UNIVERSIDAD DE BUENOS AIRES

Facultad de Ciencias Exactas y Naturales

Departamento de Matemática

Correspondencia de Dold-Kan para anillos

por José Luis Castiglioni

Director de Tesis: Guillermo H. Cortiñas

Lugar de trabajo: Departamento de Matemática, FCEyN, UBA

Trabajo de Tesis para optar al título de Doctor en Ciencias Matemáticas

Agosto 2003

ABSTRACT

The (dual) Dold-Kan correspondence says that there is an equivalence of categories $K : \mathrm{Ch}^{\geq 0} \to \mathfrak{Ab}^{\Delta}$ between nonnegatively graded cochain complexes and cosimplicial abelian groups, which is inverse to the normalization functor. We show that the restriction of K to DG-rings can be equipped with an associative product and that the resulting functor $DGR^* \to \mathrm{Rings}^{\Delta}$, although not itself an equivalence, does induce one at the level of homotopy categories. In other words both DGR^* and Rings^{Δ} are Quillen closed model categories and the total left derived functor of K is an equivalence:

$$\mathbb{L}K : \mathrm{Ho}\, DGR^* \xrightarrow{\sim} \mathrm{Ho}\,\mathrm{Rings}^{\Delta}$$

The dual of this result for chain DG and simplicial rings was obtained independently by S. Schwede and B. Shipley through different methods (*Equivalences of monoidal model categories.* Algebraic and Geometric Topology 3 (2003), 287-334). Our proof is based on a functor $Q : DGR^* \to \mathrm{Rings}^{\Delta}$, naturally homotopy equivalent to K, and which preserves the closed model structure. It also has other interesting applications. For example, we use Q to prove a noncommutative version of the Hochschild-Konstant-Rosenberg and Loday-Quillen theorems. Our version applies to the cyclic module $[n] \mapsto \coprod_R^n S$ that arises from a homomorphism $R \to S$ of not necessarily commutative rings, using the coproduct $\coprod_R$ of associative R-algebras. As another application of the properties of Q, we obtain a simple, braid-free description of a product on the tensor power $S^{\otimes_R^n}$ originally defined by P. Nuss using braids (*Noncommutative descent and nonabelian cohomology,* K-theory **12** (1997) 23-74.).

Keywords: noncommutative differential forms, cosimplicial rings, model category.

RESUMEN

La correspondencia (dual) de Dold-Kan establece que hay una equivalencia de categorías $K : \mathrm{Ch}^{\geq 0} \to \mathfrak{Ab}^{\Delta}$ entre los complejos de cocadenas no negativamente graduados y los grupos abelianos cosimpliciales, que es inverso del funtor normalización. Mostramos que la restricción de K a DGR^*, la categoría de anillos diferenciales graduados con diferencial de grado $+1$, o anillos diferenciales de cocadenas, se puede equipar con un producto asociativo, y que el funtor resultante $DGR^* \to \mathrm{Rings}^{\Delta}$, si bien no es una equivalencia, induce una a nivel de categorías de homotopía. Es decir, tanto DGR^* como Rings^{Δ} son categorías de modelo cerrado de Quillen y el funtor derivado total a izquierda de K es una equivalencia:

$$\mathbb{L}K : \mathrm{Ho}\, DGR^* \xrightarrow{\sim} \mathrm{Ho}\,\mathrm{Rings}^{\Delta}$$

El dual de este resultado para anillos diferenciales de cadenas y anillos simpliciales fue obtenido, de forma independiente, por S. Schwede and B. Shipley mediante métodos diferentes (*Equivalences of monoidal model categories.* Algebraic and Geometric Topology 3 (2003), 287-334). Nuestra demostración está basada en un funtor $Q : DGR^* \to \mathrm{Rings}^{\Delta}$, naturalmente homotópicamente equivalente a K, y que preserva la estructura de modelo cerrado. Este funtor tiene otras aplicaciones interesantes. Por ejemplo, usamos Q para probar una versión no conmutativa de los teoremas de Hochschild-Konstant-Rosenberg y Loday-Quillen. Nuestra versión se aplica al módulo cíclico $[n] \mapsto \coprod_R^n S$ que se obtiene a partir de un homomorfismo de anillos no necesariamente conmutativos $R \to S$, usando el coproducto $\coprod_R$. También como aplicación de las propiedades de Q obtenemos una descripción sencilla, que no involucra trenzas, de un producto en la potencia tensorial $S^{\otimes_R^n}$, definido originalmente por P. Nuss, utilizando trenzas (*Noncommutative descent and nonabelian cohomology,* K-theory **12** (1997) 23-74.).

Palabras clave: formas diferenciales no commutativas, anillos cosimpliciales, categoría de modelos.

UNIVERSIDAD DE BUENOS AIRES

Facultad de Ciencias Exactas y Naturales

Departamento de Matemática

Dold-Kan correspondence for rings.

by José Luis Castiglioni

Advisor: Guillermo H. Cortiñas

Academic unit: Departamento de Matemática, FCEyN, UBA

Thesis submitted in partial fulfilment of the requirements for the degree of
Doctor en Ciencias Matemáticas
August 2003

To Patricia and Loreto,
without whom I would have finished
much sooner.

ACKNOWLEDGEMENTS

I wish to express my thankfulness to Guillermo Cortiñas for accepting to be my PhD advisor. His guidance and support were not limited to this work but also made me a better mathematician. I also wish to thank the University of Buenos Aires for the financial support provided through an Open Fellowship, as part of FOMEC programme, and specially the people of the Mathematics Department of the Science School (FCEyN) for receiving me as one of them. Finally, I wish to thank my family, and particularly my wife, for their understanding and containment during this time.

TABLE OF CONTENTS

Chapter 1

Introduction

The (dual) Dold-Kan correspondence is an equivalence between the category $\mathrm{Ch}^{\geq 0}$ of non-negatively graded cochain complexes of abelian groups and the category $\mathfrak{Ab}^{\Delta}$ of cosimplicial abelian groups. This equivalence is defined by a pair of inverse functors

$$N : \mathfrak{Ab}^{\Delta} \leftrightarrows \mathrm{Ch}^{\geq 0} : \mathrm{K} \tag{1}$$

Here N is the normalized or Moore complex (see (24) below). The functor K is described in [15], 8.4.4; if $A = (A, d) \in \mathrm{Ch}^{\geq 0}$ and $n \geq 0$, then

$$K^n A = \bigoplus_{i=0}^{n} \binom{n}{i} A^i \cong \bigoplus_{i=0}^{n} A^i \otimes \Lambda^i \mathbb{Z}^n \tag{2}$$

If in addition A happens to be a DG-ring, then $K^n A$ can be equipped with a product, namely that coming from the tensor product of rings $A \otimes \Lambda \mathbb{Z}^n$:

$$(a \otimes x)(b \otimes y) = ab \otimes x \wedge y. \tag{3}$$

This product actually makes $[n] \mapsto K^n A$ into a cosimplicial ring (see section 2.4 below). Thus K can be viewed as a functor from DG- to cosimplicial rings:

$$K : DGR^* \to \mathrm{Rings}^{\Delta}, \qquad A \mapsto KA \tag{4}$$

Note that for all n, $K^n A$ is a nilpotent extension of A^0. As there are cosimplicial rings which are not codimensionwise nilpotent extensions of constant cosimplicial rings, $A \mapsto KA$ is not a category equivalence. However we prove (Thm. 4.13) that it induces one upon inverting weak equivalences. Precisely, K carries quasi-isomorphisms to maps inducing an isomorphism at the cohomotopy level, and therefore induces a functor $\mathbb{L}K$ between the localizations $\mathrm{Ho}\, DGR^*$ and $\mathrm{Ho}\, \mathrm{Rings}^{\Delta}$ obtained by formally inverting such maps, and we prove that $\mathbb{L}K$ is an equivalence:

$$\mathbb{L}K : \mathrm{Ho}\, DGR^* \xrightarrow{\sim} \mathrm{Ho}\, \mathrm{Rings}^{\Delta} \tag{5}$$

The dual of this result, that is, the equivalence between the homotopy categories of chain DG and simplicial rings, was obtained independently by Schwede and Shipley through different methods (see [13] and also Remark 4.4 below).

To prove (5) we use Quillen's formalism of closed model categories [12]. We give each of DGR^* and Rings^{Δ} a closed model structure, in which weak equivalences are as above, fibrations are surjective maps and cofibrations are appropiately defined to fit Quillen's axioms. There is a technical problem in that the functor K does not preserve cofibrations. To get around this, we replace K by a certain functor Q. As is the case of the Dold-Kan functor, Q too is defined for all cochain complexes A, even if they may not be DG-rings. If $A \in \mathrm{Ch}^{\geq 0}$ then

$$Q^n A = \bigoplus_{i=0}^{\infty} A^i \otimes T^i(\mathbb{Z}^n) \tag{6}$$

We show that any set map $\alpha : [n] \to [m]$ induces a group homomorphism $Q^n A \to Q^m A$, so that $[n] \mapsto Q^n A$ is not only a functor on Δ but on the larger category $\mathfrak{Fin}$ with the same objects, where a homomorphism $[n] \to [m]$ is just any set map. The projection $T\mathbb{Z}^n \to \Lambda\mathbb{Z}^n$ induces a homomorphism

$$\hat{p} : QA \xrightarrow{\sim} KA \tag{7}$$

We show $\hat{p}$ induces an isomorphism of cohomotopy groups. If moreover A is a DG-ring, $Q^n A$ has an obvious product coming from $A \otimes T\mathbb{Z}^n$; however this product is not well-behaved with respect to the $\mathfrak{Fin}$ nor the cosimplicial structure. In order to get a $\mathfrak{Fin}$-ring we perturb the product by a Hochschild 2-cocycle $f : A^* \otimes T^* V \to A^{*+1} \otimes T^{*+1} V$. We obtain a product $\circ$ of the form

$$(a \otimes x) \circ (b \otimes y) = ab \otimes xy + f(a \otimes x, b \otimes y) \tag{8}$$

For a definition of f see (46) below. It turns out that the map $\hat{p}$ is a ring homomorphism (see Section 2.4). This implies that the derived functors of K and of the functor $\tilde{Q}$ obtained from Q by restriction of its $\mathfrak{Fin}$-structure to a cosimplicial one, are isomorphic (see 4.3):

$$\mathbb{L}\tilde{Q} \cong \mathbb{L}K. \tag{9}$$

We show further that $\tilde{Q}$ preserves all the closed model structure (4.3) and that its derived functor is an equivalence (4.12).

Next we review other results obtained in this thesis. As mentioned above, for $A \in \mathrm{Ch}^{\geq 0}$, QA is not only a cosimplicial group but a $\mathfrak{Fin}$-group. In particular the cyclic permutation $t_n := (0 \dots n) : [n] \to [n]$ acts on $Q^n A$, and we may view QA as a cyclic module in the sense of [15], 9.6.1. Consider the associated normalized mixed complex (NQA, μ, B). We show that there is a weak equivalence of mixed complexes

$$(A, 0, d) \xrightarrow{\sim} (NQA, \mu, B) \tag{10}$$

In particular these two mixed complexes have the same Hochschild homology:

$$A^* \cong H_*(NQA, \mu) \tag{11}$$

If A happens to be a DG-ring then the shuffle product induces a graded ring structure on $H_*(NQA, \mu)$; we show in 2.5.1 that (11) is a ring isomorphism for the product of A and the shuffle product of $H_*(NQA, \mu)$.

A specially interesting case is that of the DG-ring of noncommutative differential forms $\Omega_R S$ relative to a ring homomorphism $R \to S$ (as defined in [3]). We show in 3.1.5 that $Q\Omega_R S$ is the coproduct $\mathfrak{Fin}$-ring:

$$Q\Omega_R S : [n] \mapsto \coprod_{i=0}^{n} {}_R S \tag{12}$$

In particular, by (11), there is an isomorphism of graded rings

$$\Omega_R S \xrightarrow{\cong} H_*(\textstyle\coprod_R S, \mu) \tag{13}$$

The particular case of (13) when R is commutative and $R \to S$ is central and flat was proved in 1994 by Guccione, Guccione and Majadas [6]. More generally, by (10) we have a mixed complex equivalence

$$(\Omega_R S, 0, d) \xrightarrow{\sim} (\textstyle\coprod_R S, \mu, B) \tag{14}$$

We view (13) and (14) as noncommutative versions of the Hochschild-Konstant-Rosenberg and Loday-Quillen theorems [15] 9.4.13, 9.8.7.

As another application, we give a simple formulation for a product structure defined by Nuss [11] on each term of the Amitsur complex associated to a homomorphism $R \to S$ of not necessarily commutative rings R and S:

$$\bigotimes_R S : [n] \mapsto \bigotimes_{i=0}^{n}{}_R S \tag{15}$$

Nuss constructs his product using tools from the theory of quantum groups. We show here (see Section 3.2) that the canonical Dold-Kan isomorphism maps the product (3) to that defined by Nuss. Thus

$$K\Omega_R S = KN(\bigotimes_R S) \cong \bigotimes_R S \tag{16}$$

is an isomorphism of cosimplicial rings.

The remainder of this thesis is organized as follows. Basic notations are fixed in Section 2.1. In Section 2.2 the functor Q is defined. The homotopy equivalence of the cosimplicial groups KA and QA as well as that of the mixed complexes (10) are proved in Section 2.3. The product structure of QA for a DG-ring A is introduced in Section 2.4. The graded ring isomorphism (11) is proved in Section 2.5. The isomorphism (12) and its corollaries (13) and (14) are proved in Section 3.1. The reformulation of Nuss' product is the subject of Section 3.2. The main theorem establishing the equivalence of homotopy categories (5) is the last result of Chapter 4. This chapter also contains a general technical result concerning functors between closed model categories which may be of use elsewhere (4.11). Some computations of facts stated in Chapters 2 and 3 are left to Appendix A, in order to ease the reading of these chapters. Appendix B contains a few basic facts about noncommutative differential forms which are used in Chapter 3.

Chapter 2

The functor Q

In this chapter we introduce our main tool, which is the functor Q, and prove some of its properties. This is a functor from the category of cochain complexes to $\mathfrak{Fin}$-abelian groups. The definition of the latter as well as other basic facts are recalled in section 2.1 below; Q is defined in section 2.2. Any $\mathfrak{Fin}$-abelian group gives rise to a mixed complex; we show in section 2.3 that if $A = (A, d)$ is a cochain complex then the mixed complex associated to QA is equivalent to $(A, 0, d)$. In section 2.4, for a cochain DG-ring A, we endow QA with a $\mathfrak{Fin}$-ring structure. Finally, we find in section 2.5 a left inverse for $Q : \text{DG-Rings} \to \text{Rings}^{\mathfrak{Fin}}$.

2.1 Cochain complexes and cosimplicial abelian groups

We write Δ for the simplicial category, and $\mathfrak{Fin}$ for the category with the same objects as Δ, but where the homomorphisms $[n] \to [m]$ are just the set maps. The inclusion

$$\hom_\Delta([n],[m]) \subset \mathrm{Map}([n],[m]) = \hom_{\mathfrak{Fin}}([n],[m])$$

gives a faithful embedding $\Delta \subset \mathfrak{Fin}$. If I and $\mathfrak{C}$ are categories, we shall write $\mathfrak{C}^I$ to denote the category of functors $I \to \mathfrak{C}$, to which we refer as I-objects of $\mathfrak{C}$. If $C : I \to \mathfrak{C}$ is an I-object, we write C^i for $C(i)$. We use the same letter for a map $\alpha : [n] \to [m] \in I$ as for its image under C. The canonical embedding $\Delta \subset \mathfrak{Fin}$ mentioned above makes $[n] \mapsto [n]$ into a cosimplicial object of $\mathfrak{Fin}$. We write $\partial_i : [n] \to [n+1]$, $i = 0, \dots, n+1$ and $\mu_j : [n] \to [n-1]$, $j = 0, \dots, n-1$, for the coface and codegeneracy maps. We also consider the map $\mu_n : [n] \to [n-1]$ defined by

$$\mu_n(i) = \begin{cases} i & \text{if} \quad i < n \\ 0 & \text{if} \quad i = n \end{cases} \tag{17}$$

One checks that $d_i := \mu_i : [n] \to [n-1]$ $i = 0, \dots, n$ and $s_j = \partial_{j+1} : [n] \to [n+1]$, $j = 0, \dots, n$ satisfy the simplicial identities, with the d_i as faces and the s_i as degeneracies. Thus there is a functor $\Delta^{op} \to \mathfrak{Fin}$, $[n] \mapsto [n]$. Moreover the cyclic permutation $t_n = (0 \dots n) : [n] \to [n]$ extends this simplicial structure to a cyclic one (see [15], 9.6.3). Composing with these functors and with the inclusion $\Delta \subset \mathfrak{Fin}$ mentioned above we have a canonical way of regarding any $\mathfrak{Fin}$-object in a category $\mathfrak{C}$ as either a cosimplicial, a simplicial, or a cyclic object.

If $\mathfrak{C}$ is a category with finite coproducts, and $A \in \mathfrak{C}$, we write $\coprod A$ for the functor

$$\coprod A : \mathfrak{Fin} \to \mathfrak{C}, \qquad [n] \mapsto \coprod_{i=0}^{n} A \tag{18}$$

Here $\coprod$ may be replaced by whatever sign denotes the coproduct of $\mathfrak{C}$; for example if $\mathfrak{C}$ is abelian, we write $\oplus A$ for $\coprod A$.

If $A = \oplus_{n=0}^{\infty} A_n$ and $B = \oplus_{n=0}^{\infty} B_n$ are graded abelian groups, we write

$$A \boxtimes B := \oplus_{n=0}^{\infty} A_n \otimes B_n \tag{19}$$

If A, B are graded I-abelian groups, we put $A \boxtimes B$ for the graded I-abelian group $i \mapsto A^i \boxtimes B^i$.

2.2 The functor Q

We are going to define a functor $Q : Ch^{\geq 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$; first we need some auxiliary constructions. Write $V := \ker(\oplus\mathbb{Z} \to \mathbb{Z})$ for the kernel of the canonical map to the constant $\mathfrak{Fin}$-abelian group, and $\{e_i : 0 \leq i \leq n\}$ for the canonical basis of $\oplus_{i=0}^{n}\mathbb{Z}$. Put $v_i = e_i - e_0$, $0 \leq i \leq n$. Note $v_0 = 0$ and $\{v_1, \dots, v_n\}$ is a basis of V^n. The action of a map $\alpha : [n] \to [m] \in \mathfrak{Fin}$ on V is given by

$$\alpha v_i = v_{\alpha(i)} - v_{\alpha(0)} \qquad (0 \leq i \leq n) \tag{20}$$

Applying to V the tensor algebra functor T in each codimension yields a graded $\mathfrak{Fin}$-ring TV. If $A = (A, d) \in \mathrm{Ch}^{\geq 0}$, we put

$$Q^n A := A \boxtimes TV^n \tag{21}$$

If $\alpha : [n] \to [m] \in \mathfrak{Fin}$, we set

$$\alpha(a \otimes x) = a \otimes \alpha x + da \otimes v_{\alpha(0)}\alpha x \tag{22}$$

If $\beta : [m] \to [p] \in \mathfrak{Fin}$, then

$$\begin{aligned}\beta(\alpha(a \otimes x)) =& a \otimes \beta\alpha x + da \otimes v_{\beta(0)}\beta\alpha x + da \otimes \beta(v_{\alpha(0)})\beta\alpha x\\ =&(\beta\alpha)(a \otimes x)\end{aligned}$$

Thus QA is a $\mathfrak{Fin}$-abelian group, and $Q : \mathrm{Ch}^{\geq 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$ a functor. We have a filtration on QA by $\mathfrak{Fin}$-subgroups, given by

$$\mathcal{F}_n QA = \bigoplus_{i=n}^{\infty} A^i \otimes T^i V \tag{23}$$

The associated graded $\mathfrak{Fin}$-abelian group is $G_{\mathcal{F}} QA = A \boxtimes TV$.

2.3 Comparison between Q and the Dold-Kan functor K

The Dold-Kan correspondence is a pair of inverse functors (see [15] 8.4):

$$K : \mathrm{Ch}^{\geq 0} \leftrightarrows \mathfrak{Ab}^{\Delta} : \mathrm{N}$$

If $C \in \mathfrak{Ab}^{\Delta}$ then NC can be equivalently described as the normalized complex or as the Moore complex:

$$N^n C = C^n / \sum_{i=1}^{n} \partial_i C^{n-1} \cong \cap_{i=0}^{n-1} \ker(\mu_i : C^n \to C^{n-1}) \tag{24}$$

In either version the coboundary map $N^n C \to N^{n+1} C$ is induced by

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i. \tag{25}$$

In the first version this is the same map as that induced by ∂_0. A description of the inverse functor K (in the simplicial case) is given in [15], 8.4.4; and another in [7], 1.5. Here is yet another. Let ΛV be the exterior algebra, $p : TV \to \Lambda V$ the canonical projection. One checks that $\ker(1 \otimes p) \subset QA$ is a $\mathfrak{Fin}$-subgroup. Thus

$$K^*A := A \boxtimes \Lambda V^* \tag{26}$$

inherits a $\mathfrak{Fin}$-structure. Moreover

$$\hat{p} := 1 \otimes p : QA \twoheadrightarrow KA \tag{27}$$

is a natural surjection of $\mathfrak{Fin}$-abelian groups. To see that the resulting cosimplicial abelian group KA is indeed the same as (i.e. is naturally isomorphic to) that of [15], it suffices to show that $NKA = A$. Put

$$V_j^n = \bigoplus_{i \neq j} \mathbb{Z} v_i \subset \bigoplus_{i=1}^{n} \mathbb{Z} v_i = V^n$$

We have

$$\begin{aligned} NK^nA =& A \boxtimes \Lambda V^n / \sum_{i=1}^{n} A \boxtimes \partial_i(\Lambda V^n) \\ =& A \boxtimes (\Lambda V^n / \sum_{i=1}^{n} \Lambda(V_i^n)) \\ =& A^n \otimes v_1 \wedge \cdots \wedge v_n \cong A^n \end{aligned}$$

Furthermore it is clear that the coboundary map induced by ∂_0 is $d : A^* \to A^{*+1}$. Thus our KA is the same cosimplicial abelian group as that of [15]. But since in our construction KA has a $\mathfrak{Fin}$-structure, we may also regard it as a simplicial or cyclic abelian group. From our definition of faces and degeneracies, it is clear that the normalized complex of KA considered as a simplicial group has the abelian group $N^nKA = A^n$ in each dimension. The alternating sum μ of the faces induces the trivial boundary (see lemma A.1). Thus the normalized chain complex of the simplicial group KA is $(A, 0)$. Consider the Connes operator $B : NQ^*A \to NQ^{*+1}A$,

$$B = \partial_0 \circ \sum_{i=0}^{n} (-1)^{ni} t_n^i \tag{28}$$

We show in 2.3.2 below that $\hat{p}B = D\hat{p}$, where $D := (n+1)d$ on A^n. Hence we have a map of mixed complexes

$$\hat{p} : (NQA, \mu, B) \to (A, 0, D). \tag{29}$$

We shall see in 2.3.2 below that (29) is a *rational equivalence of mixed complexes.* We recall that a map of mixed complexes is an equivalence if it induces an isomorphism at the level of Hochschild homology; this automatically implies it also induces an isomorphism at the level of cyclic, periodic cyclic and negative cyclic homologies. In 2.3.2 we also consider the map

$$l : A \to NQA, \qquad l(a) = a \otimes \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) v_{\sigma 1} \ldots v_{\sigma n} \tag{30}$$

We show in Thm. 2.3.2 below that l is an integral equivalence $(A, 0, d) \to (NQA, \mu, B)$.

Remark 2.3.1. Note that if A is a complex of $\mathbb{Q}$-vectorspaces, then $\hat{p}$ can be rescaled as $(1/n!)\hat{p}$ on NQ^nA to give a mixed complex map $(NQA,\mu,B)\to(A,0,d)$ which is left inverse to l.

Theorem 2.3.2. *Let A be a cochain complex of abelian groups, $\hat{p}: QA \twoheadrightarrow KA$ the map of $\mathfrak{Fin}$-abelian groups defined in (27) above. Then:*
i) There are a natural cochain map $j:(A,d)\to(NQA,\partial)$ such that $\hat{p}j=1_A$ and a natural cochain homotopy $h:N^*QA\to N^{*-1}QA$ such that $[h,\partial]=1-j\hat{p}$.
ii) The map (29) is a rational equivalence of mixed complexes. On the other hand the map (30) is a natural integral equivalence $l:(A,0,d)\to(NQA,\mu,B)$.

Proof. First we compute NQA. A similar argument as that given in section 2.3 to compute NKA, shows that

$$N^nQA = A\boxtimes(TV^n/\sum_{i=1}^{n}TV_i^n). \tag{31}$$

On the other hand we have a canonical identification between the rth tensor power of $V^n=\mathbb{Z}^n$ and the free abelian group on the set of all maps $\{1,\dots,r\}\to\{1,\dots,n\}$:

$$T^rV^n\cong\mathbb{Z}[\mathrm{Map}(\{1,\dots,r\},\{1,\dots,n\})] \tag{32}$$

Using (32), $T^rV^n/\sum_{j=1}^nT^rV_j^n$ becomes the free module on all surjective maps $\{1,\dots,r\}\to\{1,\dots,n\}$; we get

$$N^nQA = A\boxtimes\mathbb{Z}[\mathrm{Sur}_{*,n}]=\bigoplus_{r=n}^{\infty}A^r\otimes\mathbb{Z}[\mathrm{Sur}_{r,n}] \tag{33}$$

To prove i), regard NQA as a cochain complex. We may view NQA as the direct sum total complex of a second quadrant double complex

$$C^{p,q}=\begin{cases} A & \text{if } p=q=0,\\ A^q\otimes\mathbb{Z}[\mathrm{Sur}_{q,q+p}] & \text{if } (p,q)\neq(0,0).\end{cases}$$

Here $1\otimes\partial_0$ and $d\otimes v_1\partial_0$ are respectively the horizontal and the vertical coboundary operators. The filtration (23) is the row filtration. If we regard $A=NKA$ as a double cochain complex concentrated in the zero column, then $\hat{p}$ becomes a map of double complexes. By definition, $\hat{p}=1\otimes p$; at the nth row, p is a map:

$$p:\mathbb{Z}[\mathrm{Sur}_{n,*}]\twoheadrightarrow\mathbb{Z}[n]. \tag{34}$$

The only nonzero component of p is $p(\sigma)=\mathrm{sign}(\sigma)$. We claim (34) is a cochain homotopy equivalence. To prove this note first that because both $\mathbb{Z}[\mathrm{Sur}_{n,*}]$ and $\mathbb{Z}[n]$ are complexes of free abelian groups, to show p is a homotopy equivalence it suffices to check it is a quasi-isomorphism. Next note that

$$\begin{aligned} H^*(\mathbb{Z}[\mathrm{Sur}_{n,*}]) =& H^*(NT^nV)=\pi^*(T^nV)\\ =& T^n\pi^*(V)\\ =& T^nH^*(NV)\\ =& T^nH^*(\mathbb{Z}[1])=\mathbb{Z}[n]. \end{aligned} \tag{35}$$

Thus, to prove p is a cochain equivalence it suffices to show that

$$\ker(p:\mathbb{Z}[S_n]\to\mathbb{Z})=\partial_0(\mathbb{Z}[\mathrm{Sur}_{n,n-1}]) \tag{36}$$

The inclusion $\supset$ of (36) holds because p is a cochain map. To prove the other inclusion, proceed as follows. First note the identification

$$\mathbb{Z}[S_n]\cong\bigoplus_{\sigma\in S_n}\mathbb{Z}v_{\sigma 1}\dots v_{\sigma n}$$

Next observe that the kernel of p is generated by elements of the form

$$\dots v_1\dots v_i\dots+\dots v_i\dots v_1\dots\equiv-\partial_0(\dots v_{i-1}\dots v_{i-1}\dots)\qquad(i>1)$$

Here congruence is taken modulo $\sum_{j\geq 1}\partial_j TV$. Thus p is a surjective homotopy equivalence, as claimed. Therefore we may choose a cochain map $j':\mathbb{Z}[n]\to\mathbb{Z}[\mathrm{Sur}_{n,*}]$ such that $pj'=1$ and a cochain homotopy $h':\mathbb{Z}[\mathrm{Sur}_{n,*}]\to\mathbb{Z}[\mathrm{Sur}_{n,*-1}]$ such that $[h',\partial_0]=1-pj'$. One checks (see A.2) that the following maps satisfy the requirements of part i) of the theorem:

$$\begin{aligned}j:=&1\otimes j'+(1\otimes h')((1\otimes j')d-d\otimes v_1\partial_0 j')\\ h:=&(1\otimes h'-(1\otimes h')(d\otimes v_1\partial_0)(1\otimes h'))(1\otimes j'p-1)\end{aligned}$$

Next we prove part ii). Observe the face maps of NQA are of the form $1\otimes\mu_i$ where μ_i is the face map in TV. Hence we have a direct sum decomposition of chain complexes

$$(NQA,\mu)=\bigoplus_{n=0}^{\infty}A^n\otimes(\mathbb{Z}[\mathrm{Sur}_{n,*}],\mu) \tag{37}$$

The homology version of the argument used in (35) shows that

$$H_*(\mathbb{Z}[\mathrm{Sur}_{n,*}])=\mathbb{Z}[n].$$

In particular $L_n:=\ker(\mu:\mathbb{Z}[S_n]\to\mathbb{Z}[\mathrm{Sur}_{n,n-1}])$ is free of rank one. By definition, to prove $\hat{p}$ is a rational mixed complex equivalence, we must prove that $\hat{p}\mu=0$, which is straighforward, that $\hat{p}B=D\hat{p}$, which we leave for later, and finally that $\hat{p}=1\otimes p:(NQA,\mu)\to(A,0)$ is a rational chain equivalence, which in turn reduces to proving $p(L_n)\neq 0$ for $n\geq 1$. Consider the element

$$\epsilon_n:=\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\sigma\in\mathbb{Z}[S_n] \tag{38}$$

We have $p(\epsilon_n)=n!$; one checks further that $\epsilon_n\in L_n$. It follows that $\hat{p}:(NQA,\mu)\to(A,0)$ is a rational equivalence, as we had to prove. Moreover, as every coefficient of ϵ_n is invertible, and L_n has rank one, we have $L_n=\mathbb{Z}\epsilon_n$. It follows that the map $l':\mathbb{Z}[n]\to(\mathbb{Z}[\mathrm{Sur}_{n,*}],\mu)$ which sends $1\in\mathbb{Z}$ to ϵ_n is a quasi-isomorphism, whence a homotopy equivalence. To finish the proof, we must show that $ld=Bl$ and $\hat{p}B=D\hat{p}$. Both of these follow once one has proven the formula (39) below, which in turn is derived from the identities (40), which are proved by induction (see A.3 for details). If $\sigma\in S_n$, we denote by $1\coprod\sigma$ the coproduct map $\{1\}\coprod\{1,\dots,n\}=\{1,\dots,n+1\}\to\{1,\dots,n+1\}$.

$$B(a\otimes\sigma)=da\otimes\sum_{i=0}^{n}(-1)^{in}(1\dots n+1)^i(1\coprod\sigma) \tag{39}$$

$$
\begin{aligned}
t^i_n(v_j) =& \begin{cases} v_{i+j} - v_i & \text{if } i \leq n-j \\ v_{p-1} - v_i & \text{if } i = n-j+p \quad j \geq p \geq 1 \end{cases} \\
t^i_n(a \otimes x) =& a \otimes t^i_n x + da \otimes v_i t^i_n x \\
B(a \otimes x) =& da \otimes \sum_{i=0}^{n} (-1)^{in} v_{i+1} \partial_0 t^i x
\end{aligned}
\tag{40}
$$

□

Notation 2.3.3. Let $B = (B, d) \in \mathrm{Ch}^{\geq 0}$. Put $PB^n = B^n \oplus B^{n-1} \oplus B^n$. Equip PB with the coboundary operator $\partial : PB^* \to PB^{*+1}$ given by the matrix

$$
\partial = \begin{bmatrix} d & 0 & 0 \\ 1 & -d & -1 \\ 0 & 0 & d \end{bmatrix}
$$

We note PB comes equipped with a natural map $\epsilon = (\epsilon_0, \epsilon_1) : PB \to B \oplus B$, and that two maps $f_0, f_1 : A \to B$ are cochain homotopic if and only if there exists a cochain homomorphism $H : A \to PB$ such that $\epsilon H = (f_0, f_1)$.

Corollary 2.3.4. *Let $A, B \in \mathrm{Ch}^{\geq 0}$. Consider the functors*

$$
\begin{aligned}
(\mathrm{Ch}^{\geq 0})^{op} \times \mathrm{Ch}^{\geq 0} &\to \mathfrak{Ab} \\
(A, B) &\mapsto \hom_{\mathfrak{Ab}^\Delta}(QA, QB) \\
(A, B) &\mapsto \hom_{\mathrm{Ch}^{\geq 0}}(NQA, PNQB).
\end{aligned}
$$

There are two natural transformations

$$
\begin{aligned}
\bar{\ } : \hom_{\mathfrak{Ab}^\Delta}(QA, QB) &\to \hom_{\mathrm{Ch}^{\geq 0}}(A, B), \\
H : \hom_{\mathfrak{Ab}^\Delta}(QA, QB) &\to \hom_{\mathrm{Ch}^{\geq 0}}(NQA, PNQB).
\end{aligned}
$$

These are such that $\overline{Qg} = g$ and that the following diagram commutes

$$
\begin{tikzcd}
& \hom_{\mathrm{Ch}^{\geq 0}}(NQA, PNQB) \arrow[d, "\epsilon"] \\
\hom_{\mathfrak{Ab}^\Delta}(QA, QB) \arrow[ur, "H"] \arrow[r] & \hom_{\mathrm{Ch}^{\geq 0}}(NQA, NQB \oplus NQB) \\
f \arrow[r, mapsto] & (Nf, NQ\bar{f})
\end{tikzcd}
\tag{41}
$$

Proof. Let $f \in \hom_{\mathfrak{Ab}^\Delta}(QA, QB)$ and $j, \hat{p}$ and h be as in the theorem. Define $\bar{f} := \hat{p}N(f)j$. Because $\hat{p}j = 1$, $\overline{Qg} = g$. Using the naturality of j and $\hat{p}$, one checks further that $f \mapsto \bar{f}$ is natural. Let $\delta = N(f) - NQ(\bar{f})$ and put

$$
\kappa = \kappa_f := h\delta + \delta h - [h\delta, \partial]h
$$

One checks that $[\kappa, \partial] = \delta$, whence $H_f := (Nf, \kappa, NQ\bar{f})$ is a homomorphism $NQA \to PNQB$ with $\epsilon H_f = (Nf, NQ\bar{f})$. The naturality of $H : f \mapsto H_f$ follows from that of h. □

Simplicial powers and cosimplicial homotopies 2.3.5. Let $A \in \mathfrak{Ab}^{\Delta}$, $X \in \mathrm{Sets}^{\Delta^{op}}$. Put

$$(A^X)^n := \coprod_{x \in X_n} A^n \tag{42}$$

If $\alpha \in \hom_{\Delta}([n],[m])$ and $a \in (A^X)^n$, define $\alpha(a)_x = \alpha(a_{\alpha x})$ $(x \in X_m)$. The dual $\mathbb{Z}[X]^{\vee} : [n] \mapsto \hom_{\mathbb{Z}}(\mathbb{Z}[X_n],\mathbb{Z})$ of the simplicial free abelian group $\mathbb{Z}[X]$ is a cosimplicial group. Consider the cosimplicial tensor product $A \times \mathbb{Z}[X]^{\vee} : [n] \mapsto A^n \otimes \mathbb{Z}[X_n]^{\vee}$. There is a natural homomorphism

$$\eta : A \times \mathbb{Z}[X]^{\vee} \to A^X, \qquad \eta(a \otimes \phi)_x = a\phi(x) \tag{43}$$

In case each X_n is finite, η is an isomorphism. Dualizing the statement in [10] –next after 8.9– we get that the composite of the normalized shuffle map $NA \otimes N\mathbb{Z}[X]^{\vee} \to N(A \times \mathbb{Z}[X]^{\vee})$ with Alexander-Whitney map $N(A \times \mathbb{Z}[X]^{\vee}) \to NA \otimes N\mathbb{Z}[X]^{\vee}$ is the identity. Thus $NA \otimes N\mathbb{Z}[X]^{\vee}$ is a deformation retract of $N(A \times \mathbb{Z}[X]^{\vee})$. In particular $PNA = NA \otimes N\mathbb{Z}[\Delta[1]]$ is a deformation retract of $N(A^{\Delta[1]})$. Recall two cosimplicial maps $f_0, f_1 : A \to B$ are called *homotopic* if $(f_0, f_1) : A \to B \times B = B^{\Delta[0] \coprod \Delta[0]}$ can be lifted to a map $H : A \to B^{\Delta[1]}$. From what we have just seen it is clear that f_0, f_1 are homotopic in this sense if and only if Nf_0,Nf_1 are cochain homotopic. (The dual of this assertion is proved in [5].) Let $\mathfrak{C}$ be either of $\mathrm{Ch}^{\geq 0}$, $\mathfrak{Ab}^{\Delta}$. We write $[\mathfrak{C}]$ for the category with the same objects as $\mathfrak{C}$, but where the homomorphisms are the homotopy classes of maps in $\mathfrak{C}$.

Proposition 2.3.6. *The functor Q induces an equivalence of categories* $[\mathrm{Ch}^{\geq 0}] \to [\mathfrak{Ab}^{\Delta}]$.

Proof. If $A \in \mathfrak{Ab}^{\Delta}$, then $A = KNA$. By Thm. 2.3.2, NA is homotopy equivalent to NQA. Thus A is homotopy equivalent to $KNQA = QA$. It remains to show that the following map is a bijection

$$[Q] : \hom_{[\mathrm{Ch}^{\geq 0}]}(A,B) \to \hom_{[\mathfrak{Ab}^{\Delta}]}(QA,QB).$$

It is clear from the previous corollary that the composite of Q with

$$[N] : \hom_{[\mathfrak{Ab}^{\Delta}]}(QA,QB) \to \hom_{[\mathrm{Ch}^{\geq 0}]}(NQA,NQB) \tag{44}$$

is a bijection. But (44) is bijective by 2.3.5. $\square$

Definition 2.3.7. Give $\mathrm{Ch}^{\geq 0}$ the closed model category structure in which a map is a *fibration* if it is surjective dimensionwise, a *weak equivalence* if it is a quasi-isomorphism, and a cofibration if it has Quillen's left lifting property (*LLP*, see [12]) with respect to those fibrations which are also weak equivalences (*trivial fibrations*). All this structure carries over to $\mathfrak{Ab}^{\Delta}$ using the category equivalence $N : \mathfrak{Ab}^{\Delta} \to \mathrm{Ch}^{\geq 0}$. In the lemma below *RLP* stands for right lifting property in the sense of [12].

Notation 2.3.8. In the next lemma and further below, we use the following notation. If $n \geq 0$, we write $\mathbb{Z} < n, n+1 >$ for the mapping cone of the identity map $\mathbb{Z}[n] \to \mathbb{Z}[n]$.

Lemma 2.3.9. *Let $f : E \to B$ be a homomorphism of cosimplicial abelian groups. We have:*

i) f is a fibration if and only if for all $n \geq 1$ f has the RLP with respect to $0 \to Q\mathbb{Z} < n-1, n >$.

ii) f is a trivial fibration if and only if for all $n \geq 1$ f has the RLP with respect to the natural inclusion $Q\mathbb{Z}[n] \hookrightarrow Q\mathbb{Z} < n-1, n >$.

Proof. Let $f : C \to D$ be a cochain map. By the theorem, Kf is a retract of Qf. Thus every map having the RLP with respect to Qf also has it with respect to Kf. The lemma follows from this applied to the cochain maps $0 \to \mathbb{Z} < n-1, n >$ and $\mathbb{Z}[n] \hookrightarrow \mathbb{Z} < n-1, n >$. $\square$

2.4 Product structure

Let $A = \oplus_{n=0}^{\infty} A^n$ be a unital cochain DG-ring. By neglecting the product, A becomes a cochain complex, and thus it makes sense to consider the $\mathfrak{Fin}$-abelian group QA. We want to equip each $Q^n A$ with a product so that QA becomes a $\mathfrak{Fin}$-ring. For this we need the map $\theta : TV^* \to TV^*$,

$$\theta(v_i) = v_i^2, \qquad \theta(xy) = \theta(x)y + (-1)^{|x|} x\theta(y). \tag{45}$$

The second identity says θ is a homogeneous derivation of degree $+1$. Consider the product $\circ : Q^*A \otimes Q^*A \to Q^*A$ given by

$$(\omega \otimes x) \circ (\eta \otimes y) := \omega\eta \otimes xy + (-1)^{|x|}\omega d\eta \otimes \theta(x)y \tag{46}$$

A straightforward calculation shows that $\circ$ is associative; also straightfoward although tedious is the proof that $(QA, \circ)$ is a $\mathfrak{Fin}$-ring. See the Appendix A for a proof of both these facts. Note that each term $\mathcal{F}_n QA$ of the filtration (23) is a $\mathfrak{Fin}$-ideal. The associated graded $\mathfrak{Fin}$-ring is $A \boxtimes TV$ equipped with the product inherited from $A \boxtimes TV \subset A \otimes TV$. Thus we may view QA as a deformation of $A \boxtimes TV$. One checks that the kernel of the map $\hat{p} : QA \twoheadrightarrow KA$ of (27) is an ideal for $\circ$. Hence KA inherits a $\mathfrak{Fin}$-ring structure; using the definition of θ we get that the induced product on $KA = A \boxtimes \Lambda V$ is just that coming from $A \otimes \Lambda V$:

$$(a \otimes x)(b \otimes y) = ab \otimes x \wedge y. \tag{47}$$

2.5 Comparison with the shuffle product

Let R be a simplicial ring. Consider the direct sum of its homotopy groups

$$\pi R := \bigoplus_{n=0}^{\infty} \pi_* R. \tag{48}$$

Recall that the shuffle product $*$ makes πR into a graded ring. If moreover R is a $\mathfrak{Fin}$-ring, then the Connes operator $B : \pi_* R \to \pi_{*+1} R$ is a derivation (by [8]) so that $\pi R = (\pi R, *, B)$ becomes in fact a DG-ring. Hence we have a functor

$$\mathrm{Rings}^{\mathfrak{Fin}} \to DGR^*, \qquad R \mapsto \pi R. \tag{49}$$

Proposition 2.5.1. *Let $A \in DGR^*$. Consider the natural isomorphism of graded abelian groups induced by the map l of 2.3.2 ii)*

$$l : A \xrightarrow{\sim} \pi QA. \tag{50}$$

The map (50) *is an isomorphism of DG-rings. In particular the functor* (49) *is a left inverse of Q.*

Proof. By 2.3.2, l induces a cochain isomorphism $(A, d) \cong (\pi QA, B)$. It remains to show that the induced map is a ring homomorphism. Recall the formula for the shuffle product $*$ involves degeneracies and shuffles. Keeping in mind that the degeneracies in QA are of the form $s_i = 1 \otimes \partial_{i+1}$ with ∂_j the coface of TV, we get the following identity for $a \in A^n$, $b \in A^m$:

$$\begin{aligned} l(a)l(b) =& (a \otimes \epsilon_n) * (b \otimes \epsilon_m) \\ \equiv& ab \otimes \epsilon_n * \epsilon_m \mod N\mathcal{F}_{n+m+1}Q^{n+m}A \\ =& ab \otimes \epsilon_{n+m} = l(ab) \end{aligned}$$

This finishes the proof, since $\pi_{n+m}N\mathcal{F}_{n+m+1}QA = 0$ by the proof of 2.3.2. □

Chapter 3

Two applications of Q

In this chapter we give two applications of the functor Q. The first one is a noncommutative version of the Hochschild-Konstant-Rosenberg and Loday-Quillen Theorems. This version applies to the cyclic module $[n] \mapsto \coprod_R^n S$ that arises from a ring homomorphism $R \to S$ using the coproduct $\coprod_R$. This is the main result of section 3.1.

In section 3.2 we obtain a simple, braid-free description of a product on the tensor power $S^{\otimes_R^n}$ originally defined by P. Nuss using braids.

3.1 Noncommutative Hochschild-Konstant-Rosenberg and Loday-Quillen theorems

Recall from [15] that for every algebra S over a commutative ring R there is defined a cyclic R-module $C_*(S/R)$. Recall also that the normalization of $C_*(S/R)$ when R is central in S, is the mixed complex of noncommutative differential forms [4] $NC_*(S/R) = \Omega_R S$. The Hochschild-Konstant-Rosenberg theorem ([15], Ex. 9.4.2) says that if R and S are commutative, R noetherian, and $R \to S$ an essentially of finite type, smooth homomorphism, then the canonical map from commutative differential forms to Hochschild homology induced by the shuffle product is an isomorphism:

$$\Omega_{S/R} = \Lambda HH_1(S/R) \xrightarrow{\sim} HH_*(S/R). \tag{51}$$

In characteristic zero the inverse of (51) is induced by the homorphism

$$\Omega_R S \to \Omega_{S/R} \qquad a_0 da_1 \dots da_n \mapsto \frac{1}{n!} a_0 da_1 \wedge \dots \wedge da_n \tag{52}$$

Here the boundary operators are the Hochschild boundary b on $\Omega_R S$ and the trivial boundary on $\Omega_{S/R}$. Moreover, as (52) maps B to d, it is in fact a mixed complex equivalence

$$(\Omega_R S, b, B) \xrightarrow{\sim} (\Omega_{S/R}, 0, d)$$

We will prove a noncommutative analogue of this. Note that if S is commutative then $C_*(S/R)$ is just the coproduct $\mathfrak{Fin}$-algebra $\bigotimes_R S$ considered as a cyclic module. The analogue concerns the coproduct $\mathfrak{Fin}$-ring $\coprod_R S$ which arises from a ring homomorphism $R \to S$ of not necessarily commutative rings. We show in 3.1.6 below that there is an equivalence of mixed complexes $(\Omega_R S, 0, d) \to (N \coprod_R S, \mu, B)$, valid without restrictions on the characteristic. We deduce this from 2.3.2 and from 3.1.5 below, where we show that $\coprod_R S = Q\Omega_R S$. In particular the isomorphism $\Omega_R^* S \cong HH_*(\coprod_R S, \mu, B) = \pi_* \coprod_R S$ is (50), which is a ring homomorphism for the product of forms and the shuffle product (by 2.5.1) just like the Hochschild-Kostant-Rosenberg isomorphism (51). Note further the analogy between (52) and the rescaled map $\hat{p}$ of 2.3.1.

To prove the isomorphism $Q\Omega_R S \cong \coprod_R S$ we show first that Q has a right adjoint (3.1.4).

Lemma 3.1.1. *Let $(U,d) \in \mathrm{Ch}^{\geq 0}$. Then there is a natural isomorphism of $\mathfrak{Fin}$-rings $TQU \xrightarrow{\cong} QTU$.*

Proof. The natural inclusion $QU = U \boxtimes TV \hookrightarrow TU \boxtimes TV = QTU$ gives rise to a homomorphism of $\mathfrak{Fin}$-rings $\alpha : TQU \to QTU$. If $u \in QU$ then $\alpha(u) = u$. Hence if $u_1, \dots, u_r \in Q^n U$ are homogeneous elements and $|u| = \sum_i |u_i|$, then for the filtration (23),

$$\alpha(u_1 \dots u_r) = u_1 \circ \dots \circ u_r \equiv u_1 \dots u_r \mod (\mathcal{F}_{|u|+1})$$

Hence $\alpha(T^{\geq *}QU) \subset \mathcal{F}_*$ and the map it induces at the graded level is the identity. It follows that α is an isomorphism. $\square$

Notation 3.1.2. The following DG-rings shall be considered often in what follows

$$D(n) := T\mathbb{Z} < n, n+1 > \supset S(n) := T\mathbb{Z}[n] \tag{53}$$

Corollary 3.1.3. *Let I be a set, $n_i \geq 0$. Then*

$$Q(\coprod_{i\in I} D(n_i)) = \coprod_{i \in I} QD(n_i).$$

Proof.

$$\begin{aligned} Q(\coprod_{i\in I} D(n_i)) =& Q(\coprod_{i\in I} T(\mathbb{Z} < n_i, n_{i+1} >)) = QT(\bigoplus_{i\in I} \mathbb{Z} < n_i, n_{i+1} >) \\ =& TQ(\bigoplus_{i\in I} \mathbb{Z} < n_i, n_{i+1} >) = T(\bigoplus_{i\in I} Q\mathbb{Z} < n_i, n_{i+1} >) \\ =& \coprod_{i\in I} TQ\mathbb{Z} < n_i, n_{i+1} > = \coprod_{i\in I} QD(n_i). \end{aligned}$$

$\square$

Proposition 3.1.4. *Let* Rings *be the category of associative unital rings and DGR^* that of cochain differential graded rings. The functor $Q : DGR^* \to \mathrm{Rings}^{\mathfrak{Fin}}$ has a right adjoint.*

Proof. This is an adaptation of the proof of the dual of Freyd's Special Adjoint Theorem ([9], Chap. V,§8, Thm. 2). Let $B \in \mathrm{Rings}^{\mathfrak{Fin}}$. Put

$$DB := \coprod_{n\geq 0} \coprod_{\hom(QD(n),B)} D(n) \tag{54}$$

If $s \in \hom(QD(n), B)$, write $j_s : D(n) \to DB$ for the corresponding inclusion. Define $\alpha : QDB \to B$ by $\alpha j_s = s$. Consider the two-sided $\mathfrak{Fin}$-ideal

$$DB \rhd K := \sum \{I \lhd DB : \alpha(QI) = 0\} \tag{55}$$

Set $PB := DB/K$. Because $Q : \mathrm{Ch}^{\geq 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$ is exact, we have a natural map $\hat{\alpha}$ making the following diagram commute

$$\begin{tikzcd} QDB \arrow[r, "\alpha"] \arrow[d] & B \\ QPB \arrow[ur, "\hat{\alpha}"'] & \end{tikzcd} \tag{56}$$

Hence $(PB, \hat{\alpha})$ is an object of the category $Q \uparrow B$ (notation is as in [9]). We shall see it is final, which proves that P is right adjoint to Q. Let $(R, f) \in Q \uparrow B$. Put

$$ER := \coprod_{n \geq 0} \coprod_{\hom(D(n), R)} D(n).$$

If $r : D(n) \to R$ is a homomorphism, write $i_r : D(n) \to ER$ for the corresponding inclusion. Consider the homomorphisms $\pi : ER \to R$, $\pi i_r = r$ and $g : ER \to DB$, $g i_r = j_{fQr}$. We claim that the following diagram commutes

$$\begin{tikzcd} QER \arrow[r, "Qg"] \arrow[d, "Q\pi"'] & QDB \arrow[d, "\alpha"] \\ QR \arrow[r, "f"'] & B \end{tikzcd} \tag{57}$$

Indeed by 3.1.3, commutativity can be checked at each "cell" $Q(D(n))$ where it is clear. Using (57) together with the exactness of Q, we get that $g(\ker \pi) \subset K$. Thus g induces a map $\hat{g}$ making the following diagram commute

$$\begin{tikzcd} ER \arrow[r, "g"] \arrow[d, "\pi"'] & DB \arrow[d] \\ R \arrow[r, "\hat{g}"'] & PB \end{tikzcd} \tag{58}$$

It follows that also the following commutes

$$\begin{tikzcd} QER \arrow[r, "Qg"] \arrow[d, "Q\pi"'] & QDB \arrow[dr, "\alpha"] & \\ QR \arrow[r, "Q\hat{g}"'] & QPB \arrow[r, "\hat{\alpha}"'] & B \end{tikzcd} \tag{59}$$

Putting together the latter diagram with (56) and (57) we get that $fQ(\pi) = \hat{\alpha}Q(g)Q(\pi)$. Because π is surjective and Q exact, we conclude $f = \hat{\alpha}Q(g)$; in other words $\hat{g}$ is a homomorphism $(R, f) \to (PB, \hat{\alpha})$ in $Q \uparrow B$. Let $h : (R, f) \to (PB, \hat{\alpha})$ be another. Lift $\hat{h}$ to a map $h : ER \to DB$. Then by (59),

$$\alpha Q(h) = \hat{\alpha} Q(\hat{h}\pi) = fQ(\pi) = \alpha Q(g)$$

Hence the image of $g - h$ lands in K, and therefore $\hat{g} = \hat{h}$. $\square$

Theorem 3.1.5. *Let $R \to S$ be a ring homomorphism, $R \uparrow \mathrm{Rings}$ the category of R-algebras, $\coprod_R$ the coproduct in $R \uparrow \mathrm{Rings}$, $\coprod_R S$ the $\mathfrak{Fin}$-ring of section 2.1 above and $\Omega_R S$ the R-DG-algebra of relative noncommutative differential forms of [3](see B for a definition). Then $Q(\Omega_R S) = \coprod_R S$.*

Proof. The $\mathfrak{Fin}$-ring $\coprod_R S$ is characterized by the following property

$$\hom_{(R\uparrow\mathrm{Rings})^{\mathfrak{Fin}}}(\coprod_R S, C) = \hom_{R\uparrow\mathrm{Rings}}(S, C^0) \tag{60}$$

We must show $Q\Omega_R S$ has the same property. On the other hand we have

$$\hom_{R\uparrow DGR^*}(\Omega_R S, X) = \hom_{R\uparrow \mathrm{Rings}}(S, X^0) \tag{61}$$

Here we identify R with the DGR^* concentrated in codimension 0 with trivial derivation. Let P be the right adjoint of $Q : DGR^* \to \mathrm{Rings}^{\mathfrak{Fin}}$; its existence is guaranteed by Prop. 3.1.4. Identifying R with the constant $\mathfrak{Fin}$-ring, noting that $QR = R$ and using (61), we obtain

$$\begin{aligned}\hom_{R\uparrow(\mathrm{Rings}^{\mathfrak{Fin}})}(Q(\Omega_R S), C) &= \hom_{R\uparrow DGR^*}(\Omega_R S, PC)\\ &= \hom_{R\uparrow \mathrm{Rings}}(S, PC^0)\end{aligned}$$

Therefore to prove the corollary it suffices to show that $PC^0 = C^0$. We have

$$\begin{aligned}PC^0 &= \hom_{\mathrm{Ch}^{\geq 0}}(\mathbb{Z} < 0, 1 >, PC) && (62)\\ &= \hom_{DGR^*}(D(0), PC)\\ &= \hom_{\mathrm{Rings}^{\mathfrak{Fin}}}(QD(0), C)\\ &= \hom_{\mathrm{Rings}^{\mathfrak{Fin}}}(TQ\mathbb{Z} < 0, 1 >, C) \text{ (by 3.1.1)}\\ &= \hom_{\mathfrak{Ab}^{\mathfrak{Fin}}}(Q\mathbb{Z} < 0, 1 >, C)\end{aligned}$$

(63)

By definition

$$Q^n\mathbb{Z} < 0, 1 >= \mathbb{Z} < 0, 1 > \boxtimes TV^n = \mathbb{Z}(1 \otimes 1) \oplus \bigoplus_{i=1}^{n} \mathbb{Z}(1 \otimes v_i) \tag{64}$$

Put $e_0 = 1 \otimes 1$, $e_i = 1 \otimes v_i + e_0$ $1 \leq i \leq n$. It follows from (20) that $\alpha(e_i) = e_{\alpha(i)}$ for all $\alpha : [n] \to [m] \in \mathfrak{Fin}$. Therefore $Q\mathbb{Z} < 0, 1 >= \bigoplus \mathbb{Z}$, whence (62) equals

$$= \hom_{\mathfrak{Ab}^{\mathfrak{Fin}}}(\bigoplus \mathbb{Z}, C) = C^0$$

$\square$

Corollary 3.1.6. *View the $\mathfrak{Fin}$-ring $\coprod_R S$ as a cyclic module by restriction, and consider its associated normalized mixed complex $(N(\coprod_R S), \mu, B)$. Then the map l of Thm. 2.3.2 is a mixed complex equivalence $l : (\Omega_R S, 0, d) \to (\coprod_R S, \mu, B)$*

Remark 3.1.7. As a particular case of Theorem 3.1.5 we get a ring isomorphism

$$S \coprod_R S \cong Q^1 \Omega_R S = \Omega_R S \boxtimes TV^1 \cong \Omega_R S \tag{65}$$

Here $\Omega_R S$ is equipped with the product $\circ$ of (46). A similar isomorphism but with a different choice of $\circ$ was proved by Cuntz and Quillen in [3] Prop. 1.3, under the stated assumption that $R = \mathbb{C}$. Their choice of $\circ$ actually works whenever 2 is invertible in R, and the rings which arise from $\Omega_R S$ with our product and that of [3] are isomorphic in that case. They are not isomorphic in general, for example if $2 = 0$. Hence 3.1.5 may be viewed as a strong generalization of Cuntz-Quillen's result.

3.2 Comparison with Nuss' product

In [11], P. Nuss considers the "twist"

$$\tau : S \otimes_R S \to S \otimes_R S, \qquad \tau(s \otimes t) = st \otimes 1 + 1 \otimes st - s \otimes t$$

It is clear that $\tau^2 = 1$ and that, for the multiplication map $\mu_0 : S \otimes_R S \to S$, we have $\mu_0 \tau = \mu_0$. He shows further ([11], 1.3) that τ satisfies the Yang-Baxter equation. Using τ, he introduces a ring structure on the $n+1$ fold tensor power $S \otimes_R \cdots \otimes_R S$ for all $n \geq 1$, by a standard procedure (use Prop. 2.3 of [2] and induction). We want to reinterpret this product in a different way. For this consider the (Amitsur) cosimplicial R-bimodule

$$\bigotimes_R S : [n] \mapsto \bigotimes_{i=0}^{n} {}_R S$$

By definition of $\Omega_R S$, we have $N(\bigotimes_R S) = \Omega_R S$. Hence the Dold-Kan correspondence gives an isomorphism of cosimplicial R-bimodules

$$\bigotimes_R S \cong K \Omega_R S \tag{66}$$

On the right hand side we also have the product (47). It is noted in [11] that (66) is a ring isomorphism in codimension ≤ 1. The next Proposition shows it is actually a ring isomorphism in all codimension.

Proposition 3.2.1. *Equip $\bigotimes_R S$ with the product defined in [11] and $K\Omega_R S$ with that given by (47). Then (66) is an isomorphism of $\mathfrak{Fin}$-rings.*

Proof. Write $\bullet$ for Nuss' product. Consider the following map

$$\delta_i = \partial_{i+1}^{n-i} \partial_0^i \in \hom_\Delta([0], [n]) \qquad (0 \leq i \leq n).$$

One checks the following identities hold in $\bigotimes_R^n S$, for $a, b \in S$ (see A.11):

$$\delta_i(a) \bullet \delta_j(b) = \begin{cases} \partial_{j+1}^{n-j} \partial_{i+1}^{j-i-1} \partial_0^i (a \otimes b) & i < j \\ \delta_i(ab) & i = j \\ -\delta_j(a) \bullet \delta_i(b) + \delta_i(ab) + \delta_j(ab) & i > j \end{cases} \tag{67}$$

In particular $\delta_i : S \to \bigotimes_R^n S$ is a ring homomorphism for $\bullet$. By the universal property of $\coprod_R^n S$, we have a unique ring homomorphism $\alpha^n : \coprod_R^n S \to \bigotimes_R^n S$ satisfying $\alpha^n \delta_i = \delta_i$ for all i. By (67),

$$\begin{aligned} s_0 \otimes \cdots \otimes s_n =& \delta_0(s_0) \bullet \cdots \bullet \delta_n(s_n) \\ =& \alpha(\delta_0(s_0) \ldots \delta_n(s_n)). \end{aligned}$$

Thus α is surjective. On the other hand the composite of α with the isomorphism $Q\Omega_R S \xrightarrow{\sim} \coprod_R S$ sends $ds \otimes v_i$ to $q_i(s) := \delta_i(s) - \delta_0(s)$. But it follows from (67) that

$$q_i(a) \bullet q_j(b) = \begin{cases} -q_j(a) \bullet q_i(b) & (i \neq j) \\ 0 & (i = j) \end{cases} \tag{68}$$

Thus α descends to a ring homomorphism $\overline{\alpha} : K\Omega_R S \to \bigotimes_R S$. On the other hand we have an R-linear map $\beta : \bigotimes_R S \to K\Omega_R S$, $\beta(s_0 \otimes \cdots \otimes s_n) = \delta_0(s_0) \ldots \delta_n(s_n)$. Clearly $\alpha\beta = 1$. To finish the proof it suffices to show that β is surjective. But we have

$$\begin{aligned} a_0 da_1 \ldots da_r \otimes v_{i_1} \wedge \cdots \wedge v_{i_r} = & \delta_0(a_0)(\delta_1(a_1) - \delta_0(a_1)) \ldots (\delta_r(a_r) - \delta_0(a_r)) \\ \equiv & \delta_0(a_0) \ldots \delta_r(a_r) \mod \bigoplus_{i=0}^{r-1} \Omega_R^i S \otimes \Lambda^i V \\ = & \beta(a_0 \otimes \cdots \otimes a_r \otimes 1 \otimes \cdots \otimes 1). \end{aligned}$$

Hence it follows by induction on r, that $\Omega_R^r S \otimes \Lambda^r A$ is included in the image of β. $\square$

Chapter 4

Dold-Kan equivalence for rings

In this chapter we establish the equivalence of homotopy categories between cochain DG and cosimplicial rings (4.12). In so doing we prove some technical results concerning functors between closed model categories (4.11), of interest in themselves.

Definition 4.1. Let $f : R \to S$ be a homomorphism in DGR^*. We say that f is a *weak equivalence* if it induces an isomorphism in cohomology. We call f a *fibration* if each $f^n : R^n \to S^n$ is surjective, and a *cofibration* if it has the left lifting property (LLP) of [12] with respect to those fibrations which are also weak equivalences (trivial fibrations). Similarly, a map $g : A \to B$ of cosimplicial rings is a *weak equivalence* if it induces an isomorphism in cohomotopy, a *fibration* if each $g^n : A^n \twoheadrightarrow B^n$ is surjective and a *cofibration* if it has the LLP with respect to trivial fibrations.

Proposition 4.2. *With the notions of fibration, cofibration and weak equivalence defined in 4.1, both* DGR^* *and* Rings^Δ *are closed model categories.*

Proof. A commutative version of this is proved in [1], Thm. 4.3 for the DG case and in [14], Thm. 2.1.2 for the cosimplicial case. Essentially the same proofs work in the noncommutative case; simply substitute the coproduct $\coprod$ of Rings for $\otimes$, which is the coproduct in the category Comm of commutative rings. One only has to check that for all $n \geq 0$, the structure maps $\mathbb{Z} \to D(n) \in DGR^*$ and $\mathbb{Z} \to D'(n) := TK\mathbb{Z} < n, n+1 > \in \mathrm{Rings}^\Delta$ induce weak equivalences

$$R \xrightarrow{\sim} R \coprod D(n) \qquad (R \in DGR^*) \tag{69}$$

$$A \xrightarrow{\sim} A \coprod D'(n) \qquad (A \in \mathrm{Rings}^\Delta) \tag{70}$$

We observe that if $R, S \in DGR^*$ (respectively $\in \mathrm{Rings}^\Delta$) then there is an isomorphism of cochain complexes (resp. of cosimplicial groups)

$$R \coprod S = T(R/\mathbb{Z} \oplus S/\mathbb{Z}) \tag{71}$$

Thus to prove (69) it suffices to show that if C and D are cochain complexes and D is contractible, then $\iota : TC \to T(C \oplus D)$ is a quasi-isomorphism. But coker ι is a sum of cochain complexes each of which is isomorphic to one of the form $T_1^r C \otimes T^{s_1} D \otimes \cdots \otimes T^{s_n} D \otimes T^{r_{n+1}} C$ with $s_1 + \cdots + s_n \geq 1$. Hence it suffices to show that $D \otimes D'$ is contractible if D is. But if h is a contracting homotopy for D, then $h \otimes 1$ is a contracting homotopy for $D \otimes D'$. This proves (69). To prove (70) one reduces in the same way to showing that if D and D' are cosimplicial groups with D contractible, then $D \otimes D'$ is contractible. This latter statement follows from the following property of the cosimplicial path functor (see [14], page 30):

$$D^{\Delta[1]} \otimes D' = (D \otimes D')^{\Delta[1]}.$$

□

Lemma 4.3.
i)The functor $\tilde{Q} : DGR^* \xrightarrow{Q} \mathrm{Rings}^{\mathfrak{Fin}} \xrightarrow{forget} \mathrm{Rings}^{\Delta}$ *preserves colimits, finite limits, cofibrations, fibrations, and weak equivalences.*

ii) Let $K : DGR^* \to \mathrm{Rings}^{\Delta}$ *be the functor sending* $A \mapsto KA$ *where* KA *is equipped with the product* (47). *Then there is a natural isomorphism of left derived functors* $\mathbb{L}\tilde{Q} \xrightarrow{\cong} \mathbb{L}K$.

Proof. Limits and colimits in Rings^{Δ} are computed dimensionwise, and the same is true in $\mathrm{Rings}^{\mathfrak{Fin}}$. In particular the forgetful functor preserves limits and colimits. The functor $Q : DGR^* \to \mathrm{Rings}^{\mathfrak{Fin}}$ preserves colimits by Prop. 3.1.4. Thus $\tilde{Q}$ preserves colimits. On the other hand limits in $\mathrm{Rings}^{\mathfrak{Fin}}$ can be computed in $\mathfrak{Ab}^{\mathfrak{Fin}}$. As $Q : \mathrm{Ch}^{\geq 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$ is exact and preserves direct sums, it follows that $\tilde{Q}$ preserves finite limits. Similarly, as the forgetful functors $DGR^* \to \mathrm{Ch}^{\geq 0}$ and $\mathrm{Rings}^{\Delta} \to \mathfrak{Ab}^{\Delta}$ as well as $Q : \mathrm{Ch}^{\geq 0} \to \mathfrak{Ab}^{\mathfrak{Fin}}$ preserve fibrations and weak equivalences, it follows that $\tilde{Q}$ does. One checks, using Lemma 3.1.1, that $\tilde{Q}$ preserves the basic cofibrations $S(m) \to D(m)$, $\mathbb{Z} \to D(m)$. Because it also preserves colimits it follows that if $m_i, i \in I$ is a family of positive integers and $e_i : S(m_i) \to X$ $(i \in I)$ a family of maps, then the following maps are cofibrations:

$$\tilde{Q}(X \rightarrowtail X \coprod_{\coprod_{i\in I} S(m_i)} \coprod_{i\in I} D(m_i))$$
$$\tilde{Q}(X \rightarrowtail X \coprod \coprod_{i\in I} D(m_i))$$

But by our proof of 4.2 and the remark on page 23 of [1], every cofibration in DGR^* is a retract of one obtained as a colimit of such cofibrations. Hence $\tilde{Q}$ preserves all cofibrations. Thus i) is proved. As shown in section 2.4, the natural weak equivalence $\hat{p} : \tilde{Q}A \to KA$ of 2.3.2 is a homomorphism of cosimplicial rings. This proves ii). □

Remark 4.4. *A functor* L_* *with properties similar to those proved for* $\tilde{Q}$ *in Lemma 4.3 is considered in [13] for the dual situation of chain DG- and simplicial rings. The authors use the shuffle product to make the normalized chain complex of a simplicial ring into a chain DG-ring, thus obtaining a functor* $N_* : \mathrm{Rings}^{\Delta^{op}} \to DGR_*$. *The functor* L_* *is defined as the left adjoint of* N_*. *Dually, one can equip the normalized complex of a cosimplicial ring with the shuffle product, consider the resulting functor* $\tilde{N} : \mathrm{Rings}^{\Delta} \to DGR^*$ *and take its left adjoint* L^*. *However we point out that* L^* *and* $\tilde{Q}$ *are not isomorphic. In other words* $\tilde{Q}$ *is not left adjoint to* $\tilde{N}$. *To see this, note that, by 3.1.1, if* $A \in \mathrm{Ch}^{\geq 0}$, *then* $\hom_{\mathrm{Rings}^{\Delta}}(\tilde{Q}TA, R) = \hom_{\mathfrak{Ab}^{\Delta}}(QA, R)$, *while* $\hom_{DGR^*}(TA, \tilde{N}R) = \hom_{\mathrm{Ch}^{\geq 0}}(A, NR) = \hom_{\mathfrak{Ab}^{\Delta}}(KA, R)$. *Hence if* $\tilde{Q}$ *were left adjoint to* $\tilde{N}$, *then* K *and* Q *should be isomorphic as functors* $\mathrm{Ch}^{\geq 0} \to \mathfrak{Ab}^{\Delta}$, *which is clearly false.*

Definition 4.5. A homomorphism $f : L \to M \in DGR^*$ is called a *relative cellular complex* if it admits a –possibly infinite– factorization $f = \dots f_n f_{n-1} \dots f_0$ as a composite of maps

$f_n : M_n \to M_{n+1}$ such that $M_{-1} = L$ and M_{n+1} is obtained from M_n by a pushout diagram as the following, for some family I_n of maps $e_i : S(m_i) \to M_{n-1}$:

$$\begin{tikzcd} \coprod_{i\in I_n} D(m_i) \arrow[r] & M_n \\ \coprod_{i\in I_n} S(m_i) \arrow[u, tail] \arrow[r, "\sum e_i"'] & M_{n-1} \arrow[u, tail, "f_n"'] \end{tikzcd} \tag{72}$$

We call f *trivial* if $f = f_0$. We say that L is a *cellular complex* if $\mathbb{Z} \to L$ is a relative cellular complex.

Remark 4.6. *The definition above extends to any closed model category $\mathfrak{C}$ with any family $\mathcal{F}$ of cofibrations substituted for $\{S(m) \to D(m)\}$.*

Proposition 4.7. *Let $B \in Rings^\Delta$. Then there exists a cellular complex $R \in DGR^*$ and a trivial fibration $\tilde{Q}R \overset{\sim}{\twoheadrightarrow} B$.*

Proof. This is a variant of the small object argument of [12]. First we show that $\tilde{Q}D(m)$ and $\tilde{Q}S(m)$ are small. Let X be either of $D(m)$, $S(m)$. Then $X = TY$ for some bounded chain complex Y of finitely generated free abelian groups. By 3.1.1,

$$\begin{aligned} \hom_{\mathrm{Rings}^\Delta}(\tilde{Q}X, ?) &= \hom_{\mathfrak{Ab}^\Delta}(QY, ?) \\ &= \hom_{\mathrm{Ch}^{\geq 0}}(NQY, N?) \end{aligned}$$

It is clear from (31) that NQY is a bounded complex of finitely generated free abelian groups. Thus $\tilde{Q}X$ is small for $X = D(m), S(m)$. Next let $B \in Rings^\Delta$. Both the cellular complex R and the trivial fibration of the proposition will be defined as colimits $R = \operatorname{colim}_n R_n$ and $f = \operatorname{colim}_n f_n : \tilde{Q}R = \operatorname{colim}_n \tilde{Q}R_n \to B$. We will use induction to define R_n and f_n. Let $R_0 := \coprod_m \coprod_{\hom(\tilde{Q}D(m),B)} D(m)$, and define $f_0 : \tilde{Q}R_0 = \coprod_m \coprod_{\hom(\tilde{Q}D(m),B)} \tilde{Q}D(m) \to B$ as σ on the copy of $\tilde{Q}D(m)$ corresponding to $\sigma \in \hom(\tilde{Q}D(m), B)$. We claim f_0 is a fibration. To see this let $b \in B^n$, and choose $\tau : TK\mathbb{Z} < n, n+1 > \to B$ such that b is in the image of τ. Let $\sigma = \tau T(\hat{p}) : TQ\mathbb{Z} < n, n+1 >= \tilde{Q}D(n+1) \to B$; then b is in the image of the restriction of f to the copy of $\tilde{Q}D(n+1)$ corresponding to σ. Next assume by induction that R_n and f_n have been defined. Let $i : S(m) \rightarrowtail D(m)$ be the natural inclusion. Consider the set $\mathcal{D}_m$ of all commutative diagrams of the form

$$\begin{tikzcd} \tilde{Q}S(m) \arrow[r, "\tilde{Q}\sigma"] \arrow[d, tail, "\tilde{Q}i"'] & \tilde{Q}R_n \arrow[d, "f_n"] \\ \tilde{Q}D(m) \arrow[r] & B \end{tikzcd}$$

Define R_{n+1} as the pushout

$$\begin{tikzcd} \coprod_m \coprod_{\mathcal{D}_m} S(m) \arrow[r, "\sigma"] \arrow[d, "i"] & R_n \arrow[d] \\ \coprod_m \coprod_{\mathcal{D}_m} D(m) \arrow[r] & R_{n+1} \end{tikzcd}$$

Because $\tilde{Q}$ preserves pushouts, there is a natural map $f_{n+1} : \tilde{Q}R_{n+1} \to B$. Thus R_n and f_n are defined for all n. Because f_0 is a fibration, so is $f = \operatorname{colim}_n f_n$. Because $\tilde{Q}S(m)$ and $\tilde{Q}D(m)$ are small, the dotted arrow in the diagram below exists whenever the top horizontal arrow is in the image of $\tilde{Q} : \hom_{DGR^*}(S(m), R) \to \hom_{Rings^\Delta}(\tilde{Q}S(m), \tilde{Q}R)$.

$$\begin{tikzcd}
\tilde{Q}S(m) \arrow[r] \arrow[d] & \tilde{Q}R \arrow[d] \\
\tilde{Q}D(m) \arrow[r] \arrow[ur, dotted] & B
\end{tikzcd} \tag{73}$$

It remains to see that f is a weak equivalence, i.e. that the following map is an isomorphism for all m.

$$f : H^m N\tilde{Q}R \xrightarrow{\cong} H^m NB \tag{74}$$

We first prove that (74) is surjective. If $x \in H^m NB$ is an element, call x the map $\mathbb{Z} \to H^m NB$, $1 \mapsto x$. Choose a cochain homomorphism $\hat{x} : \mathbb{Z}[m] \to NB$ inducing x. We have an exact sequence

$$0 \longrightarrow \mathbb{Z}[m+1] \longrightarrow \mathbb{Z} < m, m+1 > \longrightarrow \mathbb{Z}[m] \longrightarrow 0$$

Because both N and Q are exact, we have a solid line commutative diagram

$$\begin{tikzcd}
NQ\mathbb{Z}[m+1] \arrow[rrr, "0"] \arrow[d, tail] & & & N\tilde{Q}R \arrow[d, two heads, "f"] \\
NQ\mathbb{Z} < m, m+1 > \arrow[r] \arrow[urrr, dotted, "h"] & NQ\mathbb{Z}[m] \arrow[rr] \arrow[dr, two heads] & & NB \\
& & \mathbb{Z}[m] \arrow[ur, "x"] &
\end{tikzcd} \tag{75}$$

To prove that the dotted arrow exists, apply the functor K to obtain a commutative diagram

$$\begin{tikzcd}
Q\mathbb{Z}[m+1] \arrow[r, "Q(0)"] \arrow[d] & \tilde{Q}R \arrow[d] \\
Q\mathbb{Z} < m, m+1 > \arrow[r] \arrow[ur, dotted] & B
\end{tikzcd} \tag{76}$$

Next use Lemma 3.1.1 to obtain a diagram of the form (73) in which the top row is in the image of Q, whence the dotted arrow exists in (73), whence also in (76) and (75). Call y the arrow $NQ\mathbb{Z}[m] \to NQR$ induced by h. Then the image of 1 through $\bar{y} : \mathbb{Z} = H^m(NQ\mathbb{Z}[m]) \to H^m N\tilde{Q}R$ maps to x under (74). This proves that (74) is surjective. To show it is also injective, let $x : \mathbb{Z}[m] \to N\tilde{Q}R$ represent an element in the kernel of (74). Then $fx : \mathbb{Z}[m] \to NB$ factors through a map $x' : \mathbb{Z} < m-1, m > \to NB$. Because $\hat{p}$ is natural we have a commutative diagram

$$\begin{tikzcd}
NQ\mathbb{Z}[m] \arrow[r, "x\hat{p}"] \arrow[d, tail] & N\tilde{Q}R \arrow[d, two heads, "f"] \\
NQ\mathbb{Z} < m-1, m > \arrow[r, "x'\hat{p}"'] & NB
\end{tikzcd}$$

Because $\hat{p}$ is an equivalence, it suffices to show that $x\hat{p}$ induces the zero map in cohomology. Next, by virtue of 2.3.2 there is a homotopy $fx\hat{p} \to fNQ(\overline{x\hat{p}})$. Because $Q\mathbb{Z}[m] \rightarrowtail Q\mathbb{Z} < m-1, m >$ is a cofibration this homotopy extends to one between $x'\hat{p}$ and some map y which fits into the following commutative diagram.

$$\begin{tikzcd} NQ\mathbb{Z}[m] \arrow[r, "NQ(\overline{x\hat{p}})"] \arrow[d, tail] & N\tilde{Q}R \arrow[d, "f"] \\ NQ\mathbb{Z} < m-1, m > \arrow[r, "y"'] \arrow[ur, dotted] & NB \end{tikzcd} \tag{77}$$

The same argument used during the course of the proof of the surjectivity of (74) shows that the dotted arrow exists. Hence $x\hat{p}$ induces the zero map in cohomology, since it is homotopic to $NQ(\overline{x\hat{p}})$, and the latter induces zero by (77). $\square$

Corollary 4.8. *For every object $B \in \operatorname{Ho}\mathrm{Rings}^{\Delta}$ there exists $X \in \operatorname{Ho}DGR^*$ such that $\mathbb{L}\tilde{Q}(X) \cong B$.*

Definition 4.9. Let $\mathfrak{C}$ be a closed model category and $B \in \mathfrak{C}$ a fibrant object. Recall from [12] that a *path object* for B is an object $B^I \in \mathfrak{C}$ together with a fibration $\epsilon : B^I \twoheadrightarrow B \times B$ and a weak equivalence $\sigma : B \xrightarrow{\sim} B^I$ such that the following diagram commutes

$$\begin{tikzcd} & B^I \arrow[d] \\ B \arrow[ur, "\sim"] \arrow[r, "diag"'] & B \times B \end{tikzcd} \tag{78}$$

Here "*diag*" is the diagonal homomorphism. If $f_0, f_1 : A \to B$ are homomorphisms in $\mathfrak{C}$, then a *(right) homotopy* $f_0 \to f_1$ relative to B^I is a map $H : A \to B^I$ such that $\epsilon H = (f_0, f_1)$. The maps f_0 and f_1 are right homotopic if there exists a path object B^I and a right homotopy from f_0 to f_1. If $f_0 = f_1 = f$ and $H = \sigma f$ we say that H is *trivial.* Fix a path object B^I. A *double path object* for B^I is an object B^J together with a fibration $\epsilon' : B^J \twoheadrightarrow B^I \times_{B\times B} B^I$ and a weak equivalence $\sigma' : B \xrightarrow{\sim} B^J$ such that $\epsilon'\sigma' = (\sigma, \sigma)$. If A is cofibrant and $f_0, f_1 : A \to B$ are homotopic, then two homotopies $G, H : A \to B^I$ from f_0 to f_1 are *homotopic* if there is a double path object $(B^J, \epsilon', \sigma')$ such that $(G, H) : A \to B^I \times_{B\times B} B^I$ factors as $(G, H) = \epsilon'\kappa$. We call such a κ a homotopy $G \to H$. If $G = H$ and $\kappa = \sigma'(G, H)$ we say that κ is *trivial.* Finally we remark that if $F : \mathfrak{C} \to \mathfrak{D}$ is a functor between closed model categories preserving fibrations and weak equivalences as well as finite limits, then F also preserves path and double path objects.

Lemma 4.10. *Let $m \geq 1$, $M \in DGR^*$, M^I a path object, M^J a double path object for M^I. We have:*

i) Let $f : \tilde{Q}D(m) \to \tilde{Q}M$ be a homomorphism in Rings^{Δ}. Then there exist $g : D(m) \to M \in DGR^*$ and a homotopy $H : \tilde{Q}D(m) \to \tilde{Q}M^I$ from f to $\tilde{Q}g$. Assume further that there exists $g' : S(m) \to M$ such that f restricts to $\tilde{Q}g'$ on $S(m)$. Then g and H can be chosen so that g extends g' and H extends the trivial homotopy $\tilde{Q}g' \to \tilde{Q}g'$.

ii) Let $g_0, g_1 : D(m) \to M \in DGR^*$ and $H : \tilde{Q}D(m) \to \tilde{Q}M^I$ a homotopy $\tilde{Q}g_0 \to \tilde{Q}g_1$. Then there exist homotopies $G : D(m) \to M^I$ from g_0 to g_1 and $\kappa : \tilde{Q}D(m) \to \tilde{Q}M^J$ from H to $\tilde{Q}G$. Assume further that $(g_0)_{|S(m)} = (g_1)_{|S(m)} =: g$ and that $H_{|S(m)}$ is the trivial homotopy $\tilde{Q}g \to \tilde{Q}g$. Then G can be chosen so that it restricts on $S(m)$ to the trivial homotopy $g \to g$.

Proof. i) By virtue of 3.1.1, f induces a cosimplicial homomorphism $Q\mathbb{Z} < m-1, m > \to \tilde{Q}M$. Let $\overline{f} : \mathbb{Z} < m-1, m > \to M$ and $H_f : NQ\mathbb{Z} < m-1, m > \to PNQM$ be as in Lemma 2.3.4. Compose H_f with the natural inclusion $PNQM \to NQM^{\Delta[1]}$ of 2.3.5 to obtain a map $NH' : NQ\mathbb{Z} < m-1, m > \to NQM^{\Delta[1]}$. Define g to be the canonical extension of $\overline{f}$ to a DG-ring homomorphism $D(m) = T\mathbb{Z} < m-1, m > \to M$. Also extend $H' : Q\mathbb{Z} < m-1, m > \to QM^{\Delta[1]}$ to a cosimplicial ring homomorphism $\tilde{Q}D(m) = TQ\mathbb{Z} < m-1, m > \to \tilde{Q}M^{\Delta[1]}$. Factor the weak equivalence $\tilde{Q}M \to \tilde{Q}M^{\Delta[1]}$ into a trivial cofibration $\tilde{Q}M \to (\tilde{Q}M)^{I'}$ followed by a trivial fibration $\pi : (\tilde{Q}M)^{I'} \to \tilde{Q}M^{\Delta[1]}$. Use the cofibrancy of $\tilde{Q}D(m)$ to lift H' along π to a map $H'' : \tilde{Q}D(m) \to (\tilde{Q}M)^{I'}$. Define H as the composite of H'' with the dotted arrow in the diagram below.

$$\begin{tikzcd}
\tilde{Q}M \arrow[r] \arrow[d, tail, "\wr"] & \tilde{Q}(M^I) \arrow[d] \\
(\tilde{Q}M)^{I'} \arrow[r] \arrow[ur, dotted] & \tilde{Q}M \times \tilde{Q}M
\end{tikzcd}$$

If f restricts to $\tilde{Q}g'$ on $\tilde{Q}S(m)$, then –by 2.3.4– $\overline{f}$ restricts to g' on $\mathbb{Z}[m]$, whence $g_{|S(m)} = g'$. Moreover –again by 2.3.4– H_f is trivial on $NQ\mathbb{Z}[m]$, whence H' is trivial on $\tilde{Q}S(m)$ by 2.3.5. To obtain H out of H' so that its restriction to $\tilde{Q}S(m)$ remains a trivial homotopy, it suffices to follow the recipe above, being careful to choose H'' in such a way that the following diagram commutes

$$\begin{tikzcd}
\tilde{Q}S(m) \arrow[r] \arrow[d, tail] & (\tilde{Q}M)^{I'} \arrow[d, "\wr"] \\
\tilde{Q}D(m) \arrow[r] \arrow[ur, dotted, "H''"] & \tilde{Q}M^{\Delta[1]}
\end{tikzcd}$$

ii) Define G as $\overline{H}$ on $\mathbb{Z} < m-1, m >$ and extend to a DG-ring homomorphism $D(m) \to M^I$. The naturality of the mapping $f \mapsto \overline{f}$ implies that G is a right homotopy $g_0 \to g_1$, and that it restricts on $S(m)$ to the trivial homotopy $g \to g$ if H restricts to the trivial homotopy $\tilde{Q}g \to \tilde{Q}g$. Because $\mathbb{Z} \to D(m)$ is a trivial cofibration, the dotted arrow in the diagram below exists:

$$\begin{tikzcd}
 & \tilde{Q}M^J \arrow[d] \\
\tilde{Q}D(m) \arrow[r, "(H,\tilde{Q}G)"'] \arrow[ur, dotted, "\kappa"] & \tilde{Q}M^I \times_{\tilde{Q}M \times \tilde{Q}M} \tilde{Q}M^I
\end{tikzcd} \tag{79}$$

□

Proposition 4.11. *Let $\mathfrak{C}$ be a closed model category, $\mathcal{F} = \{S(m) \rightarrowtail D(m) : m \in I\}$ a family of cofibrations between cofibrant objects of $\mathfrak{C}$, $L \in \mathfrak{C}$ an $\mathcal{F}$-cellular complex (4.6), and $M \in \mathfrak{C}$ fibrant. Also let $\mathfrak{D}$ be another closed model category and $F : \mathfrak{C} \to \mathfrak{D}$ a functor preserving fibrations, weak equivalences, cofibrations, colimits and finite limits. Assume further that Lemma 4.10 holds for F and the cofibrations of $\mathcal{F}$. Then the following function between sets of homotopy classes of maps is a bijection:*

$$[F] : [L, M] \xrightarrow{\sim} [FL, FM], \qquad [f] \mapsto [Ff].$$

Proof. To prove $[F]$ is surjective, we must show that if $f : FL \to FM$ is a homomorphism then there exists a morphism $g : L \to M$ and a right homotopy $H : f \to Fg$. Let M^I be a path object. We shall construct by induction maps $g_n : L_n \to M$ and $H_n : FL_n \to FM^I$ such that H_n is a homotopy $f_n := f_{|L_n} \to Fg_n$, and such that if $n \geq 1$, then $(g_n)_{|L_{n-1}} = g_{n-1}$ and $(H_n)_{|L_{n-1}} = H_{n-1}$. By Lemma 4.10 i), g_0 and H_0 are defined on L_0. Assume by induction that g_n and H_n have been defined on L_n. Because $L_n \rightarrowtail L_{n+1}$ is a cofibration, so is $FL_n \to FL_{n+1}$. Hence H_n extends to a map $H' : FL_{n+1} \to FM^I$, which is a homotopy $f_{n+1} \to f' := F(\epsilon_1)H'$. Note f' extends Fg_n. Let $D(m)$ be a cell of L_{n+1} with attaching map $S(m) \to L_n$. The restrictions of f_{n+1}, f', and H' to $D(m)$ and of g_n to $S(m)$ satisfy the hypothesis of 4.10 i). Hence there is a map $g_{n+1} : L_{n+1} \to M$ and a homotopy $H'' : FL_{n+1} \to FM^I$ from f' to Fg_{n+1} extending the trivial homotopy $Fe : Fg_n \to Fg_n$. The composite homotopy $H''' := H' \star H'' : FL_{n+1} \to F(M^I \times_M M^I)$ extends $H_n \star Fe$, which is homotopic to H_n. Choose a homotopy $K : H_n \star Fe \to H_n$. Because $FL_n \rightarrowtail FL_{n+1}$ is a cofibration, K extends to a homotopy K' between H''' and a homotopy $H_{n+1} : f_n \to F(g_n)$ which extends H_n. This finishes the induction. It remains to show that $[F]$ is injective. Let $g^0, g^1 : L \to M$ be homomorphisms in $\mathfrak{C}$ and $H : FL \to FM^I$ a homotopy $Fg_0 \to Fg_1$; we have to show that there exists a homotopy $G : L \to M^I$, from g_0 to g_1. We will prove by induction that there exists, for each $n \geq 0$, a homotopy $G_n : L_n \to M^I$ between $g^0_n := g^0_{|L_n}$ and g^1_n, such that FG_n is homotopic to $H_n = H_{|L_n}$ and such that if $n \geq 1$ then G_n extends G_{n-1}. The step $n = 0$ of the induction is clear from Lemma 4.10 ii). Assume G_n has been constructed as above. Because $L_n \rightarrowtail L_{n+1}$ is a cofibration, G_n extends to a homotopy $G' : L_{n+1} \to M^I$ from g^0_{n+1} to some map g' extending g^1_n. Let ${G'}^{-1} : g' \to g^0_{n+1}$ be the inverse homotopy. Then $T' := F{G'}^{-1} \star H_{n+1}$ is a homotopy $Fg' \to Fg^1_{n+1}$ which extends $T := FG_n^{-1} \star H_n$. By inductive assumption, the latter is homotopic to the trivial homotopy $Fe : Fg^1_n \to Fg^1_n$. Because $FL_n \rightarrowtail FL_{n+1}$ is a cofibration, there exists a homotopy $H'' : Fg' \to Fg^1_{n+1}$ which restricts to Fe on FL_n, and which is homotopic to T'. Let $D(m)$ be a cell of L_{n+1} with attaching map $S(m) \to L_n$. By Lemma 4.10 ii) there exists a homotopy $G'' : D(m) \to M^I$ such that FG'' is homotopic to H'' and such that $G''_{|S(m)}$ is the trivial homotopy $g^1_n \to g^1_n$. Let $G''' = G' \star G''$. Then G''' extends $G_n \star e$ and if we write $\sim$ for the homotopy relation between homotopies, we have

$$\begin{aligned} FG''' =& FG' \star FG'' \sim FG' \star H'' \\ & \sim FG' \star (F{G'}^{-1} \star H_{n+1}) \sim H_{n+1} \end{aligned}$$

The inductive step can be more clearly followed in the diagram below, the simbol $\Downarrow$ indicates that the triangle is homotopy commutative.

$$\begin{tikzcd}
 & & & Fg^0_{n+1} \arrow[dr, "H_{n+1}"] \arrow[dd, dotted] \arrow[d, phantom, "\Downarrow"] & \\
FG''' : & Fg^0_{n+1} \arrow[r, "FG'"] \arrow[dd] & Fg' \arrow[ur, "F{G'}^{-1}"] \arrow[rr, bend right, "FG''"] \arrow[dd] & & Fg^1_{n+1} \arrow[dd] \\
 & & & Fg^0_n \arrow[dr, "H_n"] \arrow[d, phantom, "\Downarrow"] & \\
 & Fg^0_n \arrow[r, "FG_n"] & Fg^1_n \arrow[ur, "FG_n^{-1}"] \arrow[rr, bend right, "Fe"] & & Fg^1_n
\end{tikzcd}$$

Because G''' extends $G_n \star e$, which is homotopic to G_n, and because $L_n \rightarrowtail L_{n+1}$ is a cofibration, there exists a homotopy $G_{n+1} : g^0_{n+1} \to g^1_{n+1}$ extending G_n and homotopic to G'''. Furthermore $FG_{n+1} \sim FG''' \sim H_{n+1}$. $\square$

Theorem 4.12. *The functor* $\mathbb{L}\tilde{Q} : \operatorname{Ho} DGR^* \to \operatorname{Ho} \mathrm{Rings}^\Delta$ *of 4.3 is an equivalence of categories.*

Proof. In view of Corollary 4.8, it suffices to show that $\mathbb{L}\tilde{Q}$ is fully faithful. By our proof of 4.2 and the remark on page 23 of [1], for every $A \in DGR^*$ there exists a trivial fibration $L \twoheadrightarrow A$ where L is a cellular complex. Thus it suffices to show that if $L, M \in DGR^*$ are cellular complexes, then $\tilde{Q}$ induces a bijection $[L, M] \to [\tilde{Q}L, \tilde{Q}M]$ where $[,]$ indicates the set of homotopy classes of maps. This follows from Lemma 4.10 and Prop. 4.11. $\square$

Corollary 4.13. *Let* $K : \mathrm{Ch}^{\geq 0} \to \mathrm{Rings}^\Delta$ *be the Dold-Kan functor. If* $A \in DGR^*$*, equip* KA *with the product* (47). *Then the left derived functor* $\mathbb{L}K$ *of* $DGR^* \to \mathrm{Rings}^\Delta$*,* $A \mapsto KA$ *is a category equivalence* $\operatorname{Ho} DGR^* \xrightarrow{\sim} \operatorname{Ho} \mathrm{Rings}^\Delta$.

Proof. Immediate from 4.12 and 4.3 ii). $\square$

Appendix A

This appendix contains proofs of some formulas and identities used in the main body of the thesis. Lemma A.1 proves an assertion made in section 2.3. Lemmas A.2 and A.3 are both used in the proof of Theorem 2.3.2 and Lemma A.11 in the proof of Proposition 3.2.1. From Definition A.4 to Proposition A.10, we give a detailed proof of the fact that for $A \in DGR^*$, $(QA, \circ)$ is a $\mathfrak{Fin}$-ring (section 2.4).

Lemma A.1. *Let $A \in \mathrm{Ch}^{\geq 0}$. Consider the $\mathfrak{Fin}$ structure on KA defined in 2.2 above, and let $\mu : K^*A \to K^{*-1}A$ be the alternating sum of the face maps. Then the induced boundary $N\mu : N^* = A^* \to A^{*-1}$ is the zero map.*

Proof. Let $\mu = \sum_{i=0}^{n}(-1)^i \mu_i : Q^n A \to Q^{n-1}A$. We shall write $\bar{x}$ for the image of x in ΛV^*. Since $\mu_i(0) = 0$, for all i, $\mu(a \otimes x) = a \otimes \mu(x)$. Thus $\mu(a \otimes \bar{x}) = a \otimes \overline{\mu(x)} \in K^*A$. We shall see that for every $\bar{x} \in \Lambda V^*$, $\overline{\mu(x)} = 0$. Let $\bar{x} = v_{i_1} \wedge ... \wedge v_{i_r}$ with $i_1 < ... < i_r$. If $\{j, j+1\} \subseteq \{i_1, ..., i_r\} = I$, then $\mu_j(\bar{x}) = ... \wedge v_j \wedge v_j \wedge ... = 0$. On the other hand, if $j < n$, $j \in I$ and $j+1 \notin I$, then $\mu_j(\bar{x}) = \overline{\mu_{j+1}(x)}$; and thus $(-1)^j\overline{\mu_j(x)} + (-1)^{j+1}\overline{\mu_{j+1}(x)} = 0$. Since for each j, we are in one of these two situations, μ must be zero. ☐

Lemma A.2. *The maps h and j defined during the course of the proof of theorem 2.3.2 above satisfy the requerements of part i) of that theorem.*

Proof. Write $\partial = \delta_0 + \delta_1 : N^*Q \to N^{*+1}$, where $\delta_0 = 1 \otimes \delta_0$ and $\delta_1 = d \otimes v_1\delta_0$ are the homogeneous components of degree 0 and 1, respectively. Let $j_0 = 1 \otimes j'$ and $h_0 = 1 \otimes h'$.

Note that $\hat{p}j = \hat{p}j_0$. We are now going to check that $j = j_0 + h_0(j_0 d - \delta_1 j_0)$ is a cochain map, and that $h = (h_0 - h_0\delta_1 h_0)(1 - j_0\hat{p})$ is a homotopy between $1 - j\hat{p}$ and 0.

$$\begin{aligned}
&\partial j - jd = (\delta_0 + \delta_1)j - jd = \\
&\delta_0 j_0 + \delta_0 h_0 j_0 d - \delta_0 h_0 \delta_1 j_0 + \delta_1 j_0 + \delta_1 h_0 j_0 d - \delta_1 h_0 \delta_1 j_0 - j_0 d - h_0 j_0 d^2 + h_0 \delta_1 j_0 d = \\
&0 + \delta_0 h_0 j_0 d - \delta_0 h_0 \delta_1 j_0 + \delta_1 j_0 + 0 - 0 - j_0 d - 0 + 0 = \\
&\delta_0 h_0 (j_0 d - \delta_1 j_0) + \delta_1 j_0 - j_0 d = \\
&(1 - j_0\hat{p})(j_0 d - \delta_1 j_0) + \delta_1 j_0 - j_0 d = j_0 d - \delta_1 j_0 - j_0\hat{p} j_0 d + j_0 \hat{p}\delta_1 j_0 + \delta_1 j_0 - j_0 d = \\
&j_0 d - \delta_1 j_0 + \delta_1 j_0 - j_0 d - j_0 d + j_0\hat{p}\delta_1 j_0 = \\
&- j_0 d + j_0 \hat{p}\delta_1 j_0 = 0.
\end{aligned}$$

Thus j is a cochain map.

Let us now check that $[h, \partial] = 1 - j\hat{p}$:

$$
\begin{aligned}
&[h, \partial] = \partial h + h \partial = \\
&(\delta_0 + \delta_1)(h_0 - h_0\delta_1 h_0)(1 - j_0\hat{p}) + (h_0 - h_0\delta_1 h_0)(1 - j_0\hat{p})(\delta_0 + \delta_1) = \\
&\delta_0 h_0 + h_0\delta_0 + \delta_1 h_0 + h_0\delta_1 - \delta_0 h_0\delta_1 h_0 - h_0\delta_1 h_0\delta_0 - \\
&\delta_1 h_0 j_0\hat{p} - h_0 j_0\hat{p}\delta_1 - \delta_0 h_0 j_0\hat{p} + \delta_0 h_0\delta_1 h_0 j_0\hat{p} = \\
&1 - j_0\hat{p} + \delta_1 h_0 + h_0\delta_1 - \delta_1 h_0 + h_0\delta_0\delta_1 h_0 - h_0\delta_1 + h_0\delta_1\delta_0 h_0 + \\
&h_0\delta_1 j_0\hat{p} - \delta_1 h_0 j_0\hat{p} - h_0 j_0\hat{p}\delta_1 - \delta_0 h_0 j_0\hat{p} + \delta_0 h_0\delta_1 h_0 j_0\hat{p} = \\
&1 - j_0\hat{p} + h_0\delta_1 j_0\hat{p} - \delta_1 h_0 j_0\hat{p} - h_0 j_0\hat{p}\delta_1 - \delta_0 h_0 j_0\hat{p} + \delta_0 h_0\delta_1 h_0 j_0\hat{p} = \\
&1 - j_0\hat{p} + h_0\delta_1 j_0\hat{p} - \delta_1 h_0 j_0\hat{p} - h_0 j_0\hat{p}\delta_1 + \delta_1 h_0 j_0\hat{p} - h_0\delta_0\delta_1 h_0 j_0\hat{p} = \\
&1 - j_0\hat{p} + h_0\delta_1 j_0\hat{p} - h_0 j_0\hat{p}\delta_1 = \\
&1 - (j_0 + h_0(j_0 d - \delta_1 j_0))\hat{p} = 1 - j\hat{p}.
\end{aligned}
$$

□

Lemma A.3. *Let $\sigma \in S_n$ and $a \otimes x \in Q^n A$. Write $1 \amalg \sigma$ for the coproduct map $\{1\} \amalg \{1, ..., n\} : \{1, ..., n+1\} \to \{1, ..., n+1\}$, and $\equiv$ for congruence modulo the degenerate subcomplex. Then*

(i) $t_n^i(v_j) = \begin{cases} v_{i+j} - v_i & \text{if } i \le n - j \\ v_{p-1} - v_i & \text{if } i = n - j + p \text{ and } j \ge p \ge 1 \end{cases}$

(ii) $t_n^i(a \otimes x) = a \otimes t_n^i x + da \otimes v_i t_n^i x$

(iii) $B(a \otimes x) \equiv da \otimes \sum_{i=0}^n (-1)^{in} v_{i+1}\partial_0 t^i x$

(iv) $B(a \otimes \sigma) \equiv da \otimes \sum_{i=0}^n (-1)^{in}(1, ..., n)(1 \amalg \sigma)$

Proof. In order to prove (i), write $\overline{m}$ for the remainder of the entire division of m by $n+1$. Recall $\{e_i : 0 \le i \le n\}$ is the canonical basis of $\bigoplus_{i=0}^n \mathbb{Z}$. Then $t_n(e_i) = e_{\overline{i+1}}$. Hence $t_n^i(e_j) = e_{\overline{j+i}}$ and $t_n^i(v_j) = t_n^i(e_j - e_0) = e_{\overline{j+i}} - e_{\overline{i}} = v_{\overline{i+j}} - v_{\overline{i}}$.

On the other hand, $\overline{i+j} = i + j$ if $i \le n - j$, while $\overline{(n-j+p)+p} = p - 1$. This proves (i).

As we have already seen, $t_n^i(e_0) = e_i$, $(0 \le i \le n)$. Hence,

$$t_n^i(a \otimes x) = a \otimes t_n^i x + da \otimes v_i t_n^i x.$$

This proves (ii).

We shall use (ii) to compute (iii):

$$B(a \otimes x) = \sum_{i=0}^n (-1)^{in}\partial_0 t_n^i(a \otimes x) = a \otimes \sum_{i=0}^n (-1)^{in}\partial_0 t_n^i x + da \otimes \sum_{i=0}^n (-1)^{in} v_{i+1}\partial_0 t_n^i x. \quad (80)$$

The first term is of degree less than $n+1$, whence it belongs to the degenerate subcomplex. Thus, modulo degeneracies, (80) equals

$$da \otimes \sum_{i=0}^n (-1)^{in} v_{i+1}\partial_0 t_n^i x.$$

Item (iv) is clear from (iii) and the identification between $\overline{v_{\sigma' 1} ... v_{\sigma' n+1}}$ and $\sigma' \in S_{n+1}$; in particular for $\sigma' = (1 ... n+1)^i (1 \amalg \sigma)$. □

Definition A.4. Let A be a graded ring and M a graded A-module. An abelian group homomorphism $D : A \to M$ that satisfies $D(ab) = aD(b) + (-1)^{|b|}D(a)b$, for homogeneous $b \in A$, is an *antiderivation* of A.

Remark A.5. *Note that for every derivation $D : A \to M$, $a \mapsto (-1)^{|a|}Da$ defines an antiderivation; in fact, for homogeneous $a, b \in A$,*

$$\begin{aligned} ab \mapsto (-1)^{|a|+|b|}D(ab) =& (-1)^{|a|+|b|}((-1)^{|a|}aD(b) + D(a)b) \\ &= a((-1)^{|b|}Db) + (-1)^{|b|}((-1)^{|a|}Da)b. \end{aligned}$$

Lemma A.6. *Let A and B be two graded rings, $D_0 : A \to A$ a derivation of degree +1 and $D_1 : B \to B$ an antiderivation of degree +1. We shall write $A \boxtimes B$ for the abelian group $\bigoplus_{i\geq 0} A_i \otimes B_i \subseteq A \otimes B$. Then if $\cdot$ is the usual product in $A \otimes B$, the map f given by the following formula is a 2-cocycle:*

$$f(a \otimes x, b \otimes y) = (a \otimes D_1x) \cdot (D_0b \otimes y).$$

Proof. It suffices to see that for homogeneous $a \otimes x$, $b \otimes y$ and $c \otimes z$,

$$\begin{aligned} (a \otimes x) \cdot f((b \otimes y), (c \otimes z)) - f((a \otimes x) \cdot (b \otimes y), (c \otimes z)) + \\ f((a \otimes x), (b \otimes y) \cdot (c \otimes z)) - f((a \otimes x), (b \otimes y)) \cdot (c \otimes z) = 0. \end{aligned} \quad (81)$$

Let us compute each term of the latter equation.

$$\begin{aligned} &(a \otimes x) \cdot f((b \otimes y), (c \otimes z)) = (a \otimes x) \cdot (b \otimes D_1y) \cdot (D_0c \otimes z) = \\ &abD_0c \otimes xD_1(y)z. \end{aligned} \quad (82)$$

$$\begin{aligned} &f((a \otimes x) \cdot (b \otimes y), (c \otimes z)) = (ab \otimes D_1(xy) \cdot (D_0c \otimes z) = \\ &abD_0c \otimes xD_1(y)z + (-1)^{|b|}abD_0c \otimes D_1(x)yz. \end{aligned} \quad (83)$$

$$\begin{aligned} &f((a \otimes x), (b \otimes y) \cdot (c \otimes z)) = (a \otimes D_1x) \cdot (D_0(bc) \otimes yz) = \\ &aD_0(b)c \otimes D_1(x)yz + (-1)^{|b|}abD_0(c) \otimes D_1(x)yz. \end{aligned} \quad (84)$$

$$\begin{aligned} &f((a \otimes x), ((b \otimes y)) \cdot (c \otimes z) = (a \otimes D_1x) \cdot (D_0b \otimes y) \cdot (c \otimes z) = \\ &aD_0(b)c \otimes D_1(x)yz. \end{aligned} \quad (85)$$

The alternate sum of (82), (83), (84) and (85) is zero; thus f is a cocycle. □

Proposition A.7. *The product $\circ$ defined by* (46) *is associative.*

Proof. Since θ is a derivation on TV^*, $x \mapsto (-1)^{|x|}\theta x$, for homogeneous x, defines an antiderivation on TV^* (Remark A.5). Then the map given on an homogeneous element $\omega \otimes x$ and $\eta \otimes y \in Q^*A$ by the following formula is a 2-cocycle of Q^*A (by lemma A.6).

$$(\omega \otimes x, \eta \otimes y) \mapsto (\omega \otimes (-1)^{|x|}\theta x) \cdot (d\eta \otimes y) = (-1)^{|\omega|}\omega d\eta \otimes \theta(x)y$$

In consequence $\circ$ is associative. □

Lemma A.8. *Under the hypothesis of lemma A.6, let $\alpha : A \boxtimes B \to A \boxtimes B$ be a group homomorphism such that $\alpha = \alpha_0 + \alpha_1$, and that α_i is of degree i for $i = 0, 1$. Then α is a ring homomorphism for $\circ$ iff for every homogeneous $x, y \in A \boxtimes B$, all the following identities hold:*
(i) $\alpha_0(xy) = \alpha_0(x) \cdot \alpha_0(y)$
(ii) $\alpha_1(xy) = \alpha_1(x) \cdot \alpha_0(y) + \alpha_0(x) \cdot \alpha_1(y) + f(\alpha_0(x), \alpha_0(y)) - \alpha_0(f(x, y))$
(iii) $f(\alpha_1(x), \alpha_1(y)) = 0$
(iv) $\alpha_1(f(x, y)) = \alpha_1(x) \cdot \alpha_1(y) + f(\alpha_0(x), \alpha_1(y)) + f(\alpha_0(x), \alpha_1(y))$.

Proof. Straightforward computation. □

Lemma A.9. *For every $\alpha \in Map([n], [m])$ and $x \in T\mathbb{V}^n$, $\alpha\theta(x) = \theta\alpha(x) - [v_{\alpha(0)}, \alpha(x)]$.*

Proof. Both sides of the identity we have to prove are derivations. Thus it suffices to show they agree on the generators v_i.

$$\begin{aligned}
&\theta\alpha v_i - [v_{\alpha 0}, \alpha v_i] = \theta(v_{\alpha i} - v_{\alpha 0}) - [v_{\alpha 0}, v_{\alpha i} - v_{\alpha 0}] = \\
&v_{\alpha i}^2 - v_{\alpha 0}^2 - [v_{\alpha 0}, v_{\alpha i}] + [v_{\alpha 0}, v_{\alpha 0}] = \\
&v_{\alpha i}^2 - v_{\alpha 0}^2 - v_{\alpha 0} v_{\alpha i} - v_{\alpha i} v_{\alpha 0} + 2v_{\alpha 0}^2 = \alpha v_i^2 = \alpha\theta v_i.
\end{aligned}$$

□

Proposition A.10. *For every $A \in DGR$, $(QA, \circ)$ is a $\mathfrak{Fin}$-ring.*

Proof. It suffices to verify that the abelian group map induced by α satisfies the hypothesis of lemma A.8.

Recall from (22) that $\alpha = \alpha_0 + \alpha_1$, with $\alpha_0 = 1 \otimes \alpha$ of degree 0 and $\alpha_1 = d \otimes L_{v_{\alpha(0)}}\alpha$ of degree +1. Here L_{v_i} is the operator multiplication on the left by v_i. We must now verify conditions (i) to (iv) of lemma A.8 for α and the cocycle of proposition A.7. In what follows, $a \otimes x$ and $b \otimes y$ will be two homogeneous elements in QA and we will denote the usual homogeneous product $\cdot$ just by concatenation.
Proof of i):

$$\begin{aligned}
&\alpha_0((a \otimes x)(b \otimes y)) = \alpha_0(ab \otimes xy) = ab \otimes \alpha(xy) = ab \otimes \alpha(x)\alpha(y) = \\
&\alpha_0(a \otimes x)\alpha_0(b \otimes y).
\end{aligned}$$

Proof of ii):

$$\alpha_1((a \otimes x)(b \otimes y)) = \alpha_1(ab \otimes xy) = d(ab) \otimes v_{\alpha(0)}\alpha(xy).$$

On the other hand,

$$\begin{aligned}
&\alpha_1(x)\alpha_0(y) + \alpha_0(x)\alpha_1(y) + f(\alpha_0(x), \alpha_0(y)) - \alpha_0(f(x, y)) = \\
&d(a)b \otimes v_{\alpha(0)}\alpha(xy) + ad(b) \otimes \alpha(x)v_{\alpha(0)}\alpha(y) + (-1)^{|a|} adb \otimes \theta\alpha(x)\alpha(y) - \\
&\qquad (-1)^{|a|} adb \otimes \alpha\theta(x)\alpha(y) = \\
&d(a)b \otimes v_{\alpha(0)}\alpha(xy) + ad(b) \otimes \alpha(x)v_{\alpha(0)}\alpha(y) + (-1)^{|a|} adb \otimes (\theta\alpha - \alpha\theta)(x)\alpha(y) = \\
&d(a)b \otimes v_{\alpha(0)}\alpha(xy) + ad(b) \otimes \alpha(x)v_{\alpha(0)}\alpha(y) + (-1)^{|a|} adb \otimes [v_{\alpha(0)}, \alpha(x)]\alpha(y) \\
&\qquad = d(ab) \otimes v_{\alpha(0)}\alpha(xy).
\end{aligned}$$

Proof of iii):

$$f(a\otimes x,\alpha_1(b\otimes y)=(-1)^{|a|}(a\otimes\theta(x))(d^2b\otimes v_{\alpha(0)}\alpha(y))=0. \tag{86}$$

Proof of iv): Note that by (86), there is a trivialy null term in (iv), and

$$\begin{aligned}&\alpha_1(a\otimes x)\alpha_1(b\otimes y)+f(\alpha_1(a\otimes x),\alpha_0(b\otimes y)-\alpha_1(f(a\otimes x,b\otimes y)=\\&dadb\otimes v_{\alpha(0)}\alpha(x)v_{\alpha(0)}\alpha(y)-(-1)^{|a|}dadb\otimes\theta(v_{\alpha(0)}\alpha(x))\alpha(y)\\&+(-1)^{|a|}dadb\otimes v_{\alpha(0)}\alpha\theta(x)\alpha(y)\\&=dadb\otimes(v_{\alpha(0)}\alpha(x)v_{\alpha(0)}-(-1)^{|a|}\theta(v_{\alpha(0)}\alpha(x))+(-1)^{|a|}v_{\alpha(0)}\alpha\theta(x))\alpha(y).\end{aligned} \tag{87}$$

It suffices to analyze the factor in TV^* in the last term of (87). Because θ is a derivation, $\theta v_{\alpha(0)}=v_{\alpha(0)}^2$. From lemma A.9,

$$v_{\alpha(0)}\alpha(x)v_{\alpha(0)}-(-1)^{|a|}\theta(v_{\alpha(0)}\alpha(x))+(-1)^{|a|}v_{\alpha(0)}\alpha\theta(x)=0.$$

This proves (iv). ☐

Lemma A.11. *Let $\delta_i(a)=\partial_{i+1}^{n-i}\partial_0^i\in\hom_\Delta([0],[n])$. Then*

$$\delta_i(a)\bullet\delta_j(b)=\begin{cases}\partial_{j+1}^{n-j}\partial_{i+1}^{j-i-1}\partial_0^i(a\otimes b) & \text{if } i<j\\ \delta_i(ab) & \text{if } i=j\\ -\delta_j(a)\bullet\delta_i(b)+\delta_i(ab)+\delta_j(ab) & \text{if } i>j\end{cases}$$

Proof. Following Nuss [11], we have that $\bullet$ can be written as a composition of permutations followed by a multiplication map. We denote the permutation synthetically by

$$(a_0\otimes...\otimes a_n)\otimes(b_0\otimes...\otimes b_n)\mapsto(a_0\otimes b_0'\otimes...\otimes a_i'\otimes b_i'\otimes...\otimes a_n'\otimes b_n). \tag{88}$$

We shall use the fact that $\tau(1\otimes s)=s\otimes 1$, $\tau(s\otimes 1)=1\otimes s$ and $\tau(s\otimes t)=1\otimes st+st\otimes 1-s\otimes t$, to pass each b_i over the n terms in front of it. On the other hand, we have

$$\delta_i(a)=(1\otimes...\otimes\overset{i}{a}\otimes...\otimes 1).$$

Suppose first that $i\leq j$, then the permutation sends

$$\begin{aligned}\delta_i(a)\otimes\delta_j(b)=(1\otimes...\otimes\overset{i}{a}\otimes...\otimes 1)\otimes(1\otimes...\otimes\overset{j}{b}\otimes...\otimes 1)\\ \mapsto(1\otimes...\otimes\overset{2i}{a'}\otimes...\otimes\overset{2j+1}{b'}\otimes...\otimes 1).\end{aligned}$$

Since the only element passing over a is 1, and b only passes over 1, $a'=a$ and $b'=b$. Then,

$$\delta_i(a)\bullet\delta_j(b)=(1\otimes...\otimes\overset{i}{a}\otimes...\otimes\overset{j}{b}\otimes...\otimes 1).$$

In particular, if $i=j$, $\delta_i(a)\bullet\delta_i(b)=\delta_i(ab)$.

Suppose now that $i>j$. Then, to move b to position $2j+1$, we must pass over a. We have

$$\delta_i(a)\otimes\delta_j(b)=(1\otimes...\otimes\overset{i}{a}\otimes...\otimes 1)\otimes(1\otimes...\otimes\overset{j}{b}\otimes...\otimes 1). \tag{89}$$

The permutation (88) applied to (89) is the composite

$$
\begin{aligned}
&(1 \otimes ... \otimes \overset{i}{a} \otimes ...1 \otimes ... \otimes \overset{j}{b} \otimes ... \otimes 1) \mapsto (1 \otimes ... \otimes \overset{2i}{a} \otimes b \otimes ...\otimes) \\
&\mapsto (1 \otimes ... \otimes \overset{2i}{ab} \otimes 1 \otimes ... \otimes 1) + (1 \otimes ... \otimes 1 \otimes \overset{2i+1}{ab} \otimes ... \otimes 1) - (1 \otimes ... \otimes \overset{2i}{b} \otimes \overset{2i+1}{a} \otimes ... \otimes 1) \\
&\mapsto (1 \otimes ... \otimes \overset{2j+1}{ab} \otimes 1 \otimes ... \otimes 1) + (1 \otimes ... \otimes 1 \otimes \overset{2i}{ab} \otimes ... \otimes 1) - (1 \otimes ... \otimes \overset{2j+1}{b} \otimes ... \otimes \overset{2i}{a} \otimes ... \otimes 1).
\end{aligned}
$$

Applying the multiplication map we get

$$
\begin{aligned}
(1 \otimes ... \otimes \overset{j}{ab} \otimes ... \otimes ... \otimes 1) + (1 \otimes ... \otimes ... \otimes \overset{i}{ab} \otimes ... \otimes 1) - (1 \otimes ... \otimes \overset{j}{b} \otimes ... \otimes \overset{i}{a} \otimes ... \otimes 1) \\
= \delta_{(}ab) + \delta_j(ab) - \delta_j(a) \bullet \delta_i(b).
\end{aligned}
$$

It follows that $\delta_i(a) \bullet \delta_j(b) = \delta_i(ab) + \delta_j(ab) - \delta_j(a) \bullet \delta_i(b)$. $\square$

Appendix B

Noncommutative differential forms

In this appendix we shall present some well known results and definitions ([3], [8] and [15]) concerning noncommutative differential forms. We do not intend to be exhaustive but just to give a brief introduction, complete enough to follow this thesis without the necessity of consulting the suggested bibliography.

Let $R \to S$ be a homomorphism of associative unital rings, and M an S-bimodule. By an R-derivation on M, we mean an R-linear map D that satisfies, for $a, b \in S$, the Leibniz rule

$$D(ab) = aD(b) + D(a)b.$$

Note that this definition implies that $D(R) = 0$. We shall write $Der_R(S, M)$ for the R-module of all R-derivations from S to M.

Let us consider the following universal problem:

Find an R-derivation d from S into a bimodule $\Omega^1_R S$, such that, given an R-derivation D of S into an S-bimodule M, there is a unique S-bimodule morphism $i_D : \Omega^1_R S \to M$ with $D = i_D \circ d$, i.e., making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "D"] \arrow[d, "d"'] & M \\ \Omega^1_R S \arrow[ur, "i_D"'] & \end{tikzcd} \tag{90}$$

The assignment $\phi \mapsto \phi \circ d$ defines a linear map $Hom_S(\Omega^1_R S, M) \to Der_R(S, M)$. The universal property is the assertion that this arrow is an isomorphism. Let us see that the universal problem has a solution.

Let S/R be the cokernel of $R \to S$ seen as an R-bimodule map, and define

$$\Omega^1_S S = S \otimes_R S/R \tag{91}$$

We endow $\Omega^1_R S$ with the following S-bimodule structure:

$$a(b \otimes \bar{c}) = ab \otimes \bar{c} \tag{92}$$

$$(b \otimes \bar{c})a = b \otimes \bar{c}\bar{a} - ba \otimes \bar{c} \tag{93}$$

for $a, b, c \in S$. We define an R-linear map $d : S \to \Omega^1_R S$ by

$$d(a) = 1 \otimes \bar{a}$$

with $\bar{a}$ meaning the image of a in S/R. Then we have that

Proposition B.1. $(\Omega^1_R S, d)$ *is a solution for the universal problem* (90)*.*

Proof. The linear map d obeys $d(ab) = 1 \otimes \bar{ab} = a \otimes \bar{b} + 1 \otimes \bar{ab} - a \otimes \bar{b} = a(1 \otimes \bar{b}) + (1 \otimes \bar{a})b = ad(b) + d(a)b$ and for $r \in R$, $d(r) = 1 \otimes \bar{r} = 1 \otimes 0 = 0$; thus, it is an R-derivation. On the other hand, for an R-derivation from S to an S-bimodule M, we define i_D by $i_D(a \otimes \bar{b}) = aD(b)$ and linearity. It follows that i_D is an S-bimodule map and $i_D(d(a)) = D(a)$. This shows that $(\Omega_R^1 S, d)$ is a solution for (90). □

In what follows we write da for $1 \otimes \bar{a}$ and in view of (92), adb for $a \otimes \bar{b}$.

Now we are going to construct a differential graded S-algebra $(\Omega_R S, d)$, solution of the following universal problem:

Find a differential graded ring $(\Omega_R S, d)$, such that given a differential graded ring $A = \bigoplus_{n=0}^{\infty} A^n$ with differential δ (of degree +1), and a ring homomorphism $u : S \to A^0$ such that $\delta(u(R)) = 0$, there is a unique homomorphism $u_* : \Omega_R S \to A$ extending u.

A solution of this universal problem is called the differential graded ring of relative forms.

Define the group of relative differential forms of degree n as

$$\Omega_R^n S = \underbrace{\Omega_R^1 S \otimes_S ... \otimes_S \Omega_R^1 S}_{n-times} \tag{94}$$

We can use the S-bimodule structure of $\Omega_R^1 S$ to write (94) as

$$S \otimes_R \underbrace{S/R \otimes_R ... \otimes_R S/R}_{n-times} \tag{95}$$

Equip $\Omega_R^n S$ with the S-bimodule structure induced by (92), (93) and (94). In view of (95) we make the following identification:

$$s_0 \otimes \bar{s}_1 \otimes ... \otimes \bar{s}_n = s_0 ds_1 ... ds_n$$

where we omit the $\otimes$ simbol between the elements of $\Omega_R^1 S$. Define now

$$\Omega_R S = \bigoplus_{n=0}^{\infty} \Omega_R^n S$$

equipped with the obvious S-bimodule structure and the product defined on homogeneous elements by

$$(a_0 da_1 ... da_n)(b_0 db_1 ... db_m) = ((a_0 da_1 ... da_n) b_0)(db_1 ... db_m) \in \Omega_R^{n+m} S$$

Define d by

$$d(a_0 \otimes \bar{a}_1 \otimes ... \otimes \bar{a}_n) = 1 \otimes \bar{a}_0 \otimes \bar{a}_1 \otimes ... \otimes \bar{a}_n \tag{96}$$

Since $1 \in R$, it follows that $d^2 = 0$. Using this fact, writting (96) as

$$d(a_0 da_1 ... da_n) = da_0 da_1 ... da_n$$

and doing induction on n, we get that for $\omega \in \Omega_R^n S$ and $\eta \in \Omega_R S$, $d(\omega\eta) = (-1)^n \omega d(\eta) + d(\omega)\eta$; i.e, $(\Omega_R S, d)$ is a differential graded ring.

Remark B.2. *Note that the graded ring $\Omega_R S$ just defined is $T_S(\Omega_R^1 S)$, the tensor algebra of the S-bimodule $\Omega_R^1 S$.*

Proposition B.3. $(\Omega_R S, d)$ *is a solution of the universal problem stated above.*

Proof. (Sketch) Given a differential graded ring (A, δ) and a homomorphism $u : S \to A^0$ such that $\delta u(R) = 0$, one defines $u_* : \Omega_R S \to A$ by

$$u_*(s_0 ds_1 ... ds_n) = u(s_0)\delta u(s_1)...\delta u(s_n)$$

and extends it by linearity. $\square$

BIBLIOGRAPHY

[1] A. Bousfield, V. Gugenheim, *On PL de Rham theory and rational homotopy type,* Memoirs of the AMS, **179** (1976).

[2] A. Cap, H. Schichl, J. Vanžura, *On twisted tensor products of algebras,* Comm. Algebra 23 (1995), no. 12, 4701–4735.

[3] J. Cuntz, D. Quillen, *Algebra extensions and nonsingularity,* J. Amer. Math. Soc. **8** (1995) 251-289.

[4] J. Cuntz, D. Quillen, *Cyclic homology and nonsingularity,* J. Amer. Math. Soc. **8** (1995) 373-442.

[5] A. Dold, *Homology of symmetric products and other functors of complexes,* Annals of Math. **68** (1958) 54-80.

[6] J.A. Guccione, J.J. Guccione, J. Majadas, *Noncommutative Hochschild homology* (in Spanish). Unpublished preprint, 1994.

[7] M. Karoubi, *Correspondence de Dold-Kan et formes differentielles,* J. of Algebra **198** (1997) 618-626.

[8] J. Loday, *Cyclic homology,* Springer-Verlag, Berlin, Heidelberg, New York 1992.

[9] S. Mac Lane, *Categories for the working mathematician,* Grad. Texts in Math. 5, Springer-Verlag, 1971.

[10] S. Mac Lane, *Homology.* Die Grundlehren der mathematischen Wissenschaften, Bd. 114 Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Gttingen-Heidelberg 1963.

[11] P. Nuss, *Noncommutative descent and nonabelian cohomology,* K-theory **12** (1997) 23-74.

[12] D. Quillen, *Homotopical algebra.* Lecture Notes in Math. 43, Springer-Verlag, 1967.

[13] S. Schwede, B. Shipley, *Equivalences of monoidal model categories.* Algebraic and Geometric Topology 3 (2003), 287-334.

[14] B. Toen, *Schematization of homotopy types,* http://arXiv.org/abs/math.AG/0012219

[15] C. Weibel, *An introduction to homological algebra,* Cambridge Studies in advanced mathematics 38, Cambridge University Press, 1994.

SINOPSIS

La correspondencia de Dold-Kan (dual) es una equivalencia entre la categoría de complejos de cocadenas no negativamente graduados de grupos abelianos $\mathrm{Ch}^{\geq 0}$ y la categoría $\mathfrak{Ab}^{\Delta}$ de grupos abelianos cosimpliciales. Esta equivalencia está definida por un par de funtores inversos

$$N : \mathfrak{Ab}^{\Delta} \leftrightarrows \mathrm{Ch}^{\geq 0} : \mathrm{K} \tag{S.1}$$

Si $A \in \mathfrak{Ab}^{\Delta}$, NA es el complejo de Moore o normalizado de A (ver (24)). El funtor K se describe en [15], 8.4.4; si $A = (A, d) \in \mathrm{Ch}^{\geq 0}$ y $n \geq 0$, entonces

$$K^n A = \bigoplus_{i=0}^{n} \binom{n}{i} A^i \cong \bigoplus_{i=0}^{n} A^i \otimes \Lambda^i \mathbb{Z}^n. \tag{S.2}$$

Si además A es un anillo diferencial graduado (o anillo DG, o anillo de cocadenas), se tiene que se puede equipar a $K^n A$ con un producto; concretamente, con el usual del producto tensorial de anillos $A \otimes \Lambda \mathbb{Z}^n$:

$$(a \otimes x)(b \otimes y) = ab \otimes x \wedge y. \tag{S.3}$$

Este producto hace en efecto de $[n] \mapsto K^n A$ un anillo cosimplicial (ver sección 2.4). Luego se puede pensar a K como un functor de los anillos DG en los anillos cosimpliciales:

$$K : DGR^* \to \mathrm{Rings}^{\Delta}, \qquad A \mapsto KA \tag{S.4}$$

Observemos que para todo n, $K^n A$ es una extensión nilpotente de A^0. Dado que hay anillos cosimpliciales que no son extensiones nilpotentes en cada codimensión de un anillo cosimplicial constante, concluimos que $A \mapsto KA$ no es una equivalencia de categorías. Probamos sin embargo (Thm. 4.13) que este funtor induce una si invertimos las equivalencias débiles. Más precisamente, K aplica casi-isomorfismos en flechas que inducen un isomorfismo a nivel de cohomotopía, y en consecuencia induce un funtor $\mathbb{L}K$ entre las localizaciones $\mathrm{Ho}\, DGR^*$ y $\mathrm{Ho}\,\mathrm{Rings}^{\Delta}$ que se obtienen invirtiendo formalmente dichas flechas, y demostramos que $\mathbb{L}K$ es una equivalencia:

$$\mathbb{L}K : \mathrm{Ho}\, DGR^* \xrightarrow{\sim} \mathrm{Ho}\,\mathrm{Rings}^{\Delta} \tag{S.5}$$

El dual del resultado anterior, es decir, la equivalencia entre las categorías anillos DG de cadenas y anillos simpliciales fue obtenida en forma independiente por Schwede y Shipley mediante un enfoque distinto (ver [13] y también Remark 4.4 en el cuerpo de la tesis).

Para probar (S.5) usamos el formalismo de categorías de modelo cerrado de Quillen [12]. Dotamos a DGR^* y Rings^{Δ} de la estructura de modelo cerrado en la cual las equivalencias débiles son las antes descriptas, las fibraciones son las flechas suryectivas y las cofibraciones se definen de modo que se cumplan los axiomas de Quillen. Se tiene el problema técnico de que el funtor K no preserva cofibraciones. Para solucionarlo, reemplazamos K por otro funtor adecuado Q. Como en el caso del funtor de Dold-Kan, Q también está definido para todo complejo de cocadenas A, aunque no sea un anillo. Si $A \in \mathrm{Ch}^{\geq 0}$ entonces

$$Q^n A = \bigoplus_{i=0}^{\infty} A^i \otimes T^i(\mathbb{Z}^n) \tag{S.6}$$

Mostramos que toda función $\alpha : [n] \to [m]$ induce un homomorfismo de grupos $Q^n A \to Q^m A$, de modo que $[n] \mapsto Q^n A$ no sólo es un funtor de Δ sino de la categoría más grande $\mathfrak{Fin}$, que posee los mismos objetos y donde las flechas $[n] \to [m]$ son todas las funciones entre dichos conjuntos. La proyección $T\mathbb{Z}^n \to \Lambda\mathbb{Z}^n$ induce un homomorfismo

$$\hat{p} : QA \xrightarrow{\sim} KA \tag{S.7}$$

Mostramos que $\hat{p}$ induce un isomorfismo de grupos de cohomotopía. Si además A es un anillo DG, $Q^n A$ tiene un producto obvio proveniente de $A \otimes T\mathbb{Z}^n$; sin embargo éste no respeta ni la $\mathfrak{Fin}$ estructura, ni la estructura cosimplicial. Con el objeto de tener un $\mathfrak{Fin}$-anillo perturbamos el producto con un 2-cociclo de Hochschild $f : A^* \otimes T^* V \to A^{*+1} \otimes T^{*+1} V$. Obtenemos así un producto $\circ$ de la forma

$$(a \otimes x) \circ (b \otimes y) = ab \otimes xy + f(a \otimes x, b \otimes y). \tag{S.8}$$

Para una definición de f ver (46). Se puede ver que la aplicación $\hat{p}$ es un homomorfismo de anillos (ver sección 2.4). Esto implica que los funtores derivados totales de K y del funtor $\tilde{Q}$ que se obtiene a partir de Q olvidando la $\mathfrak{Fin}$-estructura y considerando sólo la cosimplicial, son isomorfos (ver 4.3):

$$\mathbb{L}\tilde{Q} \cong \mathbb{L}K. \tag{S.9}$$

Luego mostramos que $\tilde{Q}$ preserva la totalidad de la estructura de modelo cerrado (4.3) y que su funtor derivado total a izquierda es una equivalencia (4.12).

A continuación describimos otros resultados obtenidos en esta tesis. Como se dijo anteriormente, para $A \in \mathrm{Ch}^{\geq 0}$, QA no sólo es un grupo cosimplicial sino también un $\mathfrak{Fin}$-grupo. En particular, la permutación cíclica $t_n := (0 \dots n) : [n] \to [n]$ actúa sobre $Q^n A$, y nos permite ver a QA como un módulo cíclico en el sentido de [15], 9.6.1. Consideremos su complejo mixto normalizado asociado (NQA, μ, B). Mostramos que hay una equivalencia débil de complejos mixtos

$$(A, 0, d) \xrightarrow{\sim} (NQA, \mu, B). \tag{S.10}$$

En particular, ambos complejos mixtos poseen la misma homología de Hochschild:

$$A^* \cong H_*(NQA, \mu) \tag{S.11}$$

Si A resulta ser un anillo DG entonces el producto barajado induce una estructura de anillo graduado en $H_*(NQA, \mu)$; mostramos en 2.5.1 que (S.11) es un isomorfismo de anillos para el producto de A y el producto barajado de $H_*(NQA, \mu)$.

Un caso de interés particular es aquel en el que el anillo DG es el de las las formas diferenciales no conmutativas $\Omega_R S$ relativas a un homomorfismo de anillos $R \to S$ (definido en [3] y el Apéndice B). Mostramos en 3.1.5 que $Q\Omega_R S$ es el $\mathfrak{Fin}$-anillo coproducto:

$$Q\Omega_R S : [n] \mapsto \coprod_{i=0}^{n}{}_R S \tag{S.12}$$

En particular, por (S.11), hay un isomorfismo de anillos graduados

$$\Omega_R S \xrightarrow{\cong} H_*(\coprod_R S, \mu) \tag{S.13}$$

El caso específico de (S.13) en el que R es conmutativo y $R \to S$ es central y playo fue demostrado en 1994 por Guccione, Guccione y Majadas [6]. Más generalmente, por (S.10) se tiene una equivalencia de complejos mixtos

$$(\Omega_R S, 0, d) \xrightarrow{\sim} (\coprod_R S, \mu, B). \tag{S.14}$$

Pensamos a (S.13) y (S.14) como versiones no conmutativas de los teoremas de Hochschild-Konstant-Rosenberg y Loday-Quillen ([15] 9.4.13, 9.8.7).

Otra aplicación que damos es una formulación simple de un producto definido por Nuss [11] en cada término del complejo de Amitsur asociado a un homomorfismo $R \to S$ de anillos R y S no necesariamente conmutativos:

$$\bigotimes_R S : [n] \mapsto \bigotimes_{i=0}^{n}{}_R S \tag{S.15}$$

Nuss construye su producto utilizando herramientas de la teoría de grupos cuánticos. Nosotros mostramos en esta tesis (sección 3.2) que el isomorfismo canónico de Dold-Kan aplica el producto (S.3) en aquel definido por Nuss. Luego

$$K\Omega_R S = KN(\bigotimes_R S) \cong \bigotimes_R S \tag{S.16}$$

es un isomorfismo de anillos cosimpliciales.

El resto de la tesis está organizada del siguiente modo. En la sección 2.1 fijamos la notación básica. En la sección 2.2 definimos el funtor Q. La equivalencia de homotopía entre los grupos cosimpliciales KA y QA, y la de complejos mixtos (S.10) se muestran en la sección 2.3. El producto en QA para A un anillo DG se introduce en la sección 2.4; el isomorfismo de anillos graduados (S.11) en la sección 2.5. El isomorfismo (S.12) y sus corolarios (S.13) y (S.14) se demuestran en la sección 3.1. La reformulación del producto de Nuss es el objeto de la sección 3.2. El teorema principal que establece la equivalencia de categorías de homotopía (S.5) es el último resultado del capítulo 4. El mismo capítulo contiene además un resultado técnico sobre funtores entre categorías de modelo cerrado en general que es de interés en sí mismo (4.11).

Los detalles de algunos resultados que se establecen en los capítulos 3 y 4 se dejan para el apéndice A, con el objeto de hacer más ágil su lectura. El apéndice B presenta algunos resultados básicos sobre formas diferenciales no conmutativas con la idea de hacer autocontenido al capítulo 3.