(OCHA) Open Closed Homotopy Algebras and the swiss-cheese operad

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Outline

Introduction

Our main concern of this work was to understand OCHA as a Homotopy Algebra in the context of Markl's minimal models operad.

Since the OCHA operad $\mathcal{OC}_\infty$ fits the conditions of a minimal operad, we would like to show that

$$\mathcal{OC}_\infty \to \mathcal{OC}$$

is a quasi-isomorphism, where $\mathcal{OC}$ is the operad generated by top dimensional generators of the Swiss-Cheese Operad.

We will show, however, that the above quism is only a quism of modules over $\mathcal{L}_\infty$ (the operad of L_∞-algebras).

Let us fix a ground field k of characteristic zero. Let A be a vector space endowed with a product

$$m : A \otimes A \to A$$ which is associative.

If V is any vector space and

$$A \xrightarrow{\Phi} V$$ is an isomorphism,

then V has an associative algebra structure that is recognized by the isomorphism Φ.

Now assume both A and V are (co)chain complexes. For (co)chain maps $f, g : A \to V$, we define homotopy equivalence:

$$f \sim g \Leftrightarrow f - g = d_V h + h d_A$$

where $h : A \to V$ is a (co)chain homotopy operator of degree $|h| = -|d| = \pm 1$.

We say that A and V are homotopy equivalent if there exists chain maps $\phi : A \to V$ and $\psi : V \to A$ such that:

$$\phi\psi \sim \mathrm{Id}_V \qquad \text{and} \qquad \psi\phi \sim \mathrm{Id}_A$$

Suppose the complex A has an associative product that is compatible with the differential, (i.e. A is a DG algebra) and V is homotopy equivalent to A. The complex V does not necessarily have a DG algebra structure.

Generaly speaking, a Strong Homotopy Algebra is an algebraic structure on a (co)chain complex that is invariant under homotopy equivalences.

Remark:
The precise definition of Strong Homotopy Algebras includes conditions of invariance for morphisms.

History (very brief overview)

60's - Stasheff: A_∞-algebras related to H-spaces having the homotopy type of loop spaces.

70's - Boardman and Vogt: Homotopy invariant algebraic strucutures for topological spaces.

70's - May: Operads related to iterated loop spaces.

Intense development of deformation theory of algebraic strucutures by Gerstenhaber and his shcool.

80's - Schlessinger and Stasheff: Strong homotopy Lie algebras in deformation theory.

90's - Several authors: Renaiscence of Operads: Koszul Duality for algebraic operads (Ginzburg and Kapranov).

2000 - Markl: Homotopy Algebras via Minimal Model Operads.

A_∞-algebras

An A_∞-algebra consists of a cochain complex (A, d) endowed with a familly of multilinear maps:

$$m_2 : A \otimes A \longrightarrow A, \quad m_3 : A^{\otimes 3} \longrightarrow A, \dots, \quad m_n : A^{\otimes n} \longrightarrow A, \dots$$

such that m_2 is associative up to homotopy with m_3 playing the role of homotopy operator.

The higher maps m_n $n \geqslant 3$ satisfy coherence relations up to homotopy:
The coherence relations are such that, considering

$$D = d + m_2 + m_3 + m_4 + \dots$$

as a coderivation in the tensor coalgebra $T^c(A)$,
$D \in \mathrm{Coder}(T^c(A))$ is a differential:

$$D^2 = 0$$

Associative algebras and A_∞ algebras have a geometrical/topological description in the language of Operads.

Associative algebras are algebras over the Homology little intervals operad. A_∞-algebras are algebras over the cell chain complex of the compactified configuration space of points in the closed interval.

Those compactified configuration spaces are polytopes known as Associahedra.

L_∞-Algebras

An L_∞-algebra consists of a cochain complex (L, d) endowed with a familly of multilinear maps: $l_n : L^{\otimes n} \longrightarrow L, \quad n \geqslant 1$

The maps l_n are graded symmetric and viewing

$$D = d + l_2 + l_3 + l_4 + \dots$$

as a coderivation in the symmetric coalgebra $S^c(L)$, $D \in \mathrm{Coder}(S^c(L))$ is a differential:

$$D^2 = 0$$

Remark

Degrees and sings issues are being omitted in this talk. They are crucial for computations but not for conceptual descriptions. Different degree/signs conventios are equivalent through (de)suspension. For a gentle description of A_∞ and L_∞ algebras via coderivations (with degree/signs issues considered in detail) see (Doubek, Zima, Markl, arXiv:0705.3719).

The operad of Lie algebras and the operad of L_∞ algebras also have a nice geometrical description.

Lie algebras are algebras over the suboperad of the Homology little discs operad generated by top dimensional homology classes of $D(2)$.

L_∞-algebras are algebras over the first row of the E^1 term of the spectral sequence associated to the compactified configuration space of points in the sphere.

Minimal Operads

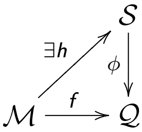
A Minimal Operad is a diferential graded operad that is free as an operad and such that the image of the differential consists of decomposable elements.

Given a DG-operad $\mathcal{P}$, a Minimal Model of $\mathcal{P}$ is an operad $\mathcal{M}_{\mathcal{P}}$ that is minimal and quasi-isomorphic to $\mathcal{P}$.

The following theorem says that minimal operads are cofibrant objects in the category of operads.

Theorem

For each quasi-isomorphism $\phi : \mathcal{S} \to \mathcal{Q}$ *and for each morphism* $f : \mathcal{M} \to \mathcal{Q}$ *from a minimal operad* $\mathcal{M}$ *into* $\mathcal{Q}$*, there exists a morphism* $h : \mathcal{M} \to \mathcal{S}$ *such that* $\phi \circ h \sim f$*.*



Corollary

Algebras over Minimal Operads are Homotopy Invariant.

Let $\mathcal{P}$ be a DG operad. According Markl (math/9907138), a Strong Homotopy $\mathcal{P}$-Algebra is an algebra over a minimal model operad $\mathcal{M}_{\mathcal{P}}$ of $\mathcal{P}$.

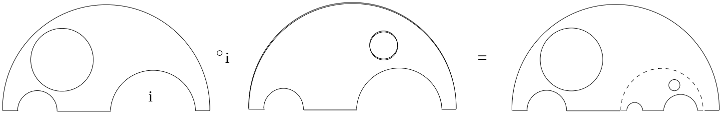
Example

The Operads for A_∞ and L_∞ algebras are minimal minimal models of the operads for associative and Lie algebras respectivelly.

The Swiss-Cheese Operad is a 2-colored operad given by discs and half discs whose composition law is similar to the composition law of the little discs operad.

Its set of colors is $\{c, o\}$. The suboperad corresponding to the color c is just the usual little discs operad, while the suboperad corresponding to the color o is defined by all possible ways of imbedding disjoint unions of discs and half discs in the standard half disc by translations and dilations.

The composition law for the Swiss-Cheese Operad is defined analogously to the composition law for the little discs operad, and is schematically described in the following figure:



Open-Closed Homotopy Algebras

An OCHA is a pair of DG spaces (L, A) with multilinear maps:

$$l_n : L^{\otimes n} \to L, \quad n \geqslant 1 \text{ and } n_{p,q} : L^{\otimes p} \otimes A^{\otimes q} \to A, \quad p+q \geqslant 1$$

The OCHA coherence relations are such that, for

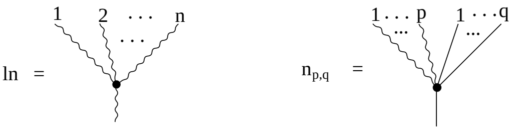
$$\mathfrak{l} = l_1 + l_2 + l_3 + \cdots + l_n + \cdots$$

$$\mathfrak{n} = n_{0,1} + n_{1,0} + n_{1,1} + n_{0,2} + \cdots + n_{p,q} + \cdots$$

$D = \mathfrak{l} + \mathfrak{n} \in \mathrm{Coder}(S^c(L) \otimes T^c(A))$, D is a differential:

$$D^2 = 0$$

The OCHA operad $\mathcal{OC}_\infty$ in defined through Partially Planar Trees:



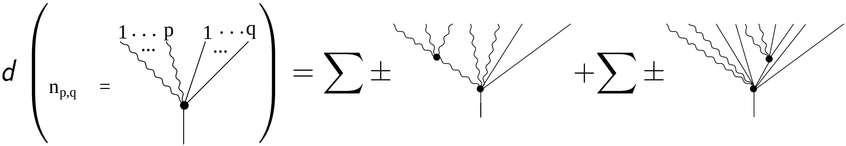
where wiggly edges are spatial and straight edges are planar.

$\mathcal{OC}_\infty$ is the DG 2-colored operad generated by all trees $\{l_n\}_{n\geqslant 2}$ and $\{n_{p,q}\}_{p+q\geqslant 1}$ and diferential operator given by:

$$dT = \sum_{T' \to T} \pm T'$$

where T' is such that, the tree T can be obtained from T' by collapsing a internal edge into a vertex.

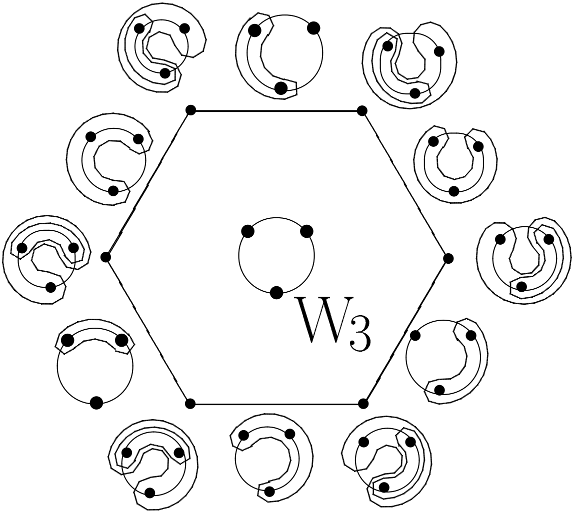
Schematically and without going into the details involving signs:



The compactified configuration space of points in the closed disc is a manifold with corners denoted $\overline{C(p,q)}$. The first row of its associated spectral sequence can be described in terms of trees with differential given by the above operator d. More precisely:

The ideal $\mathcal{N}_\infty$ of $\mathcal{OC}_\infty$ generated by trees with planar roots is isomorphic, as a cochain complex, to the first row of the E^1 term of the spectral sequence associated to $\{\overline{C(p,q)}\}_{p+q\geqslant 1}$.

The next figures illustrate the manifolds $\overline{C(p,q)}$, and their boundary strata labeled by partially planar trees, for small p and q.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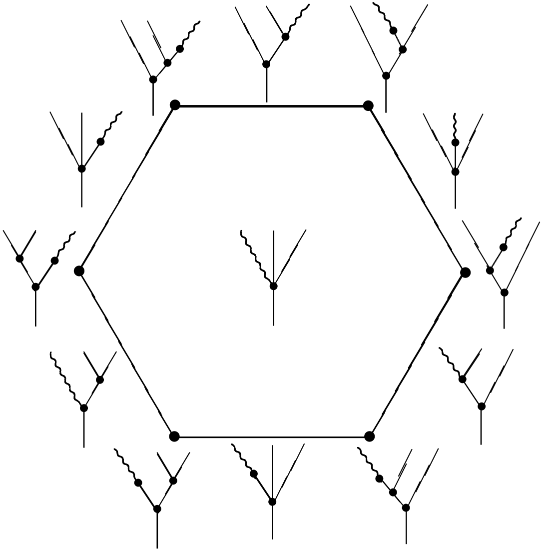
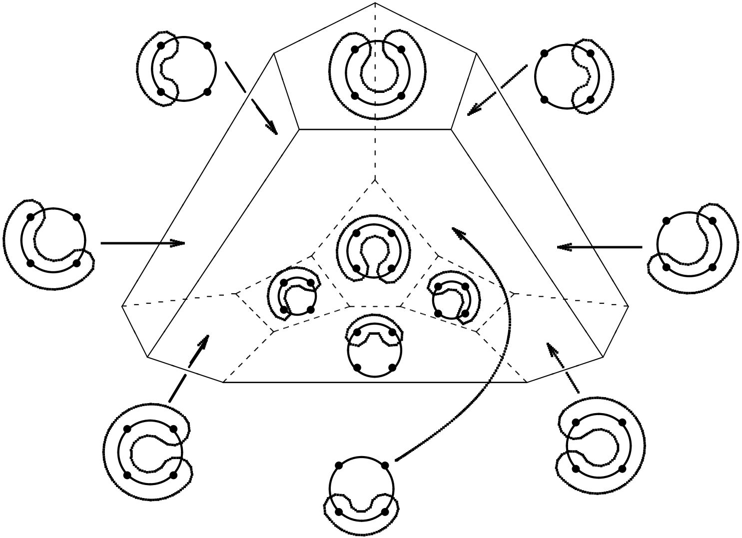
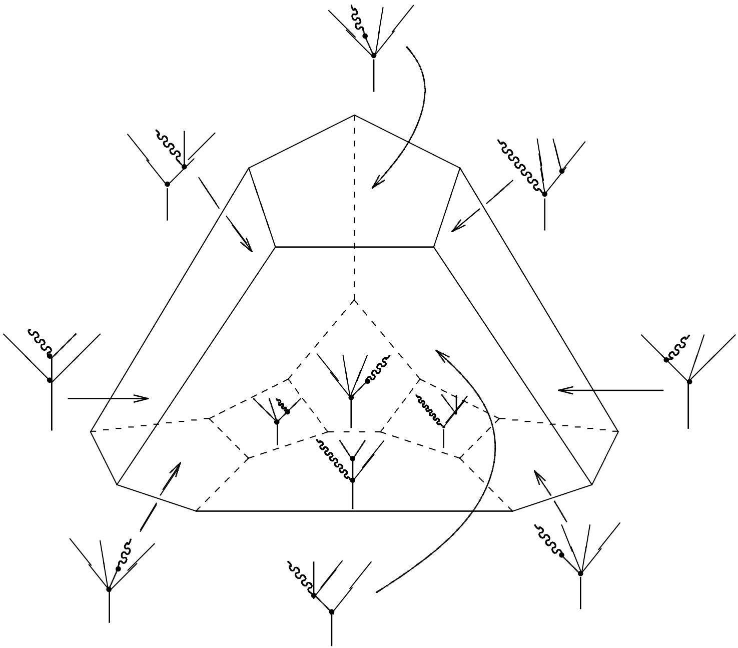
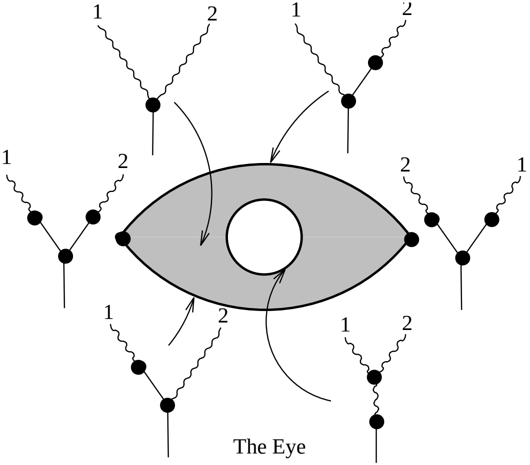
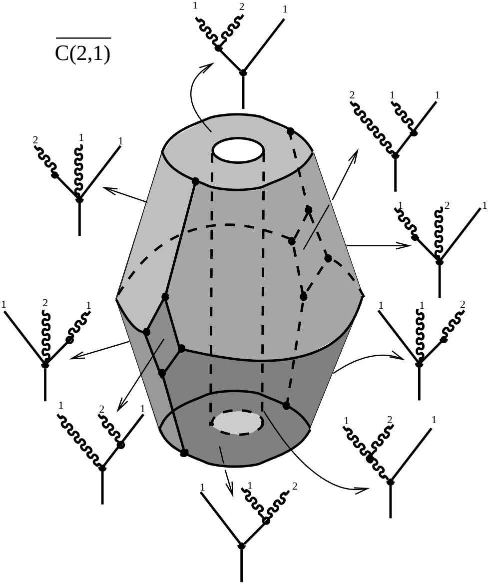








The Eye


$\overline{C(2,1)}$

New results

The following are the main results of (Hoefel, arXiv:0710.3546).

Proposition

For any $q \geqslant 0$, $H(\mathcal{N}_\infty(\_\,, q))$ and $H(\mathcal{D})$ are isomorphic as $\mathcal{L}$-modules.

Proof

- $\mathcal{N}_\infty(p, q)$ is the first row of the Spec. Seq. of $\overline{C(p,q)}$.
- $\overline{C(p,q)}$ can be deformation retracted into a stratum diffeomorphic to $\overline{C(p)} =$ {compactified config. space of points in the complex plane}.
- $\{\overline{C(p)}\}_{p \geqslant 1}$ is homotopy equivalent (as operads) to $\{\mathcal{D}(p)\}_{p \geqslant 1}$.

Corollary

The homology $H(\mathcal{N}_\infty)$ is the ideal of $H(\mathcal{OC}_\infty)$ generated by $n_{1,0}$ and $n_{0,2}$.

The top dimensional generators of the homology swiss-cheese operad are: $\{\text{Lie bracket} = l_2, n_{1,0}, n_{0,2}\}$, and $H(\mathcal{OC}_\infty) = H(\mathcal{L}_\infty \oplus \mathcal{N}_\infty) = \mathcal{L} \oplus H(\mathcal{N}_\infty)$.

From the above corollary we see that $H(\mathcal{OC}_\infty)$ is precisely the operad generated by the top dimensional classes of the Homology Swiss-Cheese operad which we denote by $\mathcal{OC}$.

Notice that $\mathcal{OC} = \langle l_2, n_{1,0}, n_{0,2} \rangle$ is a suboperad of $\mathcal{OC}_\infty$. In (arXiv:0710.3546) we exhibit a retract $\eta : \mathcal{OC}_\infty \to \mathcal{OC}$ that reduces to the identity map on cohomology but cannot be a morphism of operads.

In fact: the manifold corresponding to $n_{1,1}$ is contractible, so $\eta(n_{1,0}) = 0$, for any quism η. But, $n_{1,0} \bullet_1 l_2$ is not zero in $\mathcal{OC}$. The OCHA relation corresponding to the Manifold $\overline{C(2,0)}$ (known as "The Eye") prevents η from respecting the operad structure.

However, the retract η does preserve the $\mathcal{L}_\infty$ module structure and we can state our main result:

Theorem

The DG 2-colored operad $\mathcal{OC}_\infty$ is quasi-isomorphic to $\mathcal{OC}$ as modules over the $\mathcal{L}_\infty$ operad.

Question

Is $\eta : \mathcal{OC}_\infty \to \mathcal{OC}$ a quasi-isomorphism of Strong Homotopy Operads if we consider the DG operads $\mathcal{OC}_\infty$ and $\mathcal{OC}$ as SHoperads ?

Example of OCHA

From the fact that the OCHA operad $\mathcal{OC}_\infty$ is given by the first row of the spectral sequences of $\{\overline{C(n)}\}$ and of $\{\overline{C(p,q)}\}$, there is an OCHA structure on the space of relative 2-loops.
Let (X, A) be any pair of topological spaces with $A \subseteq X$ and a base point $* \in A$. The space of relative double based loops on (X, A)

$$\Omega^2(X, A) = \mathrm{Map}_*\{(D^2, S^1), (X, A)\}$$

and the space of double based loops on X

$$\Omega^2(X) = \mathrm{Map}_*\{(D^2, S^1), (X, *)\}$$

form a pair: $(\Omega^2(X, A), \Omega^2 X)$ that is an algebra over the swiss-cheese operad.

Since the swiss-cheese operad is homotopy equivalent (as operads) to the operad $\{\overline{C(n)}\} \cup \{\overline{C(p,q)}\}$ and the OCHA operad $\mathcal{OC}_\infty$ is defined by appropriate singular chains on $\{\overline{C(n)}\} \cup \{\overline{C(p,q)}\}$,

So, there is an OCHA structure on singular chains:

$$(C_*(\Omega^2(X,A)), C_*(\Omega^2 X))$$

with operations l_n and $n_{p,q}$ induced by the relative fundamental classes of the manifolds with corners $\{\overline{C(n)}\}$ and $\{\overline{C(p,q)}\}$.

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[4] Markl,M.; *Homotopy Algebras are Homotopy Algebras*, Forum Math. 16 (2004), no. 1, 129-160. (math/9907138).