Supercompact Cardinals Imply Reflectivity of Absolute Orthogonality Classes in Locally Presentable Categories

Carles Casacuberta, Universitat de Barcelona

Buenos Aires, November 13, 2008

Sources

For context:

1. M. Hovey, J. H. Palmieri, N. P. Strickland, *Axiomatic Stable Homotopy Theory*, Mem. Amer. Math. Soc. vol. 128, no. 610, AMS, 1997
2. J. Adámek, J. Rosický, *Locally Presentable and Accessible Categories*, London Math. Soc. Lecture Note Ser. vol. 189, CUP, 1994

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For results:

3. C. C., D. Scevenels, and J. H. Smith, Implications of large-cardinal principles in homotopical localization, *Adv. Math.* **197** (2005), 120–139
4. C. C., J. J. Gutiérrez, J. Rosický, Are all localizing subcategories of stable homotopy categories coreflective?, preprint

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5. J. Bagaria, C. C., A. R. D. Mathias, Epireflections and supercompact cardinals, *J. Pure Appl. Algebra*, to appear
6. J. Bagaria, C. C., A. R. D. Mathias, J. Rosický, Definable orthogonality classes are small, in preparation

A Very Powerful Statement

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This statement is equivalent to the following:

VP': No locally presentable category contains a large discrete full subcategory.

VP'': Given any family of objects X_i of a locally presentable category indexed by the ordinals, there exists a morphism $X_i \to X_j$ for some ordinals $i < j$.

Some Implications

- Assuming VP, every orthogonality class in a locally presentable category is small and reflective. [Adámek and Rosický, 1996]

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- Assuming VP, for every cohomology theory E^* there is a map f such that the class of f-equivalences coincides with the class of E^*-equivalences. Therefore, VP implies the existence of cohomological localizations. [C–Scevenels–Smith, 2005]

- Let $\mathcal{K}$ be a locally presentable category with a stable model category structure. Assuming VP, every triangulated subcategory of $\mathbf{Ho}(\mathcal{K})$ closed under products is reflective. [C–Gutiérrez–Rosický, 2008]

Using VP

Theorem Suppose given a functor $Q\colon \mathcal{K} \to \mathcal{T}$ where $\mathcal{K}$ is locally presentable, $\mathcal{T}$ has products, and Q is essentially surjective on cones. If VP holds, then every full subcategory $\mathcal{L} \subseteq \mathcal{T}$ closed under products is weakly reflective.

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Proof Write $\mathcal{L}$ as the union of an ascending chain of full small subcategories indexed by the ordinals, $\mathcal{L} = \bigcup_{i \in \mathrm{Ord}} \mathcal{L}_i$, and let $\overline{\mathcal{L}_i}$ be the closure of $\mathcal{L}_i$ under products. For an object X of $\mathcal{T}$, let X_i denote the product of all the codomains of morphisms from X to objects of $\mathcal{L}_i$. Then the canonical morphism $f_i\colon X \to X_i$ is a weak reflection of X onto $\overline{\mathcal{L}_i}$. Now, in order to prove that $\mathcal{L}$ is weakly reflective, it suffices to find an ordinal i such that, $(X \downarrow \mathcal{L})(f_i, f_j) \neq \emptyset$ for all $j \geq i$. Suppose the contrary. Then there is a sequence of ordinals $i_0 < i_1 < i_2 < \cdots < i_s < \cdots$, where s ranges over all the ordinals, such that $(X \downarrow \mathcal{L})(f_{i_s}, f_{i_t}) = \emptyset$ if $s < t$. Since Q is essentially surjective on cones, there is a sequence $g_i\colon K \to K_i$ in $\mathcal{K}$ such that $Qg_i \cong f_i$ for all i. Then $(K \downarrow \mathcal{K})(g_{i_s}, g_{i_t}) = \emptyset$ if $s < t$, contradicting VP.

Using VP

Theorem Let $\mathcal{K}$ be a locally presentable category with a stable model category structure. Assuming VP, every triangulated subcategory $\mathcal{L}$ of $\mathbf{Ho}(\mathcal{K})$ closed under products is reflective.

Proof Since $\mathbf{Ho}(\mathcal{K})$ has products and coproducts, idempotents split, and $\mathcal{L}$ is closed under retracts. From the previous theorem it follows that $\mathcal{L}$ is weakly reflective. Moreover, since $\mathcal{L}$ is triangulated, every pair of parallel arrows in $\mathcal{L}$ has a weak equalizer that lies in $\mathcal{L}$. In this situation, it follows by a standard argument that $\mathcal{L}$ is reflective.

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Corollary Assuming VP, if E^* is any cohomology theory in the homotopy category of spectra, then the class of E^*-local spectra (i.e., the subcategory of spectra X such that $[A, X] = 0$ for all E^*-acyclic spectra A) is reflective.

Warning

There are triangulated subcategories of triangulated categories (with models) closed under products but not reflective, even assuming VP.

Thus, the assumption that the model category is locally presentable cannot be removed. (Perhaps it is sufficient to assume that the triangulated category be well-generated.)

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Namely, consider the homotopy category of chain complexes of small modules over the large ring freely generated by all the ordinals. Then the inclusion of the full subcategory of acyclic complexes (which is closed under products and coproducts) does neither admit a right adjoint nor a left adjoint.
[C–Neeman, 2008]

Another Consequence

Theorem Let $\mathcal{K}$ be a stable combinatorial monoidal model category. If VP holds, then every localizing ideal of $\mathbf{Ho}(\mathcal{K})$ is principal and coreflective.

Trying to Get Rid of VP

VP is known to be a *large-cardinal principle*. It cannot be proved using the usual ZFC axioms of Set Theory.

It implies the existence of measurable cardinals, and is implied by the existence of huge cardinals.

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Try to prove or disprove the following claim:

In a locally presentable category, every epireflection is an f-localization for some single morphism f.

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- On the other hand, a counterexample can be displayed assuming that measurable cardinals do not exist. Namely, in the category of groups, if one localizes with respect to the morphisms $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa} \to \{0\}$, where κ ranges over all ordinals, then the resulting localization is an epireflection and it is an f-localization for a single f if and only if there exists a measurable cardinal.

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What happens in between?

Transverse Sets

Definition Let $\mathcal{A}$ be a class of objects in a category $\mathcal{C}$. A set $\mathcal{H}$ of objects of $\mathcal{C}$ is called *transverse* to $\mathcal{A}$ if every object of $\mathcal{A}$ has a subobject in $\mathcal{H} \cap \mathcal{A}$. (That is, if for every object $A \in \mathcal{A}$ there is an object $H \in \mathcal{H} \cap \mathcal{A}$ and a monomorphism $H \to A$.)

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Theorem

Let L be an epireflection on a balanced category $\mathcal{C}$.

(a) If there exists a set $\mathcal{H}$ of objects in $\mathcal{C}$ transverse to the class of objects that are not L-local, then there is a morphism f such that L is an f-localization.

(b) If $\mathcal{C}$ is co-well-powered and every morphism can be factored as an epimorphism followed by a monomorphism, then the converse holds, that is, if L is an f-localization for some f, then there is a set $\mathcal{H}$ transverse to the class of objects that are not L-local.

Sketch proof

To prove (a), let f be the coproduct of the morphisms $\{\eta_H\colon H \to LH \mid H \in \mathcal{H}\}$, where η denotes the unit of the reflection. Then the f-local objects are precisely the L-local objects; that is, L is an f-localization.

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Thus, under which assumptions on $\mathcal{C}$ and L are there transverse sets to the class of objects that are not L-local?

A Very Quick Tutorial on Set Theory

- The *language of set theory* is the first-order language whose only nonlogical symbols are equality and a binary relation symbol $\in$.
- The language consists of *formulas* built up in finitely many steps from the *atomic formulas* $x = y$ and $x \in y$, where x and y are members of a set of *variables*, using logical connectives and quantifiers.

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- The language consists of *formulas* built up in finitely many steps from the *atomic formulas* $x = y$ and $x \in y$, where x and y are members of a set of *variables*, using logical connectives and quantifiers.
- A *model* is a set or a proper class M in which the formalized ZFC axioms are true when the binary relation symbol $\in$ is interpreted as membership.
- A model M is *transitive* if every element of an element of M is an element of M.

A Very Quick Tutorial on Set Theory

- For a model M, a set or a proper class $\mathcal{C}$ is *definable* in M if there is a formula $\varphi(x, x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n$ in M such that $\mathcal{C}$ is the class of elements $c \in M$ such that $\varphi(c, a_1, \ldots, a_n)$ is satisfied in M. We then say that C is *defined by φ in M with parameters $a_1, \ldots, a_n$*.

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- A formula $\varphi(x, x_1, \ldots, x_n)$ is *absolute between two models* $N \subseteq M$ with respect to a collection of parameters $a_1, \ldots, a_n$ in N if, for each $c \in N$, $\varphi(c, a_1, \ldots, a_n)$ is satisfied in N if and only if it is satisfied in M.

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- A formula $\varphi(x, x_1, \ldots, x_n)$ is *absolute between two models* $N \subseteq M$ with respect to a collection of parameters $a_1, \ldots, a_n$ in N if, for each $c \in N$, $\varphi(c, a_1, \ldots, a_n)$ is satisfied in N if and only if it is satisfied in M. For example, formulas in which all quantifiers are *bounded* (that is, of the form $\exists x \in a$ or $\forall x \in a$) are absolute between any two transitive models.

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- A formula is called *absolute* with respect to $a_1, \ldots, a_n$ if it is absolute between any transitive model M that contains $a_1, \ldots, a_n$ and the universe V. We call a set or a proper class X *absolute* if membership of X is defined by an absolute formula with respect to some parameters.

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- A submodel N of a model M is *elementary* if all formulas are absolute between N and M with respect to every set of parameters in N.
- An embedding of V into a model M is an *elementary embedding* if its image is an elementary submodel of M.

A Very Quick Tutorial on Set Theory

- If $j\colon V \to M$ is a nontrivial elementary embedding with M transitive, then M contains all the ordinals and there is a least ordinal κ moved by j, that is, $j(\alpha) = \alpha$ for all $\alpha < \kappa$, and $j(\kappa) > \kappa$. Such a κ is called the *critical point* of j, and is necessarily a measurable cardinal.
- Conversely, if there is a measurable cardinal, then there is a nontrivial elementary embedding $j\colon V \to M$ where M is a transitive model.

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- Conversely, if there is a measurable cardinal, then there is a nontrivial elementary embedding $j \colon V \to M$ where M is a transitive model.

- If $j \colon V \to M$ is an elementary embedding, then for every set X the *restriction* $j \restriction X \colon X \to j(X)$ is the function that sends each element $x \in X$ to $j(x)$.

A Very Quick Tutorial on Set Theory

Definition

A cardinal κ is *supercompact* if and only if for every set X there is an elementary embedding j of the universe V into a transitive model M with critical point κ, such that $X \in M$, $j(\kappa) > \text{rank}(X)$, and $j \restriction X \colon X \to j(X)$ is in M.

Absolute Categories

Definition

A subcategory $\mathcal{C}$ of sets is *absolute* if there is a formula $\varphi(x, y, z, x_1, \ldots, x_n)$ which is absolute with respect to some parameters $a_1, \ldots, a_n$ and such that, for any two sets A, B and any function $f\colon A \to B$, the sentence $\varphi(A, B, f, a_1, \ldots, a_n)$ is satisfied if and only if A and B are objects of $\mathcal{C}$ and the function f is in $\mathcal{C}(A, B)$.

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Definition

We say that a subcategory $\mathcal{C}$ of sets *supports elementary embeddings* if, for every elementary embedding $j\colon V \to M$ and all objects X of $\mathcal{C}$, the restriction $j \restriction X\colon X \to j(X)$ is a morphism of $\mathcal{C}$.

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Theorem

Every locally presentable category is absolute and preserves elementary embeddings.

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Let $\mathcal{C}$ be a subcategory of sets and let $\mathcal{A}$ be a class of objects in $\mathcal{C}$. Suppose that $\mathcal{C}$ supports elementary embeddings and there are absolute formulas defining $\mathcal{C}$ and $\mathcal{A}$ with parameters whose rank is smaller than a supercompact cardinal κ. If $X \in \mathcal{A}$, then there is a subobject of X in $\mathcal{A}$ of rank less than κ.

Proof

Let φ be an absolute formula defining $\mathcal{C}$ with parameters $\vec{a} = \{a_1, \ldots, a_n\}$ of rank less than κ, and let ψ be an absolute formula defining $\mathcal{A}$ with parameters $\vec{b} = \{b_1, \ldots, b_m\}$ of rank less than κ. Fix an object $X \in \mathcal{A}$ and let $j\colon V \to M$ be an elementary embedding with critical point κ such that X and the restriction $j \restriction X$ are in M, and $j(\kappa) > \mathrm{rank}(X)$. Notice that $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ are in M, since in fact $j(a_r) = a_r$ for all r and $j(b_s) = b_s$ for all s.

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$$\exists y\, \exists f\, (f\colon y \to j(X) \wedge (f \text{ is injective}) \wedge \varphi(y, j(X), f, \vec{a}) \wedge \psi(y, \vec{b}) \wedge \mathrm{rank}(y) < j(\kappa)).$$

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As j is an elementary embedding, the following holds in V:

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$$\exists y\, \exists f\, (f\colon y \to j(X) \wedge (f \text{ is injective}) \wedge \varphi(y, j(X), f, \vec{a}) \wedge \psi(y, \vec{b}) \wedge \mathrm{rank}(y) < j(\kappa)).$$

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$$\exists y\, \exists f\, (f\colon y \to X \wedge (f \text{ is injective}) \wedge \varphi(y, X, f, \vec{a}) \wedge \psi(y, \vec{b}) \wedge \mathrm{rank}(y) < \kappa).$$

Since morphisms whose underlying function is injective are monomorphisms, this says that X has a subobject in $\mathcal{A}$ of rank less than κ.

Absolute Categories

Corollary

Let L be an epireflection on a locally presentable category $\mathcal{C}$. Suppose that both $\mathcal{C}$ and the class of L-local objects can be defined by absolute formulas with parameters whose rank is smaller than a supercompact cardinal κ. Then L is an f-localization for some single morphism f.

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Theorem

Assume the existence of arbitrarily large supercompact cardinals. Then each absolute orthogonality class $\mathcal{L}$ in a locally presentable category is small, hence reflective.

The Lévy Hierarchy

Recall that a formula is absolute if it contains no unbounded quantifiers ($\exists x$ is unbounded, while $\exists x \in a$ is bounded by a).

A formula is Σ_1 if it has the form $\exists x\ \varphi$, where φ is absolute.
A formula is Π_1 if it has the form $\forall y\ \varphi$, where φ is absolute.
A formula is Σ_2 if it has the form $\exists x\ \forall y\ \varphi$ where φ is absolute.
A formula is Π_2 if it has the form $\forall x\ \exists y\ \varphi$ where φ is absolute.
...
A formula is Σ_n if it has the form $\exists x\ \varphi$ where φ is Π_{n-1}.
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The Lévy Hierarchy

Recall that a formula is absolute if it contains no unbounded quantifiers ($\exists x$ is unbounded, while $\exists x \in a$ is bounded by a).

A formula is Σ_1 if it has the form $\exists x\ \varphi$, where φ is absolute.
A formula is Π_1 if it has the form $\forall y\ \varphi$, where φ is absolute.
A formula is Σ_2 if it has the form $\exists x\ \forall y\ \varphi$ where φ is absolute.
A formula is Π_2 if it has the form $\forall x\ \exists y\ \varphi$ where φ is absolute.
...
A formula is Σ_n if it has the form $\exists x\ \varphi$ where φ is Π_{n-1}.
A formula is Π_n if it has the form $\forall y\ \varphi$ where φ is Σ_{n-1}.

Examples Groups, rings, R-modules or simplicial sets are defined by absolute formulas. Topological spaces are defined by a Σ_2 formula. Acyclic spectra for a cohomology theory are defined by a Π_2 formula.

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Consider the following sequence of statements, where $n \geq 1$:

VP(n)**:** Given a proper class of structures of the same type defined by a Σ_n formula, there is at least an elementary embedding between two of them.

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Then:

- VP(1) is equivalent to the existence of arbitrarily large supercompact cardinals.
- VP holds if and only if VP(n) holds for all n.
- If VP(n) holds, then each full subcategory $\mathcal{L}$ defined by a Σ_n formula in a locally presentable category has a small dense subcategory. Therefore, if $\mathcal{L}$ is closed under limits, then it is reflective.