factorization.

Proposition 2.6.6 shows that, analogously as the case of conings, NH-factorizations factorize stellar exchanges via NH-manifolds. The main result of this section is that two combinatorial manifolds are combinatorially equivalent if and only if they are related by a sequence of NH-factorizations (Corollary 2.6.9). For this, we shall show that two manifolds related by general NH-factorizations are PL-isomorphic. We shall be using the following

Proposition 2.6.7 ([51, Theorem 2.1.B])**.** *Two homogeneous complexes K and L, with a common subcomplex T, are PL-isomorphic if and only if K (resp. L) can be transformed into L (resp. K) by a finite sequence of starrings and welds, which do not affect the subcomplex T.*

Proposition 2.6.8. *Let M and $\tilde{M}$ be combinatorial d-manifolds and suppose that there are NH-factorizations such that $M \overset{F(\sigma,\tau)}{\longrightarrow} N \overset{F(\rho,\eta)}{\longleftarrow} \tilde{M}$. Then $M \simeq_{PL} M'$.*

Proof. Assume that $M \overset{F(\sigma,\tau)}{\longrightarrow} N \overset{F(\rho,\eta)}{\longleftarrow} \tilde{M}$ are NH-factorizations. Hypothesis imply there exists complexes $L \subset M$ and $\tilde{L} \subset \tilde{M}$ such that $lk(\sigma, M) = \partial\tau * L$ and $lk(\rho, \tilde{M}) = \partial\eta * \tilde{L}$. Let $B = \sigma * \tau * L$ and $\tilde{B} = \rho * \eta * \tilde{L}$. It is easy to see that

$$\overline{M - st(\sigma, M)} = \overline{N - B} \qquad \text{and} \qquad \overline{\tilde{M} - st(\rho, \tilde{M})} = \overline{N - \tilde{B}}. \tag{2.4}$$

Also, by a dimension argument and the homogeneity of M and $\tilde{M}$ it follows that

$$B = \tilde{B}. \tag{2.5}$$

This shows that $\overline{M - st(\sigma, M)} = \overline{\tilde{M} - st(\rho, \tilde{M})} := T$. We claim that

$$T \cap st(\sigma, M) = T \cap st(\rho, \tilde{M})$$

By (2.4) and (2.5) it suffice to show that $T \cap st(\sigma, M) = T \cap B$ and $T \cap st(\rho, \tilde{M}) = T \cap \tilde{B}$. Since $st(\sigma, M) \subset B$ we only need to check that $T \cap B \subset st(\sigma, M)$. Let $\nu \in T \cap B$ and $\psi \in N - B$ with $\nu < \psi$. If $\tau < \nu$ then $\psi \in st(\tau, N) = B$, contradicting that $\psi \in N - B$. Hence $\nu \not< \tau$ and $\nu \in st(\sigma, M)$. This proves the claim. Finally, if $J := T \cap st(\sigma, M) = T \cap st(\rho, \tilde{M})$ then we can transform the d-ball $st(\sigma, M)$ into the d-ball $st(\rho, \tilde{M})$ without modifying the common subcomplex J by Proposition 2.6.7. Thus, we conclude

$$M_1 = T +_J B_1 \simeq_{PL} T +_J B_2 = M_2. \quad \square$$

Corollary 2.6.9. *Two combinatorial d-manifolds $M, \tilde{M}$ (with or without boundary) are PL-homeomorphic if and only if there exists a sequence*

$$M = M_1 \to N_1 \leftarrow M_2 \to N_2 \leftarrow M_3 \to \ldots \leftarrow M_{r-1} \to N_{r-1} \leftarrow M_r = \tilde{M}$$

where the N_i's are NH-manifolds, the M_i's are d-manifolds, and $M_i, M_{i+1} \to N_i$ are NH-factorizations. If M and $\tilde{M}$ are closed then the NH-factorizations may be taken to be bistellar factorizations.

Remark 2.6.10. Note that Proposition 2.6.8 provides new ways to define combinatorially equivalent moves between manifolds (or complexes in general). That is, by choosing (if possible) simplices $\rho \neq \tau$, $\eta \neq \sigma$ such that $M \overset{F(\sigma,\tau)}{\longrightarrow} N \overset{F(\rho,\eta)}{\longleftarrow} \tilde{M}$ we have a non-trivial transformation $F^{-1}(\rho, \eta)F(\sigma, \tau)$. For example, consider the 2-complex M on the vertex set $\{a, b, c, d, e\}$ generated by abc, abd, acd and cde. We compute $F(a, bcd)M = abcd + cde$. Now, this same complex is obtained by considering $F(ac, bd)N$ in the 2-complex N generated by abc, acd and cde. However, $F(bcd, a)^{-1}$ transforms $F(a, bcd)M$ in $bcd + cde$; so, $M \overset{F(\sigma,\tau)}{\longrightarrow} N \overset{F(\rho,\eta)}{\longleftarrow} \tilde{M}$ is not an NH-factorization of a stellar exchange. Moreover, it can be readily checked that the move $M \to \tilde{M}$ is not an stellar exchange, so $F^{-1}(\rho, \eta)F(\sigma, \tau)$ is indeed a different transformation between manifolds.

We finish this section by showing that NH-factorizations can be used to describe shellings. It is easy to see that direct and inverse shellings on a manifold M induce bistellar moves on ∂M. NH-factorizations provide a way to relate these Pachner moves in the opposite direction. We shall prove a slightly general result involving stellar exchanges and regular collapses and expansions. We introduce first some definitions.

Definition. Let M be a combinatorial d-manifold with boundary.

- A regular expansion $M \to M + B$ is said to be *stellar* if there are simplices $\sigma \in M$, $\tau \notin M$ and a combinatorial sphere L such that $B = \sigma * \tau * L$ and $M \cap B = \sigma * \partial\tau * L$.
- A regular collapse $M \to \overline{M - B}$ is said to be *stellar* if there are simplices $\sigma \in \partial M$, $\tau \in \overset{\circ}{M}$ and a combinatorial sphere L such that $B = \sigma * \tau * L$ and $\sigma * \partial\tau * L \in \partial M$.
- If $lk(\sigma, \partial M) = \partial\tau * L$ then a stellar exchange $\kappa(\sigma, \tau)\partial M$ is said to be *regular* if either $\tau \notin M$ or $\tau \in M^\circ$ and $\sigma * \tau * L \in M$.

Note that elementary and inverse shellings are special cases of stellar collapses and expansions.

Theorem 2.6.11. *Let M be a combinatorial d-manifold with boundary and let $\kappa(\sigma,\tau)$ be a stellar exchange in ∂M. Then $\kappa(\sigma,\tau)$ is regular if, and only if, $M \to M + F(\sigma,\tau)\partial M$ is a stellar expansion or $M \to \overline{M - F(\sigma,\tau)\partial M}$ is a stellar collapse.*

Proof. Suppose first that $\kappa(\sigma,\tau)$ is a regular stellar exchange on ∂M. Let $lk(\sigma,\partial M) = \partial\tau * L$ and $B = \sigma * \tau * L$. It is easy to see that if $\tau \notin M$ then $M \to M + B$ is a regular expansion. If $\tau \in \overset{\circ}{M}$ and $B \in M$ we shall prove that $\overline{M - B}$ is a combinatorial d-manifold by induction on d. It suffice to check the regularity of the vertices $v \in B \cap \overline{M - B}$. If $v = \tau$ then $lk(v, \overline{M - B}) = \overline{lk(v,M) - lk(v,B)}$ is a combinatorial $(d-1)$-ball by Newman's theorem. In any other case, $v \in \sigma * \partial\tau * L = st(\sigma,\partial M)$. Whether $v \in \sigma$, $v \in \partial\tau$ or $v \in L$ inductive hypothesis applies to show that $lk(v,\overline{M - B})$ is a combinatorial $(d-1)$-manifold. Since consequently $lk(v,\overline{M - B}) \to lk(v,M)$ is a regular expansion then $lk(v,\overline{M - B})$ is a combinatorial $(d-1)$-ball. This proves that v is regular.

We now prove the converse. Assume the decomposition of B as in the first part of the proof. Suppose that $M \to M + B$ is a stellar expansion. To find $\partial(M + B)$ we must study the $(d-1)$-simplices in $\partial B = \partial\sigma * \tau * L + \sigma * \partial\tau * L$. Now, if ρ is a principal simplex in $\sigma * \partial\tau * L$ then

$$lk(\rho, M + B) = lk(\rho, M) + lk(\rho, B) = \partial\Delta^1.$$

On the other hand, if ν is a principal simplex in $\partial\sigma * \tau * L$ then

$$lk(\nu, M + B) = lk(\nu, B) = \Delta^0.$$

This shows that $\partial(M + B) = \partial M - \sigma * \partial\tau * L + \partial\sigma * \tau * L = \kappa(\sigma,\tau)\partial M$.

Assume finally that $M \to \overline{M - B}$ is a stellar collapse. As in the previous case, we study the $(d-1)$-simplices of ∂B. Let $\rho \in \sigma * \partial\tau * L$ be a $(d-1)$-simplex. Since $\sigma * \partial\tau * L \subset \partial M$ and $\rho \prec B \notin \overline{M - B}$ then $\rho \notin \overline{M - B}$. On the other hand, it is easy to see that an $(d-1)$-simplex $\nu \in \partial\sigma * \tau * L$ is a face of a unique d-simplex of B. Since $\tau \in \overset{\circ}{M}$ then $\nu \in \partial\overline{M - B}$. This proves that $\partial\overline{M - B} = \partial M - \sigma * \partial\tau * L + \partial\sigma * \tau * L = \kappa(\sigma,\tau)\partial M$. $\square$

Corollary 2.6.12. *Let M be a combinatorial d-manifold with boundary. There is a 1-1 correspondence between elementary and inverse shellings on M and regular bistellar moves on ∂M.*

Resumen en castellano del Capítulo 2

En este capítulo introducimos el concepto de *variedad combinatoria no homogénea* o *NH-variedad*, las versiones no puras de las variedades poliedrales clásicas. La idea es definir una clase de complejos con propiedades estructurales similares a la de las variedades combinatorias pero sin forzar la homogeneidad. Las NH-bolas y NH-esferas, las versiones no puras de las bolas y esferas clásicas, también son definidas como un tipo especial de variedad no homogénea. Esta teoría fue en parte motivada por la noción de shellabilidad no homogénea de Björner y Wachs [7].

Las variedades combinatorias no homogéneas son complejos simpliciales cuyas realizaciones geométricas son localmente espacios euclídeos de distintas dimensiones, por lo que muchas propiedades estructurales fuertes están aún presentes en estos complejos. La Figura 2.1 muestra algunos ejemplos de esta estructura local. Las NH-bolas y NH-esferas son un tipo fundamental de NH-variedad definidas de manera de poseer propiedades características de las bolas y esferas clásicas.

Definición. Una *NH-variedad* (resp. *NH-bola*, *NH-esfera*) de dimensión 0 es una colección finita de vértices (resp. un vértice, dos vértices). Una NH-esfera de dimensión -1 es, por convención, $\{\emptyset\}$. Para $d \geq 1$, definimos por inducción

- Una *NH-variedad de dimensión d* es un complejo M de dimensión d tal que $lk(v, M)$ es una NH-bola de dimensión $0 \leq k \leq d-1$ o una NH-esfera de dimensión $-1 \leq k \leq d-1$ para todo vértice $v \in M$.

- Una *NH-bola de dimensión d* es una NH-variedad B de dimensión d colapsable; esto es, existe una subdivisión de B que colapsa simplicialmente a un vértice.

- Una *NH-esfera de dimensión d y dimensión homotópica k* es una NH-variedad S de dimensión d tal que existe una NH-bola $B \subset S$ de dimensión d y una k-bola combinatoria $L \subset S$, ambos subcomplejos generados por símplices maximales de S, tales que $B + L = S$ y $B \cap L = \partial L$. Llamamos $S = B + L$ una *descomposición* de S y notamos por $\dim_h(S)$ a la dimensión homotópica de S.

La definición de NH-bola está motivada por el Teorema de Whitehead y la de NH-esfera por el Teorema de Newman (Teoremas 1.3.1 and 1.2.6). La Figura 2.2 muestra varios ejemplos de NH-variedades, NH-bolas y NH-esferas.

Como primer resultado elemental mostramos que las NH-variedades (resp. NH-bolas, NH-esferas) son efectivamente una extensión del concepto de variedad combinatoria al contexto no homogéneo.

Resumen en castellano del Capítulo 2

Teorema 2.1.2. *Un complejo K es una NH-variedad (resp. NH-bola, NH-esfera) homogénea de dimensión d si y sólo si es una d-variedad (resp. d-bola, d-esfera) combinatoria.*

A continuación probamos que las NH-variedades satisfacen las propiedades básicas de las variedades combinatorias contenidas en la Proposición 1.2.4.

Proposición 2.1.3. *Sea M una NH-variedad de dimensión d y sea $\sigma \in M$ un k-simplex. Entonces $lk(\sigma, M)$ es una NH-bola o NH-esfera de dimensión menor a $d-k$.*

Teorema 2.1.6. *Las clases de NH-variedades, NH-bolas y NH-esferas son cerradas bajo PL-homeomorfismos.*

A continuación mostramos que el resultado esperado acerca del join simplicial entre bolas y esferas no puras se satisface en el contexto no homogéneo.

Teorema 2.1.9. *Sean B_1, B_2 NH-bolas y S_1, S_2 NH-esferas. Entonces,*

(1) $B_1 * B_2$ y $B_1 * S_2$ son NH-bolas.

(2) $S_1 * S_2$ es una NH-esfera.

Como última propiedad básica de las NH-variedades, probamos que verifican las propiedades (generalizadas) de las pseudo variedades. Para ello, definimos la noción de *NH-pseudo variedad.*

Definición. Una *NH-pseudo variedad* de dimensión d es un complejo de dimensión d tal que (1) para cada ridge $\sigma \in M$, $lk(\sigma, M)$ es un punto o una NH-esfera de dimensión homotópica 0; y (2) dados dos símplices maximales $\sigma, \tau \in M$, existe una sucesión $\sigma = \eta_1, \ldots, \eta_s = \tau$ de símplices maximales de M tal que $\eta_i \cap \eta_{i+1}$ es un ridge de η_i o η_{i+1} para cada $1 \leq i \leq s-1$.

Lema 2.1.10. *Toda NH-variedad conexa es una NH-pseudo variedad.*

Cerramos la primera sección de este Capítulo mostrando que las NH-esferas S para las cuales $\dim(S) = \dim_h(S)$ son necesariamente esferas combinatorias (Proposición 2.1.12).

En §2.2 introducimos dos nociones de "borde" en el contexto de las NH-variedades. El concepto de borde es central en la teoría clásica de variedades, y en general sólo está definido para complejos homogéneos. Gracias a la caracterización del borde de las variedades combinatorias como el subcomplejo formado por los símplices cuyos links son bolas podemos extender este concepto al contexto no homogéneo.

Definición. Sea M una NH-variedad. El *pseudo borde* de M es el conjunto de símplices $\tilde{\partial} M$ cuyo link son NH-bolas. El *borde* de M es el subcomplejo de ∂M generado por $\tilde{\partial} M$; es decir, el complejo que se obtiene al agregar las caras de los símplices en $\tilde{\partial} M$.

El pseudo borde de M no es en general un complejo, como muestran los ejemplos en la Figura 2.3. Mostramos en la Proposición 2.2.3 que $\tilde{\partial} M = \partial M$ si y sólo si M es una variedad combinatoria. Además, establecemos algunas propiedades básicas del borde de las NH-variedades en la Proposición 2.2.7.

Con la noción de borde podemos probar la existencia de *spines* para NH-variedades. Un *spine* de una d-variedad combinatoria con borde M es un subcomplejo K tal que $M \searrow K$ y $\dim(K) \leq d-1$. Los spines juegan un papel muy importante en la teoría de colapsabilidad de variedades.

Teorema 2.2.9. *Toda NH-variedad conexa con borde no vacío posee un spine.*

A continuación definimos el *complejo de anomalías* de una NH-variedad M como el subcomplejo $A(M) = \{\sigma \in M : lk(\sigma, M) \text{ es no homogéneo}\}$. En este sentido, el complejo de anomalías reúne la información acerca de las partes no puras de la NH-variedad. En la Figura 2.4 pueden verse algunos ejemplos. Las Proposiciones 2.2.10 y 2.2.11 muestran como el complejo de anomalías relaciona el borde y el pseudo borde de las NH-variedades y participa en la descomposición (única) de las NH-variedades en unión de pseudo variedades.

§2.3 está dedicada al estudio de la shellabilidad (no homogénea) en NH-variedades. La shellabilidad en el contexto no puro fue introducida por Björner y Wachs [7] en los años noventa con la motivación de analizar ejemplos provenientes de la teoría de arreglos de subespacios. Un complejo finito (no necesariamente homogéneo) es *shellable* si existe un orden $F_1, \dots, F_t$ de sus símplices maximales tal que $F_k \cap (F_1 + \cdots + F_{k-1})$ es $(\dim F_k - 1)$-homogéneo para todo $2 \leq k \leq t$. Para estudiar las NH-variedades shellables notamos primero que los conceptos de NH-bolas y NH-esferas pueden ser generalizados gracias a la presencia de borde en NH-esferas no homogéneas. Esto da lugar a los NH-bouquets.

Definición. Definimos un *NH-bouquet* G de dimensión d e *índice* k por inducción en k.

- Si $k = 0$ entonces G es una NH-bola de dimensión d.
- Si $k \geq 1$ entonces G es una NH-variedad de dimensión d tal que existe un NH-bouquet $S \subset G$ de índice $k-1$ y una bola combinatoria $L \subset G$, ambos subcomplejos generados por símplices maximales de G, tales que $S + L = G$ y $S \cap L = \partial L$.

Notar que los NH-bouquets de índice 1 son las NH-esferas. La Figura 2.6 muestra ejemplos de NH-bouquets. No es difícil ver que los NH-bouquets homogéneos tiene necesariamente índice 0 o 1 (y son por ende NH-bolas o NH-esferas) y que la clase de NH-bouquets también es cerrada por PL-homeomorfismos.

Teorema 2.3.4. *Sea M una NH-variedad shellable. Entonces para cada shelling $F_1, \dots, F_t$ de M y cada $0 \leq l \leq t$, $\mathcal{F}_l(M) = F_1 + \cdots + F_l$ es una NH-variedad. Más aún, $\mathcal{F}_l(M)$ es un NH-bouquet de índice $\sharp\{F_j \in \mathcal{T} \,|\, j \leq l\}$, donde $\mathcal{T}$ es el conjunto de símplices F_k para los cuales $F_k \cap (F_1 + \cdots + F_{k-1}) = \partial F_k$). En particular, M es un NH-bouquet de índice $\sharp\mathcal{T}$.*

§2.4 está dedicada a probar uno de los resultados más importantes de la teoría de NH-variedades: una generalización del teorema de expansiones regulares de Alexander. Recordemos que una expansión regular en una variedad combinatoria M es una expansión geométrica $M \to N = M + B^d$ tal que $M \cap B^d \subset \partial M$. El Teorema de Alexander afirma que $M + B^d$ es nuevamente una variedad combinatoria (ver Teorema 1.3.2) que además es PL-homeomorfa a la original. En esta sección presentamos dos resultados acerca de expansiones regulares. El primero muestra que puede llevarse a cabo un tipo mucho más general de expansión y aún mantener la estructura de NH-variedad. La segunda generalización involucra expansiones geométricas clásicas pero extrae consecuencias mucho más potentes: la nueva NH-variedad es equivalente (en un sentido de equivalencia no homogénea) a la variedad original.

El primero de estos teoremas, que enunciamos a continuación, es unos de los resultados principales de la primera parte de esta Tesis.

Teorema 2.4.3. *Sea M una NH-variedad y B^r una r-bola combinatoria. Supongamos que $M \cap B^r \subseteq \partial B^r$ es una NH-bola o NH-esfera generada por ridges de M o B^r y que $(M \cap B^r)^\circ \subseteq \tilde{\partial} M$. Entonces $M + B^r$ es una NH-variedad. Más aún, si M es un NH-bouquet de índice k y $M \cap B^r \neq \emptyset$ para $r \neq 0$, entonces $M + B^r$ es un NH-bouquet de índice k (si $M \cap B^r$ es una NH-bola) o $k+1$ (si $M \cap B^r$ es una NH-esfera).*

Bajo las hipótesis del Teorema 2.4.3 llamamos a un movimiento $M \to M + B^r$ una *expansión regular no homogénea.* Como la condición $(M \cap B)^\circ \subset \tilde{\partial} M$ se corresponde con $M \cap B \subset \partial M$ en el caso homogéneo vemos que las expansiones regulares clásicas son un caso especial de las expansiones regulares no homogéneas. También, este resultado nos permite extender la noción de shelling a NH-variedades de manera que este nuevo concepto sea consistente con la definición de shellabilidad de Björner y Wachs (en el sentido de la caracterización de bolas shellables).

Definición. Sea M una NH-variedad. Un *shelling inverso* es una expansión regular no homogénea $M \to M + \sigma$ que involucra un único simplex. Un *shelling elemental* es el movimiento inverso.

Corolario 2.4.5. *Una NH-bola B es shellable si y sólo si B puede transformarse en un simplex mediante una sucesión de shellings elementales.*

Para la segunda versión del teorema de expansiones regulares, introducimos una relación de equivalencia entre NH-variedades inspirada en el Teorema 1.4.1 de Pachner.

Definición. Sean M y M' dos NH-variedades. Decimos que M es *NH-equivalente* a M', denotado $M \simeq_{NH} M'$, si M' puede obtenerse de M por una sucesión finita de shellings elementales, shellings inversos y PL-isomorfismos.

Es fácil ver que $M \simeq_{NH} M'$ implica $M \nearrow\searrow M'$ y que $M \simeq_{PL} M'$ implica $M \simeq_{NH} M'$. Sin embargo, ninguna de las vueltas es válida. Notamos inmediatamente que, en el contexto homogéneo, $M \simeq_{NH} M'$ si y sólo si $M \simeq_{PL} M'$ por el Teorema de Pachner 1.4.1. Por lo tanto, la NH-equivalencia es una relación que está oculta en el contexto homogéneo y sólo se distingue de la equivalencia PL en el ámbito no puro.

Con la noción de NH-equivalencia tenemos una segunda versión del Teorema de Alexander.

Teorema 2.4.7. *Sea M una NH-variedad de dimensión d y sea B una r-bola combinatoria. Supongamos que $M \cap B \subset \partial B$ es una $(r-1)$-bola combinatoria y que $(M \cap B)^\circ \subset \tilde{\partial} M$. Entonces $M + B \simeq_{NH} M$.*

Cerramos la sección proponiendo una generalización del concepto de shelling y conjeturando que toda NH-bola puede llevarse a un único simplex mediante una sucesión de NH-shellings (una generalización del Teorema de Pachner).

En §2.5 exhibimos dos generalizaciones a NH-variedades de resultados clásicos de la teoría de PL-variedades: la existencia de entornos regulares y las ecuaciones de Dehn-Sommerville.

La teoría de entornos regulares de Whitehead es una de las herramientas más importantes de la teoría de PL-variedades. Un *entorno regular* de un subcomplejo K de una d-variedad combinatoria M es un subcomplejo $U \subset M$ tal que U es una d-variedad combinatoria tal que $U \searrow K$. Whitehead probó que todo subcomplejo de una variedad combinatoria posee siempre un entorno regular y, además, que son todos PL-homeomorfos.

Más aún, mostró que el entorno simplicial $N(K, S_K^2 M)$, que consta de todos los símplices de $S_K^2 M$ (la subdivisión baricéntrica doble de todos los símplices de $M - K$) que intersecan al complejo K, es un entorno regular de K. Nosotros probamos el mismo resultado de existencia para NH-variedades.

Teorema 2.5.4. *Sea M una NH-variedad y sea $K \subset M$. Entonces, $N(K, S_K^2 M)$ es una NH-variedad que colapsa a K.*

Por otro lado, una de las propiedades más distintivas de las variedades combinatorias es la verificación de las ecuaciones de Dehn-Sommerville. Para un complejo K de dimensión d se define el *f-vector* $f(K) = (f_0, ..., f_d)$ de K como el vector cuya coordenada f_i es la cantidad de i-simplices de K. Las ecuaciones de Dehn-Sommerville para una d-variedad combinatoria M relacionan el f-vector de M con el de su borde:

$$(1 - (-1)^{d-k}) f_k(M) + \sum_{i=k+1}^{d} (-1)^{d-k-1} \binom{k+1}{i+1} f_i(M) = f_k(\partial M) \tag{2.6}$$

Estas ecuaciones pueden escribirse matricialmente como

$$D(d)f(M) - f(\partial M) = 0,$$

donde D representa la matriz de los coeficientes de las ecuaciones en (2.6) (ver página 72).

Usando la generalización de a Chen y Yan [19] de las ecuaciones de Denh-Sommerville a poliedros para poliedros en general (ver Ecuación (2.2)) obtuvimos la siguiente versión para NH-variedades.

Teorema 2.5.6. *Para una NH-variedad M de dimensión d se satisface*

$$D(d)f(M) - f(\tilde{\partial} M) = 2f(B(M)),$$

donde

$$B(M) = \begin{cases} M_2 = \left\{\sigma \in M \mid \chi(lk(\sigma, M)) = 1 + (-1)^{\dim(\sigma)}\right\} & \textit{si } d \textit{ es par} \\ M_0 = \left\{\sigma \in M \mid \chi(lk(\sigma, M)) = 1 - (-1)^{\dim(\sigma)}\right\} & \textit{si } d \textit{ es impar} \end{cases}$$

En particular,

$$D(d)f(M) - f(\tilde{\partial} M) = 0 \pmod 2.$$

En la última sección de este capítulo estudiamos cómo las transformaciones simpliciales entre variedades combinatorias pueden descomponerse naturalmente de manera que todos los complejos involucrados en la descomposición son NH-variedades. Más concretamente, mostramos que los *starrings* y *welds* elementales y los *intercambios estelares* surgen como resultado de combinar expansiones locales de subcomplejos de la variedad y movimientos no puros de NH-variedades. Los conceptos novedosos en este contexto son los de *conings* y *NH-factorizaciones.*

Definición. Sea M una d-variedad combinatoria, $L \subset M$ un subcomplejo y $v \notin V_M$. El movimiento $M \to C_M(L) := M + v * L$ es llamado un *coning* de L.

La Figura 2.12 muestra la idea detrás de esta noción. La importancia de los conings reside en el hecho que un coning sobre una d-bola combinatoria deja como resultado una NH-variedad (Proposición 2.6.1). El resultado principal sobre la factorización de starrings y welds es el siguiente.

Resumen en castellano del Capítulo 2

Teorema 2.6.5. *Sea M una d-variedad combinatoria. Cualquier starring o weld en M puede factorizarse por una sucesión de conings y shellings (directos e inversos) no puros. En el proceso, todo complejo involucrado es una NH-variedad.*

Respecto de la factorización de intercambios estelares, introducimos la noción de *NH-factorización*.

Definición. Sea K un complejo simplicial y sea $\sigma \in K$ un simplex tal que $lk(\sigma, K) = \partial\tau * L$ con $\tau \notin K$. Una *NH-factorización* es el movimiento $K \to F(\sigma,\tau)K = K + \sigma * \tau * L$. Cuando $L = \emptyset$, la llamamos una *factorización biestelar*.

Es inmediato ver que el siguiente diagrama conmuta.

$$\begin{tikzcd} K \arrow[rr, "\kappa(\sigma,\tau)"] \arrow[dr, "F(\sigma,\tau)"'] & & \kappa(\sigma,\tau)K \arrow[dl, "F(\tau,\sigma)"] \\ & K + \sigma * \tau * L & \end{tikzcd} .$$

La Figura 2.14 muestra ejemplos de NH-factorizaciones. Al igual que para el caso de conings, se tiene que el resultado de una NH-factorización sobre una d-variedad combinatoria es una NH-variedad (Proposición 2.6.6). Mostramos en la Proposición 2.6.8 que dos d-variedades combinatorias M y M' relacionadas por una NH-factorización $M \overset{F(\sigma,\tau)}{\longrightarrow} N \overset{F(\rho,\eta)}{\longleftarrow} \tilde{M}$ son necesariamente PL-homeomorfas. Esto nos conduce al siguiente resultado.

Corolario 2.6.9. *Dos d-variedades combinatorias $M, \tilde{M}$ son PL-homeomorfas si y sólo si existe una sucesión*

$$M = M_1 \to N_1 \leftarrow M_2 \to N_2 \leftarrow M_3 \to \ldots \leftarrow M_{r-1} \to N_{r-1} \leftarrow M_r = \tilde{M}$$

donde los N_i's son NH-variedades, las M_i's son d-variedades, y $M_i, M_{i+1} \to N_i$ son NH-factorizaciones.

Es fácil ver que los shellings en una variedad M inducen movimientos biestelares en ∂M. Las NH-factorizaciones proporcionan una manera de relacionar estos movimientos de Pachner en la dirección opuesta. Definimos una clase especial de expansiones y colapso regulares llamados *expansiones y colapsos estelares* (de los cuales los shellings son un caso particular) y una clase especial de intercambios estelares llamados *intercambios regulares* y probamos el siguiente

Teorema 2.6.11. *Sea M una d-variedad combinatoria con borde y sea $\kappa(\sigma,\tau)$ un intercambio estelar en ∂M. Entonces $\kappa(\sigma,\tau)$ es regular si y sólo si $M \to M + F(\sigma,\tau)\partial M$ es una expansión estelar o $M \to \overline{M - F(\sigma,\tau)\partial M}$ es un colapso estelar.*

Como consecuencia obtenemos el siguiente

Corolario 2.6.12. *Existe una relación 1-1 entre shellings inversos y directos sobre M y movimientos biestelares regulares sobre ∂M.*

Chapter 3

Alexander Duality

One of the most famous duality theorem is due to Poincaré, relating the homology groups of a closed d-manifold with its cohomology groups. This duality can be obtained from combinatorial arguments as well as from differential arguments. Another duality theorem which is almost as old as Poincaré's is Alexander duality, relating the homology of a subspace of the d-sphere with the cohomology of its complement. For triangulable spaces, Alexander duality can be stated in a purely combinatorial form, using a simplicial homotopy-representative K^* of the complement of a complex K. In this form, Alexander duality can be proved using only simplicial methods. This form of duality is strongly related to the theory of NH-manifolds as we will show below.

In the first section of this chapter we recall the classical and simplicial versions of Alexander Duality. The rest of the chapter is original work and it is devoted to study the Alexander dual of finite simplicial complexes relative to different sets of vertices and to provide an alternative proof of a result of Dong and Santos-Sturmfels on the dual of balls and spheres. Our proofs are based on the local structure of the manifolds (in contrast to the approach of the original proofs). In the next chapter we will generalize these results.

§3.1 Classical and combinatorial Alexander duality

The study of the relationship between the topology of complementary spaces is a classical problem that goes back at least to *Jordan's curve Theorem.* This famous result states that a simple closed curve in $\mathbb{R}^2$ (or S^2) splits the space in two connected components. In 1911 Brouwer [11] gave a generalization of Jordan curve theorem to the d-dimensional euclidean space and, in 1922, Alexander [1] further generalized it to the following result.

Theorem 3.1.1 (Alexander duality)**.** *If $A \subset S^d$ is a non-empty, compact and locally contractible proper subspace of the d-dimensional sphere then*

$$H_k(A) \simeq H^{d-k-1}(S^d - A).$$

As a simple geometrical example of Alexander duality, one can readily see that the complement of two small disjoint disks about the poles of S^2 has the homology of S^1 (actually, it has the 1-dimensional sphere as a strong deformation retract in this case). The intuitive idea behind this duality is that if one has an k-dimensional hole/handle in a subspace of the d-sphere then the complement must have "complementary handle/hole" to fill the spaces in between.

Many types of duality were explored at the beginnings of the last century, motivated primarily by the work of Henri Poincaré [45], who established by 1900 what is nowadays known as "Poincaré Duality": $H_k(M) \simeq H^{d-k}(M)$ for any triangulable closed d-manifold. The work of Alexander led Lefschetz [34] to introduce in 1926 the concept of relative homology and he subsequently gave a relative version of Poincaré duality relating the homology of (M, K) with the cohomology of $M - K$ (and in particular the important case $(M, \partial M)$). All these duality theorems may be rephrased as part of the following general result about subspaces of closed manifolds (see [9, Theorem 8.3]).

Theorem 3.1.2. *Let M be a d-dimensional orientable manifold and let $L \subset K$ be compact subsets of M. Then*

$$H_k(K, L) \simeq H^{d-k}(M - L, M - K).$$

Thus, Poincaré duality corresponds to the case $M = K$ and $L = \emptyset$ and Lefschetz duality to $M = K$. Using his general version of the duality theorem, Lefschetz gave a proof of Alexander duality (which is the case $M = S^d$). A very nice geometrical proof of all these duality theorems may be found in [40] and a direct proof of the general Theorem 3.1.2 using handle decomposition may be found in [47].

The Alexander Duality Theorem admits a purely combinatorial formulation by introducing a simplicial (homotopy) representative of $S^d - A$ and can thus be stated in purely combinatorial terms. In this version, the triangulation of S^d is taken to be $\partial\Delta^{d+1}$ and, for $K \subset \partial\Delta^{d+1}$, the role of $|\partial\Delta^{d+1}| - |K|$ is played by a subcomplex K^* of $\partial\Delta^{d+1}$ called *Alexander dual* of K which we define next.

Notation. For a set of vertices V, we write $\Delta(V)$ for the simplex spanned by the vertices in V. If K is a complex, we put $\Delta_K := \Delta(V_K)$.

Definition. Let K be a simplicial complex and let $V \supseteq V_K$ be a ground set of vertices. The *Alexander dual* of K (relative to V) is the complex

$$K^{*_V} = \{\sigma \in \Delta(V) \,|\, \sigma^c \notin K\}.$$

Here the simplex $\sigma^c := \Delta(V - V_\sigma)$ is the *complement* of σ with respect to V.

Theorem 3.1.3 (Combinatorial Alexander duality)**.** *Let K be a simplicial complex and let $V \supseteq V_K$ be a ground set of vertices. Then*

$$H_i(K^{*_V}) \simeq H^{n-i-3}(K),$$

where $n = |V|$.

Note that Theorem 3.1.3 is expressed in terms of the ground set of vertices instead of the dimension of the sphere. In this setting is more practical to let the ground set of vertices $V \supset V_K$ implicitly determine the underlying dimension. Figure 3.1 shows some examples of Alexander duals of complexes.

The following lemma contains the very basic properties of K^* (all of which are easy to establish).

Lemma 3.1.4. *Let K be a finite simplicial complex and $V \supset V_K$. Then*

(1) $(\Delta(V))^{*_V} = \emptyset$ *and* $\emptyset^{*_V} = \Delta(V)$.

(2) K^{*_V} *is a simplicial complex.*

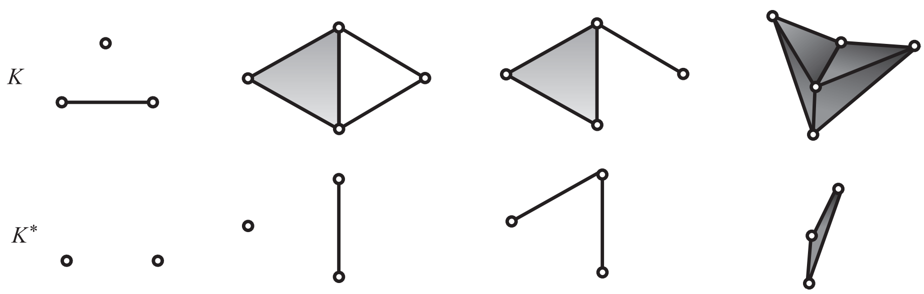


Figure 3.1: Examples of Alexander duals.

(3) $(K^{*_V})^{*_V} = K$.

(4) *If* $L \subset K$ *then* $K^{*_V} \subset L^{*_V}$.

Combinatorial Alexander duality is a direct corollary of classical Alexander duality. This follows from the fact that K^{*_V} is a deformation retract of $|\partial\Delta(V)| - |K|$. A completely simplicial proof of this form of Alexander duality can be found in Barr's article [4]. Apparently, Barr did not notice in his paper that he was producing a simplicial version of Alexander duality. In 2009, Björner and Tancer [6] presented independently a combinatorial proof Alexander duality which follows essentially the lines of [4], but without being aware of Barr's proof.

We next sketch the main ideas of the combinatorial proof of [4, 6]. Let $|V_K| = n$. There exists a short exact sequence

$$0 \to C_i(K^*) \xrightarrow{\alpha} C^{n-i-2}(\Delta_K) \xrightarrow{\beta} C^{n-i-2}(K) \to 0$$

where $\alpha(\sigma) = (\sigma^c)^*$ and β is the morphism induced by the inclusion. The injectivity of α and the surjectivity of β follow from the fact that $K, K^* \subset \Delta_K$. To see the exactness in $C^{n-i-2}(\Delta_K)$ note that $\ker(\beta) = \{\eta^* \,|\, \eta \notin K\}$, so $Im(\alpha) = \ker(\beta)$ by definition of α and K^*. Next, we see that the following square commutes up to sign

$$\begin{array}{ccc} C_i(K^*) & \xrightarrow{\alpha} & C^{n-i-2}(\Delta_K) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\delta} \\ C_{i-1}(K^*) & \xrightarrow{\alpha} & C^{n-i-1}(\Delta_K) \end{array}$$

Let $\sigma \in C_i(K^*)$ and $\tau \in C_{n-i-1}(\Delta_K)$. A simple computation shows that $\delta(\alpha(\sigma))(\tau) \neq 0$ if and only if $\tau \succ \sigma^c$ if and only if $\tau^c \prec \sigma$ if and only if $\alpha(\partial\sigma)(\tau) \neq 0$. In these cases, the morphisms equal 1 or -1, so the square effectively commutes up to sign. We conclude that α defines a chain complex morphism $\alpha : C_*(K^*) \to C^{n-*-2}(\Delta_K;)$ which gives rise to a short exact sequence of chain complexes

$$0 \to C_*(K^*) \to C^{n-*-2}(\Delta_K) \to C^{n-*-2}(K) \to 0.$$

The long exact sequence associated to this sequence is locally

$$\cdots \to H^{n-i-3}(\Delta_K) \to H^{n-i-3}(K) \to H_i(K^*) \to H^{n-i-2}(\Delta_K) \to \cdots,$$

and the duality follows from the fact that $H^*(\Delta_K) = 0$.

§3.2 The Alexander dual with respect to different ground sets of vertices

By definition, the Alexander dual of a complex K depends on the chosen ground vertex set $V \supset V_K$. Typically, a "universal" vertex set is implicitly fixed beforehand and every complex considered is supposed to have its vertices in this ground set. On the other hand, when a single complex is to be used, a natural way to chose the vertex set is $V = V_K$. In this section we study the relationship between the Alexander dual of a complex relative to its own set of vertices and to a bigger ground set of vertices. This is a natural question since geometrically it amounts to analyze the relation between the complement of a complex when seen as subspace of spheres of different dimensions. This problem seems to not have been addressed in the literature before and, although the relation between the homotopy type of different duals is probably known, the simple formula relating all possible duals of a complex was not found by the author anywhere.

The following lemma, and the immediate consequences we deduce, are central in the development of the rest of the theory.

Notation. For the rest of the work, we shall write $K^* := K^{*V_K}$ and $K^\tau := K^{*V}$ if $\tau = \Delta(V - V_K)$; that is, K^τ represents the Alexander dual of K relative to $V = V_K + V_\tau$. With this convention, if $\tau = \emptyset$ then $K^\tau = K^*$ is the Alexander dual of K relative to its own set of vertices. Note that $(\Delta^d)^* = \emptyset$ and $(\partial\Delta^{d+1})^* = \{\emptyset\}$.

Lemma 3.2.1. *Let K be a simplicial complex and let τ be a (non-empty) simplex disjoint from K. Then,*

$$K^\tau = \partial\tau * \Delta_K + \tau * K^*. \tag{3.1}$$

Here K^* is considered as a subcomplex of the simplex Δ_K.

In particular, we have the following consequences.

(1) If K is not a simplex or $\dim(\tau) \geq 1$ then $V_{K^\tau} = V_K \cup V_\tau$. If $K = \eta$ is a simplex and $\dim(\tau) = 0$ then $\eta^\tau = \eta$. In any case, $V_K \subseteq V_{K^\tau}$.

(2) If K is not a simplex or $\dim(\tau) \geq 1$ then $(K^\tau)^* = K$.

(3) If $V_{K^*} \subsetneq V_K$ and $\rho = \Delta(V_K - V_{K^*})$ then $(K^*)^\rho = K$.

(4) If K is not a simplex then $K^\tau \simeq \Sigma^t K^*$ for some $t \geq 0$.

Here $\Sigma^t K^* := \partial\Delta^t * K^*$ denotes the simplicial t-fold suspension of K^*.

Proof. Set $V = V_K \cup V_\tau$. Let $\sigma \in K^\tau$ be a principal simplex, so $\sigma^{c_V} \notin K$. If $\tau < \sigma$, say $\sigma = \tau * \eta$, then $\sigma^{c_V} = \eta^{c_{V_K}}$ and therefore $\sigma = \tau * \eta \in \tau * K^*$. Any other simplex in K^τ not containing τ lies trivially in $\partial\tau * \Delta_K$. For the other inclusion, if $\sigma = \tau * \eta$ is principal and $\eta \in K^*$ then $\sigma^{c_V} = \eta^{c_{V_K}} \notin K$, and hence $\sigma \in K^\tau$. If $\sigma \in \partial\tau * \Delta_K$ is principal then, in particular, $\Delta_K < \sigma$ and therefore $\sigma^{c_V} < \tau$. Since no vertex of τ lies in K, $\sigma^{c_V} \notin K$ and then $\sigma \in K^\tau$.

Item (1) follows directly from formula (3.1) and items (2)-(3) from the fact that for a fixed ground set V, $(K^{*V})^{*V} = K$. Finally, (4) follows from formula (3.1) since both complexes in the union are contractible (see Lemma 3.3.2 (1)). □

Note that the equation in (4) also holds for $\tau = \emptyset$ taking $t = 0$.

The following elementary lemma is a key importance for the rest of the work. It shows when $K^{**} := (K^*)^*$ (both times relative to V_K) is the original complex.

Lemma 3.2.2. *Let K be a simplicial complex of dimension d that is not a d-simplex. The following statements are equivalent.*

(1) $|V_K| = d + 2$.

(2) $V_{K^*} \neq V_K$.

(3) $K \neq K^{**}$.

Proof. Suppose that $|V_K| = d + 2$ and let $\sigma \in K$ be a d-simplex. Then the only vertex $v \in V_K - V_\sigma$ is not in V_{K^*}. Conversely, if $w \in V_K - V_{K^*}$ then $w^c \in K$. Since K is not a d-simplex then $|V_K| \geq d+2$. Since w^c is the simplex spanned by the vertices in $V_K - \{w\}$ and $\dim(K) = d$ then $|V_K| \leq d + 2$. This proves that (1) and (2) are equivalent.

(2) implies (3) since $V_{K^{**}} \subseteq V_{K^*} \subseteq V_K$. Also, (3) implies (2) since if $V_{K^*} = V_K$ then $K^{**} = K$. □

Corollary 3.2.3. *Let K be a simplicial complex and let τ be a (non-empty) simplex disjoint from K. Then,*

(1) If K is not a simplex or $\dim(\tau) \geq 1$ *then* $|V_{K^\tau}| = \dim(K^\tau) + 2$.

(2) The subcomplexes $\partial\tau * \Delta_K, \tau * K^* \subset K^\tau$ in formula (3.1) *of Lemma 3.2.1 are top generated.*

Proof. If K is not a simplex, item (1) follows directly from Lemmas 3.2.1 and 3.2.2. If $K = \eta$ is a simplex and $\dim(\tau) \geq 1$ then $\eta^\tau = \partial\tau * \eta$ which has dimension $\dim(\tau) + \dim(\eta)$ and $\dim(\tau) + 1 + |V_\eta| = \dim(\tau) + \dim(\eta) + 2$ vertices.

For (2), simply notice that $\partial\tau * \Delta_K \cap \tau * K^* = \partial\tau * K^*$ and that K^* is always properly contained in Δ_K. □

Lemma 3.2.1 and Corollary 3.2.3 state that every complex is the Alexander dual of a complex of dimension d and $d + 2$ vertices for some $d \geq 0$. As we shall see, these vertex-minimal complexes will play a distinctive role in the rest of this work. In this spirit, we present next two results that will be useful in sections to come.

Proposition 3.2.4. *If M is an NH-manifold of dimension d and $d + 2$ vertices then M is an NH-ball or NH-sphere.*

Proof. Since $|V_M| = d + 2$ there is a vertex $u \in M$ such that $M = \Delta^d + u * lk(u, M)$. It follows from Theorem 2.4.3 that M is an NH-ball or NH-sphere depending on $lk(u, M)$. □

Proposition 3.2.4 may be considered the non-pure counterpart of the characterization of d-homogeneous complexes with $d+2$ vertices: they are either the boundary of a simplex or an elementary starring of one (see Lemma 4.2.7). In particular, NH-manifolds with $d + 2$ vertices are either contractible or homotopy equivalent to a sphere. This may not be the case for general spaces with the same number of vertices.

Finally, we characterize the Alexander dual of vertex-minimal complexes.

Lemma 3.2.5. *Let K be a complex of dimension d and $d + 2$ vertices. Then, for every vertex $u \in V_K - V_{K^*}$ we have that $K^* = (lk(u, K))^\tau$ where $\tau = \Delta(V_K - V_{st(u,K)})$.*

Proof. By hypothesis we can write $K = \Delta^d + u * lk(u, K)$. Let τ be as in the statement. Then,

$$\begin{aligned}
\sigma \in (lk(u,K))^\tau &\Leftrightarrow \Delta(V_{lk(u,K)} \cup V_\tau - V_\sigma) \notin lk(u,K) \\
&\Leftrightarrow \Delta(V_{lk(u,K)} \cup (V_K - V_{st(u,K)}) - V_\sigma) \notin lk(u,K) \\
&\Leftrightarrow \Delta(V_K - \{u\} - V_\sigma) \notin lk(u,K) \\
&\Leftrightarrow u * \Delta(V_K - \{u\} - V_\sigma) \notin K \\
&\Leftrightarrow \Delta(V_K - V_\sigma) \notin K \\
&\Leftrightarrow \sigma \in K^*. \qquad \square
\end{aligned}$$

§3.3 The homotopy type of the Alexander dual of balls and spheres

Alexander duality permits to completely characterize the homology of the dual K^*. In the special cases when the complex is acyclic or has the homology of a spheres (or more generally, of a bouquet of spheres), its Alexander dual turns out to be of the same kind. However, this duality is not sufficient to characterize the homotopy type of the dual of these particular spaces. There are contractible complexes whose duals are not contractible and homotopy spheres whose duals do not have the homotopy type of a sphere. Low dimensional examples may be found in [5]. In [39] it is shown that for any finitely presented group G, there is a simply connected polyhedron K with fundamental group isomorphic to G.

In 2002, Dong [22] proved that if we restrict to the case of of simplicial spheres, then the Alexander dual of one such space has again the homotopy type of a sphere. Dong used the theory of projection of polytopes and his methods are mainly based on convexity arguments. A year later, Santos and Sturmfels [49] noticed that a similar result could be settled for simplicial balls, and they used Dong's result to show that the Alexander dual of a simplicial ball is a contractible space. Both results evidence that a locally well-behaved structure on the complex forces homotopy stability on its dual.

In this section we present a completely alternative proof of Dong's and Santos-Sturmfels' original results based on the local structure of manifolds, in contrast to the previous approaches. We also give two other proofs of Dong's result for polytopal spheres using classical results from polytope theory.

3.3.1 A new proof of the theorem of Dong and Santos-Sturmfels

The proof that we will present is very simple in nature and it relies on some elementary remarks on the relation between deletion and links of vertices. Recall that $\Sigma K = \partial\Delta^1 * K$ denotes the simplicial suspension of K.

Lemma 3.3.1. *Let $K \neq \Delta^d$ be a simplicial complex of dimension d and let $v \in V_K$. Then,*

(1) $lk(v, K^*) = (K - v)^*$.

(2) $lk(v, K) = (K^* - v)^\tau$ where $\tau = \Delta(V_{K-v} - V_{K^*-v})$.

(3) If v is not isolated and $lk(v, K)$ is not a simplex then $K^* - v \simeq \Sigma^t lk(v, K)^*$ for some $t \geq 0$.

(4) If $lk(v, K)$ is a simplex then $K^* - v$ is contractible.

Proof. For (1),

$$\sigma \in lk(v, K^*) \Leftrightarrow v * \sigma \in K^* \Leftrightarrow (v * \sigma)^c \notin K \Leftrightarrow \sigma^c \notin K - v \Leftrightarrow \sigma \in (K - v)^*.$$

To prove (2), take any $x \notin V_K$. Since $K \neq \Delta^d$ then $(K^x)^* = K$ and by (1),

$$lk(v, K) = lk(v, (K^x)^*) = (K^x - v)^*.$$

Note that $K^x = \Delta_K + x * K^*$, and then

$$K^x - v = \Delta_K - v + x * K^* - v = \Delta(V_K - v) + x * (K^* - v).$$

Now Lemma 3.2.5 implies that

$$(K^x - v)^* = lk(x, K^x - v)^\tau = (K^* - v)^\tau$$

where $\tau = \Delta(V_{K^x - v} - V_{st(x, K^x - v)}) = \Delta(V_{K-v} - V_{K^* - v})$. This proves (2).

To prove (3), apply Alexander dual to the equality given in (2) to yield

$$lk(v, K)^* = ((K^* - v)^\tau)^*.$$

When $\tau \neq \emptyset$, this equals $K^* - v$ by Lemma 3.2.1 (2), which settles the result with $t = 0$. Note that, by hypothesis, $K^* - v = \Delta^r$ and $\dim(\tau) = 0$ cannot simultaneously hold.

Suppose now that $\tau = \emptyset$. Denote $T = K^* - v$. If $\dim(T) \neq |V_T| - 2$ then $lk(v, K)^* = T^{**} = T$ by Lemma 3.2.2 and the result holds with $t = 0$. If $\dim(T) = |V_T| - 2$ then $\rho = \Delta(V_T - V_{T^*}) \neq \emptyset$ and

$$T = (T^*)^\rho = \partial\rho * \Delta_{T^*} + \rho * T^{**} = \partial\rho * \Delta_{T^*} + \rho * lk(v, K)^*.$$

Since by hypothesis $\Delta_{T^*} = \Delta_{lk(v,K)} \neq \emptyset$ and $T^{**} = lk(v, K)^* \neq \emptyset$ then

$$K^* - v = T \simeq \Sigma(\partial\rho * lk(v, K)^*) \simeq \Sigma^t lk(v, K)^*.$$

To prove (4) note that if $(K^* - v)^\tau = lk(v, K)$ is a simplex then $K^* - v$ is an NH-ball by Theorem 4.1.3. □

We note that the proof for Lemma 3.3.1 (1) is easy generalized to hold that $lk(\sigma, K^*) = (K - \{v \,|\, v \in V_\sigma\})^*$ (the deletion of all the vertices in σ) for any $\sigma \in K$.

We shall use the following standard results.

Lemma 3.3.2. *Let K be a finite simplicial complex and $A, B \subset K$ subcomplexes such that $K = A + B$.*

(1) If A and B are contractible then $K \simeq \Sigma(A \cap B)$. If, in addition, K is acyclic then K is contractible. In particular, acyclic simplicial complexes of dimension d and $d + 2$ vertices are contractible.

(2) If $A \cap B$ and B are contractible then $K \simeq A$.

Lemma 3.3.3. *Let L be a subcomplex of K. Then $K \searrow L$ if and only if $K^* \nearrow L^\rho$ where $\rho = \Delta(V_K - V_L)$. In particular, if L^* is contractible or homotopy equivalent to a sphere then so is K^*.*

Proof. It is straightforward to see that if $L = K - \{\tau, \sigma\}$ with $\sigma \prec \tau$ a free face then $K^* = L^\rho - \{\sigma^c, \tau^c\}$ with $\tau^c \prec \sigma^c$ a free face. On the other hand, by Lemma 3.2.1 (4), $K^* \simeq L^\tau \simeq \Sigma^t L^*$, so K^* is contractible or homotopy equivalent to a sphere if L^* is. □

We are now able to give an alternative proof of Dong's and Santos-Sturmfels' original results.

Theorem 3.3.4 (Dong, Santos-Sturmfels)**.** *If $B \neq \Delta^d$ is a combinatorial d-ball then B^τ is contractible. If S is a combinatorial d-sphere then S^τ is homotopy equivalent to a sphere.*

Proof. By Lemma 3.2.1 (4) it suffices to prove the result for $\tau = \emptyset$. We first prove it for a combinatorial ball B by induction on $d \geq 1$. If $d = 1$ then B collapses to a 1-ball with two edges (whose Alexander dual is a vertex) and the result follows from Lemma 3.3.3. Now, let $d \geq 2$. If $|V_B| = d + 2$, take $u \notin B^*$. If $lk(u, B)$ is not a simplex, Lemmas 3.2.5 and 3.2.1 (4) imply $B^* \simeq \Sigma^t lk(u, B)^*$, which is contractible by induction since $lk(u, B)$ is a ball. If $lk(u, B)$ is a simplex, the result follows immediately.

Suppose $|V_B| \geq d+3$ and let $v \in \partial B$. Now, $B^* - v$ is contractible by Lemma 3.3.1 (4) or Lemma 3.3.1 (3) and induction. Since $B^* = B^* - v + st(v, B^*)$ is acyclic by Alexander duality then B^* is contractible by Lemma 3.3.2 (1).

Now let S be a combinatorial sphere. We may assume that $|V_S| \geq d + 3$. We proceed again by induction on d. Let $d \geq 1$ and $v \in S$. By Lemma 3.3.1 (1), $lk(v, S^*) = (S - v)^*$ which is contractible by Newman's theorem and the previous case. Since $S^* = S^* - v + st(v, S^*)$ where $(S^* - v) \cap st(v, S^*) = lk(v, S^*)$ is contractible, then $S \simeq S^* - v \simeq \Sigma^t lk(v, S)^*$ by Lemmas 3.3.2 (2) and 3.3.1 (3). The result now follows by the inductive hypothesis on the $(d-1)$-sphere $lk(v, S)$. □

Remark 3.3.5. This theorem actually holds for simplicial (not necessarily combinatorial) balls and spheres. The more general formulation can be deduced from this result using the following argument, which Dong applies to reduce the problem to the polytopal case. Let $n = |V_S|$ and $d = \dim(S)$. If $n = d + 2$ then $S = \partial\Delta^{d+1}$ (see Lemma 4.2.7), whose Alexander dual is $\{\emptyset\}$ and hence the result holds. On the other hand, if $n - d \geq 5$ then S^* is simply connected since it contains the complete 2-skeleton of $\Delta(V_{S^*})$. This follows since the complement $(\Delta^2)^c = \Delta(V_{S^*} - V_{\Delta^2})$ of any 2-simplex Δ^2 in the vertices of V_{S^*} is a simplex with $|V_{S^*}| - |V_{\Delta^2}| = n - 3 \geq d + 2$ vertices; and hence, of dimension $\geq d + 1$. Since $\dim(S) = d$ then $(\Delta^2)^c \notin S$ and therefore $\Delta^2 \in S^*$. It is a standard result that a simply connected space with the homology of a sphere is homotopy equivalent to one.

3.3.2 A second proof of Dong's result

We now give a second proof of the result for spheres of Theorem 3.3.4 using some classical results of polytope theory. By applying Dong's argument of Remark 3.3.5 the problem is reduced to the polytopal case and the idea behind the proof is to show that two polytopal d-spheres with the same number of vertices are simply homotopy equivalent in such way that the collapses and expansions involved do not add vertices in the process. Since this implies that the Alexander duals of the spheres are also simply homotopy equivalent, the proof of Dong's result reduces to the case of finding a single representative polytopal sphere for each (needed) number of vertices.

The proof is based on the following two classic results of polytope theory.

Theorem 3.3.6 ([24, Theorem (6)]). *Given any two d-dimensional convex polytopes P and Q, there is a sequence of polytopes $P = P_0, P_1, \dots, P_r = Q$ such that for each pair (P_{i-1}, P_i) the boundary complex of one is isomorphic to a single geometric stellar (or elementary) subdivision of the boundary complex of the other.*

Theorem 3.3.7 ([42, Theorem 7]). *If P and Q are simplicial polytopes in the same dimension with the same number of vertices, then the boundary complex of one can be obtained from the boundary complex of the other by a sequence of bistellar moves that preserves de number of vertices.*

Definition. Let K and L be two simplicial complexes. We say that K is *simply homotopy equivalent without adding vertices* if no vertices are added in any of the elementary collapses or expansions. In this case, we shall write $K \underset{w.a.v.}{\nearrow\searrow} L$.

Note that, in particular, $K \underset{w.a.v.}{\nearrow\searrow} L$ implies $V_L \subset V_K$. We have the following consequence of Lemma 3.3.3.

Corollary 3.3.8. *Let K be a finite simplicial complex and let L be a complex such that L^* has the homotopy type of a sphere.*

(1) If $K \searrow L$ then K^* has the homotopy type of a sphere.

(2) If $K \underset{w.a.v.}{\nearrow} L$ then K^* has the homotopy type of a sphere.

In particular, if $K \underset{w.a.v.}{\nearrow\searrow} L$ then K^* has the homotopy type of a sphere.

Definition. Let K be a simplicial complex and $\sigma \in K$ a simplex such that $lk(\sigma, K) = \partial\tau$ for a $\tau \notin K$. A bistellar factorization is said to be *proper* if τ is a vertex (or equivalently, σ is a principal simplex and the associated bistellar move turns out to be an elementary starring).

In the following lemma we use the notion of NH-factorization $F(\sigma, \tau)$ (see §2.6.2).

Lemma 3.3.9. *$K \nearrow F(\sigma, \tau)K$. Moreover, $|V_K| = |V_{F(\sigma,\tau)K}|$ if and only if $F(\sigma, \tau)$ is not proper.*

Proof. It suffices to show that $\sigma * \tau \searrow \sigma * \partial\tau$. We proceed by induction on the dimension of σ (note that $\sigma \neq \emptyset$). If $\dim(\sigma) = 0$ then $\sigma = v$ vertex and $v * \tau \searrow v * \partial\tau$. If $\dim(\sigma) \geq 1$ then writing $\sigma = v * \sigma'$ we have $\sigma' * \tau \searrow \sigma' * \partial\tau$ by induction. Therefore, $v * \sigma' * \tau \searrow v * \sigma' * \partial\tau$.

The second assertion is clear since τ and $\partial\tau$ have the same vertices if and only if $\dim(\tau) \geq 1$. □

Note that Lemma 3.3.9 says that two bistellar equivalent complexes are simple homotopy equivalent by transient moves (see [27, §1] for definitions).

Corollary 3.3.10. *Any two polytopal d-spheres with the same number of vertices are simply homotopy equivalent without adding vertices.*

Proof. It follows directly by Theorem 3.3.7 and Lemma 3.3.9. □

Lemma 3.3.11. *Let $p, q \in \mathbb{N}$ such that $1 \leq q < p$. Then, $\Delta^p \underset{\Delta^{q-1}}{+} \Delta^q \nearrow \Delta^p \underset{\tilde{\Delta}^{p-1}}{+} \Delta^p$ without adding vertices.*

Proof. We proceed by induction on $q \geq 1$. Since $\Delta_1^p \underset{\Delta^{p-1}}{+} \Delta_2^p \searrow \Delta_1^p$ then last elementary collapse involves the removal of an edge Δ^1 from a free vertex and every other previous collapse maintains the same number of vertices. Hence, $\Delta^p \underset{\Delta^0}{+} \Delta^1$ expands to $\Delta^p \underset{\Delta^{p-1}}{+} \Delta^p$ and the case $q = 1$ is settled.

Suppose $q \geq 2$. For $v \in \Delta^p \cap \Delta^q = \Delta^{q-1}$ we have

$$lk(v, \Delta^p \underset{\Delta^{q-1}}{+} \Delta^q) = lk(v, \Delta^p) \underset{lk(v,\Delta^{q-1})}{+} lk(v, \Delta^q) = \tilde{\Delta}^{p-1} \underset{\tilde{\Delta}^{q-2}}{+} \tilde{\Delta}^{q-1}.$$

By inductive hypothesis, $lk(v, \Delta^p + \Delta^q)$ expands without adding vertices to $lk(v, \Delta^p) + \tilde{\Delta}^{p-1}$. Hence,

$$\Delta^p + \Delta^q = st(v, \Delta^p + \Delta^q) = v * lk(v, \Delta^p + \Delta^q) \nearrow v * (lk(v, \Delta^p) + \tilde{\Delta}^{p-1})$$
$$= v * lk(v, \Delta^p) + v * \tilde{\Delta}^{p-1} = st(v, \Delta^p) + \tilde{\Delta}^p = \Delta^p + \tilde{\Delta}^p.$$

□

Theorem 3.3.12. *Every polytopal d-sphere S with n vertices is simply homotopy equivalent without adding vertices to a complex of the form $\Delta^{n-2} + u * \partial\Delta^d$.*

Proof. If $n = d + 2$ then S is $\partial\Delta^{d+1} = \Delta^d + u * \partial\Delta^d$ for any $u \in S$. Let $n \geq d + 3$. By Corollary 3.3.10 it suffice to prove the following

- **Claim.** For $n \geq d + 3$ there exists a polytopal d-sphere S with n vertices having $u * \partial\Delta^d$ as a subcomplex which expands w.a.v. to a complex of the form $\Delta^{n-2} + u * \partial\Delta^d$, where $\Delta^{n-2} \cap (u * \partial\Delta^d) = \partial\Delta^d$.

We prove this claim by induction on n. For $n = d + 3$ the sphere $\Sigma\partial\Delta^d$ is easily shown to expand to $\Delta^{d+1} + u * \partial\Delta^d$. Suppose $n \geq d + 4$. By induction there exists a polytopal d-sphere $\tilde{S}$ with $n - 1$ vertices having $u * \partial\Delta^d$ as a subcomplex which expands without adding vertices to a complex of the form $\Delta^{n-3} + u * \partial\Delta^d$ with $\Delta^{n-3} \cap u * \partial\Delta^d = \partial\Delta^d$. Note that every d-simplex in $u * \partial\Delta^d$ is a free face in $\tilde{S}$ since $\Delta^{n-3} \cap u * \partial\Delta^d = \partial\Delta^d$. So, if we choose $\tilde{\Delta}^d \in u * \partial\Delta^d$ we may apply the same expansions in $\tilde{S} \nearrow \Delta^{n-3} + u * \partial\Delta^d$ to $\tilde{S} - \tilde{\Delta}^d$ and obtain

$$\tilde{S} - \tilde{\Delta}^d \underset{w.a.v.}{\nearrow} \Delta^{n-3} + (u * \partial\Delta^d) - \tilde{\Delta}^d \underset{w.a.v.}{\nearrow} \Delta^{n-3} \underset{\Delta^d}{+} (u * \Delta^d),$$

where the last (elementary) expansion is done by adding the $(d+1)$-simplex $u * \Delta^d$ from the free face $\tilde{\Delta}^d$. Now, if $n = d + 4$ then $\Delta^{n-3} + u * \Delta^d$ is of the form $\Delta^{n-3} \underset{\Delta^{n-4}}{+} \tilde{\Delta}^{n-3}$ and if $n > d + 4$ then $\Delta^{n-3} + u * \Delta^d \underset{w.a.v.}{\nearrow} \Delta^{n-3} \underset{\Delta^{n-4}}{+} \tilde{\Delta}^{n-3}$ by Lemma 3.3.11. Since $\Delta^{n-3} \underset{\Delta^{n-4}}{+} \Delta^{n-3}$ is an $(n-3)$-ball in the boundary of the $(n-2)$-simplex Δ^{n-2} spanned by the vertices of $\tilde{S}$ then $\Delta^{n-3} \underset{\Delta^{n-4}}{+} \Delta^{n-3} \underset{w.a.v.}{\nearrow} \Delta^{n-2}$. So far, we have shown that

$$\tilde{S} - \tilde{\Delta}^d \underset{w.a.v.}{\nearrow} \Delta^{n-2}.$$

Therefore, for v a vertex not in Δ^{n-2},

$$\tilde{S} - \tilde{\Delta}^d + v * \partial\tilde{\Delta}^d \underset{w.a.v.}{\nearrow} \Delta^{n-2} + v * \partial\tilde{\Delta}^d. \tag{3.2}$$

But $\overline{\tilde{S} - \tilde{\Delta}^d} + u * \partial\tilde{\Delta}^d = (\tilde{\Delta}^d, u)\tilde{S}$ is, by definition, the elementary starring of $\tilde{S}$ in $\tilde{\Delta}^d$. So, $\overline{\tilde{S} - \tilde{\Delta}^d} + u * \partial\tilde{\Delta}^d = (\tilde{\Delta}^d, u)\tilde{S}$ is a polytopal d-sphere with n vertices (it is known that a stellar subdivision of a polytopal sphere is again polytopal; see [24, (4)]). Equation (3.2) says that this sphere fulfills the inductive hypothesis we were looking for. This completes the proof. □

Second Proof of Theorem 3.3.4. By Remark 3.3.5 we only need to prove the assertion for d-spheres S with $d+3$ or $d+4$ vertices. These are polytopal by a result of Grünbaum and Mani [37]. Hence, Theorem 3.3.12 says that S is simply homotopy equivalent without adding vertices to a complex homotopy equivalent to a sphere. Hence, S^* is homotopy equivalent to a sphere by Proposition 3.3.8. □

3.3.3 A third proof of Dong's result

Finally, we sketch the final alternative proof of Dong's result using elements from the theory of vertex-decomposable complexes. Vertex-decomposability is a structural property which is stronger than shellability. It was introduced by Provan and Billera [46] (for homogeneous complexes) to study problems related to diameters of convex polyhedra, motivated by the search for a solution of the *Hirsch Conjecture*: "the graph of a simple d-dimensional polytope with n maximal simplices has diameter at most $n-d$". Actually, vertex-decomposability and shellability are the end cases of the family of k-*decomposable* complexes.

Definition. Let K be a d-dimensional simplicial complex and let $0 \leq k \leq d$. K is said to be k-*decomposable* if it is pure and if either $K = \Delta^d$ or there exists a simplex $\sigma \in K$ with $\dim(\sigma) \leq k$ such that

(1) $K - \sigma := \{\eta \in K \,|\, \sigma \not< \eta\}$ is d-dimensional and k-decomposable, and

(2) $lk(\tau, K)$ is $(d - \dim(\tau) - 1)$-dimensional and k-decomposable.

0-decomposable complexes are called *vertex decomposable* and d-decomposability is equivalent to shellability.

The Hirsch Conjecture [20] (see also [21]) has its origins in the study of the complexity of simplex algorithm of linear programming and it was first presented by W. M. Hirsch in a letter to G. B. Dantzig in 1957. A very good review of the advancements in polytope theory influenced by its relationship with linear programming may be found in [32]. A counterexample to the Hirsch Conjecture was presented by Santos in [48].

Regarding vertex-decomposability, Klee and Kleinschmidt [32] proved the following

Proposition 3.3.13 ([32, 5.7]). *If B is a simplicial d-ball with n vertices then B is vertex-decomposable for $d \leq 2$ or $n \leq d+3$. If S is a simplicial d-sphere with n vertices then S is vertex-decomposable for $d \leq 2$ or $n \leq d+4$.*

In particular, simplicial balls and spheres with few vertices are shellable. Curiously, this is the only result about shellability of complexes with few vertices: it is not known if simplicial d-balls (resp. d-spheres) with $d+4$ (resp. $d+5$) vertices are shellable.

Definition. An edge $e = \{u, v\}$ in a simplicial complex K is called *contractible* if every face $\sigma \in K$ satisfying $u, v \in lk(\sigma, K)$ also satisfies $e \in lk(\sigma, K)$.

Note that this is equivalent to the fact that $st(u, K) \cap st(v, K) = st(e, K)$. If the edge e is contractible then the *contracted complex* K/e is constructed in the following way:

- We remove the vertices u, v from V_K and add a new vertex w.
- $\sigma \in K/e$ is a simplex if either $w \notin \sigma$ and $\sigma \in K$ or $w \in \sigma$ and at least one of $\Delta(V_\sigma - \{w\} \cup \{u\})$ or $\Delta(V_\sigma - \{w\} \cup \{v\})$ is a simplex of K.

It is not hard to see that the contracted complex is simply homotopy equivalent w.a.v. to the original one.

Proposition 3.3.14. *If e is a contractible edge of K then $K \nearrow\searrow K/e$ without adding vertices.*

Sketch of proof. The proof we present here is (exactly) the same as [23, Theorem 2.4] (we simply note that the expansions involved are w.a.v.). Let $e = \{u, v\}$ and let $X = \{\sigma \in \Delta(V_K - \{u, v\}) \,|\, v * \sigma \in K,\, u * \sigma \notin K\}$. Order $X = \{\sigma_1, \ldots, \sigma_t\}$ so $\sigma_i < \sigma_j$ implies $i \leq j$ and let $K_j = K + \{u * \sigma_i, e * \sigma_i \,|\, 1 \leq i \leq j\}$. Then $K_0 = K$ and $K_j \nearrow K_{j+1}$ without adding vertices (because the vertex of every simplex in X is in V_K). Note that $lk(v, K_t)$ is a cone over u. Order now $lk(e, K_t) = \{\tau_1, \ldots, \tau_s\}$ so $\tau_i < \tau_j$ implies $j \leq i$ and let $L_j = K_t - \{v * \tau_i, e * \tau_i \,|\, 1 \leq i \leq j\}$. Then $L_0 = K_t$ and $L_j \searrow L_{j+1}$. Since $L_s = K/e$ the proof is completed. □

Since we are mainly interested in contractible edges on simplicial spheres we introduced the following concept.

Definition. Let S be a simplicial sphere and $e \in S$ a contractible edge. Then e is said to be *shrinkable* if S/e is again a simplicial sphere.

Proposition 3.3.15 ([32, 6.2])**.** *Any vertex-decomposable simplicial d-sphere with more than $d + 2$ vertices has a shrinkable edge.*

Third Proof of Theorem 3.3.4. By the same arguments as before we may restrict the problem to the cases $n = d + 3, d + 4$. A sphere S with such number of vertices is vertex-decomposable by Proposition 3.3.13; and by Proposition 3.3.15 S has a shrinkable edge e. Proposition 3.3.14 implies that the sphere S/e is simply homotopy equivalent to S without adding vertices. Therefore we obtain a sphere with $d + 2$ or $d + 3$ vertices, depending on the case. In the first case $S/e = \partial\Delta^{d+1}$ and, hence, the result holds by Corollary 3.3.8. In the other case, S/e is a vertex decomposable d-sphere by Proposition 3.3.13, and the proof is completed by repeating the same argument. □

Resumen en castellano del Capítulo 3

Entre los más importantes teoremas de dualidad se encuentra la Dualidad de Alexander, que es tan antigüa como la famosa Dualidad de Poincaré. La dualidad de Alexander relaciona la homología de un subespacio de la d-esfera con la cohomología de su complemento. Para espacios triangulables, esta dualidad admite una formulación puramente combinatoria, utilizando un representante (homotópico) simplicial K^* del complemento de un complejo K. En esta forma, la dualidad de Alexander puede probarse utilizando únicamente métodos simpliciales. Esta versión de la dualidad de Alexander está fuertemente relacionada con la teoría de NH-variedades, como veremos a continuación.

En la primera sección de este capítulo recordamos el resultado clásico de dualidad de Alexander y su versión combinatoria (Teoremas 3.1.1 y 3.1.3). La construcción fundamental en la versión simplicial de este resultado de dualidad es el *dual de Alexander* de un complejo simplicial.

Definición. Sea K un complejo simplicial y sea $V \supseteq V_K$ un conjunto base de vértices. El *dual de Alexander* de K (relativo a V) es el complejo

$$K^{*V} = \{\sigma \in \Delta(V) \,|\, \Delta(V - V_\sigma) \notin K\},$$

donde $\Delta(\mathsf{X})$ representa el simplex generado por los vértices en X.

El Lema 3.1.4 muestra las propiedades básicas del dual de Alexander de un complejo simplicial. En lo que sigue, escribimos $K^* := K^{*V_K}$ y $K^\tau := K^{*V}$ si $\tau = \Delta(V - V_K)$; esto es, K^τ representa el dual de Alexander de K relativo a $V = V_K + V_\tau$. En particular, si $\tau = \emptyset$ entonces $K^\tau = K^*$ es el dual de Alexander de K relativo a sus propios vértices. La primer sección de este capítulo termina esbozando la demostración de la dualidad de Alexander combinatoria presentada en [4, 6].

En §3.2 estudiamos la relación entre los duales de Alexander (de un mismo complejo K) relativos a distintos conjuntos base $V \supset V_K$ de vértices. Esta es una pregunta natural ya que geométricamente se corresponde a analizar la relación entre distintos tipos de complementos del complejo (visto como subespacio de esferas de distintas dimensiones). El siguiente resultado, que caracteriza la relación entre los posibles duales de Alexander de un mismo complejo, es central en el desarrollo del resto de la teoría.

Lema 3.2.1. *Sea K un complejo simplicial y sea τ un simplex no vacío disjunto de K. Entonces,*

$$K^\tau = \partial\tau * \Delta_K + \tau * K^*.$$

Aquí K^* es considerado un subcomplejo del simplex Δ_K. En particular, se tienen las siguientes consecuencias.

(1) Si K no es un simplex o $\dim(\tau) \geq 1$ entonces $V_{K^\tau} = V_K \cup V_\tau$. Si $K = \eta$ es un simplex y $\dim(\tau) = 0$ entonces $\eta^\tau = \eta$. En cualquier caso, $V_K \subseteq V_{K^\tau}$.

(2) Si K no es un simplex o $\dim(\tau) \geq 1$ entonces $(K^\tau)^* = K$.

(3) Si $V_{K^*} \subsetneq V_K$ y $\rho = \Delta(V_K - V_{K^*})$ entonces $(K^*)^\rho = K$.

(4) Si K no es un simplex entonces $K^\tau \simeq \Sigma^t K^*$ para cierto $t \geq 0$.

A continuación, el Lema 3.2.2 caracteriza los complejos para los cuales $K^{**} := (K^*)^*$ coincide con K. Para cerrar la sección, estudiamos complejos con mínima cantidad de vértices (*vertex-minimales*) y damos los siguientes dos resultados: la Proposición 3.2.4 caracteriza las NH-variedades vertex-minimales (mostrando que son necesariamente NH-bolas o NH-esferas) y el Lema 3.2.5 describe los duales de Alexander de complejos vertex-minimales.

§3.3 está destinada a proveer una demostración novedosa y completamente alternativa del resultado de Dong y Santos-Sturmfels sobre el tipo homotópico del dual de Alexander de las bolas y esferas.

Teorema 3.3.4. *Si $B \neq \Delta^d$ es una d-bola combinatoria entonces B^τ es un espacio contráctil. Si S es una d-esfera combinatoria entonces S^τ es homotópicamente equivalente a una esfera.*

Nuestra demostración del Teorema 3.3.4 está basada en la estructura local de las variedades, en contraste con los enfoques de Dong y Santos-Sturmfels que se basan en argumentos de convexidad. Esta demostración (ver página 94) se apoya básicamente en la relación entre los links y los *deletion* de vértices del complejo original y su dual (Lema 3.3.1). Cabe mencionar que los únicos casos no triviales del Teorema 3.3.4 son para bolas y esferas con pocos vértices (ver Observación 3.3.5). Dong utiliza este razonamiento para restringirse al caso de esferas politopales, lo que le permite utilizar argumentos de convexidad.

Las siguientes dos secciones de este capítulo están abocadas a proporcionar otras dos demostraciones nuevas del resultado original de Dong para esferas politopales. En §3.3.2 utilizamos algunos resultados clásicos de la teoría de polítopos (Teoremas 3.3.6 y 3.3.7) junto con las propiedades de las NH-factorizaciones introducidas en el Capítulo 2 para probar que dos esferas politopales con la misma cantidad de vértices son simplemente equivalentes sin agregar vértices en el proceso (denotado $K \underset{w.a.v.}{\nearrow\searrow} L$). El Corolario 3.3.8 afirma que si dos complejos están relacionados de esta manera entonces el dual de Alexander de uno es homotópicamente equivalente a una esfera si y sólo sí el dual de Alexander del otro también lo es.

Utilizando el Teorema 3.3.7 y el Lema 3.3.9 establecemos entonces el siguiente

Corolario 3.3.10. *Dos d-esferas politopales con la misma cantidad de vértices son simplemente equivalentes sin agregar vértices*

La segunda demostración del Teorema de Dong se sigue ahora del Teorema 3.3.12, que afirma que toda esfera politopal es simplemente equivalente a una esfera (de triangulación específica) cuyo dual de Alexander es homotópicamente equivalente a una esfera.

La tercera demostración alternativa del Teorema de Dong, en §3.3.3, está basada en la teoría de complejos *vertex-decomposable*. Un complejo K de dimensión d es *k-decomposable* si es d-homogéneo y si, o bien $K = \Delta^d$, o bien existe un simplex $\sigma \in K$ con $\dim(\sigma) \leq k$

tal que $K - \sigma := \{\eta \in K \,|\, \sigma \not< \eta\}$ es d-dimensional y k-decomposable, tal que $lk(\tau, K)$ es $(d - \dim(\tau) - 1)$-dimensional y k-decomposable. Los complejos 0-decomposables se llaman *vertex-decomposables*. Klee and Kleinschmidt [32] probaron que toda d-esfera simplicial con pocos vértices es vertex-decomposable (ver Proposición 3.3.13).

Una arista $e = \{u, v\}$ en un complejo simplicial K se dice *contráctil* si cada cara $\sigma \in K$ que satisface $u, v \in lk(\sigma, K)$ también satisface $e \in lk(\sigma, K)$. En este caso, el *complejo contraído* K/e es el que se obtiene al identificar los vértices u y v. Puede verse que el complejo contraído y el original son simplemente equivalentes sin agregar vértices. Nuestra tercera demostración del resultado de Dong se sigue ahora de la Proposición 3.3.15 (de [32]) que establece que toda esfera S vertex-decomposable posee una arista contráctil e que verifica que S/e es nuevamente una esfera.

Chapter 4

Alexander duals of non-pure balls and spheres

In this chapter we apply the theory developed in Chapters 2 and 3 to study the Alexander duals of NH-balls and NH-spheres. It turns out that non-pure balls and spheres have an intimate relationship with classical balls and spheres in the context of Alexander duality. This chapter contains three results which evidence the strong connection between the two theories. On one hand, we shall see that NH-balls and NH-spheres are closed families under taking two times Alexander dual (with respect to different ground set of vertices each time); in particular, we deduce that non-pure balls and spheres are the *Alexander double duals* of combinatorial balls and spheres. Geometrically, the complement in $S^{d'}$ of the complement in S^d ($d' \geq d$) of a ball (resp. sphere) is an NH-ball (resp. NH-sphere). On the other hand, we introduce *minimal* NH-balls and NH-spheres which are special type of non-homogeneous balls and spheres satisfying a minimality condition on the number of maximal simplices. Minimal NH-balls and NH-spheres are shown to be families of simplicial complexes whose iterated Alexander duals converge respectively to Δ^d or $\partial\Delta^{d+1}$ (for some $d \geq -1$). In the final section of this chapter we prove one of the main results of this Thesis. We extend Dong's and Santos-Sturmfels' results on the homotopy type of the Alexander dual of balls and spheres: the Alexander dual of an NH-ball is contractible and the Alexander dual of an NH-sphere is homotopy equivalent to a sphere.

§4.1 Double dual of balls and spheres

In this section we study the result of considering the complement of a complement in spheres of different dimensions in the following sense. Suppose A is a subspace of the d-sphere S^d. The complement $B = S^d - A$ is also a subspace of $S^{d'}$ for any $d' \geq d$ by embedding $S^d \subset S^{d'}$. Taking into account that $S^d - B = A$, it is natural to ask what kind of relationship exists between A and $S^{d'} - B$. In the simplicial version of Alexander duality this amounts to understand the similarities between a complex K and $(K^\tau)^\sigma$ for $V_\tau \cap V_K = \emptyset$ and $V_\sigma \cap V_{K^\tau} = \emptyset$. We call the complex $(K^\tau)^\sigma$ a *double dual* of K. When $\tau = \sigma = \emptyset$ we call $(K^*)^* = K^{**}$ the *standard double dual* of K. By Lemma 3.2.2, $K = K^{**}$ if and only if $|V_K| > \dim(K) + 2$.

Double duals share many of the properties of the original complexes. The next proposition is an example of two of them.

Proposition 4.1.1. *Let K be a simplicial complex. Then,*

(1) K is shellable if and only if $(K^\tau)^\sigma$ is shellable.

(2) If $|V_K| \geq d+3$, $(K^\tau)^\sigma \simeq \Sigma^t K$ for some $t \geq 0$.

Proof. To prove (1) we may assume $K \neq K^{**}$ so $K = (K^*)^\tau = \partial\tau * \Delta_{K^*} + \tau * K^{**}$ with $\tau = \Delta(V_K - V_{K^*})$ (see Lemma 3.1 (3)). Suppose K^{**} shellable and let $F_1, \dots, F_t$ be a shelling order. We claim a shelling order of $\partial\tau * V_{K^*}$ (which is trivially shellable) followed by $\tau * F_1, \dots, \tau * F_t$ yields a complete shelling for K. We only need to check that when we add $\tau * F_k$ in the kth step we get a $(\dim(\tau) + \dim(F_k))$-dimensional complex in the boundary of F_k. But

$$(\partial\tau * \Delta_{K^*} + (\tau * F_1 + \dots + \tau * F_{k-1})) \cap \tau * F_k = \partial\tau * F_k + \tau * ((F_1 + \dots + F_{k-1}) \cap F_k).$$

The first member has dimension $\dim(\partial\tau) + \dim(F_k) + 1 = \dim(\tau) - 1 + \dim(F_k) + 1 = \dim(\tau) + \dim(F_k)$ and the second member has dimension $\dim(\tau) + \dim((F_1 + \dots + F_{k-1}) \cap F_k) + 1 = \dim(\tau) + \dim(F_k) - 1 + 1 = \dim(\tau) + \dim(F_k)$. Thus, the claim is proven.

On the other hand, if K is shellable then the link of every vertex of K is shellable (see for example [33]). Hence, $K^{**} = lk(\tau, K)$ is shellable.

(2) follows from formula (3.1) and Lemmas 3.2.1 and 3.2.2. □

Besides the similarities exhibited in the previous lemma, strong structural properties do not generally transfer to double duals. For example, no (non-trivial) double dual of a combinatorial ball or sphere is again a combinatorial ball or sphere, respectively. More concretely, no double dual of a combinatorial manifold is in general homogeneous, let alone have a manifold structure. Nevertheless, the theory of non-pure manifolds provide a classification of the double duals of balls and spheres. We show that they are precisely the NH-balls and NH-spheres. This will follow from the fact that the class of NH-balls and NH-spheres are closed under "double duality". The result basically follows from the following

Lemma 4.1.2. *Let K be a simplicial complex. If $V_K \subseteq V$ and $\eta \neq \emptyset$ is a simplex, then*

$$L = \partial\eta * \Delta(V) + \eta * K$$

is an NH-ball (resp. NH-sphere) if and only if K is an NH-ball (resp. NH-sphere). Here K is viewed as a subcomplex of the simplex $\Delta(V)$.

Proof. Put $\Delta = \Delta(V)$. If L is an NH-ball or NH-sphere then $K = lk(\eta, L)$ is either an NH-ball or NH-sphere by Theorem 2.1.3. Since $\partial\eta * \Delta$ and $\eta * K$ are collapsible and $\partial\eta * \Delta \cap \eta * K = \partial\eta * K$ then K will be an NH-ball if L is one and an NH-sphere if L is one.

Suppose K is an NH-ball or NH-sphere. By Theorem 2.1.9, $\partial\eta * \Delta$ is a combinatorial ball, $\eta * K$ is an NH-ball and $\partial\eta * \Delta \cap \eta * K = \partial\eta * K$ is an NH-ball or NH-sphere according to K. We use Theorem 2.4.3 to prove that L is an NH-ball or NH-sphere. Note that $\partial\eta * K$ is trivially contained in $\partial(\partial\eta * \Delta)$ and it is generated by ridges of $\eta * K$. Also, if $\rho \in (\partial\eta * K)^\circ$ and $\hat{\eta}$ denotes the barycenter of η then

$$lk(\rho, \eta * K) \simeq_{PL} lk(\rho, \hat{\eta} * \partial\eta * K) = \hat{\eta} * lk(\rho, \partial\eta * K)$$

which is an NH-ball by Theorem 2.1.6. This implies that $\partial\eta * K \subset \tilde{\partial}(\eta * K)$. By Theorem 2.4.3, L is an NH-ball or NH-sphere. □

Theorem 4.1.3. *Let K be a simplicial complex and let τ be a simplex (possibly empty) disjoint from K and σ a simplex (possibly empty) disjoint from K^τ. Then K is an NH-ball (resp. NH-sphere) if and only if $(K^\tau)^\sigma$ is an NH-ball (resp. NH-sphere).*

Proof. We first prove the case $\tau = \sigma = \emptyset$. By Lemma 3.2.2 we may assume $|V_K| = \dim(K) + 2$. Let $\rho = \Delta(V_K - V_{K^*}) \neq \emptyset$ so $K = (K^*)^\rho = \partial\rho * \Delta_{K^*} + \rho * K^{**}$ by Lemma 3.2.1 (3). The result now follows from the previous lemma.

If K is a simplex and $\dim(\tau) = 0$ the result is trivial. For the remaining cases we have

$$(K^\tau)^\sigma = \begin{cases} \partial\sigma * \Delta_{K^*} + \sigma * K^{**} & \tau = \emptyset\,,\ \sigma \neq \emptyset \\ K & \tau \neq \emptyset\,,\ \sigma = \emptyset \\ \partial\sigma * \Delta_{K^\tau} + \sigma * K & \tau \neq \emptyset\,,\ \sigma \neq \emptyset \end{cases}$$

and the result follows from the previous lemma and the case $\tau, \sigma = \emptyset$. □

Corollary 4.1.4. *NH-balls are the double duals of combinatorial balls. NH-spheres are the double duals of combinatorial spheres.*

§4.2 The non-pure version of Δ^d and $\partial\Delta^d$

A simplicial complex K of dimension d is *vertex-minimal* if it is a simplex or it has $d+2$ vertices. It is not hard to see that a vertex-minimal homogeneous (or pure) complex of dimension d is either an elementary starring $(\tau, a)\Delta^d$ of a d-simplex or the boundary $\partial\Delta^{d+1}$ of a $(d+1)$-simplex (see Lemma 4.2.7 below). On the other hand, a general non-pure complex with minimum number of vertices has no precise characterization. However, since vertex-minimal pure complexes are either balls or spheres, it is natural to ask whether there is a non-pure analogue to these polyhedra within the theory of non-homogeneous balls and spheres. The purpose of this section is to introduce *minimal* NH-balls and NH-spheres, which are respectively the non-pure versions of vertex-minimal balls and spheres. Minimal NH-balls and NH-spheres are defined in terms of minimality of the number of principal simplices. This property is strictly stronger than vertex-minimality in non-pure balls and spheres and the true nature of minimality is attained with this definition. The main importance of this family of non-pure manifolds is that they completely characterize the class of Δ^d and $\partial\Delta^{d+1}$ in the equivalence relation generated by $K \sim K^*$. That is why we call minimal NH-balls and NH-spheres the *non-pure version of the simplex and the boundary of the simplex*, respectively. We next make the main result concrete.

Definition. Let K be a simplicial complex and put inductively $K^{*(0)} = K$ and $K^{*(m)} = (K^{*(m-1)})^*$. Thus, in each step $K^{*(i)}$ is computed relative to its own vertices, i.e. as a subcomplex of the sphere of minimum dimension containing it. We call $\{K^{*(m)}\}_{m\in\mathbb{N}_0}$ the *sequence of iterated Alexander duals* of K.

We will prove below the following result.

Theorem 4.2.1.

(i) There is an $m \in \mathbb{N}_0$ such that $K^{*(m)} = \partial\Delta^d$ if and only if K is a minimal NH-sphere.

(ii) There is an $m \in \mathbb{N}_0$ such that $K^{*(m)} = \Delta^d$ if and only if K is a minimal NH-ball.

If $K^* = \Delta^d$ then, letting $\tau = \Delta(V_K - V_{\Delta^d}) \neq \emptyset$, we have $K = (K^*)^\tau = \partial\tau * \Delta^d = (\tau, v)\Delta^{d+\dim(\tau)}$. This shows that Theorem 4.2.1 *(ii)* characterizes all complexes which converge to vertex-minimal balls.

4.2.1 Minimal NH-spheres

In this section we introduce the non-pure version of $\partial\Delta^d$ and prove part (i) of Theorem 4.2.1. We shall denote by $\mathsf{m}(K)$ the number of maximal simplices of K. We shall see that for a non-homogeneous sphere S, requesting minimality of $\mathsf{m}(S)$ is strictly stronger than requesting that of V_S. This is the reason why vertex-minimal NH-spheres are not necessarily *minimal* in our sense.

To introduce minimal NH-spheres we note first that any complex K with the homotopy type of a k-sphere has at least $k+2$ principal simplices. This follows from the fact that the simplicial nerve $\mathcal{N}(K)$ is homotopy equivalent to K. Recall that the nerve $\mathcal{N}(K)$ is the simplicial complex whose vertices are the principal simplices of K and whose simplices are the finite subsets of principal simplices of K with non-empty intersection.

Definition. An NH-sphere S is said to be *minimal* if $\mathsf{m}(S) = \dim_h(S) + 2$.

Equivalently, an NH-sphere S of homotopy dimension k is minimal if and only if $\mathcal{N}(S) = \partial\Delta^{k+1}$. This is because $\mathcal{N}(S) \simeq S$ and the only simplicial k-sphere with $k+2$ vertices is the boundary of a $(k+1)$-simplex. Figure 4.1 show examples of minimal NH-spheres.

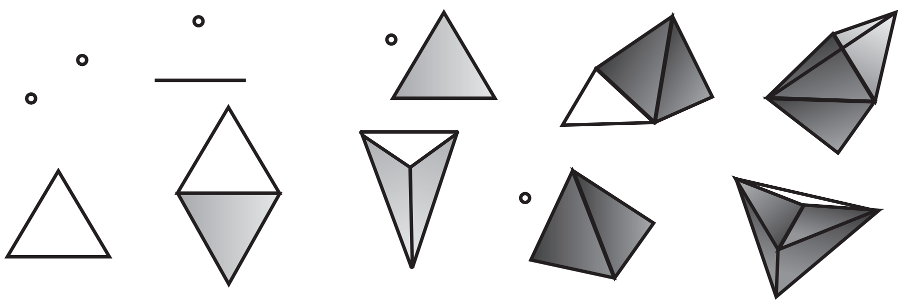


Figure 4.1: Minimal NH-spheres.

Remark 4.2.2. Suppose $S = B+L$ is a decomposition of a minimal NH-sphere of homotopy dimension k and let $v \in V_L$. Then $lk(v,S)$ is an NH-sphere of homotopy dimension $\dim_h(lk(v,S)) = k-1$ and $lk(v,S) = lk(v,B) + lk(v,L)$ is a valid decomposition (see Lemma 2.2.8). In particular, $\mathsf{m}(lk(v,S)) \geq k+1$. Also, $\mathsf{m}(lk(v,S)) < k+3$ since $\mathsf{m}(S) < k+3$ and $\mathsf{m}(lk(v,S)) \neq k+2$ since otherwise S is a cone. Therefore, $\mathsf{m}(lk(v,S)) = k+1 = \dim_h(lk(v,S)) + 2$, which shows that $lk(v,S)$ is also a minimal NH-sphere.

We next prove that minimal NH-spheres are vertex-minimal.

Proposition 4.2.3. *If S is a d-dimensional minimal NH-sphere then $|V_S| = d+2$.*

Proof. Let $S = B+L$ be decomposition of S and set $k := \dim_h(S)$. We shall prove that $|V_S| \leq d+2$ by induction on k. The case $k=0$ is straightforward, so assume $k \geq 1$. Let $\eta \in B$ be a principal simplex of minimal dimension and let Ω denote the intersection of all principal simplices of S different from η. Note that $\Omega \neq \emptyset$ since $\mathcal{N}(S) = \partial\Delta^{k+1}$ and let $u \in \Omega$ be a vertex. Since $\eta \notin L$ then $\Omega \subset L$ and $u \in L$. By Remark 4.2.2, $lk(u,S)$ is a minimal NH-sphere of dimension $d' \leq d-1$ and homotopy dimension $k-1$. By inductive hypothesis, $|V_{lk(u,S)}| \leq d'+2 \leq d+1$. Hence, $st(u,S)$ is a top generated subcomplex of S with $k+1$ principal simplices and at most $d+2$ vertices. By construction, $S = st(u,S)+\eta$.

We claim that $V_\eta \subset V_{st(u,S)}$. Since $B = st(u,B) + \eta$, by strong connectivity there is a ridge $\sigma \in B$ in $st(u,B) \cap \eta$ (see Lemma 2.1.10). By the minimality of η we must have $\eta = w * \sigma$ for some vertex w. Now, $\sigma \in st(u,B) \cap \eta \subset st(u,S) \cap \eta$; but $st(v,S) \cap \eta \neq \sigma$ since, otherwise, $S = st(u,S) + \eta \searrow st(u,S) \searrow u$, contradicting the fact that S has the homotopy type of a sphere. We conclude that $w \in st(u,S)$ since every face of η different from σ contains w. Thus, $|V_S| = |V_{st(u,S)} \cup V_\eta| = |V_{st(u,S)}| \leq d+2$. □

This last proposition shows that, in the non-pure setting, requesting the minimality of $\mathsf{m}(S)$ is strictly more restrictive than requesting that of $|V_S|$. For example, a vertex-minimal NH-sphere can be constructed from *any* NH-sphere S and a vertex $u \notin S$ by the formula $\tilde{S} := \Delta_S + u * S$. It is easy to see that if S is not minimal, neither is $\tilde{S}$.

Remark 4.2.4. By Proposition 4.2.3, a d-dimensional minimal NH-sphere S may be written $S = \Delta^d + u * lk(u,S)$ for some $u \notin \Delta^d$. Note that for any decomposition $S = B + L$, the vertex u must lie in L (since this last complex is top generated). In particular, $lk(u,S)$ is a minimal NH-sphere by Remark 4.2.2.

To prove Theorem 4.2.1 (i) we derive first the following corollary of Proposition 4.2.3.

Corollary 4.2.5. *If S is a minimal NH-sphere then $|V_{S^*}| < |V_S|$ and $\dim(S^*) < \dim(S)$.*

Proof. $V_{S^*} \subsetneq V_S$ follows by Proposition 4.2.3 since if $S = \Delta^d + u * lk(u,S)$ then $u \notin S^*$. In particular, this implies that $\dim(S^*) \neq \dim(S)$ since S^* is not a simplex by Alexander duality. □

Theorem 4.2.6. *Let K be a finite simplicial complex and let τ be a simplex (possibly empty) disjoint from K. Then, K is a minimal NH-sphere if and only if K^τ is a minimal NH-sphere. That is, the class of minimal NH-spheres is closed under taking Alexander dual.*

Proof. Assume first that K is a minimal NH-sphere and set $d = \dim(K)$. We proceed by induction on d. By Proposition 4.2.3, we can write $K = \Delta^d + u * lk(u,K)$ for $u \notin \Delta^d$. If $\tau = \emptyset$ then, by Lemma 3.2.5, $K^* = lk(u,K)^\rho$ for $\rho = \Delta(V_K - V_{st(u,K)})$. By Remarks 4.2.4 and 4.2.2, $lk(u,K)$ is a minimal NH-sphere. Therefore, $K^* = lk(u,K)^\rho$ is a minimal NH-sphere by inductive hypothesis. If $\tau \neq \emptyset$, $K^\tau = \partial\tau * \Delta_K + \tau * K^*$ by Lemma 3.2.1. In particular, K^τ is an NH-sphere by Lemma 4.1.2 and the case $\tau = \emptyset$. Now, by Alexander duality,

$$\dim_h(K^\tau) = |V_K \cup V_\tau| - \dim_h(K) - 3 = |V_K| + |V_\tau| - \dim_h(K) - 3 = \dim_h(K^*) + |V_\tau|.$$

On the other hand,

$$\mathsf{m}(K^\tau) = \mathsf{m}(\partial\tau * \Delta_K + \tau * K^*) = \mathsf{m}(\partial\tau) + \mathsf{m}(K^*) = |V_\tau| + \dim_h(K^*) + 2,$$

where the last equality follows from the case $\tau = \emptyset$. This shows that S^τ is minimal.

Assume now that K^τ is a minimal NH-sphere. If $\tau \neq \emptyset$ then $K = (K^\tau)^*$ and if $\tau = \emptyset$ then $K = (K^*)^{\Delta(V_K - V_{K^*})}$. In any case, the result follows immediately from the previous implication. □

Proof of Theorem 4.2.1 (i). Suppose first that K is a minimal NH-sphere. By Theorem 4.2.6, every non-empty complex in the sequence $\{K^{*(m)}\}_{m \in \mathbb{N}_0}$ is a minimal NH-sphere. By Corollary 4.2.5, $|V_{K^{*(m+1)}}| < |V_{K^{*(m)}}|$ for all m such that $K^{*(m)} \neq \{\emptyset\}$. Therefore, $K^{*(m_0)} = \{\emptyset\}$ for some $m_0 < |V_K|$ and hence $K^{*(m_0-1)} = \partial\Delta^d$ for some $d \geq 1$.

Assume now that $K^{*(m)} = \partial\Delta^d$ for some $m \in \mathbb{N}_0$ and $d \geq 1$. We proceed by induction on m. The case $m = 0$ corresponds to the trivial case $K = \partial\Delta^d$. For $m \geq 1$, the result follows immediately from Theorem 4.2.6 and the inductive hypothesis. □

4.2.2 Minimal NH-balls

We now develop the theory of minimal NH-balls. The definition in this case is a little less straightforward that in the case of spheres because there is no piece-wise-linear equivalence argument in the construction of non-pure balls. The following result motivates its definition.

Lemma 4.2.7. *Let B be a combinatorial d-ball. The following statements are equivalent.*

(1) $|V_B| \leq d + 2$ (i.e. B is vertex-minimal).

(2) B is an elementary starring of Δ^d.

(3) There is a combinatorial d-ball L such that $B + L = \partial\Delta^{d+1}$.

Proof. We first prove that (1) implies (2) by induction on d. Since Δ^d is trivially a starring of any of its vertices, we may assume $|V_B| = d + 2$ and write $B = \Delta^d + u * lk(u, B)$ for $u \notin \Delta^d$. Since $lk(u, B)$ is necessarily a vertex-minimal $(d-1)$-combinatorial ball then $lk(u, B) = (\tau, a)\Delta^{d-1}$ by inductive hypothesis. It follows from an easy computation that B is isomorphic to $(u * \tau, a)\Delta^d$.

We next prove that (2) implies (3). We have

$$B = (\tau, a)\Delta^d = a * \partial\tau * lk(\tau, \Delta^d) = a * \partial\tau * \Delta^{d-\dim(\tau)-1} = \partial\tau * \Delta^{d-\dim(\tau)}.$$

Letting $L := \tau * \partial\Delta^{d-\dim(\tau)}$ we get the statement of (3).

The other implication is trivial. □

Definition. An NH-ball B is said to be *minimal* if there exists a minimal NH-sphere S that admits a decomposition $S = B + L$.

Note that if B is a minimal NH-ball and $S = B + L$ is a decomposition of a minimal NH-sphere then, by Remark 4.2.2, $lk(v, B)$ is a minimal NH-ball for every $v \in B \cap L$ (see Lemma 2.2.8). Note also that the intersection of all the principal simplices of B is non-empty since $\mathcal{N}(B) \subsetneq \mathcal{N}(S) = \partial\Delta^{k+1}$. Therefore, $\mathcal{N}(B)$ is a simplex. The converse, however, is easily seen to be false. Figure 4.2 shows examples of minimal NH-balls.

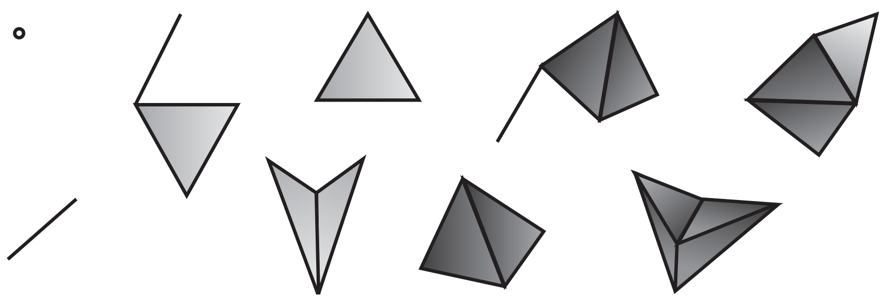


Figure 4.2: Minimal NH-balls.

The proof of Theorem 4.2.1 (*ii*) follows the same lines as the version for NH-spheres.

Proposition 4.2.8. *If B is a d-dimensional minimal NH-ball then $|V_B| \leq d+2$.*

Proof. This follows immediately from Proposition 4.2.3 since $\dim(B) = \dim(S)$ for any decomposition $S = B + L$ of an NH-sphere. □

Corollary 4.2.9. *If B is a minimal NH-ball then $|V_{B^*}| < |V_B|$ and $\dim(B^*) < \dim(B)$.*

Proof. We may assume $B \neq \Delta^d$. $V_{B^*} \subsetneq V_B$ by the same reasoning made in the proof of Corollary 4.2.5. Also, if $\dim(B) = \dim(B^*)$ then $B^* = \Delta^d$. By Lemma 3.2.1, $B = (B^*)^\rho = \partial\rho * \Delta^d$ where $\rho = \Delta(V_B - V_{B^*})$, which is a contradiction since $|V_B| = d+2$. □

Remark 4.2.10. The same construction that we made for minimal NH-spheres shows that vertex-minimal NH-balls need not be minimal. Also, similarly to the case of non-pure spheres, if $B = \Delta^d + u * lk(u, B)$ is a minimal NH-ball which is not a simplex, then for any decomposition $S = B + L$ of a minimal NH-sphere, we have $u \in L$. In particular, since $lk(u, S) = lk(u, B) + lk(u, L)$ is a valid decomposition of a minimal NH-sphere, then $lk(u, B)$ is a minimal NH-ball (see Remark 4.2.4).

Theorem 4.2.11. *Let K be a finite simplicial complex and let τ be a simplex (possibly empty) disjoint from K. Then, K is a minimal NH-ball if and only if K^τ is a minimal NH-ball. That is, the class of minimal NH-balls is closed under taking Alexander dual.*

Proof. Assume first that K is a minimal NH-ball and proceed by induction on $d = \dim(K)$. The case $\tau = \emptyset$ follows the same reasoning as the proof of Theorem 4.2.6 using the previous remarks. Suppose then $\tau \neq \emptyset$. Since by the previous case K^* is a minimal NH-ball, there exists a decomposition $\tilde{S} = K^* + \tilde{L}$ of a minimal NH-sphere. We claim that $S := K^\tau + \tau * \tilde{L}$ is a valid decomposition of a minimal NH-sphere, thus proving the implication. On one hand, Lemma 4.1.2 implies that K^τ is an NH-ball and that

$$S = \partial\tau * \Delta_K + \tau * K^* + \tau * \tilde{L} = \partial\tau * \Delta_K + \tau * \tilde{S}$$

is an NH-sphere. Also,

$$\begin{aligned}
K^\tau \cap (\tau * \tilde{L}) &= (\partial\tau * \Delta_K + \tau * K^*) \cap (\tau * \tilde{L}) \\
&= \partial\tau * \tilde{L} + \tau * (K^* \cap \tilde{L}) \\
&= \partial\tau * \tilde{L} + \tau * \partial\tilde{L} \\
&= \partial(\tau * \tilde{L}).
\end{aligned}$$

This shows that $S = K^\tau + \tau * \tilde{L}$ is valid decomposition of an NH-sphere. On the other hand,

$$\mathsf{m}(S) = \mathsf{m}(\partial\tau) + \mathsf{m}(\tilde{S}) = \dim(\tau) + 1 + \dim(\tilde{L}) + 2 = \dim_h(S) + 2,$$

which proves that S is minimal.

The other implication is analogous to the corresponding part of the proof of Theorem 4.2.6. □

Proof of Theorem 4.2.1 (ii). It follows the same reasoning as the proof of Theorem 4.2.1 (i) (replacing $\{\emptyset\}$ with $\emptyset$). □

4.2.3 Further properties of minimal NH-balls and NH-spheres

Let us finish this section with a brief discussion of some characteristic properties of minimal NH-balls and NH-spheres. To start with, Theorems 4.2.6 and 4.2.11 state that the families of minimal NH-spheres and NH-balls are closed under taking Alexander dual.

As stated above, Theorem 4.2.1 characterizes the classes of Δ^d and $\partial\Delta^d$ in the equivalence relation generated by $K \sim K^*$. More precisely, let $\sim$ be the equivalence relation on finite simplicial complexes defined by $K \sim L$ if and only if there exists a finite sequence of complexes $K = T_0, T_1, \dots, T_r = L$ such that either $T_i = T_{i-1}^*$ or $T_i = T_{i+1}^*$. We then have the following

Corollary 4.2.12. *Let $d \in \mathbb{N}_0$. The equivalence class of $\partial\Delta^d$ is exactly the class of minimal NH-spheres. The equivalence class of Δ^d is exactly the class of minimal NH-balls.*

It is trivial to check that vertex-minimal balls and spheres are closed under taking links and under deletion of vertices. We next prove the same results for their non-homogeneous counterpart.

Proposition 4.2.13. *In a minimal NH-ball or NH-sphere, the link of every simplex is a minimal NH-ball or NH-sphere.*

Proof. Let K be a minimal NH-ball or NH-sphere of dimension d and let $\sigma \in K$. We may assume $K \neq \Delta^d$. Since for a non-trivial decomposition $\sigma = w * \eta$ we have $lk(\sigma, S) = lk(w, lk(\eta, S))$, by an inductive argument it suffices to prove the case $\sigma = v$ vertex. We proceed by induction on d. We may assume $d \geq 1$. Write $K = \Delta^d + u * lk(u, K)$ where, as shown before, $lk(u, K)$ is either a minimal NH-ball or a minimal NH-sphere. Note that this in particular settles the case $v = u$. Suppose then $v \neq u$. If $v \notin lk(u, K)$ then $lk(v, K) = \Delta^{d-1}$. Otherwise, $lk(v, K) = \Delta^{d-1} + u * lk(v, lk(u, K))$. By inductive hypothesis, $lk(v, lk(u, K))$ is a minimal NH-ball or NH-sphere. By Lemma 3.2.5,

$$lk(v, K)^* = lk(v, lk(u, K))^\rho,$$

and the result follows from Theorems 4.2.6 and 4.2.11. □

Corollary 4.2.14. *In a minimal NH-ball or NH-sphere, the deletion of any vertex is a minimal NH-ball or NH-sphere.*

Proof. Let $v \in V_K$. We proceed by induction on $\dim(K)$. The result is trivial for low-dimensional cases. By Lemma 3.3.1 (1), $(K - v)^* = lk(v, K^*)$. Since by Theorems 4.2.6 or 4.2.11 and Corollaries 4.2.5 or 4.2.5 K^* is a minimal NH-ball or NH-sphere of dimension less that K then $lk(v, K^*)$ is an minimal NH-ball or NH-sphere by the previous proposition. Hence, $K - v$ is a minimal NH-ball or NH-sphere by Theorems 4.2.6 or 4.2.11 again. □

The non-pure notion of vertex-decomposability was also introduced by Björner and Wachs [8] in the nineties.

Definition. A complex K is *vertex-decomposable* if

(1) K is a simplex or $K = \{\emptyset\}$

(2) there exists a vertex v such that $K - v$ and $lk(v, K)$ are vertex-decomposable and no principal simplex of $lk(v, K)$ is principal in $K - v$.

As a trivial consequence of Remark 4.2.10 we have the following

Corollary 4.2.15. *Minimal NH-balls are vertex-decomposable.*

Finally, we make use of Theorems 4.2.6 and 4.2.11 to compute the number of minimal NH-spheres and NH-balls in each dimension.

Proposition 4.2.16. *Let $0 \leq k \leq d$.*

(1) There are exactly $\binom{d}{k}$ minimal NH-spheres of dimension d and homotopy dimension k. In particular, there are exactly 2^d minimal NH-spheres of dimension d.

(2) There are exactly 2^d minimal NH-balls of dimension d.

Proof. We first prove (1). An NH-sphere with $d = k$ is homogeneous by Proposition 2.1.12, in which case the result is obvious. Assume then $0 \leq k \leq d - 1$ and proceed by induction on d. Let $\mathcal{S}_{d,k}$ denote the set of minimal NH-spheres of dimension d and homotopy dimension k. If $S \in \mathcal{S}_{d,k}$ it follows from Theorem 4.2.6, Corollary 4.2.5 and Alexander duality that S^* is a minimal NH-sphere with $\dim(S^*) < d$ and $\dim_h(S^*) = d - k - 1$. Therefore, there is a well defined application

$$\mathcal{S}_{d,k} \xrightarrow{f} \bigcup_{i=d-k-1}^{d-1} \mathcal{S}_{i,d-k-1}$$

sending S to S^*. We claim that f is a bijection. To prove injectivity, suppose $S_1, S_2 \in \mathcal{S}_{d,k}$ are such that $S_1^* = S_2^*$. Let $\rho_i = \Delta(V_{S_i} - V_{S_i^*})$ $(i = 1, 2)$. Since $|V_{S_1}| = d + 2 = |V_{S_2}|$ then $\dim(\rho_1) = \dim(\rho_2)$ and, hence, $S_1 = (S_1^*)^{\rho_1} = (S_2^*)^{\rho_2} = S_2$. To prove surjectivity, let $\tilde{S} \in \mathcal{S}_{j,d-k-1}$ with $d - k - 1 \leq j \leq d - 1$. Taking $\tau = \Delta^{d-j-1}$ we have $\tilde{S}^\tau \in \mathcal{S}_{d,k}$ and $f(\tilde{S}^\tau) = \tilde{S}$ (see §2.3). Thus, the claim is proved. Finally, using the inductive hypothesis,

$$|\mathcal{S}_{d,k}| = \sum_{i=d-k-1}^{d-1} |\mathcal{S}_{i,d-k-1}| = \sum_{i=d-k-1}^{d-1} \binom{i}{d-k-1} = \binom{d}{k}.$$

For (2), let $\mathcal{B}_d$ denote the set of minimal NH-balls of dimension d and proceed again by induction on d. The very same reasoning as above gives a well defined bijection

$$\mathcal{B}_d - \{\Delta^d\} \xrightarrow{f} \bigcup_{i=0}^{d-1} \mathcal{B}_i.$$

Therefore, using the inductive hypothesis,

$$|\mathcal{B}_d - \{\Delta^d\}| = \sum_{i=0}^{d-1} |\mathcal{B}_i| = \sum_{i=0}^{d-1} 2^i = 2^d - 1. \quad \square$$

§4.3 A generalization of Dong's and Santos-Sturmfels' results

In this final section we generalize Dong's and Santos-Sturmfels' results on the Alexander dual of simplicial spheres and balls to the more general setting of NH-spheres and NH-balls. This is one of the main results of this work, involving a consistent use of the developed theory of non-pure manifolds and their interaction with combinatorial Alexander duality. We shall attack each case separately.

4.3.1 The Alexander dual of non-pure balls

First we prove the generalization of Santos and Sturmfels' result.

Theorem 4.3.1. *Let B be an NH-ball and let τ be a simplex (possibly empty). Then, B^τ is contractible.*

Let us mention that the same proof as for combinatorial balls in Theorem 3.3.4 would work *if we could guarantee that every NH-ball B has a vertex in $\tilde{\partial}B$.* Instead of pursuing that goal, we take a different approach. Note first that, by Lemma 3.2.1 (4), it suffices to prove the case $\tau = \emptyset$. Now, the very same reasoning of Remark 3.3.5 shows that for any complex K with $|V_K| - \dim(K) \geq 5$ K^* is simply connected. Since acyclic simply connected spaces are contractible, it suffices to focus on the cases $d+2 \leq n \leq d+4$ for an NH-ball B with $d = \dim(B)$ and $n = |V_B|$ (B^* is acyclic by Alexander duality). Not strangely, our approach to prove Theorem 4.3.1 will be to show that any NH-ball B with few vertices collapses to a complex K with $|V_K| = |V_B|$ and $|V_K| - \dim(T) \geq 5$; under this situation, K^* is contractible by the above reasoning and hence is B^* contractible by Lemma 3.3.3. Now, the case $n = d+4$ will follow directly from Theorem 2.2.9 as we shall show. For the case $d+3$ we need some preliminary results. We begin characterizing the d-homogeneous subcomplex of one such NH-ball. We need the following known result on manifolds with few vertices.

Theorem 4.3.2 ([10, Theorem A])**.** *Let M be a boundaryless combinatorial d-manifold with n vertices. If*

$$n < 3\left\lceil \frac{d}{2} \right\rceil + 3$$

then M is a combinatorial d-sphere. Also, if $d = 2$ and $n = 6$ then M is either PL-homeomorphic to a 2-sphere or combinatorially equivalent to the projective plane $\mathbb{R}P^2$.

The following is an immediate consequence of this result.

Corollary 4.3.3. *Let M be a combinatorial d-manifold with boundary with n vertices. If*

$$n < min\left\{3\left\lceil \frac{d-1}{2} \right\rceil + 3, 3\left\lceil \frac{d}{2} \right\rceil + 2\right\}$$

then M is a combinatorial d-ball. The result is also valid if $d = 3$ and $n = 6$.

Proof. By Theorem 4.3.2 ∂M is a combinatorial $(d-1)$-sphere. This includes the case $d = 3$ and $n = 6$ since $\mathbb{R}P^2$ cannot be the boundary of a compact manifold. Take $u \notin M$ and build $N = M + u * \partial M$ where $M \cap u * \partial M = \partial M$. It is easy to see that N is a boundaryless combinatorial d-manifold. Now, since $|V_N| < 3\lceil \frac{d}{2} \rceil + 3$ then N is a combinatorial d-sphere by Theorem 4.3.2 and $M = \overline{N - u * \partial M}$ is a combinatorial d-ball by Newman's theorem. □

We will also use the following result on collapsibility of complexes with few vertices.

Theorem 4.3.4 ([3, Theorem 1]). *For $n \leq 7$, any $\mathbb{Z}_2$-acyclic simplicial complex with n vertices is collapsible.*

Proposition 4.3.5. *Let B be an NH-ball of dimension d and $n \leq d+3$ vertices. Then, the d-homogeneous subcomplex $Y^d \subset B$ is a combinatorial d-ball.*

Proof. Since B is acyclic, by Lemma 2.1.10 Y^d is a weak d-pseudo manifold with boundary. We may assume $d \geq 2$ and $|V_{Y^d}| = d+3$ since the cases $d = 0, 1$ and $|V_{Y^d}| = d+1$ are trivial and, if $|V_{Y^d}| = d+2$, Y^d is an elementary starring of a simplex by 4.2.7. Note that Y^d is necessarily connected. We first prove that Y^d is a combinatorial manifold. Let $v \in Y^d$. By the same reasoning as above we may assume $|V_{lk(v,B)}| = d+2$. If $lk(v,B)$ is an NH-ball then $lk(v, Y^d)$ is a combinatorial $(d-1)$-ball by inductive hypothesis since $lk(v, Y^d)$ is the $(d-1)$-homogeneous part of $lk(v,B)$. Suppose $lk(v,B)$ is an NH-sphere. If $\dim_h(lk(v,B)) = d-1$ then $lk(v,B) = lk(v,Y^d)$ is a combinatorial $(d-1)$-sphere by Proposition 2.1.12. Otherwise, $lk(v,Y^d)$ is the $(d-1)$-homogeneous part of the NH-ball in any decomposition of $lk(v,B)$ and the result follows again by induction. This shows that Y^d is a combinatorial d-manifold.

Suppose $d = 2$. Note that Y^d is $\mathbb{Z}_2$-acyclic since it is connected, it has non-empty boundary and it is contained in the acyclic complex B. On the other hand, any $\mathbb{Z}_2$-acyclic complex with 5 vertices is collapsible by Theorem 4.3.4.

For $d \geq 3$, Y^d is a combinatorial d-ball by Corollary 4.3.3. □

Proposition 4.3.6. *Any NH-ball B of dimension $d \geq 2$ and $d+3$ vertices collapses to a complex of dimension $d-2$.*

Proof. We first show that all the principal $(d-1)$-simplices in B can be collapsed. Let Y^{d-1} be the subcomplex of B generated by the principal $(d-1)$-simplices and let Y^d be the d-homogeneous part of B. By the previous proposition, Y^d is a combinatorial ball. Suppose that not all the $(d-1)$-simplices in Y^{d-1} can be collapsed. Let K be the subcomplex of Y^{d-1} generated by these $(d-1)$-simplices. By assumption, $K \neq \emptyset$. Note that K is a weak $(d-1)$-pseudo manifold with boundary by Lemma 2.1.10 but it has no free $(d-2)$-faces in B. Then $\partial K \subset Y^d$. Therefore, if c denotes the formal sum of the $(d-1)$-simplices of K then $c \in H_{d-1}(B, Y^d)$. Since B and Y^d are contractible then $H_{d-1}(B, Y^d) = 0$. This implies that c is not a generating cycle, which is a contradiction since the $(d-1)$-simplices of c are maximal. This shows that we can collapse all the principal $(d-1)$-simplices in B. On the other hand, since Y^d is a combinatorial d-ball with $d+3$ vertices or less, it is vertex-decomposable by Proposition 3.3.13. In particular Y^d is collapsible with no need of further subdivision. Then we can perform the collapses in order of decreasing dimension and collapse the d-simplices and the $(d-1)$-simplices of Y^d afterwards to obtain a $(d-2)$-dimensional complex. □

Corollary 4.3.7. *Any NH-ball of dimension $d \geq 3$ and $d+2$ vertices collapses to a complex of dimension $d-3$.*

Proof. We proceed by induction on d. If $d = 3$ then B is collapsible by Theorem 4.3.4. Let $d \geq 4$ and write $B = \Delta^d + st(u,B)$ where $u \notin \Delta^d$. Now, $\Delta^d \cap st(u,B) = lk(u,B) \subset \Delta^d$ is an NH-ball since B is one. Also, $\dim(lk(u,B)) \leq d-1$ and $|V_{lk(u,B)}| \leq d+1$. Let $m = |V_{lk(u,B)}| - \dim(lk(u,B))$. If $m = 1$ then $lk(u,B)$ is a simplex and $B \searrow \Delta^d \searrow 0$.

For $m = 2, 3, 4$ we use the inductive hypothesis, Proposition 4.3.6 or Corollary 2.2.9 respectively to show that $lk(u, B)$ collapses to a complex of dimension $\dim(lk(u, B)) - (5 - m) = |V_{lk(u,B)}| - m - (5 - m) = |V_{lk(u,B)}| - 5 \leq d + 1 - 5 = d - 4$. Therefore, $u * lk(u, B) = st(u, B)$ collapses to a complex of dimension $d - 3$. Finally, if $m \geq 5$ then $\dim(lk(u, B)) \leq |V_{lk(u,B)}| - 5 \leq d - 4$ and $\dim(st(u, B)) \leq d - 3$. In any case we can collapse afterwards the i-simplices of Δ^d ($i = d, d - 1, d - 2$) in order of decreasing dimension to obtain a $(d - 3)$-dimensional complex. □

We are ready to prove now the first of our main results.

Proof of 4.3.1. As stated above, it suffice to prove that B^* is simply connected. We can assume that $d \geq 2$ since in lower dimensions all NH-balls are combinatorial. Let $d+2 \leq n \leq d+4$. If $n \leq 7$, B^* is collapsible by Theorem 4.3.4. For $n \geq 8$, by Proposition 4.3.6 and Corollaries 2.2.9 and 4.3.7 there exists a subcomplex $K \subset B$ such that $B \searrow K$ with $V_K = V_B$ and $|V_K| - \dim(K) = 5$. Therefore $B^* \nearrow K^*$, and since $|V_K| - \dim(K) = 5$, K^* is simply connected. □

4.3.2 The Alexander dual of non-pure spheres

Our next goal is to prove the second of our main results: a generalization of Dong's theorem.

Theorem 4.3.8. *Let S be an NH-sphere and let τ be a simplex (possibly empty). Then, S^τ is homotopy equivalent to a sphere.*

Like in the proof for NH-balls, we only need to prove the case $\tau = \emptyset$ and $2 \leq |V_S| - \dim(S) \leq 4$ since, as before, if $|V_S| - \dim(S) \geq 5$, then S^* is simply connected, and a simply connected space with the homology of a sphere is homotopy equivalent to one. We can also assume that $\dim_h(S) < \dim(S)$ by Proposition 2.1.12 and Theorem 3.3.4. The 1-dimensional case is easy to verify.

We first exhibit some examples to show that the property of being NH-sphere is important for Theorem 4.3.8 to hold.

Examples 4.3.9. Let P be the triangulation of the Poincaré homology 3-sphere with f-vector $(16, 106, 180, 90)$ given in [5, §5]. It is easy to see that P^* has dimension 13, 16 vertices and it is homotopy equivalent to a 10-sphere.

(1) For $S = P^*$, $|V_S| - \dim(S) = 3$ and $S^* = P$ is not homotopy equivalent to a sphere.

(2) For $u \notin P$ we have that $|V_{P^u}| - \dim(P^u) = 2$ and $(P^u)^* = P$.

(3) From the explicit triangulation of P given in [5, §5] notice that the vertex w numbered 10 is adjacent to every other vertex of the triangulation. For every non-edge $\{u, v\}$ of P (there are 14 of them) perform the expansion $\{\{u, v, w\}, \{u, v\}\}$. The result is a complex P' that has every possible edge in V_P. Therefore, $(P')^* \nearrow P^*$ has the homotopy of a sphere, $|V_{P'}| - \dim(P') = 4$ and $(P')^{**} = P' \searrow P$ does not.

The proof of Theorem 4.3.8 will be divided in the following four cases. Let $d = \dim(S) \geq 2$, $n = |V_S|$ and $k = \dim_h(S)$. We handle each case separately.

(A) $n = d + 2$ and $k = d - 1$.

(B) $n = d + 2$ and $k = d - 2$.

(C) $n = d + 3$ and $k = d - 1$.

(D) Remaining cases.

Proof of Case (D). We will show that $S \searrow K$ with $|V_K| - \dim(K) = 5$. The result will follow immediately from Lemma 3.3.3 and the fact that K^* is simply connected. The case $n = d + 4$ follows directly from Corollary 2.2.9 by collapsing (only) the d-simplices of S.

Suppose now that $n = d + 2$ or $d + 3$ and let $S = B + L$ be a decomposition. We first analyze the case $n = 5$. In this situation, $L = *$. If $d = 2$ then B is acyclic with four vertices and if $d = 3$ then $B = \Delta^3$. Similarly as in the 1-dimensional case, $S \searrow S^0$ and the result follows from Lemma 3.3.3.

Suppose $n = d+3$ with $n \geq 6$. The complex B in the decomposition of S is an NH-ball of dimension d and $|V_B| \in \{d + 1, d + 2, d + 3\}$. In any case, B collapses to a $(d - 2)$-dimensional complex T whether because $B = \Delta^d$ or by Corollary 4.3.7 or Proposition 4.3.6. Moreover, since $d \geq 3$, we can arrange the collapses in order of decreasing dimension to get $V_T = V_B$ by collapsing only the d and $(d-1)$-dimensional simplices. Since $\dim(L) \leq d-2$ and it is top generated, the collapses in $B \searrow T$ can be carried out in S and therefore $S \searrow K = T + L$, which is a complex with the desired properties.

The case $n = d + 2$ with $n \geq 6$ follows similarly as the previous case by showing that S collapses to a $(d - 3)$-dimensional complex with the same vertices. □

Proof of Case (A). We proceed by induction. Write $S = \Delta^d + u * lk(u, S)$ with $u \notin \Delta^d$. Note that $lk(u, S)$ is an NH-sphere of homotopy dimension $d - 2$ and dimension $d - 2$ or $d - 1$. By Lemmas 3.2.5 and 3.2.1 (4) it suffices to show that $lk(u, S)^*$ is homotopy equivalent to a sphere. If $\dim(lk(u, S)) = d-2$ then $lk(u, S)$ is homogeneous by Proposition 2.1.12 and the result follows from Theorem 3.3.4. If $\dim(lk(u, S)) = d-1$ then $|V_{lk(u,S)}| = d + 1$ and the result follows by the inductive hypothesis. □

In order to prove the cases (B) and (C) we need some preliminary results.

Lemma 4.3.10. *Let $S = B + L$ be a decomposition of an NH-sphere. If $v \in L$ then $S - v$ is contractible.*

Proof. If $v \in L^\circ$ then $L - v$ deformation retracts to $\partial L \subset B$ and, hence, $S - v \simeq B \simeq *$. Otherwise, $v \in \partial L \cap \tilde{\partial} B$ and $S - v = (B - v) + (L - v)$ with $(B - v) \cap (L - v) = \partial L - v$. Since $v \in \partial L \cap \tilde{\partial} B$, then $B - v$ and $L - v$ are contractible. On the other hand, $\partial L - v$ is contractible by Newman's theorem. Hence, $S - v$ is contractible. □

Lemma 4.3.11. *Let $S = B + L$ be a decomposition of an NH-sphere of dimension $d \geq 1$ satisfying the hypotheses of case* (C). *If $lk(v, S)$ is a combinatorial $(d - 2)$-sphere then $S - v$ is an NH-ball.*

Proof. We proceed by induction in d. The case $d = 1$ is straightforward. Let $d \geq 2$. We prove first that $S - v$ is an NH-manifold.

Let $w \in S - v$. We have to show that its link is an NH-sphere or an NH-ball. If $w \notin st(v, S)$ then $lk(w, S - v) = lk(w, S)$ which is an NH-ball or NH-sphere. Suppose $w \in st(v, S)$. We will show first that $lk(w, S)$ is an NH-sphere of homotopy dimension $d - 2$. We prove this in various steps. Note that this is clear if $w \in L$, so we may suppose $w \notin L$.

Step 1. We first prove that if $v \notin L$ then there is a d-simplex in $st(w, S)$ which is adjacent to a $(d - 1)$-simplex of L. Write $\Delta^d = \{v, w\}^c$. Since $v, w \notin L$ then $L \subset \Delta^d$

and therefore $\Delta^d \notin S$ because L is top generated in S. Since $\dim(S) = d$ and $st(v, S)$ is $(d-1)$-homogeneous then w is a face of d-simplex ρ not containing v. Since any two $(d-1)$-faces of Δ^d are adjacent then ρ is adjacent to some $(d-1)$-simplex of L.

Step 2. We now prove that the inclusion induces an isomorphism $H_{d-1}(S - w) \simeq H_{d-1}(B - w)$. On one hand, the induced homomorphism $H_{d-1}(B - w) \to H_{d-1}(S - w)$ is injective since $(S - w) - (B - w) = L - w$ is $(d-1)$-dimensional. To prove that it is also surjective we show that any $(d-1)$-cycle in $S - w$ cannot contain a $(d-1)$-simplex of L. Suppose $\sigma \in L$ is a non-trivial factor in a $(d-1)$-cycle c^{d-1} of $S - w$. Then every $(d-1)$-simplex in L appears in c^{d-1} since c^{d-1} is a cycle and L is a top generated combinatorial $(d-1)$-ball. If $v \in L$ then every $(d-1)$-simplex of $st(v, S)$ appears in c^{d-1} since $st(v, S)$ is also a top generated $(d-1)$-ball. In this case, at least one $(d-1)$-simplex of $st(v, S)$ belongs to $st(w, S)$, contradicting the fact that c^{d-1} is a cycle in $S - w$. On the other hand, if $v \notin L$ then there exists by step 1 a principal $(d-1)$-simplex $\tau \in L$ with a boundary $(d-2)$-face $\eta < \rho \in st(w, S)$ with $\dim(\rho) = d$. Let $z = lk(\eta, \tau)$. Note that there are no d-simplices outside $st(w, S)$ containing η since neither v, w nor z may belong to such d-simplex and $|V_S| = d + 3$. Since S is an NH-manifold, τ is the only principal $(d-1)$-simplex containing η, and then $\partial c^{d-1} \neq 0$ in $S - w$, which is a contradiction.

Step 3. We prove that $lk(w, S)$ is an NH-sphere of homotopy dimension $d - 2$. We claim first that $H_{d-1}(S-w) = 0$. By step 2 it suffices to show that $H_{d-1}(B-w) = 0$. From the Mayer-Vietoris sequence applied to $B = B - w + st(w, S)$ and the fact that $lk(w, B) = lk(w, S)$ (here we use that $w \notin L$), it follows that $H_{d-1}(B - w) \simeq H_{d-1}(lk(w, S))$. If $H_{d-1}(lk(w, S)) \neq 0$ then $lk(w, S)$ is $(d-1)$-homogeneous by Proposition 2.1.12, which is a contradiction since $st(w, S)$ contains at least a $(d-1)$-simplex. Thus, the claim is proved.

If we now consider the Mayer-Vietoris sequence for $S = S - w + st(w, S)$ in degree $d-1$ one has that $\mathbb{Z} \simeq H_{d-1}(S) \to H_{d-2}(lk(w, S))$ is injective, so $H_{d-2}(lk(w, S)) \neq 0$ and therefore $lk(w, S)$ is an NH-sphere of homotopy dimension $d - 2$.

Finally if $\dim(lk(w, S)) = d - 2$ then $lk(w, S)$ is a combinatorial $(d-2)$-sphere by Proposition 2.1.12 and therefore, $lk(w, S - v) = lk(w, S) - v$ is a combinatorial $(d-2)$-ball by Newman's theorem. Suppose that $\dim(lk(w, S)) = d - 1$. If $|V_{lk(w,S)}| = d + 1$ then we may write $lk(w, S) = \Delta^{d-1} + st(v, lk(w, S))$ since v is not a vertex of a d-simplex in S. In this case $lk(w, S) - v = \Delta^{d-1}$. If $|V_{lk(w,S)}| = d + 2$ we may apply the inductive hypothesis since $lk(v, lk(w, S)) = lk(w, lk(v, S))$ is a combinatorial $(d-3)$-sphere, and conclude that $lk(w, S) - v$ is an NH-ball. This proves that $S - v$ is an NH-manifold.

We prove now that $S - v$ is an NH-ball. Note that $\dim(S - v) = d$ and $|V_{S-v}| = d + 2$, so by Proposition 3.2.4 it suffice to discard that $S - v$ is an NH-sphere. For this, we show that it is acyclic. Considering the Mayer-Vietoris sequence of the decomposition $S = S - v + st(v, S)$ we readily see that $H_i(S - v) = 0$ for every $i \neq d-1, d-2$. For the cases $i = d - 1, d - 2$, the sequence reduces to

$$0 \to H_{d-1}(S - v) \to \mathbb{Z} \to \mathbb{Z} \to H_{d-2}(S - v) \to 0$$

Since $H_i(S - v)$ is either zero or infinite cyclic then the only two possibilities are $H_{d-1}(S - v) = H_{d-2}(S - v) = 0$ or $H_{d-1}(S - v) = H_{d-2}(S - v) = \mathbb{Z}$. Since S is either an NH-ball or NH-sphere then this last case cannot happen, and so we conclude that $S - v$ is acyclic. □

Lemma 4.3.12. *Let $S = B + L$ be a decomposition of an NH-sphere satisfying the hypotheses of case (C). If there is a vertex v in L such that $\dim(S - v) = d$ and there is a non-edge $\{u, w\}$ of S with $u, w \neq v$ then $(S - v)^*$ is contractible.*

Proof. By Lemma 3.2.2, $|V_{S^*}| = d+3$. By hypothesis $\{u,w\}^c \in S^*$ is a d-simplex and since $v \neq u, w$ then $v \in \{u,w\}^c$. Therefore, $\dim(lk(v,S^*)) = d-1$. On the other hand, there exists a d-simplex $\eta \in S$ with $v \notin \eta$; hence $\{v,a\} := \eta^c \notin S^*$. Therefore, $|V_{lk(v,S^*)}| \leq d+1$. If $|V_{lk(v,S^*)}| = d$ then $(S-v)^* = lk(v,S^*)$ is a $(d-1)$-simplex. If $|V_{lk(v,S^*)}| = d+1$ then $lk(v,S^*) = (S-v)^*$ is acyclic by Lemma 4.3.10 and Alexander duality, and therefore contractible by Lemma 3.3.2 (1). □

Lemma 4.3.13. *Let S be an NH-sphere satisfying the hypotheses of case (C). Then, for any decomposition $S = B + L$ there exists $z \in V_L$ such that $(S-z)^*$ is contractible.*

Proof. We proceed by induction in d. The 1-dimensional case is straightforward. Let $d \geq 2$ and let $u \in V_L$. If $\dim(lk(u,S)) = d-2$ then $lk(u,S)$ is a combinatorial $(d-2)$-sphere and the result follows from Lemma 4.3.11 and Theorem 4.3.1. Suppose $\dim(lk(u,S)) = d-1$. We analyze the two possible cases $|V_{lk(u,S)}| = d+1$ or $|V_{lk(u,S)}| = d+2$.

If $|V_{lk(u,S)}| = d+1$, let $w \in S$ such that $\{u,w\} \notin S$. Let Δ^d be a d-simplex containing u and let $v = V_S - V_{\Delta^d} - \{w\}$. Since L is top generated then either $v \in L$ or $w \in L$. If $v \in L$ then Lemma 4.3.12 implies that $(S-v)^* \simeq *$. Assume then that $v \notin L$ (and hence $w \in L$). We may assume $\dim(lk(w,S)) = d-1$ since otherwise w is the desired vertex by Lemma 4.3.11 and Theorem 4.3.1 again. Let $\tilde{\Delta}^d$ be a d-simplex containing w. Since L is top generated, $w \in L$ and $v \notin L$ then $\tilde{\Delta}^d = w * v * \Delta^{d-2}$ with $\Delta^{d-2} \prec \Delta^d - u$. Let $x \neq u$ be the only vertex in Δ^{d-2} not in $\tilde{\Delta}^d$. Then, $x \in L$ and it fulfills the hypotheses of Lemma 4.3.12. Therefore $(S-x)^*$ is contractible.

Suppose finally that $|V_{lk(u,S)}| = d+2$. From the decomposition $lk(u,S) = lk(u,B) + lk(u,L)$ there exists $y \in lk(u,L)$ such that $(lk(u,S)-y)^* \simeq *$ by the inductive hypothesis. If $u \notin (S-y)^*$ then $u^c \in S-y$; i.e. $S-y-u = \Delta^d$. In this case, we can write $S-y = \Delta^d + u * lk(u,S-y)$ and we have $(S-y)^* = (lk(u,S)-y)^\tau$ by Lemma 3.2.5. If $lk(u,S)-y$ is not a simplex then $(lk(u,S)-y)^\tau \simeq \Sigma^t(lk(u,S)-y)^* \simeq *$ by Lemma 3.2.1 (4) and if $lk(u,S)-y = \Delta^r$ then $\tau \neq \emptyset$ and $(lk(u,S)-y)^\tau = \partial\tau * \Delta^r \simeq *$. In either case, y is the desired vertex. Assume $u \in (S-y)^*$. Then we have a non-trivial decomposition

$$(S-y)^* = (S-y)^* - u \underset{lk(u,(S-y)^*)}{+} st(u,(S-y)^*).$$

Since neither $S-y$ nor $lk(u,S)-y$ are simplices and $u \in S-y$ is not isolated then $(S-y)^* - u \simeq \Sigma^t lk(u,S-y)^* \simeq *$ by Lemma 3.3.1 (3). The result then follows by Lemmas 3.3.2 (1) and 4.3.10. □

Proof of Cases (B) and (C). We prove (B) and (C) together by induction in d. Let $S = B + L$ be a decomposition.

If $d = 2$, B is collapsible since it is acyclic and has few vertices. Then $S \searrow S^0$ for (B) and $S \searrow S^1$ for (C). The results then follow in both cases from Lemma 3.3.3.

Let $d \geq 3$. Suppose first that S satisfies the hypotheses of (B). Write $S = \Delta^d + v * lk(v,S)$. Then $S^* = lk(v,S)^\tau$ for $\tau = \Delta(V_S - V_{st(v,S)})$ by Lemma 3.2.5. Since $lk(v,S)$ is an NH-sphere of dimension $\leq d-1$ then the result follows from Theorem 3.3.4, cases (A) and (D) or the inductive hypothesis on (B) and (C).

Finally suppose S satisfies the hypotheses of (C). By Lemma 4.3.13 there exists $v \in V_L$ such that $(S-v)^* \simeq *$. Write $S^* = S^* - v + st(v,S^*)$ where $(S^*-v) \cap st(v,S^*) = lk(v,S^*) = (S-v)^* \simeq *$. By Lemmas 3.3.2 (2) and 3.3.1 (3), $S^* \simeq S^* - v \simeq \Sigma^t lk(v,S)^*$ (note that v is not isolated nor $lk(v,S)$ is a simplex because $v \in L$). Since $lk(v,S)$ is an NH-sphere of dimension $\leq d-1$ then $lk(v,S)^*$ is homotopy equivalent to a sphere by Theorem 3.3.4, cases (A) and (D) or inductive hypothesis on (B) and (C). □

Resumen en castellano del Capítulo 4

En este capítulo final aplicamos la teoría desarrollada en los capítulos anteriores para estudiar el dual de Alexander de bolas y esferas simpliciales. Mostramos que las NH-bolas y NH-esferas tienen una íntima relación con las bolas y esferas clásicas en el contexto de la dualidad de Alexander combinatoria. Este capítulo contiene tres resultados que exhiben la fuerte conexión entre las teorías pura y no pura. Por un lado, veremos que las NH-bolas y NH-esferas son familias cerradas bajo la acción de tomar dos veces dual de Alexander (respecto a conjuntos base de vértices distintos cada vez); en particular, concluimos que las bolas y esferas no puras son los *doble duales de Alexander* de las bolas y esferas combinatorias. Geométricamente, esto dice que el complemento en $S^{d'}$ del complemento en S^d ($d' \geq d$) de una bola (resp. esfera) es una NH-bola (resp. NH-esfera). Por otro lado, introducimos las *NH-bolas y NH-esferas minimales*, un tipo especial de bolas y esferas no homogéneas que satisfacen una condición de minimalidad en la cantidad de símplices maximales. Demostramos que las NH-bolas y NH-esferas minimales caracterizan completamente a los complejos simpliciales cuyos duales de Alexander iterados convergen respectivamente a Δ^d o $\partial\Delta^{d+1}$ (para algún $d \geq -1$). En la sección final del capítulo probamos, como uno de los resultados principales de esta Tesis, una generalización de los resultados de Dong y Santos-Sturmfels sobre el tipo homotópico del dual de Alexander de bolas y esferas: el dual de Alexander de una NH-bola es un espacio contráctil y el dual de Alexander de una NH-esfera es homotópicamente equivalente a una esfera.

En §4.1 estudiamos el resultado de considerar el complemento del complemento en esferas de distintas dimensiones. Supongamos que A es un subespacio de la d-esfera S^d. El complemento $B = S^d - A$ es también un subespacio de $S^{d'}$ para cualquier $d' \geq d$ considerando un embedding $S^d \subset S^{d'}$. Teniendo en cuenta que $S^d - B = A$, es natural preguntarse qué tipo de relación existe entre A y $S^{d'} - B$. En la versión simplicial de la dualidad de Alexander, esto se reduce a entender la similitudes entre K y $(K^\tau)^\sigma$ para $V_\tau \cap V_K = \emptyset$ y $V_\sigma \cap V_{K^\tau} = \emptyset$. Llamamos al complejo $(K^\tau)^\sigma$ un *doble dual* de K. Cuando $\tau = \sigma = \emptyset$, llamamos a $(K^*)^* = K^{**}$ el *doble dual estándar* de K. Por el Lema 3.2.2, $K = K^{**}$ si y sólo si $|V_K| > \dim(K) + 2$.

Los doble duales comparten muchas de las propiedades de los complejos originales. La Proposición 4.1.1 muestra dos ejemplos de esto. Sin embargo, características estructurales fuertes no se mantienen por doble dualidad. Por ejemplo, el doble dual de una bola o esfera no es en general una bola o una esfera. Más aún, el doble dual de una variedad combinatoria no es generalmente un complejo homogéneo. Sin embargo, la teoría de NH-variedades proporciona una clasificación de los doble duales de las bolas y esferas combinatorias: son NH-bolas y NH-esferas (respectivamente). Esto se sigue del hecho que las clases de bolas y esferas no homogéneas son cerradas por doble dualidad.

Teorema 4.1.3. *Sea K un complejo simplicial y sea τ un simplex (posíblemente vacío) disjunto de K y σ un simplex (posiblemente vacío) disjunto de K^τ . Entonces K es una NH-bola (resp. NH-esfera) si y sólo si $(K^\tau)^\sigma$ es una NH-bola (resp. NH-esfera).*

Corolario 4.1.4. *Las NH-bolas son los doble duales de las bolas combinatorias. Las NH-esferas son los doble duales de las esferas combinatorias.*

En §4.2 estudiamos los duales de Alexander de las bolas y esferas combinatorias con mínima cantidad de vértices. Un complejo simplicial K de dimensión d es *vertex-minimal* si es un d-simplex o si tiene $d+2$ vértices. No es difícil ver que un complejo vertex-minimal d-homogéneo es un starring elemental $(\tau, a)\Delta^d$ de un d-simplex o el borde $\partial\Delta^{d+1}$ de un $(d+1)$-simplex (ver Lemma 4.2.7). Por otro lado, un complejo general con mínima cantidad de vértices no tiene una caracterización precisa. Sin embargo, dado que los complejos vertex-minimales puros son bolas o esferas, es natural preguntarse si existe una noción análoga a estos poliedros en la teoría de bolas y esferas no puras. En esta sección introducimos las NH-bolas y NH-esferas *minimales*, que son respectivamente las versiones no homogéneas de las bolas y esferas vertex-minimales. Las NH-bolas y NH-esferas minimales se definen en términos de la minimalidad de la cantidad de símplices maximales. Esta propiedad es estrictamente más fuerte que la minimalidad de vértices para bolas y esferas no puras. La principal importancia de esta familia de NH-variedades es que caracterizan completamente la clase de Δ^d y $\partial\Delta^{d+1}$ en la relación de equivalencia generada por $K \sim K^*$.

Definición. Una NH-esfera se dice *minimal* si la cantidad de símplices maximales $\mathsf{m}(S)$ es exactamente $\dim_h(S) + 2$.

La definición se basa en el hecho que cualquier complejo simplicial con el tipo homotópico de una k-esfera tiene al menos $k+2$ símplices maximales. En la Proposición 4.2.3 probamos que las NH-esferas minimales son vertex-minimales, lo que evidencia que la minimalidad de $\mathsf{m}(S)$ es más fuerte que la de V_S. En particular se deduce del Lema 3.2.2 que, si S es minimal, entonces $|V_{S^*}| < |V_S|$ y $\dim(S^*) < \dim(S)$ (Corolario 4.2.5). La propiedad más sobresaliente de la clase de NH-esferas minimales es que es cerrada por tomar dual de Alexander.

Teorema 4.2.6. *Sea K un complejo simplicial finito y sea τ un simplex (posiblemente vacío) disjunto de K. Entonces, K es una NH-esfera minimal si y sólo si K^τ es una NH-esfera minimal.*

La teoría para NH-bolas minimales es paralela a la de NH-esferas minimales, aunque se apoya en esta para la definición.

Definición. Una NH-bola B se dice *minimal* si existe una NH-esfera minimal S que admite una descomposición $S = B + L$.

La definición de NH-bola minimal está inspirada en las equivalencias para bolas vertex-minimales del Lema 4.2.7. Los mismos resultados probados para NH-esferas minimales siguen siendo válidos para NH-bolas minimales: Lema 4.2.8, Corolario 4.2.9 y el siguiente

Teorema 4.2.11. *Sea K un complejo simplicial finito y sea τ un simplex (posiblemente vacío) disjunto de K. Entonces, K es una NH-bola minimal si y sólo si K^τ es una NH-bola minimal.*

Con estos resultados sobre NH-bolas y NH-esferas minimales podemos probar el resultado principal de la sección.

Definición. Sea K un complejo simplicial y definamos inductivamente $K^{*(0)} = K$ y $K^{*(m)} = (K^{*(m-1)})^*$. Esto es, en cada paso $K^{*(i)}$ es relativo a sus propios vértices; es decir, como subcomplejo de la esfera de mínima dimensión que lo contiene. Llamamos $\{K^{*(m)}\}_{m\in\mathbb{N}_0}$ la *sucesión de duales de Alexander iterados* de K.

Teorema 4.2.1.

(i) Existe $m \in \mathbb{N}_0$ tal que $K^{*(m)} = \partial\Delta^d$ si y sólo si K es una NH-esfera minimal.

(ii) Existe $m \in \mathbb{N}_0$ tal que $K^{*(m)} = \Delta^d$ si y sólo si K es una NH-bola minimal.

Dado que el dual de Alexander de un complejo K es un simplex si y sólo si K es una bola vertex-minimal, el Teorema 4.2.1 (*ii*) efectivamente caracteriza todos los complejos que convergen a bolas vertex-minimales.

Corolario 4.2.12. *Sea $d \in \mathbb{N}_0$ y consideremos la relación de equivalencia en los complejos simpliciales finitos dada por $K \sim L$ si y sólo si existe una sucesión finita de complejos $K = T_0, T_1, \ldots, T_r = L$ tal que o bien $T_i = T_{i-1}^*$ o bien $T_i = T_{i+1}^*$. Entonces, la clase de equivalencia de $\partial\Delta^d$ es exactamente la clase de NH-esferas minimales y la clase de equivalencia de Δ^d es exactamente la clase de NH-bolas minimales.*

En la última parte de la sección discutimos algunas propiedades características de las NH-bolas y NH-esferas minimales y utilizamos el Teorema 4.2.1 para contar la cantidad de NH-bolas y NH-esferas minimales en cada dimensión.

Proposición 4.2.16. *Sea $0 \leq k \leq d$.*

(1) Hay exactamente $\binom{d}{k}$ NH-esferas minimales de dimensión d y dimensión homotópica k. En particular, hay exactamente 2^d NH-esferas minimales de dimensión d.

(2) Hay exactamente 2^d NH-bolas minimales de dimensión d.

La sección final §4.3 de esta Tesis está dedicada a generalizar los resultados de Dong and Santos-Sturmfels sobre el dual de Alexander de las esferas y bolas al contexto de las NH-esferas y NH-bolas. Este es uno de los principales resultados de nuestro trabajo, involucrando un uso consistente de la teoría de NH-variedades desarrollada y su interacción con la dualidad de Alexander combinatoria.

Cada caso (NH-bolas y NH-esferas) se ataca por separado. El resultado para NH-bolas es el siguiente.

Teorema 4.3.1. *Sea B una NH-bola y sea τ un simplex (posiblemente vacío). Entonces B^τ es un espacio contráctil.*

La demostración de este resultado se reduce a analizar los casos donde $\tau = \emptyset$ (por Lema 3.2.1 (4)) y donde la cantidad de vértices n de B es $d+2$, $d+3$ o $d+4$ (debido a la Observación 3.3.5).

El caso $n = d + 4$ se sigue de la existencia de spines (Teorema 2.2.9), mientras que los otros dos casos requieren algunos resultados preliminares para caracterizar el complejo d-homogéneo de las NH-bolas de dimensión d y $d + 3$ vértices (Proposición 4.3.5). Este resultado está basado en un teorema clásico de Brehm y W. Kühnel sobre PL-variedades cerradas con pocos vértices (Teorema 4.3.2), de donde deducimos un corolario para variedades con borde (Corolario 4.3.3). Las característica fundamental de las NH-bolas con pocos vértices que permiten demostrar el teorema son las siguientes resultados.

Proposición 4.3.6. *Cualquier NH-bola B de dimensión $d \geq 2$ y $d+3$ vértices colapsa a un complejo de dimensión $d - 2$.*

Corolario 4.3.7. *Cualquier NH-bola de dimensión $d \geq 3$ y $d + 2$ vértices colapsa a un complejo de dimensión $d - 3$.*

Por otro lado, el resultado para NH-esferas es el siguiente.

Teorema 4.3.8. *Sea S una NH-esfera y sea τ un simplex (posiblemente vacío). Entonces, S^τ es homotópicamente equivalente a una esfera.*

Al igual que para la demostración para el caso de NH-bolas, podemos restringirnos al caso $\tau = \emptyset$ y $2 \leq |V_S| - \dim(S) \leq 4$, ya que un espacio simplemente conexo con la homología de una esfera es homotópicamente equivalente a una esfera.

La demostración del Teorema 4.3.8 está dividida en cuatro casos. Denotando $d = \dim(S) \geq 2$, $n = |V_S|$ y $k = \dim_h(S)$:

(A) $n = d + 2$ and $k = d - 1$.

(B) $n = d + 2$ and $k = d - 2$.

(C) $n = d + 3$ and $k = d - 1$.

(D) El resto de los casos.

El caso (D) se desprende inmediatamente del Teorema 4.3.1 y el caso (A) sale fácilmente de algunas consideraciones elementales gracias a la estructura de los complejos vertex-minimales.

Para probar los casos (B) y (C) necesitamos introducir varios resultados previos sobre NH-esferas satisfaciendo las hipótesis de (C): Lemas 4.3.10, 4.3.11, 4.3.12 y 4.3.13. Entre ellos, los más importantes son el segundo, que establece que el deletion de un vértice de link homogéneo deja como resultado una NH-bola, y el último, que garantiza la existencia de un vértice z de la NH-esfera de manera que el dual del deletion $S - z$ sea contráctil. Los casos (B) y (C) se prueban finalmente en conjunto por medio de un argumento inductivo combinado.

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