Galois extensions of the $K(n)$-local sphere

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Good categories of spectra

- We need to work in a good category of spectra with strictly associative and unital smash product *before passage to its derived category*. The category of S-modules $\mathscr{M}_S$ has this property. Both $\mathscr{M}_S$ and the derived category $\mathscr{D}_S$ are symmetric monoidal under $\wedge = \wedge_S$ with S as unit.
- Note that S is not cofibrant in $\mathscr{M}_S$, so to define cellular objects we use use free objects $\mathbb{F}S^n$ as cofibrant spheres. We write X for $\mathbb{F}X$, and if Y is a space, write $\mathbb{F}Y$ for $\mathbb{F}\Sigma^\infty Y$.
- A monoid object R in $\mathscr{M}_S$ is an *S-algebra*, and a commutative monoid object is a *commutative S-algebra*. Such an S-algebra gives rise to a ring spectrum in $\mathscr{D}_S$.
- Given a commutative S-algebra R we can also define R-modules leading to a category $\mathscr{M}_R$ which is also symmetric monoidal with a smash product $\wedge_R$. A monoid object A in $\mathscr{M}_R$ is an *R-algebra*, and if it is commutative it is a *commutative R-algebra*. The category of commutative R-algebras $\mathscr{C}_R$ is also a model category and has a derived category.

Galois Theory in the sense of John Rognes

Let A be commutative S-algebra and let B be a commutative A-algebra, we view this as a pair or extension B/A. Suppose that a finite group acts faithfully as a group of algebra automorphisms of B/A.

The B/A is a *G-Galois extension* if it satisfies

$$A \xrightarrow{\sim} B^{hG},$$

$$\Phi\colon B \wedge_A B \xrightarrow{\sim} F(G_+, B) \cong \prod_G B, \qquad \text{(unramified condition)}$$

where Φ is the adjoint of $\mathrm{id} \wedge \mathrm{mult}\colon B \wedge_A (G_+ \wedge B) \longrightarrow B$. This can be extended to topological group-like monoids, and also to Bousfield localisations of $\mathscr{C}_A$. These equivalences can be interpreted as isomorphisms in the derived category of commutative A-algebras.

Examples

- If T/R is a G-Galois extension of commutative rings then there is a G-Galois extension HT/HR, where $H(-)$ is the Eilenberg-Mac Lane spectrum functor suitably rigidified.
- KU/KO is a C_2-Galois extension. (Theorem of Reg Wood) Here π_*KU/π_*KO is not a Galois extension of rings since π_*KU is not projective over π_*KO.
- If A is a commutative S-algebra and B_*/π_*A is a G-Galois extension of commutative rings, then there is a G-Galois B/A realising it.
- S has no Galois extensions with finite Galois groups. The proof uses the fact that every finite Galois extension of $\mathbb{Q}$ is ramified at some prime: for example $\mathbb{Q}(i)/\mathbb{Q}$ ramifies at 2, so $\mathbb{Z}(i)/\mathbb{Z}$ is not a Galois extension of rings, but $\mathbb{Z}[1/2](i)/\mathbb{Z}[1/2]$ is a C_2-Galois extension.

Lubin-Tate spectra

For each prime p and $0 < n < \infty$, there are morphisms of commutative ring spectra

$$S_{(p)} \longrightarrow E(n) \longrightarrow \widehat{E(n)} \longrightarrow E_n \longrightarrow E_n^{\mathrm{nr}},$$

where $E(n)$ is a classical Johnson-Wilson spectrum. Here

$$\begin{aligned} \pi_* E(n) &= \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}][v_n, v_n^{-1}], \\ \pi_* E_n &= W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u, u^{-1}], \\ \pi_* E_n^{\mathrm{nr}} &= W\overline{\mathbb{F}}_p[[u_1, \ldots, u_{n-1}]][u, u^{-1}]. \end{aligned}$$

The following is a composite result proved using machinery of Goerss, Hopkins, Miller, Richter, Robinson:

Theorem

There are essentially unique morphisms of commutative S-algebras

$$S_p \longrightarrow \widehat{E(n)} \longrightarrow E_n \longrightarrow E_n^{\mathrm{nr}}.$$

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E_n has a kind of residue field, an extended form of Morava K-theory, namely an E_n-algebra K_n with

$$\pi_* K_n = \mathbb{F}_{p^n}[u, u^{-1}].$$

Similarly E_n^{nr} has a residue field which is an E_n^{nr}-algebra K_n^{nr} with

$$\pi_* K_n^{\mathrm{nr}} = \overline{\mathbb{F}}_p[u, u^{-1}].$$

Bousfield localisation with respect to each of these is essentially localisation with respect to Morava K-theory $K(n)$ itself and we denote this by $(-)_K$. Note that E_n and E_n^{nr} are known to be $K(n)$-local.

Theorem

There are essentially unique morphisms of commutative S-algebras

$$S_K \longrightarrow E_n \longrightarrow E_n^{\mathrm{nr}}.$$

In fact, the extension E_n/S_K is itself a suitable kind of Galois extension where the group is a version of the Morava stabiliser group $\mathbb{G}_n = \mathbb{O}_n^\times \rtimes C_n$ (this is a profinite group). There is a homotopy fixed point spectral sequence

$$\mathrm{E}_2^{s,t} = H^s(\mathbb{G}_n; \pi_{-t} E_n) \quad \Longrightarrow \quad \pi_{s-t} S_K,$$

which is a disguised form of an Adams-Novikov spectral sequence.

The extension E_n^{nr}/E_n is a lifting of the algebraic closure $\overline{\mathbb{F}}_p/\mathbb{F}_{p^n}$ which is a profinite abelian extension with group $\widehat{\mathbb{Z}}$.

Obvious question: Are there any connected Galois extensions of E_n^{nr} with finite Galois group?
If the answer is *No*, this would imply that E_n^{nr}/S_K is a sort of connected algebraic closure, or equivalently a maximal unramified extension of S_K.

Main Theorem

Theorem

For an odd prime p, let B/E_n^{nr} be a finite Galois extension with non-trivial Galois group. Then B is not connected. Hence E_n^{nr} is a maximal connected Galois extension of E_n.

For $p = 2$ any finite Galois extension B/E_n^{nr} whose Galois group has a cyclic quotient is not connected.

So far, we are unable to prove that there are no non-trivial connected Galois extensions of E_n^{nr} at $p = 2$ with Galois group having only finite simple non-abelian quotients.

Outline of proof

First we need some technical results on Galois extensions. Where necessary we always assume appropriate cofibrancy conditions on S-algebras.

Lemma

Let B be a cofibrant commutative A-algebra.
(i) Let $\pi_*(B)/\pi_*(A)$ be a G-Galois extension and let C be an associative A-algebra whose coefficient ring $\pi_*(C)$ is a graded commutative $\pi_*(A)$-algebra. Then $\pi_*(C \wedge_A B)/\pi_*(C)$ is also a G-Galois extension.
(ii) Let B/A be a G-Galois extension of commutative S-algebras, and let C be an associative A-algebra that is a retract of a finite cell A-module spectrum and for which $\pi_*(C)$ is a graded field. Then $\pi_*(C \wedge_A B)/\pi_*(C)$ is a G-Galois extension.

We will use this when $A = E_n^{\mathrm{nr}} = E$ and $C = K_n^{\mathrm{nr}} = K$.

Some numerology

Since $\pi_{\mathrm{odd}}(K) = 0$, for a finite dimensional $\pi_*(K)$-module V_* we can consider

$$d_0(V_*) = \dim_{\pi_{\mathrm{even}}(K)} V_{\mathrm{even}} = \dim_{\overline{\mathbb{F}}_p} V_0,$$
$$d_1(V_*) = \dim_{\pi_{\mathrm{even}}(K)} V_{\mathrm{odd}} = \dim_{\overline{\mathbb{F}}_p} V_1.$$

Let M be a cofibrant E-module spectrum for which

$$d_0 = \dim_{\overline{\mathbb{F}}_p} \pi_0(K \wedge_E M), \quad d_1 = \dim_{\overline{\mathbb{F}}_p} \pi_1(K \wedge_E M)$$

are finite and not both zero.

Lemma

Suppose that for some finite set X of cardinality $|X| = m$,

$$M \wedge_E M \simeq \prod_X M.$$

Then the dimensions d_0 and d_1 satisfy one of the following conditions:

- $d_1 = 0$ *and* $d_0 = m$.
- $d_1 \neq 0$, m *is even and* $d_0 = m/2 = d_1$.

In particular, if m is odd, then the first condition holds.

p odd

Now we can prove

- Let G be an arbitrary finite group and p an odd prime. Then for every G-Galois extension B of E there is a weak equivalence of commutative E-algebras

$$B \simeq \prod_G E.$$

First we show that $\pi_*(K \wedge_E B)$ is concentrated in even degrees using separability of Galois extensions. In degree 0, we have a (finite dimesional) separable extension of $\overline{\mathbb{F}}_p$ and this splits into a product of $|G|$ copies of $\overline{\mathbb{F}}_p$. The idempotents can be realised as E-algebra maps $B \longrightarrow B$ using 'I-adic tower' arguments.

$p = 2$

We can now prove a more general statement.

- Let B/E be a G-Galois extension where G is a finite group with a cyclic quotient of prime order. Then B is not connected.
  In particular, every G-Galois extension B of E with finite solvable Galois group G is not connected. In this sense, the commutative E_n-algebra E is a maximal connected solvable Galois extension of E_n.

When $p = 2$ this gives the second part of our Main Theorem. The proof involves analysing the cases where there is a prime order quotient of form $G/N \cong C_\ell$ ($\ell \neq p$) or $G/N \cong C_p$. The above numerology is required for the second case.

Some references

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