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Categorías de K-teoría algebraica bivariante y un espectro para la K-teoría algebraica bivariante G-equivariante

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Categorías de K-teoría algebraica bivariante y un espectro para la K-teoría algebraica bivariante G-equivariante

Este trabajo se enfoca en el estudio de K-teorías algebraicas bivariantes universales. Cortiñas y Thom construyeron en [2] una K-teoría bivariante, invariante por homotopía, escisiva, M_∞-estable y universal en la categoría Alg_ℓ de álgebras sobre un anillo unital ℓ. Más precisamente, construyeron una categoría triangulada kk junto con un funtor $j :$ $\mathrm{Alg}_\ell \to kk$ que verifica:

1. j manda morfismos (polinomialmente) homotópicos en morfismos iguales;

2. j manda sucesiones exactas cortas en Alg_ℓ que se parten como sucesiones de ℓ-módulos en triángulos distinguidos en kk;

3. $j(A \to M_\infty A)$ es un isomorfismo, para toda álgebra A.

El funtor j es universal en el sentido de que cualquier otro funtor $\mathrm{Alg}_\ell \to \mathscr{T}$ con las mismas propiedades —donde $\mathscr{T}$ es una categoría triangulada— se factoriza por j de manera única. Independientemente de [2], Garkusha construyó en [6] distintas teorías de homología bivariantes, invariantes por homotopía, escisivas y universales en Alg_ℓ. Todas estas teorías verifican (1) y (2), pero satisfacen condiciones de estabilidad distintas de (3). Los métodos usados por Garkusha son bien distintos de los usados por Cortiñas-Thom: el primero construye sus categorías de K-teoría bivariante derivando una categoría de Brown mientras que los segundos dan una descripción más explícita de la categoría kk en términos de clases de homotopía de morfismos de ind-álgebras. En esta tesis combinamos resultados de [5] con ideas desarrolladas por Cortiñas-Thom en [2] y damos nuevas descripciones de las categorías de K-teoría bivariante definidas por Garkusha en [6]. Nuestra construcción de la *categoría de homotopía estable por lazos* sigue de cerca a la construcción hecha en [3, Section 6.3] en el contexto topológico. En el camino, calculamos los grupos de homotopía del espacio de morfismos $\mathrm{Hom}_{\mathrm{Alg}_k}(A, B^\Delta)$ para cualquier par de álgebras A y B, generalizando [2, Theorem 3.3.2]. Como aplicación de esto último, damos una demostración simplificada de [5, Comparison Theorem A] sin usar localización de Bousfield de categorías de modelos. Por último, usando el espectro de K-teoría bivariante definido por Garkusha en [5], construímos un espectro simplicial que representa a la K-teoría algebraica bivariante G-equivariante kk^G definida por Eugenia Ellis en [4]. Además, mostramos que el teorema de Green-Julg [4, Theorem 5.2.1] y la adjunción entre inducción y restricción [4, Theorem 6.14] se levantan a equivalencias débiles de espectros.

Palabras clave: K-teoría algebraica bivariante, teorías de homología bivariantes, espectros de K-teoría bivariante, teoría de homotopía de álgebras, categorías trianguladas.

Bivariant algebraic K-theory categories and a spectrum for G-equivariant bivariant algebraic K-theory

This work is focused on the study of universal bivariant algebraic K-theories. Cortiñas and Thom constructed in [2] a universal bivariant, homotopy invariant, excisive and M_∞-stable homology theory in the category Alg_ℓ of algebras over a unital ring ℓ. More precisely, they constructed a triangulated category kk together with a functor $j : \mathrm{Alg}_\ell \to kk$ that has the following properties:

1. j sends (polynomially) homotopic morphisms to the same morphism;

2. j sends short exact sequences in Alg_ℓ that split in the category of ℓ-modules to distinguished triangles in kk;

3. $j(A \to M_\infty A)$ is an isomorphism, for any algebra A.

The functor j is universal in the sense that any other functor $\mathrm{Alg}_\ell \to \mathscr{T}$ with the above properties —where $\mathscr{T}$ is a triangulated category— factors uniquely trough j. Independently of [2], Garkusha constructed in [6] various universal bivariant, homotopy invariant and excisive homology theories in Alg_ℓ. All these theories have properties (1) and (2), but they satisfy different stability conditions instead of (3). The methods used by Garkusha are very different from the ones used by Cortiñas-Thom: the former constructs his bivariant K-theory categories by means of deriving a Brown category and the latter give a more explicit description of kk in terms of homotopy classes of morphisms of ind-algebras. In this work we combine results from [5] with the ideas developed by Cortiñas-Thom in [2] to give new descriptions of the bivariant K-theory categories defined by Garkusha in [6]. Our construction of the *loop-stable homotopy category* closely follows that of the suspension-stable homotopy category given in [3, Section 6.3] in the topological setting. Along the way, we compute the homotopy groups of the simplicial mapping space $\mathrm{Hom}_{\mathrm{Alg}_k}(A, B^\Delta)$ for any pair of algebras A and B, generalizing [2, Theorem 3.3.2]. As an application of the latter, we give a simplified proof of [5, Comparison Theorem A] that avoids the use of Bousfield localization of model categories. Finally, using the bivariant K-theory spectrum defined by Garkusha in [5], we construct a simplicial spectrum that represents the G-equivariant bivariant algebraic K-theory kk^G defined by Eugenia Ellis in [4]. Moreover, we show that the Green-Julg theorem [4, Theorem 5.2.1] and the adjunction between induction and restriction [4, Theorem 6.14] lift to weak equivalences of spectra.

Keywords: bivariant algebraic K-theory, bivariant homology theories, bivariant K-theory spectra, homotopy theory of algebras, triangulated categories.

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Introducción

Sea ℓ un anillo conmutativo con unidad y sea Alg_ℓ la categoría de ℓ-álgebras no necesariamente unitales. Una *extensión* en Alg_ℓ es una sucesión exacta corta de álgebras

$$\mathscr{E} : \quad A \longrightarrow B \longrightarrow C \tag{1}$$

que se parte en la categoría de ℓ-módulos. Sea $(\mathscr{T}, L)$ una categoría triangulada —en realidad, queremos decir la categoría opuesta de lo que usualmente se entiende por *categoría triangulada*, de manera que los triángulos en $\mathscr{T}$ serán de la forma: $LZ \to X \to Y \to Z$. Siguiendo a Cortiñas-Thom [2], una *teoría de homología escisiva* a valores en $\mathscr{T}$ consiste de:

(i) Un funtor $X : \mathrm{Alg}_\ell \to \mathscr{T}$;

(ii) Un morfismo $\delta_{\mathscr{E}} \in \mathscr{T}(LX(C), X(A))$ por cada extensión (1).

Estos datos están sujetos a las siguientes condiciones:

(a) Para cada extensión (1), el triángulo que sigue es distinguido:

$$LX(C) \xrightarrow{\delta_{\mathscr{E}}} X(A) \longrightarrow X(B) \longrightarrow X(C)$$

(b) Los morfismos $\delta_{\mathscr{E}}$ son naturales con respecto a morfismos de extensiones.

Ejemplos de teorías de homología escisivas fueron construídos por Cortiñas-Thom [2], Garkusha [6] y Ellis [4]. Todas estas teorías son *invariantes por homotopía* —i.e. identifican morfismos (polinomialmente) homotópicos— y poseen cierta propiedad universal. La mayoría satisface algún tipo de estabilidad por matrices.

Cortiñas y Thom [2] introdujeron la *kk-teoría algebraica*; esta es una teoría de homología escisiva e invariante por homotopía tal que, para todo $A \in \mathrm{Alg}_\ell$, la inclusión $s_\infty : A \to M_\infty A$ en la coordenada $(1, 1)$ es inversible en kk. La kk-teoría algebraica es además universal con estas propiedades: cualquier otra teoría de homología con las mismas propiedades se factoriza por kk de manera única. Una propiedad importante de la kk-teoría es que se relaciona con la K-teoría homotópica de Weiblel, KH. Más precisamente, para todo $A \in \mathrm{Alg}_\ell$, hay un isomorfismo natural $kk(\ell, \Omega^n A) \cong KH_n A$ [2, Theorem 8.2.1].

Para definir la kk-teoría, Cortiñas y Thom introdujeron un enriquecimiento simplicial de álgebras: Para toda ℓ-álgebra A y todo conjunto simplicial K, definieron una ℓ-álgebra A^K; esta puede pensarse como el álgebra de funciones polinomiales en K con

coeficientes en A. Para cada par de ℓ-álgebras A y B, definieron un espacio de morfismos $\mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$. Sin embargo, este no es un enriquecimiento simplicial en el sentido de [9, Chapter 4] porque no se satisface la ley exponencial $(A^K)^L \cong A^{K \times L}$. Otra herramienta técnica importante para la definición de kk es la noción de homotopía entre morfismos de ind-álgebras.

Sea G un grupo. Basándose en el trabajo de Cortiñas-Thom [2], Ellis [4] contruyó una versión G-equivariante de kk, que llamó kk^G. Esta es una teoría de homología universal, escisiva, invariante por homotopía y G-estable en la categoría de ℓ-álgebras con una acción de G. Sólo esta definida cuando G es a lo sumo numerable.

Usando métodos completamente distintos, Garkusha [6] construyó varias categorías de K-teoría algebraica bivariante. Todas estas son universales, escisivas e invariantes por homotopía, pero satisfacen distintas condiciones de estabilidad por matrices. La *K-teoría de Kasparov inestable* $D(\mathfrak{R}, \mathfrak{F})$ no satisface ninguna condición de estabilidad por matrices. La *K-teoría de Kasparov Morita-estable* $D_{\mathrm{mor}}(\mathfrak{R}, \mathfrak{F})$ es M_n-estable para todo $n \in \mathbb{N}$. La *K-teoría de Kasparov estable* $D_{\mathrm{st}}(\mathfrak{R}, \mathfrak{F})$ es M_∞-estable —y, por lo tanto, es naturalmente isomorfa a la kk-teoría de Cortiñas-Thom.

Garkusha probó en [5] que las categorías de K-teoría de Kasparov definidas en [6] son representables por ciertos *espectros de K-teoría de Kasparov*. Por ejemplo, para todo par de álgebras A y B, contruyó un espectro $\mathbb{K}(A, B)$ tal que $\pi_n \mathbb{K}(A, B) \cong D(\mathfrak{R}, \mathfrak{F})(A, \Omega^n B)$ [5, Comparison Theorem B] —aquí, Ω es el functor de traslación en la categoría triangulada $D(\mathfrak{R}, \mathfrak{F})$. En [5, Comparison Theorem A], se calcula el grupo $\pi_0 \mathbb{K}(A, B)$ en términos de clases de homotopía de morfismos de ind-álgebras; la fórmula obtenida es prácticamente igual a la definición de $kk(A, B)$ de Cortiñas-Thom —sin tener en cuenta la M_∞-estabilidad. La demostración de este resultado, sin embargo, usa herramientas del álgebra homotópica tales como la localización de Bousfield.

En esta tesis, usamos los métodos desarrollados por Cortiñas-Thom [2] para dar nuevas construcciones de las categorías de K-teoría de Kasparov definidas por Garkusha en [6] y de otras categorías de K-teoría bivariante. En el camino, calculamos los grupos de homotopía del espacio de morfismos $\mathrm{Hom}_{\mathrm{Alg}_k}(A, B^\Delta)$ —generalizando [2, Theorem 3.3.2]— y damos una demostración simplificada de [5, Comparison Theorem A] que evita el uso de la localización de Bousfield. En el último capítulo, definimos un espectro que representa a la K-teoría algebraica bivariante G-equivariante [4], y mostramos que el teorema de Green-Julg [4, Theorem 5.2.1] y la adjunción entre inducción y restricción [4, Theorem 6.14] se levantan a equivalencias débiles de espectros.

La organización de esta tesis es la siguiente:

En el capítulo 1 presentamos definiciones y resultados usados en el resto de la tesis. Los temas tratados aquí son: categorías de diagramas dirigidos, enriquecimiento simplicial de álgebras y homotopía algebraica (polinomial). Nada de este material es nuevo.

En el capítulo 2 probamos la siguiente generalización de [2, Theorem 3.3.2]: para todo par de ℓ-álgebras A y B y todo $n \geq 0$, hay una biyección natural:

$$\pi_n \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta) \cong [A, B_\bullet^{\mathfrak{S}^n}] \tag{2}$$

Aquí, $B_\bullet^{\mathfrak{S}^n}$ es la ind-álgebra de polinomios en el cubo n-dimensional que se anulan en el borde del cubo, y los corchetes en el lado derecho de la igualdad denotan al conjunto

de clases de homotopía de morfismos. La demostración de este resultado tiene dos ingredientes clave: uno de ellos es [5, Hauptlemma (2)] y el otro es el Lema 2.3.2 —este último es un resultado en la línea de [5, Hauptlemma (3)], que compara las nociones de homotopía simplicial y algebraica.

En el capítulo 3 construímos, usando los métodos desarrollados por Cortiñas-Thom en [2], una teoría de homología universal, escisiva e invariante por homotopía —por lo tanto, naturalmente isomorfa a la categoría $D(\mathfrak{R}, \mathfrak{F})$ de Garkusha [6, Theorem 2.6 (2)]. La llamamos la *categoría de homotopía estable por lazos* y la denotamos por $\mathfrak{K}$. Nuestra definición de $\mathfrak{K}$ sigue de cerca a la construcción hecha en [3, Chapter 6] en el contexto topológico. Hay, sin embargo, algunas dificultades técnicas. Usamos (2) para dotar a $[A, B_\bullet^{\mathfrak{S}_n}]$ de una estructura de grupo abeliano; esto es necesario para manejar los signos que aparecen al permutar las coordenadas [3, Section 6.3]. Otra dificultad es que $(B^{\mathfrak{S}_m})^{\mathfrak{S}_n} \not\cong B^{\mathfrak{S}_{m+n}}$; en lugar de un isomorfismo, tenemos un *morfismo de multiplicación* $(B^{\mathfrak{S}_m})^{\mathfrak{S}_n} \to B^{\mathfrak{S}_{m+n}}$.

En el capítulo 4 recordamos la definición del espectro $\mathbb{K}(A, B)$ de Garkusha, y damos una demostración simplificada de [5, Comparison Theorem A]. Probamos que, para un conjunto simplicial finito K, los grupos $\mathfrak{K}(A^K, (B, n))$ y $\mathfrak{K}(A, (B^K, -n))$ son, respectivamente, el n-ésimo grupo de homología y el n-ésimo grupo de cohomología de K con coeficientes en $\mathbb{K}(A, B)$ —en el sentido de [12]. Como consecuencia de lo anterior, toda equivalencia débil $K \xrightarrow{\sim} L$ entre conjuntos simpliciales finitos induce un isomorfismo $X(A^L) \xrightarrow{\cong} X(A^K)$ en cualquier teoría de homología X escisiva e invariante por homotopía. Probamos que en $\mathfrak{K}$ se cumple la ley exponencial $(B^K)^L \cong B^{K \times L}$, para K y L finitos.

En el capítulo 5 definimos, para cualquier conjunto infinito $\mathcal{X}$, una teoría de homología universal, escisiva, invariante por homotopía y $M_\mathcal{X}$-estable, que llamamos $\mathfrak{K}_\mathcal{X}$. Probamos que, para toda ℓ-álgebra A, hay un isomorfismo natural $\mathfrak{K}_\mathcal{X}(\ell, A) \cong KH_0(A)$, extendiendo [2, Theorem 8.2.1]. Para cualquier grupo G, definimos una teoría de homología universal, escisiva, invariante por homotopía y G-estable en la categoría de ℓ-álgebras con una acción de G; denotamos a esta teoría por $\mathfrak{K}^G$. Aquí seguimos de cerca a [4], reemplazando a kk por $\mathfrak{K}_\mathcal{X}$ para deshacernos de la restricción sobre el cardinal de G. Finalmente, definimos un espectro que representa a $\mathfrak{K}^G$ y mostramos que los teoremas de adjunción [4, Theorem 5.2.1] y [4, Theorem 6.14] se levantan a equivalencias débiles de espectros.

Introduction

Let ℓ be a commutative ring with unit and write Alg_ℓ for the category of (not necessarily unital) ℓ-algebras and ℓ-algebra homomorphisms. An *extension* in Alg_ℓ is a short exact sequence of algebras

$$\mathscr{E} : \quad A \longrightarrow B \longrightarrow C \tag{1}$$

that splits in the category of ℓ-modules. Let $(\mathscr{T}, L)$ be a triangulated category —by this, we actually mean the opposite category of what is usually understood by *triangulated category*, so that triangles in $\mathscr{T}$ will be of the form: $LZ \to X \to Y \to Z$. Following Cortiñas-Thom [2], an *excisive homology theory* with values in $\mathscr{T}$ consists of:

(i) A functor $X : \mathrm{Alg}_\ell \to \mathscr{T}$;

(ii) A morphism $\delta_{\mathscr{E}} \in \mathscr{T}(LX(C), X(A))$ for every extension (1).

These data are subject to the following conditions:

(a) For every extension (1), the triangle below is distinguished:

$$LX(C) \xrightarrow{\delta_{\mathscr{E}}} X(A) \longrightarrow X(B) \longrightarrow X(C)$$

(b) The morphisms $\delta_{\mathscr{E}}$ are natural with respect to morphisms of extensions.

Examples of excisive homology theories were given by Cortiñas-Thom [2], Garkusha [6] and Ellis [4]. All these theories are *homotopy invariant* —i.e. they identify (polynomially) homotopic morphisms— and are characterized by a certain universal property. Most of them satisfy some kind of matrix-stability.

Cortiñas-Thom [2] introduced *algebraic kk-theory*; this is an excisive and homotopy invariant homology theory such that, for any $A \in \mathrm{Alg}_\ell$, the inclusion $s_\infty : A \to M_\infty A$ into the $(1, 1)$-place becomes invertible in kk. Algebraic kk-theory is moreover universal with respect to these properties: any other homology theory with the same properties factors uniquely through kk. An important property of kk-theory is that it relates to Weibel's homotopy K-theory KH. More precisely, for any $A \in \mathrm{Alg}_\ell$, there is a natural isomorphism $kk(\ell, \Omega^n A) \cong KH_n A$ [2, Theorem 8.2.1].

In order to define kk-theory, Cortiñas-Thom introduced a simplicial enrichment of algebras: For an ℓ-algebra A and a simplicial set K, they defined an ℓ-algebra A^K; this is to be thought of as the algebra of polynomial functions on K with coefficients in A. For two ℓ-algebras A and B, they defined a simplicial mapping space $\mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$.

However, this is not a simplicial enrichment in the sense of [9, Chapter 4] because the exponential law $(A^K)^L \cong A^{K \times L}$ fails to hold. Another important technical tool involved in the definition of kk is the notion of homotopy between ind-algebra homomorphisms.

Let G be a group. Based on the work by Cortiñas-Thom in [2], Ellis [4] constructed a G-equivariant version of kk, denoted by kk^G. This is a universal excisive, homotopy invariant and G-stable homology theory in the category of ℓ-algebras with an action of G. It is only defined when G is countable.

Using completely different methods, Garkusha constructed in [6] several bivariant algebraic K-theory categories. All of these are universal, excisive and homotopy invariant homology theories, but they differ from each other in their matrix-stability conditions. The *unstable Kasparov K-theory* $D(\mathfrak{R}, \mathfrak{F})$ is not matrix-stable at all. The *Morita stable Kasparov K-theory* $D_{\mathrm{mor}}(\mathfrak{R}, \mathfrak{F})$ is M_n-stable for all $n \in \mathbb{N}$. The *stable Kasparov K-theory* $D_{\mathrm{st}}(\mathfrak{R}, \mathfrak{F})$ is M_∞-stable —hence, it is naturally isomorphic to the kk-theory of Cortiñas-Thom.

Garkusha proved in [5] that the Kasparov K-theory categories defined in [6] are representable by certain *Kasparov K-theory spectra*. For example, for any pair of algebras A and B, he constructed a spectrum $\mathbb{K}(A, B)$ such that $\pi_n \mathbb{K}(A, B) \cong D(\mathfrak{R}, \mathfrak{F})(A, \Omega^n B)$ [5, Comparison Theorem B] —here, Ω is the translation functor in the triangulated category $D(\mathfrak{R}, \mathfrak{F})$. In [5, Comparison Theorem A], the group $\pi_0 \mathbb{K}(A, B)$ is computed in terms of homotopy classes of morphisms of ind-algebras; the formula is almost equal to the definition of $kk(A, B)$ by Cortiñas-Thom —without taking into account M_∞-stability. The proof of this result, however, involves techniques from homotopical algebra such as Bousfield localization.

In this thesis, we use the methods developed by Cortiñas-Thom [2] to give new constructions of the Kasparov K-theory categories defined by Garkusha in [6] and other bivariant K-theory categories. The first important result is the following:

Theorem 1 (Theorem 2.3.3)**.** *Let C be either Alg_ℓ or the category $G\mathrm{Alg}_\ell$ of ℓ-algebras with an action of G. Then, for any pair (A, B) of objects of C and any $n \geq 0$, there is a natural bijection:*

$$\pi_n \mathrm{Hom}_C(A, B^\Delta) \cong [A, B_\bullet^{\mathfrak{S}_n}]_C$$

Here, $B_\bullet^{\mathfrak{S}_n}$ is the ind-algebra of polynomial functions on the n-dimensional cube that vanish at the boundary of the cube, and the square brackets on the right-hand side stand for the set of homotopy classes of morphisms. This theorem is a generalization to arbitrary dimensions of [2, Theorem 3.3.2], which addresses the cases $n \leq 1$.

As an easy application of Theorem 1, we give a simplified proof Garkusha's computation of the homotopy groups of $\mathbb{K}(A, B)$ in terms of homotopy classes of morphisms [5, Corollary 7.1]; our proof doesn't involve Bousfield localization:

Theorem 2 (Theorem 4.3.3; cf. [5, Corollary 7.1])**.** *For any pair of algebras A and B and any $n \in \mathbb{Z}$, we have a natural isomorphism:*

$$\pi_n \mathbb{K}(A, B) \cong \mathrm{colim}_v [J^v A, B_\bullet^{\mathfrak{S}_{n+v}}]_C$$

Another consecuence of Theorem 1 is that, for $n \geq 2$, we have a natural abelian group structure on the set $[A, B_\bullet^{\mathfrak{S}_n}]_C$; this is relevant for the construction that we proceed

to describe. Mimicking the definition of the *suspension-stable homotopy category* in the topological setting of bornological algebras [3, Chapter 6], we define a category $\mathfrak{K}^C$ as follows: The objects of $\mathfrak{K}^C$ are pairs (A, m) with $A \in C$ and $m \in \mathbb{Z}$. The hom-sets are defined by a certain filtering colimit of *groups*:

$$\mathrm{Hom}_{\mathfrak{K}^C}((A, m), (B, n)) := \underset{v}{\mathrm{colim}}[J^{m+v}A, B_\bullet^{\mathfrak{S}_{n+v}}]_C \tag{2}$$

The definition of the composition law that makes these data into a category is technically involved and occupies the whole section 3.7. The group structure on the right-hand side of (2) is fundamental for the definition of this composition law, in order to handle the signs that appear when permuting coordinates —see, for example, Lemmas 3.7.4 and 3.7.5. We have a natural functor $j : C \to \mathfrak{K}^C$ such that $j(A) = (A, 0)$ for every $A \in C$. After showing that $\mathfrak{K}^C$ is a triangulated category (Section 3.12), we prove the main result of Chapter 3:

Theorem 3 (Theorem 3.13.12). *The functor $j : C \to \mathfrak{K}^C$ is a universal excisive and homotopy invariant homology theory.*

A universal excisive and homotopy invariant homology theory was already constructed by Garkusha [6, Theorem 2.6 (2)] using completely different methods. Of course, both constructions are naturally isomorphic, since they satisfy the same universal property. It is easily seen that a functor $F : C \to C$ that preserves extensions and homotopy induces a triangulated functor $\bar{F} : \mathfrak{K}^C \to \mathfrak{K}^C$. We prove the following similar statement about natural transformations:

Theorem 4 (Theorem 3.13.14). *Let $F, G : C \to C$ be two functors that preserve extensions and homotopy and let $\eta : F \to G$ be a natural transformation. Then η induces a unique graded natural transformation $\bar{\eta} : \bar{F} \to \bar{G}$ such that $\bar{\eta}_{j(A)} = j(\eta_A)$ for all $A \in C$.*

Let X be an infinite set and let M_X be the ℓ-algebra of finite matrices with coefficients in ℓ indexed on $X \times X$. Based on our construction of $\mathfrak{K}^C$, we construct a triangulated category $\mathfrak{K}_X^C$ endowed with a functor $j_X : C \to \mathfrak{K}_X^C$ and prove the following result:

Theorem 5 (Theorem 5.2.16). *The functor $j_X : C \to \mathfrak{K}_X^C$ is a universal excisive, homotopy invariant and M_X-stable homology theory.*

In the special case $X = \mathbb{N}$, we recover the M_∞-stable homology theories constructed by Cortiñas-Thom [2, Theorem 6.6.2] and Garkusha [5, Theorem 9.3.2]. We prove the following theorem, relating $\mathfrak{K}_X$ to Weibel's *homotopy K-theory*, KH:

Theorem 6 (Theorem 5.2.20; cf. [2, Theorem 8.2.1]). *Let X be any infinite set and let $A \in \mathrm{Alg}_\ell$. Then there is a natural isomorphism:*

$$\mathfrak{K}_X(\ell, A) \cong KH_0(A)$$

Theorem 6 is a generalization of [2, Theorem 8.2.1], which addresses the case $X = \mathbb{N}$. Finally, we use our category $\mathfrak{K}_X$ to generalize the definition of G-equivariant bivariant algebraic K-theory [4] to an arbitrary group G —the definition in [4] requires G to be countable.

We also make some computations concerning the bivariant K-theory spectra:

Theorem 7 (Propositions 4.4.1 and 4.4.5)**.** *Let X be a finite simplicial set and $A, B \in \mathrm{Alg}_\ell$. Then there are natural weak equivalences of spectra:*

$$\widetilde{\mathbb{K}}(A, B) \wedge X_+ \overset{\sim}{\longrightarrow} \mathbb{K}(A^X, B)$$

$$\mathbb{K}(A, B^X) \overset{\sim}{\longrightarrow} \mathrm{Map}(X, \mathbb{K}(A, B))$$

Here, $\widetilde{\mathbb{K}}(A, B)$ is a cofibrant replacement of $\mathbb{K}(A, B)$ in the stable model category.

Let G be a group and let $\mathfrak{K}^G$ denote the *G-equivariant bivariant K-theory* category. For any pair (A, B) of ℓ-algebras with an action of G, we define a spectrum $\mathbb{K}^G(A, B)$ representing $\mathfrak{K}^G$:

Theorem 8 (Theorem 5.3.11)**.** *Let $A, B \in G\mathrm{Alg}_\ell$ and let $n \in \mathbb{Z}$. Then there is a natural isomorphism:*

$$\pi_n \mathbb{K}^G(A, B) \cong \mathfrak{K}^G(A, (B, n))$$

We prove that the Green-Julg theorem [4, Theorem 5.2.1] and the adjunction between induction and restriction [4, Theorem 6.14] lift to weak equivalences of spectra:

Theorem 9 (Theorem 5.3.15)**.** *Let G be a finite group of n elements and suppose that n is invertible in ℓ. Let $A \in \mathrm{Alg}_\ell$ and let $B \in G\mathrm{Alg}_\ell$. Then there is a weak equivalence of spectra as follows, that induces the Green-Julg isomorphism upon taking π_0:*

$$\mathbb{K}^G(A^\tau, B) \overset{\sim}{\longrightarrow} \mathbb{K}_\infty(A, B \rtimes G)$$

Here, A^τ denotes the ℓ-algebra A with trivial G-action and $\mathbb{K}_\infty$ denotes the spectrum representing M_∞-stable bivariant K-theory.

Theorem 10 (Theorem 5.3.18)**.** *Let G be a countable group and let $H \subseteq G$ be a subgroup. Let $B \in H\mathrm{Alg}_\ell$ and let $C \in G\mathrm{Alg}_\ell$. Then there is a weak equivalence of spectra as follows:*

$$\mathbb{K}^G(\overline{\mathrm{Ind}}_H^G B, C) \overset{\sim}{\longrightarrow} \mathbb{K}^H(M_H B, \mathrm{Res}_G^H C)$$

This thesis is organized as follows:

In Chapter 1 we present definitions and results used in the rest of the thesis. The topics covered here are: categories of directed diagrams, simplicial enrichment of algebras and algebraic (polynomial) homotopy. None of this material is new.

At the beginning of Chapter 2 we define, for every pair of finite simplicial sets K and L and every $B \in \mathrm{Alg}_\ell$, an algebra homomorphism $(B^K)^L \to B^{K \times L}$ that we call *multiplication morphism* (Lemma 2.2.1). The main result of this chapter is Theorem 2.3.3, where we compute the homotopy groups of the simplicial mapping space $\mathrm{Hom}_{\mathrm{Alg}_k}(A, B^\Delta)$. This is a generalization of [2, Theorem 3.3.2], as explained above. The proof of Theorem 2.3.3 has two key ingredients, both comparing the notions of simplicial and algebraic homotopy: The first one is [5, Hauptlemma (2)], whose statement we recall in Lemma 2.3.1. The second one is Lemma 2.3.2 —this is a result in the line of [5, Hauptlemma (3)], which follows immediately from the existence of the multiplication morphisms.

In Chapter 3 we construct, using the methods developed by Cortiñas-Thom in [2], a universal excisive and homotopy invariant homology theory (Theorem 3.13.12). This theory is naturally isomorphic to Garkusha's $D(\mathfrak{R}, \mathfrak{F})$ [6, Theorem 2.6 (2)], since they both satisfy the same universal property. We call it the *loop-stable homotopy category* and denote it by $\mathfrak{R}^C$.

In Chapter 4 we recall the definition of Garkusha's unstable Kasparov K-theory spectrum $\mathbb{K}(A, B)$ (Definition 3.6.4) and we give a simplified proof of [5, Comparison Theorem A] (Theorem 4.3.3). Let K be a finite simplicial set. We prove that the groups $\mathfrak{R}(A^K, (B, n))$ and $\mathfrak{R}(A, (B^K, -n))$ are, respectively, the n-th homology and n-th cohomology groups of K with coefficients in $\mathbb{K}(A, B)$ (Propositions 4.4.1 and 4.4.5). As a consecuence of this, any weak equivalence $K \xrightarrow{\sim} L$ between finite simplicial sets induces an isomorphism $X(A^L) \xrightarrow{\cong} X(A^K)$ in any excisive and homotopy invariant homology theory X (Corollary 4.4.2). We prove that the exponential law $(B^K)^L \cong B^{K \times L}$ holds in $\mathfrak{R}$, for finite K and L (Corollary 4.4.3).

In Chapter 5 we define, for any infinite set X, a universal excisive, homotopy invariant and M_X-stable homology theory, that we denote by $\mathfrak{R}_X$ (Theorem 5.2.16). In Theorem 5.2.20 we prove that, for any ℓ-algebra A, there is a natural isomorphism $\mathfrak{R}_X(\ell, A) \cong KH_0(A)$; this extends [2, Theorem 8.2.1]. For any group G, we define a universal excisive, homotopy invariant and G-stable homology theory in the category of ℓ-algebras with an action of G; we denote this theory by $\mathfrak{R}^G$ (Theorem 5.3.8). Here we closely follow [4], replacing kk by $\mathfrak{R}_X$ to get rid of the restriction on the cardinality of G. Finally, we define a spectrum representing $\mathfrak{R}^G$ (Theorem 5.3.11) and show that the adjunction theorems [4, Theorem 5.2.1] and [4, Theorem 6.14] lift to weak equivalences of spectra (Theorems 5.3.15 and 5.3.18).

Chapter 1

Preliminaries

Resumen del capítulo

En este capítulo presentamos definiciones y resultados que usaremos más adelante; nada de este material es nuevo. En las secciones 1.1 y 1.3 principalmente fijamos notación. En la sección 1.2 estudiamos diferentes nociones de morfismo entre diagramas dirigidos y analizamos algunas relaciones entre ellas (Lema 1.2.3.1). La sección 1.4 está dedicada al enriquecimiento simplicial de álgebras introducido en [2]. Para cada par de ℓ-álgebras A y B, se define un espacio de morfismos $\mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$. Para una ℓ-álgebra B y un conjunto simplicial X, describimos una ℓ-álgebra B^X, que puede pensarse como el álgebra de polinomios en X con coeficientes en B. En la sección 1.5 estudiamos la noción de homotopía algebraica.

Chapter summary

In this chapter we present definitions and results that will be used later on; none of this material is new. In sections 1.1 and 1.3 we mainly fix notation used throughout the text. In section 1.2 we discuss different notions of morphism between directed diagrams and explore some relations among them (Lemma 1.2.3.1). Section 1.4 is concerned with the simplicial enrichment of algebras developed in [2]. For ℓ-algebras A and B, a simplicial mapping space $\mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$ is defined. For an ℓ-algebra B and a simplicial set X, we describe an ℓ-algebra B^X, which is to be thought of as the algebra of polynomials on X with coefficients in B. Section 1.5 discusses the notion of algebraic homotopy.

1.1   Conventions

Throughout this text, ℓ is a commutative ring with unit and G is a group. We only consider not necessarily unital ℓ-algebras. A G-ℓ-algebra is, by definition, an ℓ-algebra with an action of G. The letter C denotes either the category Alg_ℓ of ℓ-algebras or the category $G\mathrm{Alg}_\ell$ of G-ℓ-algebras. Simplicial objects in C can be considered as simplicial sets using

the forgetful functor $C \to \mathrm{Set}$; this is usually done without further mention. The symbol $\otimes$ indicates tensor product over $\mathbb{Z}$.

1.2 Categories of directed diagrams

Let $\mathfrak{C}$ be a category. A *directed diagram* in $\mathfrak{C}$ is a functor $X : I \to \mathfrak{C}$, where I is a filtering partially ordered set. We often write (X, I) or $X_\bullet$ for such a functor. We shall consider different categories whose objects are directed diagrams:

1.2.1 Fixing the filtering poset

Let I be a filtering poset. We will write $\mathfrak{C}^I$ for the category whose objects are the functors $X : I \to \mathfrak{C}$ and whose morphisms are the natural transformations.

If J is another filtering poset, the cartesian product $I \times J$ is a filtering poset with the product order and there is an isomorphism of categories $(\mathfrak{C}^I)^J \cong \mathfrak{C}^{I \times J}$ given by the exponential law.

1.2.2 Varying the filtering poset

We will write $\vec{\mathfrak{C}}$ for the category whose objects are the directed diagrams in $\mathfrak{C}$ and whose morphisms are defined as follows: Let (X, I) and (Y, J) be two directed diagrams. A morphism from (X, I) to (Y, J) consists of a pair (f, θ) where $\theta : I \to J$ is a functor and $f : X \to Y \circ \theta$ is a natural transformation.

For a fixed filtering poset I, there is a faithful functor $a : \mathfrak{C}^I \to \vec{\mathfrak{C}}$ that acts as the identity on objects and that sends a natural transformation f to the morphism (f, id_I).

1.2.3 The category of ind-objects

The category $\mathfrak{C}^{\mathrm{ind}}$ of ind-objects of $\mathfrak{C}$ is defined as follows: The objects of $\mathfrak{C}^{\mathrm{ind}}$ are the directed diagrams in $\mathfrak{C}$. The hom-sets are defined by:

$$\mathrm{Hom}_{\mathfrak{C}^{\mathrm{ind}}}((X, I), (Y, J)) := \lim_{i \in I} \operatorname*{colim}_{j \in J} \mathrm{Hom}_{\mathfrak{C}}(X_i, Y_j)$$

There is a functor $\vec{\mathfrak{C}} \to \mathfrak{C}^{\mathrm{ind}}$ that acts as the identity on objects and that sends a morphism $(f, \theta) : (X, I) \to (Y, J)$ to the morphism:

$$(f_i : X_i \to Y_{\theta(i)})_{i \in I} \in \lim_{i \in I} \operatorname*{colim}_{j \in J} \mathrm{Hom}_{\mathfrak{C}}(X_i, Y_j)$$

Lemma 1.2.3.1. *Let I be a filtering poset and let $\mathfrak{C}$ be a category. There are functors $\vec{a} : (\vec{\mathfrak{C}^I}) \to \vec{\mathfrak{C}}$ and $a^{\mathrm{ind}} : (\mathfrak{C}^I)^{\mathrm{ind}} \to \mathfrak{C}^{\mathrm{ind}}$ such that, for every filtering poset J, the following diagram commutes:*

$$
\begin{array}{ccccc}
(\mathfrak{C}^I)^J & \longrightarrow & (\vec{\mathfrak{C}^I}) & \longrightarrow & (\mathfrak{C}^I)^{\mathrm{ind}} \\
{\scriptstyle \cong} \downarrow & & {\scriptstyle \vec{a}} \downarrow & & {\scriptstyle a^{\mathrm{ind}}} \downarrow \\
\mathfrak{C}^{I \times J} & \longrightarrow & \vec{\mathfrak{C}} & \longrightarrow & \mathfrak{C}^{\mathrm{ind}}
\end{array}
\tag{1}
$$

Proof. Let us define the functor $\vec{a}$. If $X : J \to \mathfrak{C}^I$ is an object of $(\vec{\mathfrak{C}^I})$ and the diagram (1) commutes, then $\vec{a}(X)$ should be the functor $X : I \times J \to \mathfrak{C}$ obtained by the exponential law; this defines $\vec{a}$ on objects. Let $(f, \theta) : (X, J) \to (Y, K)$ be a morphism in $(\vec{\mathfrak{C}^I})$, i.e. $\theta : J \to K$ is a functor and $f : X \to Y \circ \theta$ is a natural transformation of functors $J \to \mathfrak{C}^I$. Then $\mathrm{id}_I \times \theta : I \times J \to I \times K$ is a functor and $f : X \to Y \circ (\mathrm{id}_I \times \theta)$ is a natural transformation of functors $I \times J \to \mathfrak{C}$; this defines $\vec{a}$ on morphisms. It is clear that the definitions above determine a functor $\vec{a}$ that makes the left square in (1) commute.

Let us define the functor a^{ind}. Again, it is clear how to define the functor on objects: one should use the exponential law. Let $X : J \to \mathfrak{C}^I$ and $Y : K \to \mathfrak{C}^I$ be two objects of $(\mathfrak{C}^I)^{\mathrm{ind}}$. We have to define a function:

$$\lim_{j \in J} \operatorname*{colim}_{k \in K} \int_{i \in I} \operatorname{Hom}_{\mathfrak{C}}(X_j(i), Y_k(i)) \longrightarrow \lim_{(\tilde{i}, \tilde{j}) \in I \times J} \operatorname*{colim}_{(\hat{i}, \hat{k}) \in I \times K} \operatorname{Hom}_{\mathfrak{C}}(X_{\tilde{j}}(\tilde{i}), Y_{\hat{k}}(\hat{i}))$$

This is equivalent to defining compatible functions:

$$\pi_{(\tilde{i}, \tilde{j})} : \lim_{j \in J} \operatorname*{colim}_{k \in K} \int_{i \in I} \operatorname{Hom}_{\mathfrak{C}}(X_j(i), Y_k(i)) \longrightarrow \operatorname*{colim}_{(\hat{i}, \hat{k}) \in I \times K} \operatorname{Hom}_{\mathfrak{C}}(X_{\tilde{j}}(\tilde{i}), Y_{\hat{k}}(\hat{i}))$$

Let $\pi_{(\tilde{i}, \tilde{j})}$ be the composite of the following solid morphisms, whose definition we proceed to explain:

$$
\begin{array}{ccc}
\displaystyle\lim_{j \in J} \operatorname*{colim}_{k \in K} \int_{i \in I} \operatorname{Hom}_{\mathfrak{C}}(X_j(i), Y_k(i)) & \xrightarrow{\pi_{(\tilde{i}, \tilde{j})}} & \displaystyle\operatorname*{colim}_{(\hat{k}, \hat{i}) \in K \times I} \operatorname{Hom}_{\mathfrak{C}}(X_{\tilde{j}}(\tilde{i}), Y_{\hat{k}}(\hat{i})) \\
\downarrow & & \uparrow \\
\displaystyle\operatorname*{colim}_{k \in K} \int_{i \in I} \operatorname{Hom}_{\mathfrak{C}}(X_{\tilde{j}}(i), Y_k(i)) & \longrightarrow & \displaystyle\operatorname*{colim}_{k \in K} \operatorname{Hom}_{\mathfrak{C}}(X_{\tilde{j}}(\tilde{i}), Y_k(\tilde{i}))
\end{array}
$$

The vertical map on the left is the projection from $\lim_{j \in J}$; the horizontal map on the bottom is the $\operatorname{colim}_{k \in K}$ of the projections

$$\int_{i \in I} \operatorname{Hom}_{\mathfrak{C}}(X_{\tilde{j}}(i), Y_k(i)) \longrightarrow \operatorname{Hom}_{\mathfrak{C}}(X_{\tilde{j}}(\tilde{i}), Y_k(\tilde{i}));$$

and the vertical map on the right is the one induced by the structural morphisms:

$$\operatorname{Hom}_{\mathfrak{C}}(X_{\tilde{j}}(\tilde{i}), Y_k(\tilde{i})) \longrightarrow \operatorname*{colim}_{(\hat{i}, \hat{k}) \in I \times K} \operatorname{Hom}_{\mathfrak{C}}(X_{\tilde{j}}(\tilde{i}), Y_{\hat{k}}(\hat{i}))$$

It is straightforward but tedious to verify that this definition determines a functor a^{ind} that makes the diagram (1) commute. $\square$

1.3 Simplicial sets

The category of simplicial sets is denoted by $\mathbb{S}$ [9, Chapter 3]. Let $\operatorname{Map}(?, ??)$ be the internal-hom in $\mathbb{S}$; we often write Y^X instead of $\operatorname{Map}(X, Y)$.

1.3.1   The iterated last vertex map

Let $\mathrm{sd} : \mathbb{S} \to \mathbb{S}$ be the subdivision functor. There is a natural transformation $\gamma : \mathrm{sd} \to \mathrm{id}_{\mathbb{S}}$ called the *last vertex map* [7, Section III. 4]. Put $\gamma^1 := \gamma$ and define inductively γ_X^n to be the following composite:

$$\mathrm{sd}^n X = \mathrm{sd}(\mathrm{sd}^{n-1}X) \xrightarrow{\ \gamma^1_{\mathrm{sd}^{n-1}X}\ } \mathrm{sd}^{n-1}X \xrightarrow{\ \gamma^{n-1}_X\ } X$$

It is immediate that $\gamma^n : \mathrm{sd}^n \to \mathrm{id}_{\mathbb{S}}$ is a natural transformation. Let $\mathrm{sd}^0 : \mathbb{S} \to \mathbb{S}$ be the identity functor and let $\gamma^0 : \mathrm{sd}^0 \to \mathrm{id}_{\mathbb{S}}$ be the identity natural transformation.

Lemma 1.3.1.1. *For any $p, q \geq 0$ and any $X \in \mathbb{S}$ we have:*

$$\gamma_X^{p+q} = \gamma_X^p \circ \mathrm{sd}^p\left(\gamma_X^q\right) = \gamma_X^p \circ \gamma_{\mathrm{sd}^p X}^q$$

Proof. It follows from a straightforward induction on $n = p + q$. $\square$

1.3.2   Simplicial cubes

Let $I := \Delta^1$ and let $\partial I := \{0, 1\} \subset I$. For $n \geq 1$, let $I^n := I \times \cdots \times I$ be the n–fold direct product and let ∂I^n be the following simplicial subset of I^n:

$$\partial I^n := [(\partial I) \times I \times \cdots \times I] \cup [I \times (\partial I) \times \cdots \times I] \cup \cdots \cup [I \times \cdots \times I \times (\partial I)]$$

Let $I^0 := \Delta^0$ and let $\partial I^0 := \emptyset$. We identify $I^{m+n} = I^m \times I^n$ and $\partial(I^{m+n}) = [(\partial I^m) \times I^n] \cup [I^m \times (\partial I^n)]$ using the associativity and unit isomorphisms of the direct product in $\mathbb{S}$.

1.3.3   Iterated loop spaces

Let $(X, *)$ be a pointed fibrant simplicial set. Recall from [7, Section I.7] that the loopspace ΩX is defined as the fiber of a natural fibration $\pi_X : PX \to X$, where PX has trivial homotopy groups. By the long exact sequence associated to a fibration, we have pointed bijections $\pi_{n+1}(X, *) \cong \pi_n(\Omega X, *)$ for $n \geq 0$ that are group isomorphisms for $n \geq 1$. Iterating the loopspace construction we get:

$$\pi_0(\Omega^n X) \cong \pi_1(\Omega^{n-1}X, *) \cong \cdots \cong \pi_n(X, *)$$

Thus, $\pi_0 \Omega^n X$ is a group for $n \geq 1$ and this group is abelian for $n \geq 2$. Moreover, a morphism $\varphi : X \to Y$ of pointed fibrant simplicial sets induces group homomorphisms $\varphi_* : \pi_0\Omega^n X \to \pi_0\Omega^n Y$ for $n \geq 1$. Let inc denote the inclusion $\partial I^n \to I^n$. It is easy to see that the iterated loop functor Ω^n on pointed fibrant simplicial sets can be alternatively defined by the following pullback of simplicial sets:

$$\begin{array}{ccc} \Omega^n X & \xrightarrow{\ \iota_{n,X}\ } & \mathrm{Map}(I^n, X) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{inc}^*} \\ \Delta^0 & \xrightarrow{\ *\ } & \mathrm{Map}(\partial I^n, X) \end{array} \qquad (2)$$

We will always use this description of Ω^n. Occasionally we will need to compare Ω^n for different integers n; for this purpose we will explicitly describe how the diagram (2) arises from successive applications of the functor Ω. We start defining ΩX by the following pullback in $\mathbb{S}$:

$$\begin{array}{ccc} \Omega X & \xrightarrow{\iota_{1,X}} & \mathrm{Map}(I, X) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{inc}^*} \\ \Delta^0 & \xrightarrow{\quad * \quad} & \mathrm{Map}(\partial I, X) \end{array}$$

For $n \geq 1$, define inductively $\iota_{n+1,X} : \Omega^{n+1}X \to \mathrm{Map}(I^{n+1}, X)$ as the following composite:

$$\Omega\left(\Omega^n X\right) \xrightarrow{\iota_{1,\Omega^n X}} \mathrm{Map}\left(I, \Omega^n X\right) \xrightarrow{(\iota_{n,X})_*} \mathrm{Map}\left(I, \mathrm{Map}(I^n, X)\right) \cong \mathrm{Map}\left(I^n \times I, X\right)$$

It is easily verified that (2) is a pullback. Moreover, $\iota_{m+n,X}$ equals the following composite:

$$\Omega^n\left(\Omega^m X\right) \xrightarrow{\iota_{n,\Omega^m X}} \mathrm{Map}\left(I^n, \Omega^m X\right) \xrightarrow{(\iota_{m,X})_*} \mathrm{Map}\left(I^n, \mathrm{Map}(I^m, X)\right) \cong \mathrm{Map}\left(I^m \times I^n, X\right)$$

Thus, under the identification of diagram (2), each time we apply Ω the new I-coordinate appears to the right.

1.4   Simplicial enrichment of algebras

We proceed to recall some of the details of the simplicial enrichment of Alg_ℓ introduced in [2, Section 3]. Let $\mathbb{Z}^\Delta$ be the simplicial ring defined by:

$$[p] \mapsto \mathbb{Z}^{\Delta^p} := \mathbb{Z}[t_0, \dots, t_p]/\langle 1 - \textstyle\sum t_i \rangle$$

An order-preserving function $\varphi : [p] \to [q]$ induces a ring homomorphism $\mathbb{Z}^{\Delta^q} \to \mathbb{Z}^{\Delta^p}$ by the formula:

$$t_i \mapsto \sum_{\varphi(j)=i} t_j$$

Now let $B \in \mathrm{Alg}_\ell$ and define a simplicial ℓ-algebra B^Δ by:

$$[p] \mapsto B^{\Delta^p} := B \otimes \mathbb{Z}^{\Delta^p} \tag{3}$$

If A is another ℓ-algebra, the simplicial set $\mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$ is called the *simplicial mapping space* from A to B. For $X \in \mathbb{S}$, put $B^X := \mathrm{Hom}_{\mathbb{S}}(X, B^\Delta)$; it is easily verified that B^X is an ℓ-algebra with the operations defined pointwise. When $X = \Delta^p$, this definition of B^{Δ^p} coincides with (3). We have a natural isomorphism as follows, where the limit is taken over the category of simplices of X:

$$B^X \xrightarrow{\ \cong\ } \lim_{\Delta^p \downarrow X} B^{\Delta^p} \tag{4}$$

For $A, B \in \mathrm{Alg}_\ell$ and $X \in \mathbb{S}$ we have the following adjunction isomorphism:

$$\mathrm{Hom}_\mathbb{S}(X, \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)) \cong \mathrm{Hom}_\mathbb{S}\left(\operatorname*{colim}_{\Delta^p \downarrow X} \Delta^p, \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right)$$

$$\cong \lim_{\Delta^p \downarrow X} \mathrm{Hom}_\mathbb{S}\left(\Delta^p, \mathrm{Hom}_{\mathrm{Alg}_\ell}\left(A, B^\Delta\right)\right)$$

$$\cong \lim_{\Delta^p \downarrow X} \mathrm{Hom}_{\mathrm{Alg}_\ell}\left(A, B^{\Delta^p}\right)$$

$$\cong \mathrm{Hom}_{\mathrm{Alg}_\ell}\left(A, \lim_{\Delta^p \downarrow X} B^{\Delta^p}\right)$$

$$\cong \mathrm{Hom}_{\mathrm{Alg}_\ell}\left(A, B^X\right)$$

The category $G\mathrm{Alg}_\ell$ has a simplicial enrichment as well [4, Section 2.3]; we proceed to recall some of the details. Let $B \in G\mathrm{Alg}_\ell$. For any $X \in \mathbb{S}$, consider $\mathbb{Z}^X$ as a G-ring with the trivial action of G. Consider $B^{\Delta^p} = B \otimes \mathbb{Z}^{\Delta^p}$ as a G-ℓ-algebra with the diagonal action of G. Now the assignment $[p] \mapsto B^{\Delta^p}$ defines a simplicial G-ℓ-algebra B^Δ. If A is another G-ℓ-algebra, the simplicial set $\mathrm{Hom}_{G\mathrm{Alg}_\ell}(A, B^\Delta)$ is called the *simplicial mapping space* from A to B. For $X \in \mathbb{S}$, the ℓ-algebra $B^X = \mathrm{Hom}_\mathbb{S}(X, B^\Delta)$ is now a G-ℓ-algebra with the G-action defined pointwise. The morphism (4) is in this case a G-ℓ-algebra isomorphism. Again, for $A, B \in G\mathrm{Alg}_\ell$ and $X \in \mathbb{S}$ there is an adjunction isomorphism:

$$\mathrm{Hom}_\mathbb{S}\left(X, \mathrm{Hom}_{G\mathrm{Alg}_\ell}\left(A, B^\Delta\right)\right) \cong \mathrm{Hom}_{G\mathrm{Alg}_\ell}\left(A, B^X\right)$$

Remark 1.4.1. Let X and Y be simplicial sets. In general $(B^X)^Y \not\cong B^{X \times Y}$ —this already fails when X and Y are standard simplices; see [2, Remark 3.1.4].

Remark 1.4.2. The simplicial ring $\mathbb{Z}^\Delta$ is commutative and hence the same holds for the rings $\mathbb{Z}^X = \mathrm{Hom}_\mathbb{S}\left(X, \mathbb{Z}^\Delta\right)$, for any $X \in \mathbb{S}$. Thus, the multiplication in $\mathbb{Z}^X$ induces a ring homomorphism $\mathbf{m}_X : \mathbb{Z}^X \otimes \mathbb{Z}^X \to \mathbb{Z}^X$. Note that $\mathbf{m}_X$ is natural in X.

1.5   Algebraic homotopy

Two morphisms $f_0, f_1 : A \to B$ in C are *elementary homotopic* if there exists $f : A \to B^{\Delta^1}$ such that the following diagram commutes for $i = 0, 1$:

$$\begin{array}{ccc}
A & \xrightarrow{\;\;f\;\;} & B^{\Delta^1} \\
{\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle (d^i)^*} \\
B & \xrightarrow{\;\cong\;} & B^{\Delta^0}
\end{array}$$

Here the $d^i : \Delta^0 \to \Delta^1$ are the coface maps. Elementary homotopy $\sim_e$ is a reflexive and symmetric relation, but it is not transitive. Let $\sim$ be the transitive closure of $\sim_e$. Two morphisms $f_0, f_1 : A \to B$ in C are *homotopic* if $f_0 \sim f_1$. It can be shown that $f_0 \sim f_1$ iff

there exist $r \in \mathbb{N}$ and $f : A \to B^{\mathrm{sd}^r \Delta^1}$ such that the following diagrams commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B^{\mathrm{sd}^r \Delta^1} \\
{\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle (d^i)^*} \\
B & \xrightarrow{\ \cong\ } & B^{\mathrm{sd}^r \Delta^0}
\end{array}
$$

Let $[A, B]_C := \mathrm{Hom}_C(A, B)/ \sim$; we will often drop C from the notation and write $[A, B]$ instead of $[A, B]_C$. It can be shown that $\sim$ is compatible with composition; i.e. $f \sim g$ implies $h \circ f \sim h \circ g$ and $f \circ k \sim g \circ k$. Thus, we have a category $[C]$ whose objects are the objects of C and whose hom-sets are the sets $[A, B]_C$. We also have an obvious functor $C \to [C]$.

Definition 1.5.1 ([2, Definition 3.1.1]). Let (A, I) and (B, J) be two directed diagrams in C and let $f, g \in \mathrm{Hom}_{C^{\mathrm{ind}}}((A, I), (B, J))$. We say that f and g are *homotopic* if they correspond to the same morphism upon applying the functor $C^{\mathrm{ind}} \to [C]^{\mathrm{ind}}$. We also write:

$$
[A_\bullet, B_\bullet]_C := \mathrm{Hom}_{[C]^{\mathrm{ind}}}((A, I), (B, J)) = \lim_{i \in I} \mathrm{colim}_{j \in J} [A_i, B_j]_C
$$

Chapter 2

Homotopy groups of the simplicial mapping space

Resumen del capítulo

Sean A y B dos ℓ-álgebras. En este capítulo calculamos los grupos de homotopía del espacio de morfismos $\mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$. Más precisamente, en el Teorema 2.3.3 probamos que hay una biyección natural:

$$\pi_n Ex^\infty \mathrm{Hom}_C(A, B^\Delta) \cong [A, B_\bullet^{\mathfrak{S}_n}]_C$$

Aquí, $B_\bullet^{\mathfrak{S}_n}$ es la ind-álgebra de polinomios en el cubo n-dimensional a coeficientes en B que se anulan en el borde del cubo. Este resultado es una generalización de [2, Theorem 3.3.2]. En la sección 2.2 definimos, para cada par de conjuntos simpliciales finitos K y L, un morfismo de álgebras $\mu^{K,L} : (B^K)^L \to B^{K \times L}$; llamamos a estos morfismos *morfismos de multiplicación*. En la sección 2.3 probamos el Lema 2.3.2; este es un resultado en la línea de [5, Hauptlemma (3)] que se deduce inmediatamente de la existencia de los morfismos de multiplicación. Finalmente, usamos el Lema 2.3.2 y [5, Hauptlemma (2)] para probar el Teorema 2.3.3.

Chapter summary

Let A and B be two ℓ-algebras. In this chapter we compute the homotopy groups of the simplicial mapping space $\mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$. More precisely, in Theorem 2.3.3, we prove that there is a natural bijection:

$$\pi_n Ex^\infty \mathrm{Hom}_C(A, B^\Delta) \cong [A, B_\bullet^{\mathfrak{S}_n}]_C$$

Here, $B_\bullet^{\mathfrak{S}_n}$ is the ind-algebra of polynomials on the n-dimensional cube with coefficients in B vanishing at the boundary of the cube. This result is a generalization of [2, Theorem 3.3.2]. In section 2.2 we define, for every pair of finite simplicial sets K and L, an algebra homomorphism $\mu^{K,L} : (B^K)^L \to B^{K \times L}$; we call these morphisms *multiplication morphisms*. In section 2.3 we prove Lemma 2.3.2, which is a result in the line of [5, Hauptlemma (3)]

that follows immediately from the existence of multiplication morphisms. Finally, we use Lemma 2.3.2 and [5, Hauptlemma (2)] to prove Theorem 2.3.3.

2.1   Functions vanishing on a subset

A *simplicial pair* is a pair (K, L) where K is a simplicial set and $L \subseteq K$ is a simplicial subset. A *morphism of pairs* $f : (K', L') \to (K, L)$ is a morphism of simplicial sets $f : K' \to K$ such that $f(L') \subseteq L$. A simplicial pair (K, L) is *finite* if K is a finite simplicial set. We will only consider finite simplicial pairs, omitting the word "finite" from now on. Let (K, L) be a simplicial pair, let $B \in C$ and let $r \geq 0$. Put:

$$B_r^{(K,L)} := \ker \left(B^{\mathrm{sd}^r K} \longrightarrow B^{\mathrm{sd}^r L} \right) \in C$$

The last vertex map induces morphisms $B_r^{(K,L)} \to B_{r+1}^{(K,L)}$ and we usually consider $B_\bullet^{(K,L)}$ as a directed diagram in C:

$$B_\bullet^{(K,L)} : B_0^{(K,L)} \longrightarrow B_1^{(K,L)} \longrightarrow B_2^{(K,L)} \longrightarrow \cdots$$

Notice that a morphism $f : (K', L') \to (K, L)$ induces a morphism $f^* : B_\bullet^{(K,L)} \to B_\bullet^{(K',L')}$ of $\mathbb{Z}_{\geq 0}$-diagrams.

Lemma 2.1.1 (cf. [2, Proposition 3.1.3])**.** *Let (K, L) be a simplicial pair and let $B \in C$. Then $\mathbb{Z}_r^{(K,L)}$ is a free abelian group and there is a natural isomorphism in C:*

$$B \otimes \mathbb{Z}_r^{(K,L)} \xrightarrow{\cong} B_r^{(K,L)} \tag{1}$$

In the case $C = G\mathrm{Alg}_\ell$, we consider $\mathbb{Z}_r^{(K,L)}$ as a G-ring with the trivial action of G, and the domain of (1) as a G-ℓ-algebra with the diagonal G-action.

Proof. The following sequence is exact by definition of $\mathbb{Z}_r^{(K,L)}$ and [2, Lemma 3.1.2]:

$$0 \longrightarrow \mathbb{Z}_r^{(K,L)} \longrightarrow \mathbb{Z}^{\mathrm{sd}^r K} \longrightarrow \mathbb{Z}^{\mathrm{sd}^r L} \longrightarrow 0 \tag{2}$$

The group $\mathbb{Z}^{\mathrm{sd}^r L}$ is free abelian by [2, Proposition 3.1.3] and thus the sequence (2) splits. It follows that $\mathbb{Z}_r^{(K,L)}$ is free because it is a direct summand of the free abelian group $\mathbb{Z}^{\mathrm{sd}^r K}$. Moreover, the following sequence is exact:

$$0 \longrightarrow B \otimes \mathbb{Z}_r^{(K,L)} \longrightarrow B \otimes \mathbb{Z}^{\mathrm{sd}^r K} \longrightarrow B \otimes \mathbb{Z}^{\mathrm{sd}^r L} \longrightarrow 0$$

To finish the proof we identify $B \otimes \mathbb{Z}^{\mathrm{sd}^r K} \xrightarrow{\cong} B^{\mathrm{sd}^r K}$ using the natural isomorphism of [2, Proposition 3.1.3]. It is immediate to check that the isomorphism (1) respects the G-action in the equivariant setting. □

Important example 2.1.2. Let $(K, L) = (I^n, \partial I^n)$. Following [5, Section 7.2], we will write $B_\bullet^{\mathfrak{S}_n}$ instead of $B_\bullet^{(K,L)}$. Notice that $B_\bullet^{\mathfrak{S}_0}$ is the constant $\mathbb{Z}_{\geq 0}$-diagram B.

2.2 Multiplication morphisms

Let (K, L) and (K', L') be simplicial pairs. It follows from Lemma 2.1.1 that $\mathbb{Z}_r^{(K,L)} \otimes \mathbb{Z}_s^{(K',L')}$ identifies with a subring of $\mathbb{Z}^{\mathrm{sd}^r K} \otimes \mathbb{Z}^{\mathrm{sd}^s K'}$. Let $\mu^{K,K'}$ be the composite of the following ring homomorphisms:

$$
\begin{array}{ccc}
\mathbb{Z}^{\mathrm{sd}^r K} \otimes \mathbb{Z}^{\mathrm{sd}^s K'} & \xrightarrow{\ (\gamma^s)^* \otimes (\gamma^r)^*\ } & \mathbb{Z}^{\mathrm{sd}^{r+s} K} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s} K'} \\
{\scriptstyle \mu^{K,K'}}\Big\downarrow & & \Big\downarrow {\scriptstyle (\mathrm{pr}_1)^* \otimes (\mathrm{pr}_2)^*} \\
\mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')} & \xleftarrow{\quad \mathbf{m} \quad} & \mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')}
\end{array}
$$

Here γ^j is the iterated last vertex map defined in section 1.3.1, pr_j is the projection of the direct product into its j-th factor and $\mathbf{m}$ is the map described in Remark 1.4.2.

Lemma 2.2.1. *The morphism $\mu^{K,K'}$ defined above induces a ring homomorphism:*

$$
\mu^{(K,L),(K',L')} : \mathbb{Z}_r^{(K,L)} \otimes \mathbb{Z}_s^{(K',L')} \longrightarrow \mathbb{Z}_{r+s}^{(K \times K', (K \times L') \cup (K' \times L))}
$$

Moreover, $\mu^{(K,L),(K',L')}$ is natural in both variables with respect to morphisms of simplicial pairs. We call $\mu^{(K,L),(K',L')}$ a multiplication morphism.

Proof. Let ε be the restriction of $\mu^{K,K'}$ to $\mathbb{Z}_r^{(K,L)} \otimes \mathbb{Z}_s^{(K',L')}$; we have to show that ε is zero when composed with the morphism:

$$
\mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')} \longrightarrow \mathbb{Z}^{\mathrm{sd}^{r+s}((K \times L') \cup (L \times K'))}
$$

Since the functor $\mathbb{Z}^{\mathrm{sd}^{r+s}(?)} : \mathbb{S} \to \mathrm{Alg}_{\mathbb{Z}}^{\mathrm{op}}$ commutes with colimits, it will be enough to show that ε is zero when composed with the projections to $\mathbb{Z}^{\mathrm{sd}^{r+s}(K \times L')}$ and to $\mathbb{Z}^{\mathrm{sd}^{r+s}(K' \times L)}$; this is a straightforward check. For example, the following commutative diagram shows that ε is zero when composed with the projection to $\mathbb{Z}^{\mathrm{sd}^{r+s}(L \times K')}$; we write i for the inclusion $L \subseteq K$.

$$
\begin{array}{ccc}
\mathbb{Z}_r^{(K,L)} \otimes \mathbb{Z}_s^{(K',L')} & & \\
\Big\downarrow & \searrow^{\ 0} & \\
\mathbb{Z}^{\mathrm{sd}^r K} \otimes \mathbb{Z}^{\mathrm{sd}^s K'} & \xrightarrow{\ i^* \otimes 1\ } & \mathbb{Z}^{\mathrm{sd}^r L} \otimes \mathbb{Z}^{\mathrm{sd}^s K'} \\
{\scriptstyle (\gamma^s)^* \otimes (\gamma^r)^*}\Big\downarrow & & \Big\downarrow {\scriptstyle (\gamma^s)^* \otimes (\gamma^r)^*} \\
\mathbb{Z}^{\mathrm{sd}^{r+s} K} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s} K'} & \xrightarrow{\ i^* \otimes 1\ } & \mathbb{Z}^{\mathrm{sd}^{r+s} L} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s} K'} \\
{\scriptstyle (\mathrm{pr}_1)^* \otimes (\mathrm{pr}_2)^*}\Big\downarrow & & \Big\downarrow {\scriptstyle (\mathrm{pr}_1)^* \otimes (\mathrm{pr}_2)^*} \\
\mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')} & \xrightarrow{\ i^* \otimes i^*\ } & \mathbb{Z}^{\mathrm{sd}^{r+s}(L \times K')} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s}(L \times K')} \\
{\scriptstyle \mathbf{m}}\Big\downarrow & & \Big\downarrow {\scriptstyle \mathbf{m}} \\
\mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')} & \xrightarrow{\quad i^* \quad} & \mathbb{Z}^{\mathrm{sd}^{r+s}(L \times K')}
\end{array}
$$

The assertion about naturality is clear. $\qquad\square$

Remark 2.2.2. We can consider $\mathbb{Z}_\bullet^{(K,L)} \otimes \mathbb{Z}_\bullet^{(K',L')}$ as a directed diagram of rings indexed over $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Let $\theta : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be defined by $\theta(r, s) = r + s$; it is clear that θ is a functor. Then the morphisms of Lemma 2.2.1 assemble into a morphism in $(\vec{\mathrm{Alg}}_\mathbb{Z})$:

$$\left(\mu^{(K,L),(K',L')}, \theta\right) : \mathbb{Z}_\bullet^{(K,L)} \otimes \mathbb{Z}_\bullet^{(K',L')} \longrightarrow \mathbb{Z}_\bullet^{(K\times K',(K\times L')\cup(K'\times L))}$$

We will often consider $\mu^{(K,L),(K',L')}$ in this way, omitting θ from the notation.

Remark 2.2.3. Upon tensoring $\mu^{(K,L),(K',L')}$ with $B \in C$ and using (1) we obtain a morphism in C:

$$\mu_B^{(K,L),(K',L')} : \left(B_r^{(K,L)}\right)_s^{(K',L')} \longrightarrow B_{r+s}^{(K\times K',(L\times K')\cup(K\times L'))}$$

This morphism is natural in both variables with respect to morphisms of simplicial pairs. Again, defining θ as in Remark 2.2.2, we have a morphism in $\vec{C}$:

$$\left(\mu_B^{(K,L),(K',L')}, \theta\right) : \left(B_\bullet^{(K,L)}\right)_\bullet^{(K',L')} \longrightarrow B_\bullet^{(K\times K',(L\times K')\cup(K\times L'))}$$

Important example 2.2.4. Let $(K, L) = (I^m, \partial I^m)$ and let $(K', L') = (I^n, \partial I^n)$. By Remark 2.2.3, we have a morphism in C:

$$\mu_B^{(K,L),(K',L')} : \left(B_r^{\mathfrak{S}_m}\right)_s^{\mathfrak{S}_n} \longrightarrow B_{r+s}^{\mathfrak{S}_{m+n}}$$

In this case, we will write $\mu_B^{m,n}$ instead of $\mu_B^{(K,L),(K',L')}$. It is straightforward to verify that these maps are associative; i.e. that the following diagram in C commutes:

$$
\begin{array}{ccc}
\left(\left(B_r^{\mathfrak{S}_l}\right)_s^{\mathfrak{S}_m}\right)_t^{\mathfrak{S}_n} & \xrightarrow{\mu_{B_r^{\mathfrak{S}_l}}^{m,n}} & \left(B_r^{\mathfrak{S}_l}\right)_{s+t}^{\mathfrak{S}_{m+n}} \\
\left(\mu_B^{l,m}\right)_t^{\mathfrak{S}_n} \downarrow & & \downarrow \mu_B^{l,m+n} \\
\left(B_{r+s}^{\mathfrak{S}_{l+m}}\right)_t^{\mathfrak{S}_n} & \xrightarrow{\mu_B^{l+m,n}} & B_{r+s+t}^{\mathfrak{S}_{l+m+n}}
\end{array}
$$

Indeed, asociativity holds for the maps $\mu_B^{(K,L),(K',L')}$ of Remark 2.2.3.

Example 2.2.5. For any $n \geq 0$ and any $B \in C$ we have a morphism $\iota : B \to B^{\Delta^n}$ induced by $\Delta^n \to *$. It is well known that ι is a homotopy equivalence, as we proceed to explain. Let $v : B^{\Delta^n} \to B$ be the restriction to the 0-simplex 0. Explicitly, we have $v(t_i) = 0$ for $i > 0$ and $v(t_0) = 1$. It is easily verified that $v \circ \iota = \mathrm{id}_B$. Now let $H : B^{\Delta^p} \to B^{\Delta^n}[u]$ be the elementary homotopy defined by $H(t_i) = ut_i$ for $i > 0$ and $H(t_0) = t_0 + (1-u)(t_1 + \cdots + t_n)$. We have $\mathrm{ev}_1 \circ H = \mathrm{id}_{B^{\Delta^n}}$ and $\mathrm{ev}_0 \circ H = \iota \circ v$. This shows that $\iota \circ v = \mathrm{id}_{B^{\Delta^n}}$ in $[C]$.

The homotopy H constructed above is natural with respect to the inclusion of faces of Δ^n that contain the 0-simplex 0. More precisely: if $f : [m] \to [n]$ is an injective order-preserving map such that $f(0) = 0$, then the following diagram commutes:

$$
\begin{array}{ccc}
B^{\Delta^n} & \xrightarrow{H} & B^{\Delta^n}[u] \\
f^* \downarrow & & \downarrow f^*[u] \\
B^{\Delta^m} & \xrightarrow{H} & B^{\Delta^m}[u]
\end{array}
$$

Now let $p, q \geq 0$. Recall from the proof of [9, Lemma 3.1.8] that the simplices of $\Delta^p \times \Delta^q$ can be identified with the chains in $[p] \times [q]$ with the product order. The nondegenerate $(p + q)$-simplices of $\Delta^p \times \Delta^q$ are identified with the maximal chains in $[p] \times [q]$; there are exactly $\binom{p+q}{p}$ of these. Following [9], let $c(i)$ for $1 \leq i \leq \binom{p+q}{p}$ be the complete list of maximal chains of $[p] \times [q]$. Then $\Delta^p \times \Delta^q$ is the coequalizer in $\mathbb{S}$ of the two natural morphisms of simplicial sets f and g induced by the inclusions $c(i) \cap c(j) \subseteq c(i)$ and $c(i) \cap c(j) \subseteq c(j)$ respectively:

$$f, g : \coprod_{1 \leq i < j \leq \binom{p+q}{p}} \Delta^{n_{c(i) \cap c(j)}} \rightrightarrows \coprod_{1 \leq i \leq \binom{p+q}{p}} \Delta^{n_{c(i)}}$$

Here n_c is the number of edges in c; that is, the dimension of the nondegenerate simplex corresponding to c. Since $B^? : \mathbb{S}^{\mathrm{op}} \to C$ preserves limits, it follows that $B^{\Delta^p \times \Delta^q}$ is the equalizer of the following diagram in C:

$$f^*, g^* : \prod_{1 \leq i \leq \binom{p+q}{p}} B^{\Delta^{n_{c(i)}}} \rightrightarrows \prod_{1 \leq i < j \leq \binom{p+q}{p}} B^{\Delta^{n_{c(i) \cap c(j)}}} \tag{3}$$

Moreover, since $\mathbb{Z}[u]$ is a flat ring, $? \otimes \mathbb{Z}[u]$ preserves finite limits and $B^{\Delta^p \times \Delta^q}[u]$ is the equalizer of the following diagram:

$$f^*[u], g^*[u] : \prod_{1 \leq i \leq \binom{p+q}{p}} B^{\Delta^{n_{c(i)}}}[u] \rightrightarrows \prod_{1 \leq i < j \leq \binom{p+q}{p}} B^{\Delta^{n_{c(i) \cap c(j)}}}[u] \tag{4}$$

Notice that every maximal chain of $[p] \times [q]$ starts at $(0, 0)$. This implies, by the discussion above on the naturality of H, that the following diagram commutes for every i and j:

$$
\begin{array}{ccc}
B^{\Delta^{n_{c(i)}}} & \xrightarrow{\ H\ } & B^{\Delta^{n_{c(i)}}}[u] \\
\downarrow & & \downarrow \\
B^{\Delta^{n_{c(i) \cap c(j)}}} & \xrightarrow{\ H\ } & B^{\Delta^{n_{c(i) \cap c(j)}}}[u]
\end{array}
$$

Then the homotopy H on the different $B^{\Delta^{n_{c(i)}}}$ gives a morphism of diagrams from (3) to (4) that induces $H : B^{\Delta^p \times \Delta^q} \to B^{\Delta^p \times \Delta^q}[u]$. Let $\iota : B \to B^{\Delta^p \times \Delta^q}$ be the morphism induced by $\Delta^p \times \Delta^q \to *$ and let $v : B^{\Delta^p \times \Delta^q} \to B$ be the restriction to the 0-simplex $(0, 0)$. It is easily verified that $\mathrm{ev}_1 \circ H$ is the identity of $B^{\Delta^p \times \Delta^q}$ and that $\mathrm{ev}_0 \circ H = \iota \circ v$; this shows that ι is a homotopy equivalence.

Finally, consider the following commutative diagram. Since each ι is a homotopy equivalence, it follows that $\mu^{\Delta^p, \Delta^q} : (B^{\Delta^p})^{\Delta^q} \to B^{\Delta^p \times \Delta^q}$ is a homotopy equivalence too.

$$
\begin{array}{ccc}
B^{\Delta^p} & \xrightarrow{\ \iota\ } & (B^{\Delta^p})^{\Delta^q} \\
\uparrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \mu} \\
B & \xrightarrow{\ \iota\ } & B^{\Delta^p \times \Delta^q}
\end{array}
$$

The author does not know whether $\mu^{K,L} : (B^K)^L \to B^{K \times L}$ is a homotopy equivalence for general K and L. Later on we will prove that $\mu^{K,L}$ becomes invertible in the bivariant algebraic K-theory categories (Corollary 4.4.3).

2.3   Main theorem

Let $A, B \in C$ and let $n \geq 0$. In this section we prove that there is a natural bijection:

$$\pi_n Ex^\infty \mathrm{Hom}_C(A, B^\Delta) \cong [A, B_\bullet^{\mathfrak{S}_n}]_C$$

This result is a generalization of [2, Theorem 3.3.2].

  We start by recalling a result from [5] that allows us to compare simplicial and algebraic homotopy. Following [5, Section 7.2], put $\widetilde{B}_\bullet^{\mathfrak{S}_n} := B_\bullet^{(I^n \times I, \partial I^n \times I)}$. The coface maps $d^i : \Delta^0 \to I$ induce morphisms $(d^i)^* : \widetilde{B}_\bullet^{\mathfrak{S}_n} \to B_\bullet^{\mathfrak{S}_n}$.

Lemma 2.3.1 (Garkusha). *Let $f : A \to \widetilde{B}_r^{\mathfrak{S}_n}$ be a morphism in C. Then the following composites are algebraically homotopic; i.e. they belong to the same class in $[A, B_r^{\mathfrak{S}_n}]_C$:*

$$A \xrightarrow{\ f\ } \widetilde{B}_r^{\mathfrak{S}_n} \xrightarrow{\ (d^i)^*\ } B_r^{\mathfrak{S}_n} \quad (i = 0, 1)$$

Proof. In the case $C = \mathrm{Alg}_\ell$ this is [5, Hauptlemma (2)]; the proof given there works verbatim in the G-equivariant setting. $\square$

Lemma 2.3.2 (cf. [5, Hauptlemma (3)]). *Let $H : A \to (B_r^{\mathfrak{S}_n})^{\mathrm{sd}^s I}$ be a morphism in C. Then there exists a morphism $\widetilde{H} : A \to \widetilde{B}_{r+s}^{\mathfrak{S}_n}$ in C such that the following diagram commutes for $i = 0, 1$:*

$$
\begin{array}{ccc}
A & \xrightarrow{\ H\ } (B_r^{\mathfrak{S}_n})^{\mathrm{sd}^s I} \xrightarrow{\ (d^i)^*\ } & B_r^{\mathfrak{S}_n} \\
{\scriptstyle \widetilde{H}} \downarrow & & \downarrow {\scriptstyle (\gamma^s)^*} \\
\widetilde{B}_{r+s}^{\mathfrak{S}_n} & \xrightarrow{\qquad (d^i)^* \qquad} & B_{r+s}^{\mathfrak{S}_n}
\end{array}
$$

Proof. Let $\widetilde{H}$ be the composite:

$$A \xrightarrow{\qquad H \qquad} (B_r^{(I^n, \partial I^n)})_s^{(I, \emptyset)} \xrightarrow{\ \mu^{(I^n, \partial I^n),(I, \emptyset)}\ } B_{r+s}^{(I^n \times I, (\partial I^n) \times I)}$$

It is immediate from the naturality of μ that $\widetilde{H}$ satisfies the required properties. $\square$

Theorem 2.3.3 (cf. [2, Theorem 3.3.2]). *Let $A, B \in C$ and let $n \geq 0$. Then there is a natural bijection:*

$$\pi_n Ex^\infty \mathrm{Hom}_C(A, B^\Delta) \cong [A, B_\bullet^{\mathfrak{S}_n}]_C \tag{5}$$

Proof. We will show that $\pi_0 \Omega^n Ex^\infty \mathrm{Hom}_C(A, B^\Delta) \cong [A, B_\bullet^{\mathfrak{S}_n}]_C$. Consider $\mathrm{Hom}_C(A, B^\Delta)$ as a simplicial set pointed at the zero morphism. For every $p \geq 0$ we have a pullback of sets:

$$
\begin{array}{ccc}
\left(\Omega^n Ex^\infty \mathrm{Hom}_C(A, B^\Delta)\right)_p & \longrightarrow & \mathrm{Map}\left(I^n, Ex^\infty \mathrm{Hom}_C(A, B^\Delta)\right)_p \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathrm{Map}\left(\partial I^n, Ex^\infty \mathrm{Hom}_C(A, B^\Delta)\right)_p
\end{array}
\tag{6}
$$

For a finite simplicial set K we have:

$$\text{Map}\left(K, Ex^\infty\text{Hom}_C(A, B^\Delta)\right)_p = \text{Hom}_\mathbb{S}\left(K \times \Delta^p, Ex^\infty\text{Hom}_C(A, B^\Delta)\right)$$

$$\cong \underset{r}{\text{colim}}\, \text{Hom}_\mathbb{S}\left(K \times \Delta^p, Ex^r\text{Hom}_C(A, B^\Delta)\right)$$

$$\cong \underset{r}{\text{colim}}\, \text{Hom}_\mathbb{S}\left(\text{sd}^r(K \times \Delta^p), \text{Hom}_C(A, B^\Delta)\right)$$

$$\cong \underset{r}{\text{colim}}\, \text{Hom}_C\left(A, B^{\text{sd}^r(K \times \Delta^p)}\right)$$

It follows from these identifications, from (6) and from the fact that filtered colimits of sets commute with finite limits, that we have the following bijections:

$$\left(\Omega^n Ex^\infty\text{Hom}_C(A, B^\Delta)\right)_0 \cong \underset{r}{\text{colim}}\, \text{Hom}_C(A, B_r^{\mathfrak{S}_n}) \tag{7}$$

$$\left(\Omega^n Ex^\infty\text{Hom}_C(A, B^\Delta)\right)_1 \cong \underset{r}{\text{colim}}\, \text{Hom}_C(A, \widetilde{B}_r^{\mathfrak{S}_n}) \tag{8}$$

Using (7) we get a surjection:

$$\left(\Omega^n Ex^\infty\text{Hom}_C(A, B^\Delta)\right)_0 \cong \underset{r}{\text{colim}}\, \text{Hom}_C(A, B_r^{\mathfrak{S}_n}) \longrightarrow [A, B_\bullet^{\mathfrak{S}_n}]_C$$

We claim that this function induces the desired bijection. The fact that it factors through π_0 follows from the identification (8) and Lemma 2.3.1. The injectivity of the induced function from π_0 follows from Lemma 2.3.2. □

Remark 2.3.4. Let $A, B \in C$ and let $n \geq 1$. Consider the set $[A, B_\bullet^{\mathfrak{S}_n}]_C$ together with the group structure for which (5) is a group isomorphism. This group structure is abelian if $n \geq 2$. Moreover, if $f : A \to A'$ and $g : B \to B'$ are morphisms in $[C]$, then the following functions are group homomorphisms:

$$f^* : [A', B_\bullet^{\mathfrak{S}_n}]_C \longrightarrow [A, B_\bullet^{\mathfrak{S}_n}]_C$$

$$g_* : [A, B_\bullet^{\mathfrak{S}_n}]_C \longrightarrow [A, (B')_\bullet^{\mathfrak{S}_n}]_C$$

In the sequel we will always consider $[A, B_\bullet^{\mathfrak{S}_n}]_C$ as a group with this group structure.

Example 2.3.5. Recall that $B^{\Delta^1} = B[t_0, t_1]/\langle 1 - t_0 - t_1 \rangle$. Let ω be the automorphism of B^{Δ^1} defined by $\omega(t_0) = t_1$, $\omega(t_1) = t_0$; it is clear that ω induces an automorphism of $B_0^{\mathfrak{S}_1} = \ker(B^{\Delta^1} \longrightarrow B^{\partial\Delta^1})$. Let $f : A \to B_0^{\mathfrak{S}_1}$ be a morphism in C and let $[f]$ be its class in $[A, B_\bullet^{\mathfrak{S}_1}]_C$. We claim that $[\omega \circ f] = [f]^{-1} \in [A, B_\bullet^{\mathfrak{S}_1}]_C$. In order to prove this claim, we proceed to recall the definition of the group law $*$ in $\pi_1 Ex^\infty\text{Hom}_C(A, B^\Delta)$. Consider f and $\omega \circ f$ as 1-simplices of $Ex^\infty\text{Hom}_C(A, B^\Delta)$ using the identification:

$$\left(Ex^\infty\text{Hom}_C(A, B^\Delta)\right)_1 \cong \underset{r}{\text{colim}}\, \text{Hom}_C(A, B^{\text{sd}^r\Delta^1})$$

According to [7, Section I.7], if we find $\alpha \in \left(Ex^\infty\text{Hom}_C(A, B^\Delta)\right)_2$ such that

$$\begin{cases} d_0\alpha = \omega \circ f \\ d_2\alpha = f \end{cases} \tag{9}$$

then we have $[f] * [\omega \circ f] = [d_1\alpha]$. Let $\varphi : B^{\Delta^1} \to B^{\Delta^2}$ be the morphism in C defined by $\varphi(t_0) = t_0 + t_2$, $\varphi(t_1) = t_1$. Let α be the 2-simplex of $Ex^\infty \mathrm{Hom}_C(A, B^\Delta)$ induced by the composite:

$$A \xrightarrow{\ f\ } B^{\Delta^1} \xrightarrow{\ \varphi\ } B^{\Delta^2}$$

It is easy to verify that the equations (9) hold and that $d_1\alpha$ is the zero path.

Example 2.3.6. Let $A, B \in C$ and let $m, n \geq 1$. Let $c : I^m \times I^n \xrightarrow{\cong} I^n \times I^m$ be the commutativity isomorphism. It is easily verified that c induces an isomorphism $c^* : B_\bullet^{\mathfrak{S}_{n+m}} \to B_\bullet^{\mathfrak{S}_{m+n}}$. We claim that the following function is multiplication by $(-1)^{mn}$:

$$c^* : [A, B_\bullet^{\mathfrak{S}_{n+m}}]_C \longrightarrow [A, B_\bullet^{\mathfrak{S}_{m+n}}]_C$$

Indeed, this follows from Theorem 2.3.3 and the well known fact that permuting two coordinates in Ω^{m+n} induces multiplication by (-1) upon taking π_0.

Chapter 3

The loop-stable homotopy category

Resumen del capítulo

Construímos la *categoría de homotopía estable por lazos* —denotada por $\mathfrak{K}^C$— que es una teoría de homología universal, escisiva e invariante por homotopía en el sentido de [2, Theorem 6.6.2]. La existencia de una teoría con estas características ya había sido probada por Garkusha en [6, Theorem 2.6 (2)] usando métodos completamente distintos. Seguimos de cerca la construcción de la categoría de homotopía estable por suspensiones hecha en [3, Chapter 6] para álgebras bornológicas. Desde luego, es necesario hacer algunos cambios para traducir los resultados del contexto topolgógico al algebraico; aquí utilizamos métodos e ideas de Cortiñas-Thom [2]. En la sección 3.1 estudiamos las extensiones de álgebras y sus morfismos clasificantes; este material puede encontrarse en [2], [5] y [4]. Hay dos funtores de lazos J y $(?)^{\mathfrak{S}_1}$ que se vuelven equivalencias (naturalmente isomorfas) en $\mathfrak{K}^C$; estos funtores se estudian en las secciones 3.1 y 3.3. En la sección 3.6 damos la definición de $\mathfrak{K}^C$, que se obtiene de la categoría de homotopía $[C]$ invirtiendo formalmente a los funtores J y $(?)^{\mathfrak{S}_1}$. En la sección 3.12 probamos que $\mathfrak{K}^C$ es una categoría triangulada. Al igual que en el contexto topológico [3, Theorem 6.63], probamos que la triangulación de $\mathfrak{K}^C$ puede definirse usando tanto extensiones como *mapping path algebras* (Proposición 3.12.12). En la sección 3.13 mostramos que el funtor $j : C \to \mathfrak{K}^C$ es una teoría de homología universal, escisiva e invariante por homotopía. Por lo tanto, cualquier funtor $F : C \to C$ que preserve extensiones y homotopía induce un funtor triangulado $\bar{F} : \mathfrak{K}^C \to \mathfrak{K}^C$. En el Teorema 3.13.14 mostramos que cualquier transformación natural $\eta : F \to G$ entre funtores con dichas propiedades induce una transformación natural (graduada) $\bar{\eta} : \bar{F} \to \bar{G}$.

Chapter summary

We construct the *loop-stable homotopy category* —denoted by $\mathfrak{K}^C$— which is a universal excisive and homotopy invariant homology theory of algebras in the sense of [2, Theorem 6.6.2]. The existence of such a theory was already proved by Garkusha in [6, Theorem 2.6 (2)] using completely different methods. We closely follow the construction of the suspension-stable homotopy category of bornological algebras [3, Chapter 6]. We make,

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of course, appropiate changes to translate the arguments from the topological to the algebraic setting, using methods and ideas developed by Cortiñas-Thom in [2]. In section 3.1 we discuss the extensions of algebras and their classifying maps; this material is not new and can be found in [2], [5] and [4]. There are two loop functors J and $(?)^{\mathfrak{S}_1}$ that become (naturally isomorphic) equivalences in $\mathfrak{K}^C$; these functors are dealt with in sections 3.1 and 3.3. In section 3.6 we give the definition of $\mathfrak{K}^C$, which is obtained from the homotopy category $[C]$ by adding formal deloopings of J and $(?)^{\mathfrak{S}_1}$. In section 3.12 we prove that $\mathfrak{K}^C$ is a triangulated category. As in the topological setting [3, Theorem 6.63], we show that the distinguished triangles in $\mathfrak{K}^C$ can be defined using either extensions or mapping path algebras (Proposition 3.12.12). In section 3.13 we show that the functor $j : C \to \mathfrak{K}^C$ is the universal excisive and homotopy invariant homology theory with values in a triangulated category. Thus, any functor $F : C \to C$ that preserves extensions and homotopy induces a triangulated functor $\bar{F} : \mathfrak{K}^C \to \mathfrak{K}^C$. In Theorem 3.13.14 we show that any natural transformation $\eta : F \to G$ between functors with these properties induces a (graded) natural transformation $\bar{\eta} : \bar{F} \to \bar{G}$.

3.1 Extensions and classifying maps

Let Mod_ℓ be the category of ℓ-modules and let $G\mathrm{Mod}_\ell$ be the category of G-ℓ-modules. Put:

$$\mathcal{U}(C) := \begin{cases} \mathrm{Mod}_\ell & \text{if } C = \mathrm{Alg}_\ell \\ G\mathrm{Mod}_\ell & \text{if } C = G\mathrm{Alg}_\ell \end{cases}$$

Write $F : C \to \mathcal{U}(C)$ for the forgetful functor. An *extension* in C is a diagram in C

$$\mathscr{E} : \quad A \longrightarrow B \longrightarrow C \tag{1}$$

that becomes a split short exact sequence upon applying F. A *morphism of extensions* is a morphism of diagrams in C. We usually consider specific splittings for the extensions we work with and we sometimes write $(\mathscr{E}, s)$ to emphasize that we are considering an extension $\mathscr{E}$ with splitting s. Let $(\mathscr{E}, s)$ and $(\mathscr{E}', s')$ be two extensions with specified splittings; a *strong morphism of extensions* $(\mathscr{E}', s') \to (\mathscr{E}, s)$ is a morphism of extensions $(\alpha, \beta, \gamma) : \mathscr{E}' \to \mathscr{E}$ that is compatible with the splittings; i.e. such that the folowing diagram commutes:

$$
\begin{array}{ccc}
FB' & \xleftarrow{\ s'\ } & FC' \\
{\scriptstyle F\beta}\downarrow & & \downarrow{\scriptstyle F\gamma} \\
FB & \xleftarrow{\ s\ } & FC
\end{array}
$$

The functor $F : C \to \mathcal{U}(C)$ admits a right adjoint $\widetilde{T} : \mathcal{U}(C) \to C$; see [4, Section 2.4] for details. Let T be the composite functor $\widetilde{T} \circ F : C \to C$. Let $A \in C$ and let $\eta_A : TA \to A$ be the counit of the adjunction. Notice that $F\eta_A$ is a retraction which has the unit map $\sigma_A : FA \to F\widetilde{T}(FA) = FTA$ as a section. Let $JA := \ker \eta_A$. The *universal extension* of A is the extension:

$$\mathscr{U}_A : \quad JA \longrightarrow TA \xrightarrow{\ \eta_A\ } A \tag{2}$$

We will always consider σ_A as a splitting for $\mathscr{U}_A$.

Proposition 3.1.1 (cf. [2, Proposition 4.4.1]). *Let* (1) *be an extension in* C *with splitting s and let $f : C' \to C$ be a morphism in* C. *Then there exists a unique strong morphism of extensions $\mathcal{U}_{C'} \to (\mathcal{E}, s)$ extending f:*

$$
\begin{array}{ccccccc}
\mathcal{U}_{C'} & & JC' & \longrightarrow & TC' & \xrightarrow{\eta_{C'}} & C' \\
\exists! \downarrow & & \xi \downarrow & & \downarrow & & \downarrow f \\
(\mathcal{E}, s) & & A & \longrightarrow & B & \longrightarrow & C
\end{array}
\tag{3}
$$

Proof. It follows easily from the adjointness of $\widetilde{T}$ and F. $\square$

The morphism ξ in (3) is called the *classifying map of f with respect to the extension* $(\mathcal{E}, s)$. When $C' = C$ and $f = \mathrm{id}_C$ we call ξ the *classifying map of* $(\mathcal{E}, s)$.

Proposition 3.1.2 (cf. [2, Proposition 4.4.1]). *In the hypothesis of Proposition 3.1.1, the homotopy class of the classifying map ξ does not depend upon the splitting s.*

Proof. See, for example, [5, Section 3]. $\square$

Because of Proposition 3.1.2, it makes sense to speak of the classifying map of (1) as a homotopy class $JC \to A$ without specifying a splitting for (1).

Proposition 3.1.3 ([2, Proposition 4.4.2]). *Let $\mathcal{E}_i : A_i \to B_i \to C_i$ be an extension in C with classifying map ξ_i. Let $(a, b, c) : \mathcal{E}_1 \to \mathcal{E}_2$ be a morphism of extensions. Then the following diagram commutes in* $[C]$*:*

$$
\begin{array}{ccc}
JC_1 & \xrightarrow{J(c)} & JC_2 \\
\xi_1 \downarrow & & \downarrow \xi_2 \\
A_1 & \xrightarrow{a} & A_2
\end{array}
$$

Moreover, if we consider specific splittings for the $\mathcal{E}_i$ and (a, b, c) is a strong morphism of extensions then the diagram above commutes in C.

Proof. See, for example, [5, Section 3]. $\square$

Example 3.1.4. Let (K, L) be a simplicial pair and let $A \in C$. Then the following diagram is an extension in C, as we proceed to explain:

$$
(\mathcal{U}_A)_r^{(K,L)} : \ (JA)_r^{(K,L)} \longrightarrow (TA)_r^{(K,L)} \xrightarrow{\eta_A^{(K,L)}} A_r^{(K,L)}
$$

The sequence is exact since it is obtained from (2) upon tensoring with $\mathbb{Z}_r^{(K,L)}$; see Lemma 2.1.1. The splitting of $\eta_A^{(K,L)}$ in $\mathcal{U}(C)$ is given by

$$
\sigma_A \otimes \mathbb{Z}_r^{(K,L)} : A \otimes \mathbb{Z}_r^{(K,L)} \longrightarrow TA \otimes \mathbb{Z}_r^{(K,L)}
$$

under the identification of Lemma 2.1.1. The last vertex map induces strong morphisms of extensions $(\mathcal{U}_A)_r^{(K,L)} \to (\mathcal{U}_A)_{r+1}^{(K,L)}$. A morphism $A \to B$ in C induces strong morphisms of extensions $(\mathcal{U}_A)_r^{(K,L)} \to (\mathcal{U}_B)_r^{(K,L)}$. A morphism of simplicial pairs $(K', L') \to (K, L)$ induces strong morphisms of extensions $(\mathcal{U}_A)_r^{(K,L)} \to (\mathcal{U}_A)_r^{(K',L')}$.

Lemma 3.1.5. *The functor $J : \mathcal{C} \to \mathcal{C}$ sends homotopic morphisms to homotopic morphisms. Thus, it defines a functor $J : [\mathcal{C}] \to [\mathcal{C}]$.*

Proof. It is explained in [2] in the discussion following [2, Corollary 4.4.4.].   □

3.2   Path extensions

Let us define a class of extensions that will be useful later on. Let $B \in \mathcal{C}$ and let $n \geq 0$. Put:

$$P(n, B)_\bullet := B_\bullet^{(I^{n+1}, (\partial I^n \times I) \cup (I^n \times \{1\}))}$$

We will often write $(PB)_\bullet$ instead of $P(0, B)_\bullet$. The diagram of simplicial pairs

$$(I^{n+1}, \partial I^{n+1}) \supseteq (I^{n+1}, (\partial I^n \times I) \cup (I^n \times \{1\})) \supseteq (I^n \times \{0\}, \partial I^n \times \{0\})$$

induces the following sequence of $\mathbb{Z}_{\geq 0}$-diagrams:

$$\mathscr{P}_{n,B} : B_r^{\mathfrak{S}_{n+1}} \longrightarrow P(n, B)_r \xrightarrow{\quad p_{n,B} \quad} B_r^{\mathfrak{S}_n} \tag{4}$$

We claim that (4) is an extension in $\mathcal{C}$. Exactness at $P(n, B)_r$ holds because the functors $B^{\mathrm{sd}^r(-)} : \mathbb{S} \to \mathcal{C}^{\mathrm{op}}$ preserve pushouts and we have:

$$\partial I^{n+1} = [(\partial I^n \times I) \cup (I^n \times \{1\})] \cup (I^n \times \{0\})$$

Exactness at $B_r^{\mathfrak{S}_{n+1}}$ follows from the fact that both $B_r^{\mathfrak{S}_{n+1}}$ and $P(n, B)_r$ are subalgebras of $B^{\mathrm{sd}^r I^{n+1}}$. We proceed to construct a splitting of (4) in $\mathcal{U}(\mathcal{C})$. Consider the element $t_0 \in \mathbb{Z}^{\Delta^1}$; t_0 is actually in $\mathbb{Z}_0^{(I,\{1\})}$ since $d_0(t_0) = 0$. Let $s_{n,B}$ be the composite:

$$B_r^{\mathfrak{S}_n} \xrightarrow{\; ? \otimes t_0 \;} B_r^{\mathfrak{S}_n} \otimes \mathbb{Z}_0^{(I,\{1\})} \cong \left(B_r^{\mathfrak{S}_n}\right)_0^{(I,\{1\})} \xrightarrow{\;\mu\;} B_r^{(I^{n+1}, (\partial I^n \times I) \cup (I^n \times \{1\}))}$$

Here μ is the morphism defined in Remark 2.2.3. It is straightforward to check that $s_{n,B}$ is a section of $p_{n,B}$ in $\mathcal{U}(\mathcal{C})$. We will always consider $s_{n,B}$ as a splitting for (4). It is clear that (4) is natural in B with respect to morphisms in $\mathcal{C}$.

Example 3.2.1. By naturality of μ (see Remark 2.2.3), there is a strong morphism of extensions:

$$
\begin{array}{ccccc}
\mathscr{P}_{m,B_s^{\mathfrak{S}_n}} & \left(B_s^{\mathfrak{S}_n}\right)_r^{\mathfrak{S}_{m+1}} & \longrightarrow & P(m, B_s^{\mathfrak{S}_n})_r & \longrightarrow & \left(B_s^{\mathfrak{S}_n}\right)_r^{\mathfrak{S}_m} \\
\downarrow & {\scriptstyle \mu_B^{n,m+1}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mu_B^{n,m}} \\
\mathscr{P}_{n+m,B} & B_{s+r}^{\mathfrak{S}_{n+m+1}} & \longrightarrow & P(n + m, B)_{s+r} & \longrightarrow & B_{s+r}^{\mathfrak{S}_{n+m}}
\end{array}
$$

Example 3.2.2. For $n = 0$, the extension (4) takes the form:

$$\mathscr{P}_{0,B} : B_r^{\mathfrak{S}_1} \longrightarrow (PB)_r \longrightarrow B_r^{\mathfrak{S}_0} \cong B \tag{5}$$

Here, the morphisms are induced by the following diagram of simplicial pairs:

$$(I, \partial I) \xleftarrow{\;\mathrm{inc}\;} (I, \{1\}) \xleftarrow{\;(d^1, \emptyset)\;} (\Delta^0, \emptyset)$$

This is the *loop extension* of [2, Section 4.5]; we will write λ_B for its classifying map.

Example 3.2.3. Define $\widetilde{P}(n, B)_\bullet := B_\bullet^{(I^{1+n}, (I \times \partial I^n) \cup (\{1\} \times I^n))}$. The diagram of simplicial pairs

$$(I^{1+n}, \partial I^{1+n}) \supseteq (I^{1+n}, (I \times \partial I^n) \cup (\{1\} \times I^n)) \supseteq (\{0\} \times I^n, \{0\} \times \partial I^n)$$

induces a sequence of $\mathbb{Z}_{\geq 0}$–diagrams:

$$\widetilde{\mathscr{P}}_{n,B} : B_r^{\mathfrak{S}_{1+n}} \longrightarrow \widetilde{P}(n, B)_r \longrightarrow B_r^{\mathfrak{S}_n} \tag{6}$$

It is shown that this sequence is an extension by proceeding in analogy to what was done above for (4). In fact, the commutativity isomorphism $c : I^n \times I \xrightarrow{\cong} I \times I^n$ induces a strong isomorphism of extensions:

$$
\begin{array}{ccccccc}
\widetilde{\mathscr{P}}_{n,B} & & B_r^{\mathfrak{S}_{1+n}} & \longrightarrow & \widetilde{P}(n, B)_r & \longrightarrow & B_r^{\mathfrak{S}_n} \\
\cong \Big\downarrow & & \cong \Big\downarrow {\scriptstyle c^*} & & \cong \Big\downarrow {\scriptstyle c^*} & & \Big\downarrow {\scriptstyle \mathrm{id}} \\
\mathscr{P}_{n,B} & & B_r^{\mathfrak{S}_{n+1}} & \longrightarrow & P(n, B)_r & \longrightarrow & B_r^{\mathfrak{S}_n}
\end{array}
$$

Example 3.2.4. It will be useful to have a more explicit description of (5). Consider the following isomorphisms:

$$\begin{cases} B^{\Delta^0} = B[t_0]/\langle 1 - t_0 \rangle \cong B; \\ B^{\Delta^1} = B[t_0, t_1]/\langle 1 - t_0 - t_1 \rangle \cong B[t], \quad t_1 \leftrightarrow t. \end{cases} \tag{7}$$

Under these identifications, the face morphisms $B^{\Delta^1} \to B^{\Delta^0}$ coincide with the evaluations $\mathrm{ev}_i : B[t] \to B$ for $i = 0, 1$. We have:

$$(PB)_0 = \ker\left(B^{\Delta^1} \xrightarrow{d_0} B^{\Delta^0}\right) \cong \ker\left(B[t] \xrightarrow{\mathrm{ev}_1} B\right) = (t - 1)B[t]$$

$$B_0^{\mathfrak{S}_1} = \ker\left((PB)_0 \xrightarrow{d_1} B^{\Delta^0}\right) \cong \ker\left((t - 1)B[t] \xrightarrow{\mathrm{ev}_0} B\right) = (t^2 - t)B[t]$$

Hence, the extension (5) is isomorphic to:

$$(t^2 - t)B[t] \xrightarrow{\mathrm{inc}} (t - 1)B[t] \xrightarrow{\mathrm{ev}_0} B \tag{8}$$

The section in (5) identifies with the morphism $B \to (t - 1)B[t]$, $b \mapsto b(1 - t)$.

We now want a description of (5) once a subdivision has been made. Recall that $\mathrm{sd}\Delta^1$ fits into the following pushout:

$$
\begin{array}{ccc}
\Delta^0 & \xrightarrow{d^0} & \Delta^1 \\
{\scriptstyle d^0} \Big\downarrow & & \Big\downarrow \\
\Delta^1 & \longrightarrow & \mathrm{sd}\Delta^1
\end{array}
$$

Since the functor $B^? : \mathbb{S}^{\mathrm{op}} \to C$ preserves limits, we have:

$$\begin{aligned} B^{\mathrm{sd}\Delta^1} &\cong \{(x, y) \in B^{\Delta^1} \times B^{\Delta^1} : d_0(x) = d_0(y)\} \\ &\cong \{(p, q) \in B[t] \times B[t] : p(1) = q(1)\} = B[t] \, _{\mathrm{ev}_1}\!\times_{\mathrm{ev}_1} B[t] \end{aligned}$$

Under this identification, the endpoints of $\mathrm{sd}\Delta^1$ are the images of the coface maps d^1 : $\Delta^0 \to \Delta^1$ whose codomains are each of the two copies of Δ^1 that are contained in $\mathrm{sd}\Delta^1$. We get:

$$(PB)_1 = \ker\left(B^{\mathrm{sd}\Delta^1} \longrightarrow B^{\mathrm{sd}\{1\}}\right) \cong B[t] \,_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} tB[t]$$

$$B_1^{\mathfrak{S}_1} = \ker\left(B^{\mathrm{sd}\Delta^1} \longrightarrow B^{\mathrm{sd}(\partial\Delta^1)}\right) \cong tB[t] \,_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} tB[t]$$

In this description of $(PB)_1$ a choice has been made, since the two endpoints of $\mathrm{sd}\Delta^1$ are indistinguishable. The extension (5) is isomorphic to:

$$tB[t] \,_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} tB[t] \xrightarrow{\ \mathrm{inc}\ } B[t] \,_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} tB[t] \xrightarrow{\ \mathrm{ev}_0\circ\mathrm{pr}_1\ } B \qquad (9)$$

Here $\mathrm{pr}_1 : B[t] \,_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} tB[t] \longrightarrow B[t]$ is the projection into the first factor. The section in (5) identifies with the morphism:

$$B \to B[t] \,_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} tB[t], \quad b \mapsto (b(1-t), 0).$$

The last vertex map induces a strong morphism of extensions from (8) to (9); this morphism has the following components:

$$(t^2 - t)B[t] \to tB[t] \,_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} tB[t], \quad p \mapsto (p, 0);$$

$$(t - 1)B[t] \to B[t] \,_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} tB[t], \quad p \mapsto (p, 0).$$

Lemma 3.2.5. *Let $B \in C$, $n \geq 1$ and $r \geq 0$. Then $P(n, B)_r$ is contractible.*

Proof. Let $\vartheta : I \times I \to I$ be the unique morphism of simplicial sets that satisfies:

$$(0, 0) \xmapsto{\quad \vartheta \quad} 0$$
$$(0, 1), (1, 0), (1, 1) \xmapsto{\ \vartheta\ } 1$$

It is easily verified that the following square commutes, where the vertical morphisms are inclusions:

$$
\begin{array}{ccc}
I^n \times I \times I & \xrightarrow{\quad I^n \times \vartheta \quad} & I^n \times I \\
\uparrow & & \uparrow \\
(\partial I^n \times I \times I) \cup (I^n \times \{1\} \times I) & \longrightarrow & (\partial I^n \times I) \cup (I^n \times \{1\})
\end{array}
$$

Thus $I^n \times \vartheta$ induces a morphism in C:

$$f : P(n, B)_r \longrightarrow B_r^{(I^n \times I \times I, (\partial I^n \times I \times I) \cup (I^n \times \{1\} \times I))}$$

The coface maps $I^{n+1} \times d^i : I^{n+1} \cong I^{n+1} \times \Delta^0 \to I^{n+1} \times I$ induce morphisms in C:

$$B_r^{(I^n \times I \times I, (\partial I^n \times I \times I) \cup (I^n \times \{1\} \times I))} \xrightarrow{\ \delta_i\ } P(n, B)_r$$

Notice that $\delta_0 \circ f = 0$ and $\delta_1 \circ f = \mathrm{id}_{P(n,B)_r}$. By [5, Hauptlemma (2)], $\delta_0 \circ f$ and $\delta_1 \circ f$ represent the same class in $[P(n, B)_r, P(n, B)_r]$. Indeed, both morphisms represent the same class in $[P(n, B)_r, B^{\mathrm{sd}^r I^{n+1}}]$ but the polynomial homotopies constructed in op. cit. preserve our boundary conditions. $\qquad\qquad\square$

3.3 Exchanging loop functors

Let $B \in C$ and let $m, n \geq 0$. We proceed to define a natural transformation:

$$\kappa_B^{n,m} : J^n(B_r^{\mathfrak{S}_m}) \longrightarrow (J^n B)_r^{\mathfrak{S}_m}$$

Recall from Example 3.1.4 that we have an extension:

$$(\mathscr{U}_B)^{\mathfrak{S}_m} : (JB)_r^{\mathfrak{S}_m} \longrightarrow (TB)_r^{\mathfrak{S}_m} \xrightarrow{(\eta_B)^{\mathfrak{S}_m}} B_r^{\mathfrak{S}_m} \tag{10}$$

Let $\kappa_B^{1,m} : J(B_r^{\mathfrak{S}_m}) \to (JB)_r^{\mathfrak{S}_m}$ be the classifying map of (10). It follows from Example 3.1.4 and Proposition 3.1.3 that $\kappa_B^{1,m}$ can be considered as a morphism $J(B_\bullet^{\mathfrak{S}_m}) \to (JB)_\bullet^{\mathfrak{S}_m}$ in $C^{\mathbb{Z}_{\geq 0}}$. For $n \geq 1$, define inductively $\kappa_B^{n+1,m}$ as the composite:

$$J^{n+1}(B_r^{\mathfrak{S}_m}) \xrightarrow{J(\kappa_B^{n,m})} J\left((J^n B)_r^{\mathfrak{S}_m}\right) \xrightarrow{\kappa_{J^n B}^{1,m}} (J^{n+1} B)_r^{\mathfrak{S}_m}$$

The $\kappa_B^{n,m}$ are easily seen to be natural morphisms $J^n(B_\bullet^{\mathfrak{S}_m}) \to (J^n B)_\bullet^{\mathfrak{S}_m}$ in $C^{\mathbb{Z}_{\geq 0}}$. Let J^0 be the identity functor of C and let $\kappa_B^{0,m}$ be the identity of $B_\bullet^{\mathfrak{S}_m} \in C^{\mathbb{Z}_{\geq 0}}$. The next result follows from an easy induction on $n = p + q$.

Lemma 3.3.1. *Let $p, q \in \mathbb{Z}_{\geq 0}$ and let $B \in C$. Then $\kappa_B^{p+q,m} = \kappa_{J^p B}^{q,m} \circ J^q\left(\kappa_B^{p,m}\right)$.*

Lemma 3.3.1 should be interpreted as follows: Let $n = p + q$. The morphism $\kappa^{n,m}$ exchanges J^n and $(?)^{\mathfrak{S}_m}$. We have $J^n = J^q \circ J^p$. Thus, in order to exchange J^n and $(?)^{\mathfrak{S}_m}$, we can first exchange J^p and $(?)^{\mathfrak{S}_m}$ and then exchange J^q and $(?)^{\mathfrak{S}_m}$:

$$
\begin{array}{ccc}
J^{p+q}(B^{\mathfrak{S}_m}) & \xrightarrow{\kappa^{p+q,m}} & (J^{p+q}B)^{\mathfrak{S}_m} \\
{\scriptstyle J^q(\kappa^{p,m})} \searrow & & \nearrow {\scriptstyle \kappa^{q,m}} \\
& J^q\left((J^p B)^{\mathfrak{S}_m}\right) &
\end{array}
$$

Remark 3.3.2. For any finite simplicial set K we have a classifying map $J(B^K) \to (JB)^K$. Imitating what we did above, we can define morphisms $J^n(B^K) \to (J^n B)^K$ and prove Lemma 3.3.1 in this setting:

$$
\begin{array}{ccc}
J^{p+q}(B^K) & \longrightarrow & (J^{p+q}B)^K \\
\searrow & & \nearrow \\
& J^q\left((J^p B)^K\right) &
\end{array}
$$

The following result is an analog of Lemma 3.3.1. Its statement is, however, more complicated since $(B^{\mathfrak{S}_p})^{\mathfrak{S}_q} \not\cong B^{\mathfrak{S}_{p+q}}$.

Lemma 3.3.3. *Let $B \in C$. Then the following diagram in C commutes:*

$$
\begin{array}{ccc}
J^n\left(\left(B_r^{\mathfrak{S}_p}\right)_s^{\mathfrak{S}_q}\right) & \xrightarrow{\kappa_{B_r^{\mathfrak{S}_p}}^{n,q}} \left(J^n\left(B_r^{\mathfrak{S}_p}\right)\right)_s^{\mathfrak{S}_q} \xrightarrow{\left(\kappa_B^{n,p}\right)_s^{\mathfrak{S}_q}} & \left((J^n B)_r^{\mathfrak{S}_p}\right)_s^{\mathfrak{S}_q} \\
{\scriptstyle J^n(\mu_B^{p,q})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu_{J^n B}^{p,q}} \\
J^n\left(B_{r+s}^{\mathfrak{S}_{p+q}}\right) & \xrightarrow{\qquad\qquad \kappa_B^{n,p+q} \qquad\qquad} & (J^n B)_{r+s}^{\mathfrak{S}_{p+q}}
\end{array}
$$

Proof. We proceed by induction on n. The case $n = 1$ follows from Proposition 3.1.3 applied to the following strong morphism of extensions:

$$
\begin{array}{ccccc}
\left(\mathcal{U}_{B_r^{\mathfrak{S}_p}}\right)_s^{\mathfrak{S}_q} & & \left(J\left(B_r^{\mathfrak{S}_p}\right)\right)_s^{\mathfrak{S}_q} \longrightarrow \left(T\left(B_r^{\mathfrak{S}_p}\right)\right)_s^{\mathfrak{S}_q} \longrightarrow & & \left(B_r^{\mathfrak{S}_p}\right)_s^{\mathfrak{S}_q} \\
\downarrow & & {\scriptstyle \left(\kappa_B^{1,p}\right)_s^{\mathfrak{S}_q}}\downarrow \qquad\qquad \downarrow \qquad\qquad & & \downarrow{\scriptstyle 1} \\
\left((\mathcal{U}_B)_r^{\mathfrak{S}_p}\right)_s^{\mathfrak{S}_q} & & \left((JB)_r^{\mathfrak{S}_p}\right)_s^{\mathfrak{S}_q} \longrightarrow \left((TB)_r^{\mathfrak{S}_p}\right)_s^{\mathfrak{S}_q} \longrightarrow & & \left(B_r^{\mathfrak{S}_p}\right)_s^{\mathfrak{S}_q} \\
\downarrow & & {\scriptstyle \mu_{JB}^{p,q}}\downarrow \qquad\qquad {\scriptstyle \mu_{TB}^{p,q}}\downarrow \qquad\qquad & & \downarrow{\scriptstyle \mu_B^{p,q}} \\
(\mathcal{U}_B)_{r+s}^{\mathfrak{S}_{p+q}} & & (JB)_{r+s}^{\mathfrak{S}_{p+q}} \longrightarrow (TB)_{r+s}^{\mathfrak{S}_{p+q}} \longrightarrow & & B_{r+s}^{\mathfrak{S}_{p+q}}
\end{array}
$$

Now suppose that the diagram commutes for n; we will show it also commutes for $n + 1$. The following diagram commutes by inductive hypothesis and naturality of $\kappa_?^{n,1}$; we omit the subindices r and s to alleviate notation:

$$
\begin{array}{c}
J^{n+1}\left(\left(B^{\mathfrak{S}_p}\right)^{\mathfrak{S}_q}\right) \\[4pt]
{\scriptstyle J^n\left(\kappa_{B^{\mathfrak{S}_p}}^{1,q}\right)}\Big\downarrow \qquad\qquad\qquad\qquad \xrightarrow{\quad J^{n+1}(\mu_B^{p,q}) \quad} \\[4pt]
J^n\left(\left(J\left(B^{\mathfrak{S}_p}\right)\right)^{\mathfrak{S}_q}\right) \qquad\qquad\qquad\qquad J^{n+1}\left(B^{\mathfrak{S}_{p+q}}\right) \\[4pt]
{\scriptstyle \kappa_{J\left(B^{\mathfrak{S}_p}\right)}^{n,q}}\Big\downarrow \quad \xrightarrow{J^n\left(\left(\kappa_B^{1,p}\right)^{\mathfrak{S}_q}\right)} \qquad\qquad \Big\downarrow{\scriptstyle J^n\left(\kappa_B^{1,p+q}\right)} \\[4pt]
\left(J^{n+1}\left(B^{\mathfrak{S}_p}\right)\right)^{\mathfrak{S}_q} \qquad J^n\left(\left((JB)^{\mathfrak{S}_p}\right)^{\mathfrak{S}_q}\right) \xrightarrow{J^n(\mu_{JB}^{p,q})} J^n\left((JB)^{\mathfrak{S}_{p+q}}\right) \\[4pt]
\xrightarrow{\left(J^n\left(\kappa_B^{1,p}\right)\right)^{\mathfrak{S}_q}} \qquad {\scriptstyle \kappa_{(JB)^{\mathfrak{S}_p}}^{n,q}}\Big\downarrow \qquad\qquad \Big\downarrow \\[4pt]
\left(J^n\left((JB)^{\mathfrak{S}_p}\right)\right)^{\mathfrak{S}_q} \qquad\qquad {\scriptstyle \kappa_{JB}^{n,p+q}}\Big\downarrow \\[4pt]
{\scriptstyle \left(\kappa_{JB}^{n,p}\right)^{\mathfrak{S}_q}}\Big\downarrow \\[4pt]
\left((J^{n+1}B)^{\mathfrak{S}_p}\right)^{\mathfrak{S}_q} \xrightarrow{\mu_{J^{n+1}B}^{p,q}} (J^{n+1}B)^{\mathfrak{S}_{p+q}}
\end{array}
$$

Moreover, the following equalities hold by Lemma 3.3.1, proving the result:

$$
\kappa_{JB}^{n,p+q} \circ J^n\left(\kappa_B^{1,p+q}\right) = \kappa_B^{n+1,p+q}
$$

$$
\left(\kappa_{JB}^{n,p}\right)^{\mathfrak{S}_q} \circ \left(J^n\left(\kappa_B^{1,p}\right)\right)^{\mathfrak{S}_q} = \left(\kappa_B^{n+1,p}\right)^{\mathfrak{S}_q}
$$

$$
\kappa_{J\left(B^{\mathfrak{S}_p}\right)}^{n,q} \circ J^n\left(\kappa_{B^{\mathfrak{S}_p}}^{1,q}\right) = \kappa_{B^{\mathfrak{S}_p}}^{n+1,q} \qquad\qquad \square
$$

3.4 Extending constructions to the ind-homotopy category

Let $(I, \leq)$ be a filtering poset and let $F : C \to C^I$ be a functor. Then F induces a functor $F^{\text{ind}} : C^{\text{ind}} \to (C^I)^{\text{ind}}$; composing this with the functor a^{ind} of Lemma 1.2.3.1 we get a functor $C^{\text{ind}} \to C^{\text{ind}}$ that we still denote F. This happens, for example, in the following situations:

(i) $I = \{*\}$ and $F = J : C \to C$;

(ii) $I = \{*\}$ and $F = (?)^X : C \to C$ for any $X \in \mathbb{S}$;

(iii) $I = \mathbb{Z}_{\geq 0}$ and $F = (?)_\bullet^{(K,L)} : C \to C^{\mathbb{Z}_{\geq 0}}$ for any simplicial pair (K, L);

(iv) I any poset and $F =? \otimes C_\bullet : C \to C^I$, with $C_\bullet \in (\text{Alg}_\mathbb{Z})^I$.

In these examples, F has the aditional property of being *homotopy invariant*: if f and g are two homotopic morphisms in C then, for all $i \in I$, $F(f)_i$ and $F(g)_i$ are homotopic morphisms in C. Because of this, F induces a functor $F : [C] \to [C]^I$ and thus a functor $F : [C]^{\text{ind}} \to [C]^{\text{ind}}$; here we are using Lemma 1.2.3.1 once more. It is easy to see that the following diagram commutes:

$$
\begin{array}{ccc}
C^{\text{ind}} & \xrightarrow{\ F\ } & C^{\text{ind}} \\
\downarrow & & \downarrow \\
[C]^{\text{ind}} & \xrightarrow{\ F\ } & [C]^{\text{ind}}
\end{array}
$$

Thus, we can apply functors like (i)-(iv) to objects and morphisms in $[C]^{\text{ind}}$.

By the discussion above, we can regard $((?)_\bullet^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}$ and $(?)_\bullet^{\mathfrak{S}_{m+n}}$ as endfunctors of $[C]^{\text{ind}}$; we would like to consider $\mu_?^{m,n}$ as a natural transformation between these endofunctors. For this purpose, we proceed to explain how certain morphisms from $F : C \to C^I$ to $G : C \to C^J$ induce a natural transformation between the associated functors $[C]^{\text{ind}} \to [C]^{\text{ind}}$ —here, I and J may be different filtering posets, and F and G are homotopy invariant functors.

Let $F : C \to C^I$ and $G : C \to C^J$ be two homotopy invariant functors. Consider a pair (ν, θ) where $\theta : I \to J$ is a functor and $\nu : F \to \theta^* G$ is a natural transformation of functors $C \to C^I$. This means that:

(a) For each $A \in C$ we have $\nu_A : F(A) \to G(A) \circ \theta \in C^I$;

(b) For each morphism $f : A \to A'$ in C, the following diagram in C^I commutes:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\ \nu_A\ } & G(A) \circ \theta \\
{\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle (\theta^* G)(f)} \\
F(A') & \xrightarrow{\ \nu_{A'}\ } & G(A') \circ \theta
\end{array}
$$

Let $(C, K) \in [C]^{\text{ind}}$. Define a morphism $\nu_{C_\bullet} \in [F(C_\bullet)_\bullet, G(C_\bullet)_\bullet]$ as follows: For each pair $(i, k) \in I \times K$, let $(\nu_{C_\bullet})_{(i,k)}$ be the class of the morphism $(\nu_{C_k})_i : F(C_k)_i \to G(C_k)_{\theta(i)}$ in:

$$[F(C_k)_i, G(C_\bullet)_\bullet] = \operatornamewithlimits{colim}_{(j,k')} \left[F(C_k)_i, G(C_{k'})_j \right]$$

It is easily verified that the $(\nu_{C_\bullet})_{(i,k)}$ are compatible and assemble into a morphism:

$$\nu_{C_\bullet} = \left\{ (\nu_{C_\bullet})_{(i,k)} \right\} \in \operatornamewithlimits{lim}_{(i,k)} [F(C_k)_i, G(C_\bullet)_\bullet] = [F(C_\bullet)_\bullet, G(C_\bullet)_\bullet]$$

Lemma 3.4.1. *The construction above determines a natural transformation $\nu : F \to G$ of functors $[C]^{\text{ind}} \to [C]^{\text{ind}}$. That is, for every morphism $f \in [C_\bullet, D_\bullet]$, the following diagram in $[C]^{\text{ind}}$ commutes:*

$$
\begin{array}{ccc}
F(C_\bullet)_\bullet & \xrightarrow{\ \nu_{C_\bullet}\ } & G(C_\bullet)_\bullet \\
{\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\
F(D_\bullet)_\bullet & \xrightarrow{\ \nu_{D_\bullet}\ } & G(D_\bullet)_\bullet
\end{array}
$$

Proof. It is a straightforward verification.   □

Example 3.4.2. Regard $\kappa_?^{n,m} : J^n((?)_\bullet^{\mathfrak{S}_m}) \to (J^n(?))_\bullet^{\mathfrak{S}_m}$ as a natural transformation between (homotopy invariant) functors $C \to C^{\mathbb{Z}_{\geq 0}}$. By Lemma 3.4.1, we can also regard $\kappa_?^{n,m}$ a natural transformation:

$$\kappa_?^{n,m} : J^n((?)_\bullet^{\mathfrak{S}_m}) \longrightarrow (J^n(?))_\bullet^{\mathfrak{S}_m} : [C]^{\text{ind}} \longrightarrow [C]^{\text{ind}}$$

Example 3.4.3. Consider the (homotopy invariant) functors $F : C \to C^{\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}}$, $F(B) = (B_\bullet^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}$, and $G : C \to C^{\mathbb{Z}_{\geq 0}}$, $G(B) = B_\bullet^{\mathfrak{S}_{m+n}}$. Define $\theta : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by $\theta(r, s) = r + s$. Then θ is a functor and $\mu_?^{m,n} : F \to \theta^* G$ is a natural transformation. By Lemma 3.4.1, the pair $(\mu_?^{m,n}, \theta)$ induces a natural transformation:

$$\mu_?^{m,n} : ((?)_\bullet^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n} \longrightarrow (?)_\bullet^{\mathfrak{S}_{m+n}} : [C]^{\text{ind}} \longrightarrow [C]^{\text{ind}}$$

Remark 3.4.4. We have just seen that it makes sense to apply J and $(?)_\bullet^{\mathfrak{S}_n}$ to objects and morphisms in $[C]^{\text{ind}}$. Moreover, we can consider $\kappa_?^{n,m}$ and $\mu_?^{n,m}$ as natural transformations between functors $[C]^{\text{ind}} \to [C]^{\text{ind}}$. In the sequel, we will do this without further mention.

Let $A, B \in C$ and let $n \geq 1$. Recall from Remark 2.3.4 that the set $[A, B_\bullet^{\mathfrak{S}_n}]_C$ has a natural group structure, that is abelian if $n \geq 2$. We proceed to show that this assertion remains true if we replace A and B by arbitrary ind-objects in $[C]$.

Let $A \in C$ and let $(B, J) \in [C]^{\text{ind}}$. We have a bijection:

$$[A, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C \cong \operatornamewithlimits{colim}_j [A, (B_j)_\bullet^{\mathfrak{S}_n}]_C \tag{11}$$

By Remark 2.3.4, the transition functions of the colimit in (11) are group homomorphisms. Since filtering colimits of groups are computed as filtering colimits of sets, the right hand side of (11) is the underlying set of the colimit in the category of groups.

Consider the set $[A, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C$ together with the group structure for which (11) is a group isomorphism. This group structure is abelian if $n \geq 2$. Moreover, it is easily verified that if $f : A \to A'$ is a morphism in $[C]$, then $f^* : [A', (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C \to [A, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C$ is a group homomorphism. Now let $g \in [B_\bullet, B'_\bullet]_C$; we will show that the function $g_* : [A, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C \to [A, (B'_\bullet)_\bullet^{\mathfrak{S}_n}]_C$ is a group homomorphism. Let $j \in J$ and let $g_j : B_j \to B'_{g(j)}$ be a component of the morphism g. The following diagram of sets clearly commutes, where the vertical functions are the structural morphisms into the colimit:

$$
\begin{array}{ccc}
[A, (B_j)_\bullet^{\mathfrak{S}_n}] & \xrightarrow{(g_j)_*} & [A, (B'_{g(j)})_\bullet^{\mathfrak{S}_n}] \\
\Big\downarrow{\iota_j} & & \Big\downarrow{\iota_{g(j)}} \\
[A, (B_\bullet)_\bullet^{\mathfrak{S}_n}] & \xrightarrow{g_*} & [A, (B'_\bullet)_\bullet^{\mathfrak{S}_n}]
\end{array}
$$

Then $g_* \circ \iota_j$ is a group homomorphism, since the vertical functions and $(g_j)_*$ are. This shows that g_* is a group homomorphism because j is arbitrary. From now on, we will consider $[A, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C$ as a group with this group structure.

Now let $(A, I), (B, J) \in [C]^{\mathrm{ind}}$. We have an inverse system of group homomorphisms:

$$
\left\{ [A_{i'}, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C \longrightarrow [A_i, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C \right\}_{i \leq i'} \tag{12}
$$

Since limits of groups are computed as limits of sets, the set

$$
[A_\bullet, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C = \lim_i [A_i, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C \tag{13}
$$

is the underlying set of the limit of (12) in the category of groups. From now on, we consider (13) as a group; it is clear that (13) is abelian if $n \geq 2$. It is easily verified that if $f \in [A_\bullet, A'_\bullet]_C$ and $g \in [B_\bullet, B'_\bullet]_C$, then the following functions are group homomorphisms:

$$
f^* : [A'_\bullet, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C \longrightarrow [A_\bullet, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C
$$

$$
g_* : [A_\bullet, (B_\bullet)_\bullet^{\mathfrak{S}_n}]_C \longrightarrow [A_\bullet, (B'_\bullet)_\bullet^{\mathfrak{S}_n}]_C
$$

The discussion above can be summarized in the following result, which is actually a corollary of Theorem 2.3.3.

Lemma 3.4.5. *Let* $(B, J) \in [C]^{\mathrm{ind}}$ *and let* $n \geq 1$. *Then* $(B_\bullet)_\bullet^{\mathfrak{S}_n}$ *is a group object in* $[C]^{\mathrm{ind}}$, *which is abelian if* $n \geq 2$. *Moreover, a morphism* $g \in [B_\bullet, B'_\bullet]_C$ *induces a morphism of group objects* $g_* \in [(B_\bullet)_\bullet^{\mathfrak{S}_n}, (B'_\bullet)_\bullet^{\mathfrak{S}_n}]_C$.

3.5 Some technical results

Lemma 3.5.1. *Let* $A_\bullet, B_\bullet \in [C]^{\mathrm{ind}}$ *and let* $m, n \geq 1$. *Then the following composite function is a group homomorphism:*

$$
[A_\bullet, (B_\bullet)_\bullet^{\mathfrak{S}_m}]_C \xrightarrow{J^n} [J^n(A_\bullet), J^n((B_\bullet)_\bullet^{\mathfrak{S}_m})]_C \xrightarrow{\left(\kappa_{B_\bullet}^{n,m}\right)_*} [J^n(A_\bullet), (J^n(B_\bullet))_\bullet^{\mathfrak{S}_m}]_C \tag{14}
$$

Proof. Write $\psi^{n,m}_{A_\bullet,B_\bullet}$ for the composition of the functions in (14). It suffices to consider the case $A_\bullet = A \in C$ and $B_\bullet = B \in C$, as it is easily verified that $\psi^{n,m}_{A_\bullet,B_\bullet}$ equals the function:

$$\lim_i \operatorname{colim}_j [A_i, (B_j)^{\mathfrak{S}_m}_\bullet]_C \xrightarrow{\lim \operatorname{colim} \psi^{n,m}_{A_i,B_j}} \lim_i \operatorname{colim}_j [J^n(A_i), (J^n(B_j))^{\mathfrak{S}_m}_\bullet]_C$$

We proceed by induction on n. To prove the case $n = 1$, we will show that there is a morphism of simplicial sets $\varphi : \operatorname{Hom}_C(A, B^\Delta) \to \operatorname{Hom}_C(JA, (JB)^\Delta)$ that induces $\psi^{1,m}_{A,B}$ under the identification of Theorem 2.3.3. Define φ by

$$\varphi : \operatorname{Hom}_C(A, B^{\Delta^p}) \to \operatorname{Hom}_C(JA, (JB)^{\Delta^p}), \quad f \mapsto \xi_{\Delta^p} \circ J(f),$$

where $\xi_K : J(B^K) \to (JB)^K$ is the classifying map of $(\mathscr{U}_B)^K$; see Example 3.1.4. It is easily verified that the following diagram commutes:

$$
\begin{array}{ccc}
\operatorname{Hom}_{\mathbb{S}}(K, \operatorname{Hom}_C(A, B^\Delta)) & \xrightarrow{\cong} & \operatorname{Hom}_C(A, B^K) \\
\varphi_* \downarrow & & \downarrow \xi_K \circ J(?) \\
\operatorname{Hom}_{\mathbb{S}}(K, \operatorname{Hom}_C(JA, (JB)^\Delta)) & \xrightarrow{\cong} & \operatorname{Hom}_C(JA, (JB)^K)
\end{array}
$$

Here, the horizontal bijections are the adjunction isomorphisms described in section 1.4. This implies that the following diagram commutes, proving the case $n = 1$.

$$
\begin{array}{ccc}
\left(\Omega^m Ex^\infty \operatorname{Hom}_C(A, B^\Delta)\right)_0 & \xrightarrow{\cong} & \operatorname{Hom}_{C^{\mathrm{ind}}}(A, B^{\mathfrak{S}_m}) \\
(\Omega^m Ex^\infty \varphi)_0 \downarrow & & \downarrow \kappa^{1,m}_B \circ J(?) \\
\left(\Omega^m Ex^\infty \operatorname{Hom}_C(JA, (JB)^\Delta)\right)_0 & \xrightarrow{\cong} & \operatorname{Hom}_{C^{\mathrm{ind}}}(A, (JB)^{\mathfrak{S}_m})
\end{array}
$$

The inductive step is straightforward once we notice that $\psi^{n+1,m}_{A,B} = \psi^{1,m}_{J^n A, J^n B} \circ \psi^{n,m}_{A,B}$. $\qquad\square$

Lemma 3.5.2. *Let $A, B \in C$, let $C_\bullet \in (\mathrm{Alg}_\mathbb{Z})^K$ and let $m \geq 1$. Then the following composite function is a group homomorphism:*

$$[A, B^{\mathfrak{S}_m}_\bullet]_C \xrightarrow{? \otimes C_\bullet} [A \otimes C_\bullet, B^{\mathfrak{S}_m}_\bullet \otimes C_\bullet]_C \cong [A \otimes C_\bullet, (B \otimes C_\bullet)^{\mathfrak{S}_m}_\bullet]_C \qquad (15)$$

Here, the bijection on the right is induced by the obvious isomorphism of $K \times \mathbb{Z}_{\geq 0}$-diagrams $B^{\mathfrak{S}_m}_\bullet \otimes C_\bullet \cong (B \otimes C_\bullet)^{\mathfrak{S}_m}_\bullet$.

Proof. Write $\tau_{C_\bullet}$ for the composition of the functions in (15). We begin with the special case $C_\bullet = C \in \mathrm{Alg}_\mathbb{Z}$. Let t_p be the following composite function:

$$\operatorname{Hom}_C(A, B^{\Delta^p}) \xrightarrow{? \otimes C} \operatorname{Hom}_C(A \otimes C, B^{\Delta^p} \otimes C) \cong \operatorname{Hom}_C(A \otimes C, (B \otimes C)^{\Delta^p})$$

It is easily verified that, for varying p, the functions t_p assemble into a morphism of simplicial sets $\operatorname{Hom}_C(A, B^\Delta) \to \operatorname{Hom}_C(A \otimes C, (B \otimes C)^\Delta)$ that induces τ_C upon taking $\pi_m Ex^\infty$ and making the identifications of Theorem 2.3.3

Now let $C_\bullet \in (\mathrm{Alg}_\mathbb{Z})^K$ be any ind-ring. We have:

$$[A \otimes C_\bullet, (B \otimes C_\bullet)_\bullet^{\mathfrak{S}_m}]_C = \lim_k [A \otimes C_k, (B \otimes C_\bullet)_\bullet^{\mathfrak{S}_m}]_C$$

Let $\pi_k : [A \otimes C_\bullet, (B \otimes C_\bullet)_\bullet^{\mathfrak{S}_m}]_C \to [A \otimes C_k, (B \otimes C_\bullet)_\bullet^{\mathfrak{S}_m}]_C$ be the projection from the limit. To prove that $\tau_{C_\bullet}$ is a group homomorphism it suffices to show that $\pi_k \circ \tau_{C_\bullet}$ is a group homomorphism for all $k \in K$. Let ι_k be the natural morphism into the colimit:

$$[A \otimes C_k, (B \otimes C_k)_\bullet^{\mathfrak{S}_m}] \xrightarrow{\iota_k} \operatorname*{colim}_{k'}[A \otimes C_k, (B \otimes C_{k'})_\bullet^{\mathfrak{S}_m}] = [A \otimes C_k, (B \otimes C_\bullet)_\bullet^{\mathfrak{S}_m}]$$

It is easily verified that $\pi_k \circ \tau_{C_\bullet}$ factors as:

$$[A, B_\bullet^{\mathfrak{S}_m}]_C \xrightarrow{\tau_{C_k}} [A \otimes C_k, (B \otimes C_k)_\bullet^{\mathfrak{S}_m}]_C \xrightarrow{\iota_k} [A \otimes C_k, (B \otimes C_\bullet)_\bullet^{\mathfrak{S}_m}]_C$$

This shows that $\pi_k \circ \tau_{C_\bullet}$ is a group homomorphism since both τ_{C_k} and ι_k are. $\qquad\square$

Lemma 3.5.3. *Let $A_\bullet, B_\bullet \in [C]^{\mathrm{ind}}$ and let $n \geq 1$. Then the following function is a group homomorphism:*

$$[A_\bullet, ((B_\bullet)_\bullet^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}]_C \xrightarrow{\left(\mu_{B_\bullet}^{m,n}\right)_*} [A_\bullet, (B_\bullet)_\bullet^{\mathfrak{S}_{m+n}}]_C \qquad (16)$$

Proof. The general case reduces to the case $A_\bullet = A \in C$, as it is easily verified that the function (16) equals:

$$\lim_i [A_i, ((B_\bullet)_\bullet^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}]_C \xrightarrow{\lim\left(\mu_{B_\bullet}^{m,n}\right)_*} \lim_i [A_i, (B_\bullet)_\bullet^{\mathfrak{S}_{m+n}}]_C$$

From now on, suppose that $A_\bullet = A \in C$. We have:

$$[A, ((B_\bullet)_\bullet^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}] = \operatorname*{colim}_{(r,j)}[A, ((B_j)_r^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}]$$

Let $\iota_{(r,j)} : [A, ((B_j)_r^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}] \to [A, ((B_\bullet)_\bullet^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}]$ be the natural morphism into the colimit. To prove the result, it suffices to show that $\left(\mu_{B_\bullet}^{m,n}\right)_* \circ \iota_{(r,j)}$ is a group homomorphism for every pair (r, j). It is easily verified that there is a commutative diagram as follows, where the vertical function is the structural morphism into the colimit:

$$
\begin{array}{ccc}
[A, ((B_j)_r^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}]_C & \xrightarrow{\left(\mu_{B_\bullet}^{m,n}\right)_* \circ \iota_{(r,j)}} & [A, (B_\bullet)_\bullet^{\mathfrak{S}_{m+n}}]_C \\
& \searrow{\scriptstyle \left(\mu_{B_j}^{m,n}\right)_*} & \uparrow \\
& & [A, (B_j)_\bullet^{\mathfrak{S}_{m+n}}]_C
\end{array}
\qquad (17)
$$

Thus, it is enough to show that $\left(\mu_{B_j}^{m,n}\right)_*$ is a group homomorphism. For fixed r, j, s and p, consider the dotted function that makes the following diagram commute:

$$
\begin{array}{ccc}
\left(Ex^s \mathrm{Hom}_C(A, ((B_j)_r^{\mathfrak{S}_m})^\Delta)\right)_p & \xrightarrow{\cong} & \mathrm{Hom}_C\left(A, ((B_j)_r^{\mathfrak{S}_m})_s^{(\Delta^p, \emptyset)}\right) \\
\vdots & & \downarrow{\scriptstyle \left(\mu_{B_j}^{(I^m, \partial I^m),(\Delta^p, \emptyset)}\right)_*} \\
\left(\Omega^m Ex^{r+s} \mathrm{Hom}_C(A, (B_j)^\Delta)\right)_p & \xrightarrow{\cong} & \mathrm{Hom}_C\left(A, (B_j)_{r+s}^{(I^m \times \Delta^p, \partial I^m \times \Delta^p)}\right)
\end{array}
$$

This dotted function is natural in p and thus induces a morphism of simplicial sets:

$$Ex^s \mathrm{Hom}_C \left(A, ((B_j)_r^{\mathfrak{S}_m})^\Delta \right) \longrightarrow \Omega^m Ex^{r+s} \mathrm{Hom}_C \left(A, (B_j)^\Delta \right)$$

The latter is in turn natural in s and thus induces the following morphism ψ upon taking colimit:

$$\psi : Ex^\infty \mathrm{Hom}_C \left(A, ((B_j)_r^{\mathfrak{S}_m})^\Delta \right) \longrightarrow \Omega^m Ex^\infty \mathrm{Hom}_C \left(A, (B_j)^\Delta \right)$$

Recall our conventions about iterated loop spaces from section 1.3.3. It is easily verified that ψ fits into the following commutative diagram —indeed, the commutativity of the diagram ultimately reduces to the naturality of the morphism μ of Remark 2.2.3.

$$
\begin{array}{ccc}
\mathrm{Hom}_C \left(A, ((B_j)_r^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n} \right) & \xrightarrow{\left(\mu_{B_j}^{m,n} \right)_* \circ \iota_r} & \mathrm{Hom}_C \left(A, (B_j)_\bullet^{\mathfrak{S}_{m+n}} \right) \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} \\
 & & \left(\Omega^{m+n} Ex^\infty \mathrm{Hom}_C(A, (B_j)^\Delta) \right)_0 \\
 & & \Big\downarrow{\scriptstyle\cong} \\
\left(\Omega^n Ex^\infty \mathrm{Hom}_C(A, ((B_j)_r^{\mathfrak{S}_m})^\Delta) \right)_0 & \xrightarrow{\Omega^n(\psi)} & \left(\Omega^n \Omega^m Ex^\infty \mathrm{Hom}_C(A, (B_j)^\Delta) \right)_0
\end{array}
$$

It follows that, under the bijection in Theorem 2.3.3, the function $\left(\mu_{B_j}^{m,n} \right)_*$ in (17) identifies with the group homomorphism:

$$\pi_n(\psi) : \pi_n Ex^\infty \mathrm{Hom}_C(A, ((B_j)_r^{\mathfrak{S}_m})^\Delta) \longrightarrow \pi_n \Omega^m Ex^\infty \mathrm{Hom}_C(A, (B_j)^\Delta)$$

This finishes the proof.   □

Lemma 3.5.4. *Let $A, B \in C$ and $m, n \geq 1$. Then the following composite function is a group homomorphism:*

$$[A, B_\bullet^{\mathfrak{S}_m}]_C \xrightarrow{(?)^{\mathfrak{S}_n}} [A_\bullet^{\mathfrak{S}_n}, (B_\bullet^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}]_C \xrightarrow{\left(\mu_{B_\bullet}^{m,n} \right)_*} [A_\bullet^{\mathfrak{S}_n}, B_\bullet^{\mathfrak{S}_{m+n}}]_C$$

Proof. The functors $? \otimes \mathbb{Z}_\bullet^{\mathfrak{S}_n}$ and $(?)_\bullet^{\mathfrak{S}_n}$ are naturally isomorphic. Then, by Lemma 3.5.2 applied to $C_\bullet = \mathbb{Z}_\bullet^{\mathfrak{S}_n}$, we have a group homomorphism:

$$\tau_{\mathbb{Z}_\bullet^{\mathfrak{S}_n}} : [A, B_\bullet^{\mathfrak{S}_m}]_C \longrightarrow [A_\bullet^{\mathfrak{S}_n}, (B_\bullet^{\mathfrak{S}_n})_\bullet^{\mathfrak{S}_m}]_C$$

Let $c : I^m \times I^n \xrightarrow{\cong} I^n \times I^m$ be the commutativity isomorphism. It is easily verified that the following diagram commutes:

$$
\begin{array}{ccc}
[A, B_\bullet^{\mathfrak{S}_m}]_C & \xrightarrow{(?)^{\mathfrak{S}_n}} [A_\bullet^{\mathfrak{S}_n}, (B_\bullet^{\mathfrak{S}_m})_\bullet^{\mathfrak{S}_n}]_C \xrightarrow{\left(\mu_B^{m,n} \right)_*} & [A_\bullet^{\mathfrak{S}_n}, B_\bullet^{\mathfrak{S}_{m+n}}]_C \\
\Big\downarrow{\scriptstyle \tau_{\mathbb{Z}_\bullet^{\mathfrak{S}_n}}} & & \Big\uparrow{\scriptstyle c^*} \\
[A_\bullet^{\mathfrak{S}_n}, (B_\bullet^{\mathfrak{S}_n})_\bullet^{\mathfrak{S}_m}]_C & \xrightarrow{\left(\mu_B^{n,m} \right)_*} & [A_\bullet^{\mathfrak{S}_n}, B_\bullet^{\mathfrak{S}_{n+m}}]_C
\end{array}
$$

The function $\left(\mu_B^{n,m} \right)_*$ is a group homomorphism by Lemma 3.5.3 and the function c^* is multiplication by $(-1)^{mn}$. The result follows.   □

3.6 The loop-stable homotopy category

Let $f : A \to B_r^{\mathfrak{S}_n}$ be a morphism in C. By Proposition 3.1.1, there exists a unique strong morphism of extensions $\mathscr{U}_A \to \mathscr{P}_{n,B}$ that extends f:

$$
\begin{array}{ccccccc}
\mathscr{U}_A & & JA & \longrightarrow & TA & \longrightarrow & A \\
\exists! \downarrow & & \Lambda^n(f) \downarrow & & \downarrow & & \downarrow f \\
\mathscr{P}_{n,B} & & B_r^{\mathfrak{S}_{n+1}} & \longrightarrow & P(n,B)_r & \longrightarrow & B_r^{\mathfrak{S}_n}
\end{array}
$$

We will write $\Lambda^n(f)$ for the classifying map of f with respect to $\mathscr{P}_{n,B}$.

Remark 3.6.1. We have $\Lambda^n(f) = \Lambda^n(\mathrm{id}_{B^{\mathfrak{S}_n}}) \circ J(f)$. Indeed, this follows from the uniqueness statement in Proposition 3.1.1 and the fact that the following diagram exhibits a strong morphism of extensions $\mathscr{U}_A \to \mathscr{P}_{n,B}$ that extends f:

$$
\begin{array}{ccccccc}
\mathscr{U}_A & & JA & \longrightarrow & TA & \longrightarrow & A \\
\downarrow & & J(f) \downarrow & & T(f) \downarrow & & \downarrow f \\
\mathscr{U}_{B_r^{\mathfrak{S}_n}} & & J(B_r^{\mathfrak{S}_n}) & \longrightarrow & T(B_r^{\mathfrak{S}_n}) & \longrightarrow & B_r^{\mathfrak{S}_n} \\
\downarrow & & \Lambda^n(\mathrm{id}_{B^{\mathfrak{S}_n}}) \downarrow & & \downarrow & & \downarrow \mathrm{id} \\
\mathscr{P}_{n,B} & & B_r^{\mathfrak{S}_{n+1}} & \longrightarrow & P(n,B)_r & \longrightarrow & B_r^{\mathfrak{S}_n}
\end{array}
$$

Remark 3.6.2. We have $\Lambda^n(f) = \mu_B^{n,1} \circ (f)_0^{\mathfrak{S}_1} \circ \lambda_A$. Indeed, this follows from the uniqueness statement in Proposition 3.1.1 and the fact that the following diagram exhibits a strong morphism of extensions $\mathscr{U}_A \to \mathscr{P}_{n,B}$ that extends f:

$$
\begin{array}{ccccccc}
\mathscr{U}_A & & JA & \longrightarrow & TA & \longrightarrow & A \\
\downarrow & & \lambda_A \downarrow & & \downarrow & & \downarrow \mathrm{id} \\
\mathscr{P}_{0,A} & & A_0^{\mathfrak{S}_1} & \longrightarrow & P(0,A)_0 & \longrightarrow & A \\
\downarrow & & (f)_0^{\mathfrak{S}_1} \downarrow & & P(0,f)_0 \downarrow & & \downarrow f \\
\mathscr{P}_{0,B_r^{\mathfrak{S}_n}} & & (B_r^{\mathfrak{S}_n})_0^{\mathfrak{S}_1} & \longrightarrow & P(0,B_r^{\mathfrak{S}_n})_0 & \longrightarrow & B_r^{\mathfrak{S}_n} \\
\text{Example 3.2.1} \downarrow & & \mu_B^{n,1} \downarrow & & \downarrow & & \downarrow \mathrm{id} \\
\mathscr{P}_{n,B} & & B_r^{\mathfrak{S}_{n+1}} & \longrightarrow & P(n,B)_r & \longrightarrow & B_r^{\mathfrak{S}_n}
\end{array}
$$

If $f,g : A \to B_r^{\mathfrak{S}_n}$ are homotopic morphisms, then $\Lambda^n(f)$ and $\Lambda^n(g)$ are homotopic too. Thus, we can regard Λ^n as a function $\Lambda_{A,B}^n : [A, B_r^{\mathfrak{S}_n}] \to [JA, B_r^{\mathfrak{S}_{n+1}}]$. We proceed to explain how to define $\Lambda_{A_\bullet,B_\bullet}^n$ for ind-algebras $A_\bullet$ and $B_\bullet$.

Let $A \in C$ and let $(B, J) \in [C]^{\mathrm{ind}}$. An easy verification shows that, for $j \leq j' \in J$, the

following diagrams commute:

$$
\begin{array}{ccc}
[A,(B_j)_r^{\mathfrak{S}_n}] & \xrightarrow{\Lambda^n_{A,B_j}} & [JA,(B_j)_r^{\mathfrak{S}_{n+1}}] \\
\downarrow & & \downarrow \\
[A,(B_j)_{r+1}^{\mathfrak{S}_n}] & \xrightarrow{\Lambda^n_{A,B_j}} & [JA,(B_j)_{r+1}^{\mathfrak{S}_{n+1}}]
\end{array}
\qquad
\begin{array}{ccc}
[A,(B_j)_r^{\mathfrak{S}_n}] & \xrightarrow{\Lambda^n_{A,B_j}} & [JA,(B_j)_r^{\mathfrak{S}_{n+1}}] \\
\downarrow & & \downarrow \\
[A,(B_{j'})_r^{\mathfrak{S}_n}] & \xrightarrow{\Lambda^n_{A,B_{j'}}} & [JA,(B_{j'})_r^{\mathfrak{S}_{n+1}}]
\end{array}
$$

Then, it makes sense to define $\Lambda^n_{A,B_\bullet} : [A,(B_\bullet)_\bullet^{\mathfrak{S}_n}] \to [JA,(B_\bullet)_\bullet^{\mathfrak{S}_{n+1}}]$ as the function:

$$
\operatorname*{colim}_{(r,j)}[A,(B_j)_r^{\mathfrak{S}_n}] \xrightarrow{\operatorname{colim}\Lambda^n_{A,B_j}} \operatorname*{colim}_{(r,j)}[JA,(B_j)_r^{\mathfrak{S}_{n+1}}]
$$

Now let $(A,I) \in [C]^{\mathrm{ind}}$. It is easily verified that, for $i \le i' \in I$, the following diagram commutes:

$$
\begin{array}{ccc}
[A_{i'},(B_\bullet)_\bullet^{\mathfrak{S}_n}] & \xrightarrow{\Lambda^n_{A_{i'},B_\bullet}} & [JA_{i'},(B_\bullet)_\bullet^{\mathfrak{S}_{n+1}}] \\
\downarrow & & \downarrow \\
[A_i,(B_\bullet)_\bullet^{\mathfrak{S}_n}] & \xrightarrow{\Lambda^n_{A_i,B_\bullet}} & [JA_i,(B_\bullet)_\bullet^{\mathfrak{S}_{n+1}}]
\end{array}
$$

Then, it makes sense to define $\Lambda^n_{A_\bullet,B_\bullet} : [A_\bullet,(B_\bullet)_\bullet^{\mathfrak{S}_n}] \to [J(A_\bullet),(B_\bullet)_\bullet^{\mathfrak{S}_{n+1}}]$ as the function:

$$
\lim_i[A_i,(B_\bullet)_\bullet^{\mathfrak{S}_n}] \xrightarrow{\lim\Lambda^n_{A_i,B_\bullet}} \lim_i[JA_i,(B_\bullet)_\bullet^{\mathfrak{S}_{n+1}}]
$$

When $A_\bullet$ and $B_\bullet$ are clear from the context, we will write Λ^n instead of $\Lambda^n_{A_\bullet,B_\bullet}$.

Lemma 3.6.3. *Let $A_\bullet, B_\bullet \in [C]^{\mathrm{ind}}$ and let $n \ge 1$. Then the functions*

$$
\Lambda^n_{A_\bullet,B_\bullet} : [A_\bullet,(B_\bullet)_\bullet^{\mathfrak{S}_n}]_C \longrightarrow [J(A_\bullet),(B_\bullet)_\bullet^{\mathfrak{S}_{n+1}}]_C
$$

are group homomorphisms.

Proof. We easily reduce to the case $A_\bullet = A \in C$ and $B_\bullet = B \in C$. Consider the following chain of strong morphisms of extensions:

$$
\begin{array}{cccccc}
\mathscr{U}_{B_r^{\mathfrak{S}_n}} & & J(B_r^{\mathfrak{S}_n}) \longrightarrow & T(B_r^{\mathfrak{S}_n}) \longrightarrow & B_r^{\mathfrak{S}_n} \\
\downarrow & & {\kappa_B^{1,n}}\downarrow & \downarrow & \downarrow^{\mathrm{id}} \\
(\mathscr{U}_B)^{\mathfrak{S}_n} & & (JB)_r^{\mathfrak{S}_n} \longrightarrow & (TB)_r^{\mathfrak{S}_n} \longrightarrow & B_r^{\mathfrak{S}_n} \\
\downarrow & & {(\lambda_B)^{\mathfrak{S}_n}}\downarrow & \downarrow & \downarrow^{\mathrm{id}} \\
(\mathscr{P}_{0,B})^{\mathfrak{S}_n} & & (B_0^{\mathfrak{S}_1})_r^{\mathfrak{S}_n} \longrightarrow & (PB)_r^{\mathfrak{S}_n} \longrightarrow & B_r^{\mathfrak{S}_n} \\
\downarrow & & {\mu_B^{1,n}}\downarrow & \downarrow & \downarrow^{\mathrm{id}} \\
\widetilde{\mathscr{P}}_{n,B} & & B_r^{\mathfrak{S}_{1+n}} \longrightarrow & \widetilde{P}(n,B)_r \longrightarrow & B_r^{\mathfrak{S}_n} \\
{\scriptstyle\text{Example 3.2.3}}\downarrow & & {c^*}\downarrow{\scriptstyle\cong} & {c^*}\downarrow{\scriptstyle\cong} & \downarrow^{\mathrm{id}} \\
\mathscr{P}_{n,B} & & B_r^{\mathfrak{S}_{n+1}} \longrightarrow & P(n,B)_r \longrightarrow & B_r^{\mathfrak{S}_n}
\end{array}
$$

By the uniqueness statement in Proposition 3.1.1, we have:

$$\Lambda^n(\mathrm{id}_{B^{\mathfrak{S}_n}}) = c^* \circ \mu_B^{1,n} \circ (\lambda_B)^{\mathfrak{S}_n} \circ \kappa_B^{1,n}$$

Then, by Remark 3.6.1, $\Lambda_{A,B}^n$ equals the following composite:

$$[A, B_\bullet^{\mathfrak{S}_n}] \xrightarrow{\kappa_B^{1,n} \circ J(?)} [JA, (JB)_\bullet^{\mathfrak{S}_n}] \xrightarrow{(\lambda_B)_*} [JA, (B_0^{\mathfrak{S}_1})_\bullet^{\mathfrak{S}_n}] \xrightarrow{(\mu_B^{1,n})_*} [JA, B_\bullet^{\mathfrak{S}_{1,n}}] \xrightarrow{c^*} [JA, B_\bullet^{\mathfrak{S}_{n+1}}]$$

This implies that $\Lambda_{A,B}^n$ is a group homomorphism by Lemma 3.5.1, Lemma 3.5.3 and Example 2.3.6. □

Definition 3.6.4 (cf. [3, Section 6.3]). We proceed to define a category $\mathfrak{K}^C$, that we will call the *loop-stable homotopy category*. The objects of $\mathfrak{K}^C$ are the pairs (A, m) where A is an object of C and $m \in \mathbb{Z}$. For two objects (A, m) and (B, n), put:

$$\mathrm{Hom}_{\mathfrak{K}^C}((A, m), (B.n)) := \mathrm{colim}_v [J^{m+v} A, B_\bullet^{\mathfrak{S}_{n+v}}]_C$$

Here, the colimit is taken over the morphisms Λ^{n+v} of Lemma 3.6.3 and v runs over the integers such that both $m + v \geq 0$ and $n + v \geq 0$. The composition in $\mathfrak{K}^C$ is defined as follows: Represent elements of $\mathrm{Hom}_{\mathfrak{K}^C}((A, m), (B, n))$ and $\mathrm{Hom}_{\mathfrak{K}^C}((B, n), (C, k))$ by $f \in [J^{m+v}A, B_\bullet^{\mathfrak{S}_{n+v}}]$ and $g \in [J^{n+w}B, C_\bullet^{\mathfrak{S}_{k+w}}]$ respectively. To simplify notation, write:

$$N_1 := m + v, \quad N_2 := n + v, \quad N_3 := n + w \quad \text{and} \quad N_4 := k + w.$$

Let $g \star f \in [J^{N_1+N_3}A, C_\bullet^{\mathfrak{S}_{N_2+N_4}}]$ be the unique homotopy class that makes the following diagram in $[C]^{\mathrm{ind}}$ commute:

We will show in Lemma 3.7.7 that the class of $g \star f$ in $\mathrm{Hom}_{\mathfrak{K}^C}((A, m), (C, k))$ does not depend upon the choice of the representatives f and g. Then, in Theorem 3.7.8, we will prove that $\star$ defines a composition that makes $\mathfrak{K}^C$ into a category.

3.7 Well-definedness of the composition

In this section we prove that the composition law described in Definition 3.6.4 is well-defined and makes $\mathfrak{K}^C$ into a category. We will closely follow [3, Section 6.3], making appropiate changes to translate the proof into the algebraic setting. We start with the following two lemmas, whose proofs are straightforward verifications.

Lemma 3.7.1. *Let $A_\bullet, B_\bullet \in [C]^{\mathrm{ind}}$ and let $g \in [A_\bullet, (B_\bullet)_\bullet^{\mathfrak{S}_n}]$.*

  (i) If $f \in [A'_\bullet, A_\bullet]$, then $\Lambda^n(g \circ f) = \Lambda^n(g) \circ J(f) \in [J(A'_\bullet), (B_\bullet)_\bullet^{\mathfrak{S}_{n+1}}]$.

  (ii) If $h \in [B_\bullet, B'_\bullet]$, then $\Lambda^n(h^{\mathfrak{S}_n} \circ g) = h^{\mathfrak{S}_{n+1}} \circ \Lambda^n(g) \in [J(A_\bullet), (B'_\bullet)_\bullet^{\mathfrak{S}_{n+1}}]$.

Lemma 3.7.2. *Let $A_\bullet, B_\bullet \in [C]^{\mathrm{ind}}$ and let $f \in [A_\bullet, ((B_\bullet)_\bullet^{\mathfrak{S}_n})_\bullet^{\mathfrak{S}_m}]$. Then:*

$$\Lambda^{n+m}\left(\mu_{B_\bullet}^{n,m} \circ f\right) = \mu_{B_\bullet}^{n,m+1} \circ \Lambda^m(f) \in [J(A_\bullet), (B_\bullet)_\bullet^{\mathfrak{S}_{n+m+1}}]$$

Lemma 3.7.3 (cf. [3, Lemma 6.30]). *Let $B \in C$. Then the following diagram in $[C]^{\mathrm{ind}}$ commutes:*

$$
\begin{array}{ccc}
J^2 B & \xrightarrow{\ J(\lambda_B)\ } & J(B_\bullet^{\mathfrak{S}_1}) \\
 & {\scriptstyle (\lambda_{JB})^{-1}}\searrow & \downarrow{\scriptstyle \kappa_B^{1,1}} \\
 & & (JB)_\bullet^{\mathfrak{S}_1}
\end{array}
\qquad (18)
$$

Here, $(\lambda_{JB})^{-1}$ is the inverse of λ_{JB} in the group $[J^2 B, (JB)_\bullet^{\mathfrak{S}_1}]$.

Proof. Let $A \in C$. Recall from (2) and (5) that there are extensions:

$$(\mathscr{U}_A, \sigma_A): \quad JA \longrightarrow TA \xrightarrow{\ \eta_A\ } A$$

$$\mathscr{P}_{0,A}: \quad A_r^{\mathfrak{S}_1} \longrightarrow (PA)_r \longrightarrow A$$

Recall from Example 3.2.4 that, for $r = 0$, the extension $\mathscr{P}_{0,A}$ is isomorphic to:

$$\mathscr{P}_{0,A}: \quad (t^2 - t)A[t] \xrightarrow{\ \mathrm{inc}\ } (t - 1)A[t] \xrightarrow{\ \mathrm{ev_0}\ } A$$

In the rest of the proof we will identify these extensions without further mention. Define:

$$I := \ker\left(t(TB)[t] \xrightarrow{\ \mathrm{ev_1}\ } TB \xrightarrow{\ \eta_B\ } B \right)$$

$$E := \{(p, q) \in t(TB)[t] \times t(JB)[t] : p(1) = q(1)\}$$

It is easily verified that the following diagram is an extension in C, that has a section $s: B \to t(TB)[t]$ defined by $s(b) = \sigma_B(b)t$:

$$(\mathscr{E}, s): \quad I \xrightarrow{\ \mathrm{inc}\ } t(TB)[t] \xrightarrow{\ \eta_B \circ \mathrm{ev_1}\ } B$$

Note that $\mathrm{ev_1}: I \to TB$ factors through JB and let $s': I \to E$, $s'(p) = (p, p(1)t)$. It is easily verified that the following diagram is an extension in C, where $\mathrm{pr_1}$ is the projection into the first factor:

$$(\mathscr{E}', s'): (JB)_0^{\mathfrak{S}_1} = (t^2 - t)(JB)[t] \xrightarrow{\ (0,\mathrm{inc})\ } E \xrightarrow{\ \mathrm{pr_1}\ } I$$

Let $\chi : JB \to I$ be the classifying map of $(\mathscr{E}, s)$. Consider the following diagram, that exhibits a strong morphism of extensions from $\mathscr{U}_B$ into itself extending id_B:

$$
\begin{array}{ccccc}
\mathscr{U}_B & JB & \longrightarrow & TB & \longrightarrow & B \\
\downarrow & \chi\downarrow & & \downarrow & & \downarrow\mathrm{id} \\
(\mathscr{E}, s) & I & \xrightarrow{\ \mathrm{inc}\ } & t(TB)[t] & \xrightarrow{\ \eta_B\circ\mathrm{ev}_1\ } & B \\
\downarrow & \mathrm{ev}_1\downarrow & & \downarrow\mathrm{ev}_1 & & \downarrow\mathrm{id} \\
\mathscr{U}_B & JB & \xrightarrow{\ \mathrm{inc}\ } & TB & \xrightarrow{\ \eta_B\ } & B
\end{array}
$$

It follows that $\mathrm{ev}_1 \circ \chi = \mathrm{id}_{JB}$. Let ω be the automorphism of $(JB)_0^{\mathfrak{S}_1} = (t^2 - t)(JB)[t]$ defined by $\omega(t) = 1 - t$. Now consider the following diagram, which exhibits a strong morphism of extensions $\mathscr{U}_{JB} \to \mathscr{P}_{0,JB}$ extending id_{JB}:

$$
\begin{array}{ccccccc}
\mathscr{U}_{JB} & J^2B & \longrightarrow & T(JB) & \longrightarrow & JB \\
\downarrow & \downarrow & & \downarrow & & \downarrow\chi \\
(\mathscr{E}', s') & (t^2 - t)(JB)[t] & \xrightarrow{(0,\mathrm{inc})} & E & \xrightarrow{\ \mathrm{pr}_1\ } & I \\
\downarrow & \mathrm{id}\downarrow & & \downarrow\mathrm{pr}_2 & & \downarrow\mathrm{ev}_1 \\
\mathscr{E}'' & (t^2 - t)(JB)[t] & \xrightarrow{\ \mathrm{inc}\ } & t(JB)[t] & \xrightarrow{\ \mathrm{ev}_1\ } & JB \\
\downarrow & \omega\downarrow & & \downarrow t\mapsto 1-t & & \downarrow\mathrm{id} \\
\mathscr{P}_{0,JB} & (t^2 - t)(JB)[t] & \xrightarrow{\ \mathrm{inc}\ } & (t - 1)(JB)[t] & \xrightarrow{\ \mathrm{ev}_0\ } & JB
\end{array}
$$

Since $\omega^{-1} = \omega$, it follows that the classifying map of χ with respect to $(\mathscr{E}', s')$ equals the composite:

$$
J^2B \xrightarrow{\ \lambda_{JB}\ } (t^2 - t)(JB)[t] \xrightarrow{\ \omega\ } (t^2 - t)(JB)[t]
$$

Now recall from Example 3.2.4 that, for $r = 1$, the extension $\mathscr{P}_{0,A}$ is isomorphic to:

$$
\mathscr{P}_{0,A} : \ tA[t] \ {}_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} tA[t] \xrightarrow{\ \mathrm{inc}\ } A[t] \ {}_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} tA[t] \xrightarrow{\ \mathrm{ev}_0\circ\mathrm{pr}_1\ } A
$$

In the rest of the proof we will identify these extensions without further mention. Define a morphism $\theta : E \to (TB)_1^{\mathfrak{S}_1} = t(TB)[t] \ {}_{\mathrm{ev}_1}\times_{\mathrm{ev}_1} t(TB)[t]$ by the formula $(p, q) \mapsto (q, p)$. Consider the following commutative diagram in $\mathcal{C}$:

$$
\begin{array}{ccccccc}
\mathscr{U}_{JB} & J^2B & \longrightarrow & T(JB) & \longrightarrow & JB \\
\downarrow & \omega\circ\lambda_{JB}\downarrow & & \downarrow & & \downarrow\chi \\
\mathscr{E}' & (JB)_0^{\mathfrak{S}_1} & \xrightarrow{(0,\mathrm{inc})} & E & \xrightarrow{\ \mathrm{pr}_1\ } & I \\
\downarrow & (\mathrm{inc},0)\downarrow & & \downarrow\theta & & \downarrow(0,\eta_B) \\
(\mathscr{U}_B)^{\mathfrak{S}_1} & (JB)_1^{\mathfrak{S}_1} & \xrightarrow{(\mathrm{inc})^{\mathfrak{S}_1}} & (TB)_1^{\mathfrak{S}_1} & \xrightarrow{(\eta_B)^{\mathfrak{S}_1}} & B_1^{\mathfrak{S}_1}
\end{array}
$$

Note that the morphism $\mathscr{E}' \to (\mathscr{U}_B)^{\mathfrak{S}_1}$ is not compatible with the sections. Let $\psi :=$ $(0, \eta_B) \circ \chi : JB \to B_1^{\mathfrak{S}_1}$. By Proposition 3.1.3 applied to the diagram above, the following diagram in $[C]$ commutes:

$$
\begin{array}{ccc}
J^2B & \xrightarrow{\quad J(\psi) \quad} & J(B_1^{\mathfrak{S}_1}) \\
\text{id} \downarrow & & \downarrow \kappa_B^{1,1} \\
J^2B \xrightarrow{\lambda_{JB}} (JB)_0^{\mathfrak{S}_1} & \xrightarrow{\ \omega\ } (JB)_0^{\mathfrak{S}_1} \xrightarrow{\ \gamma^*\ } & (JB)_1^{\mathfrak{S}_1}
\end{array}
$$

Here $\gamma^* = (\text{inc}, 0) : (t^2 - t)(JB)[t] \to t(JB)[t]\ _{\text{ev}_1}\times_{\text{ev}_1} t(JB)[t]$ is the morphism induced by the last vertex map; see Example 3.2.4. The proof will be finished if we show that $J(\psi)$ equals $J(\lambda_B)$ in $[C]$. By Lemma 3.1.5 it suffices to show that ψ equals λ_B in $[C]$. Consider the following diagram; it exhibits a strong morphism of extensions $\mathscr{U}_B \to \mathscr{P}_{0,B}$ extending the identity of B:

$$
\begin{array}{ccccc}
\mathscr{U}_B & \quad JB \longrightarrow TB \longrightarrow B \\
\downarrow & \quad \chi \downarrow \qquad\qquad \downarrow \qquad\quad \downarrow \text{id} \\
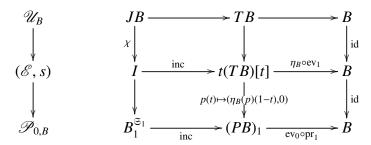(\mathscr{E}, s) & \quad I \xrightarrow{\text{inc}} t(TB)[t] \xrightarrow{\eta_B \circ \text{ev}_1} B \\
\downarrow & \quad \downarrow \qquad p(t) \mapsto (\eta_B(p)(1-t),0) \downarrow \quad\quad \downarrow \text{id} \\
\mathscr{P}_{0,B} & \quad B_1^{\mathfrak{S}_1} \xrightarrow{\text{inc}} (PB)_1 \xrightarrow{\text{ev}_0 \circ \text{pr}_1} B
\end{array}
$$

It follows that λ_B equals the composite:

$$
\begin{array}{ccc}
JB \xrightarrow{\quad \chi \quad} I \longrightarrow & B_1^{\mathfrak{S}_1} = tB[t]\ _{\text{ev}_1}\times_{\text{ev}_1} tB[t] \\
p(t) \longmapsto & (\eta_B(p)(1 - t), 0)
\end{array}
$$

This is easily seen to be homotopic to $\psi = (0, \eta_B) \circ \chi$. $\qquad\qquad\qquad\qquad \square$

Lemma 3.7.4. *Let* $B \in C$ *and let* $\varepsilon_n = (-1)^n$. *Then the following diagram in* $[C]^{\text{ind}}$ *commutes:*

$$
\begin{array}{ccc}
J^{n+1}B & \xrightarrow{J^n(\lambda_B)} & J^n(B_\bullet^{\mathfrak{S}_1}) \\
& (\lambda_{J^nB})^{\varepsilon_n} \searrow & \downarrow \kappa_B^{n,1} \\
& & (J^nB)_\bullet^{\mathfrak{S}_1}
\end{array}
$$

Proof. We prove the result by induction on n. The case $n = 1$ is Lemma 3.7.3. Suppose that the result holds for $n \geq 1$. We have:

$$
\begin{aligned}
\kappa_B^{n+1,1} \circ J^{n+1}(\lambda_B) &= \kappa_{J^nB}^{1,1} \circ J(\kappa_B^{n,1}) \circ J^{n+1}(\lambda_B) && \text{(by Lemma 3.3.1)} \\
&= \kappa_{J^nB}^{1,1} \circ J(\kappa_B^{n,1} \circ J^n(\lambda_B)) \\
&= \kappa_{J^nB}^{1,1} \circ J((\lambda_{J^nB})^{\varepsilon_n}) && \text{(by hypothesis)} \\
&= \left[\kappa_{J^nB}^{1,1} \circ J(\lambda_{J^nB})\right]^{\varepsilon_n} && \text{(by Lemma 3.5.1)} \\
&= \left[(\lambda_{J^{n+1}B})^{-1}\right]^{\varepsilon_n} = (\lambda_{J^{n+1}B})^{\varepsilon_{n+1}} && \text{(by the case } n = 1)
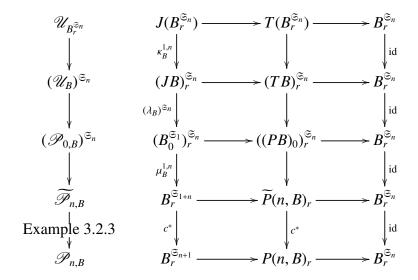\end{aligned}
$$

Then the result holds for $n + 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Lemma 3.7.5 (cf. [3, Lemma 6.29])**.** *Let $B \in C$ and let $n \geq 0$. Then the following diagram in $[C]^{\mathrm{ind}}$ commutes:*

$$
\begin{array}{ccc}
J(B_{\bullet}^{\mathfrak{S}_n}) & \xrightarrow{\ \kappa_B^{1,n}\ } & (JB)_{\bullet}^{\mathfrak{S}_n} \\
{\scriptstyle (-1)^n \Lambda^n(\mathrm{id}_{B^{\mathfrak{S}_n}})} \downarrow & & \downarrow {\scriptstyle (\lambda_B)^{\mathfrak{S}_n}} \\
B_{\bullet}^{\mathfrak{S}_{n+1}} & \xleftarrow[\mu_B^{1,n}]{} & (B_{\bullet}^{\mathfrak{S}_1})_{\bullet}^{\mathfrak{S}_n}
\end{array}
$$

Proof. We have to show the equality of two morphisms in $[C]^{\mathrm{ind}}$; since $[J(B_{\bullet}^{\mathfrak{S}_n}), B_{\bullet}^{\mathfrak{S}_{n+1}}]_C = \lim_r [J(B_r^{\mathfrak{S}_n}), B_{\bullet}^{\mathfrak{S}_{n+1}}]_C$, it will be enough to show that both morphisms are equal when projected to $[J(B_r^{\mathfrak{S}_n}), B_{\bullet}^{\mathfrak{S}_{n+1}}]_C$, for every r. Recall the definition of the extension $\widetilde{\mathscr{P}}_{n,B}$ from Example 3.2.3. The following diagram exhibits a strong morphism of extensions $\mathscr{U}_{B_r^{\mathfrak{S}_n}} \to \mathscr{P}_{n,B}$ that extends the identity of $B_r^{\mathfrak{S}_n}$:

$$
\begin{array}{ccccc}
\mathscr{U}_{B_r^{\mathfrak{S}_n}} & & J(B_r^{\mathfrak{S}_n}) \longrightarrow T(B_r^{\mathfrak{S}_n}) \longrightarrow B_r^{\mathfrak{S}_n} \\
\downarrow & & {\scriptstyle \kappa_B^{1,n}}\downarrow \qquad \downarrow \qquad \downarrow{\scriptstyle \mathrm{id}} \\
(\mathscr{U}_B)^{\mathfrak{S}_n} & & (JB)_r^{\mathfrak{S}_n} \longrightarrow (TB)_r^{\mathfrak{S}_n} \longrightarrow B_r^{\mathfrak{S}_n} \\
\downarrow & & {\scriptstyle (\lambda_B)^{\mathfrak{S}_n}}\downarrow \qquad \downarrow \qquad \downarrow{\scriptstyle \mathrm{id}} \\
(\mathscr{P}_{0,B})^{\mathfrak{S}_n} & & (B_0^{\mathfrak{S}_1})_r^{\mathfrak{S}_n} \longrightarrow ((PB)_0)_r^{\mathfrak{S}_n} \longrightarrow B_r^{\mathfrak{S}_n} \\
\downarrow & & {\scriptstyle \mu_B^{1,n}}\downarrow \qquad \downarrow \qquad \downarrow{\scriptstyle \mathrm{id}} \\
\widetilde{\mathscr{P}}_{n,B} & & B_r^{\mathfrak{S}_{1+n}} \longrightarrow \widetilde{P}(n,B)_r \longrightarrow B_r^{\mathfrak{S}_n} \\
\text{Example 3.2.3}\downarrow & & {\scriptstyle c^*}\downarrow \qquad \downarrow{\scriptstyle c^*} \qquad \downarrow{\scriptstyle \mathrm{id}} \\
\mathscr{P}_{n,B} & & B_r^{\mathfrak{S}_{n+1}} \longrightarrow P(n,B)_r \longrightarrow B_r^{\mathfrak{S}_n}
\end{array}
$$

It follows that $\Lambda^n(\mathrm{id}_{B_r^{\mathfrak{S}_n}})$ equals the composite of the vertical morphisms on the left. The appeareance of th sign $(-1)^n$ is explained in Example 2.3.6. $\qquad\square$

Lemma 3.7.6. *Let $B \in C$. Then, we have:*

$$(-1)^n \Lambda^m(\kappa_B^{n,m}) = \kappa_B^{n,m+1} \circ J^n \Lambda^m(\mathrm{id}_{B^{\mathfrak{S}_m}}) \in [J^{n+1}(B_{\bullet}^{\mathfrak{S}_m}), (J^n B)_{\bullet}^{\mathfrak{S}_{m+1}}]_C$$

Proof. The following diagram in $[C]^{\mathrm{ind}}$ commutes by Lemmas 3.3.3 and 3.7.4:

$$
\begin{array}{ccccc}
J^{n+1}(B_{\bullet}^{\mathfrak{S}_m}) & \xrightarrow{J^n(\lambda_{B^{\mathfrak{S}_m}})} & J^n((B_{\bullet}^{\mathfrak{S}_m})_{\bullet}^{\mathfrak{S}_1}) & \xrightarrow{J^n(\mu_B^{m,1})} & J^n(B_{\bullet}^{\mathfrak{S}_{m+1}}) \\
& {\scriptstyle (\lambda_{J^n(B^{\mathfrak{S}_m})})^{\mathfrak{S}_n}} \searrow & \downarrow{\scriptstyle \kappa_{B^{\mathfrak{S}_m}}^{n,1}} & & \\
& & (J^n(B_{\bullet}^{\mathfrak{S}_m}))_{\bullet}^{\mathfrak{S}_1} & & \downarrow{\scriptstyle \kappa_B^{n,m+1}} \\
& & {\scriptstyle (\kappa_B^{n,m})^{\mathfrak{S}_1}}\downarrow & & \\
& & ((J^n B)_{\bullet}^{\mathfrak{S}_m})_{\bullet}^{\mathfrak{S}_1} & \xrightarrow{\mu_{J^n B}^{m,1}} & (J^n B)_{\bullet}^{\mathfrak{S}_{m+1}}
\end{array}
$$

On the other hand, by Remark 3.6.2, we have:

$$\kappa_B^{n,m+1} \circ J^n(\mu_B^{m,1}) \circ J^n(\lambda_{B^{\varepsilon_m}}) = \kappa_B^{n,m+1} \circ J^n\Lambda^m(\mathrm{id}_{B^{\varepsilon_m}})$$

$$\mu_{J^nB}^{m,1} \circ (\kappa_B^{n,m})^{\mathfrak{S}_1} \circ (\lambda_{J^n(B^{\varepsilon_m})})^{\varepsilon_n} = (-1)^n \Lambda^m(\kappa_B^{n,m})$$

Note that in the second equation we are also using Lemma 3.5.3 to handle the sign $(-1)^n$. The result follows. $\qquad\square$

Lemma 3.7.7 (cf. [3, Lemma 6.32]). *Let $f \in [J^{N_1}A, B_\bullet^{\mathfrak{S}_{N_2}}]$ and $g \in [J^{N_3}B, C_\bullet^{\mathfrak{S}_{N_4}}]$. Then:*

$$\Lambda^{N_4}(g) \star f = \Lambda^{N_2+N_4}(g \star f) = g \star \Lambda^{N_2}(f) \in [J^{N_1+N_3+1}A, C_\bullet^{\mathfrak{S}_{N_2+N_4+1}}]$$

Proof. First, we have:

$$
\begin{aligned}
(-1)^{N_2 N_3}\Lambda^{N_2+N_4}(g \star f) &= \Lambda^{N_2+N_4}\left(\mu_C^{N_4,N_2} \circ g^{\mathfrak{S}_{N_2}} \circ \kappa_B^{N_3,N_2} \circ J^{N_3}(f)\right) \\
&= \Lambda^{N_2+N_4}\left(\mu_C^{N_4,N_2} \circ g^{\mathfrak{S}_{N_2}} \circ \kappa_B^{N_3,N_2}\right) \circ J^{N_3+1}(f) \\
&= \mu_C^{N_4,N_2+1} \circ \Lambda^{N_2}\left(g^{\mathfrak{S}_{N_2}} \circ \kappa_B^{N_3,N_2}\right) \circ J^{N_3+1}(f) \\
&= \mu_C^{N_4,N_2+1} \circ g^{\mathfrak{S}_{N_2+1}} \circ \Lambda^{N_2}\left(\kappa_B^{N_3,N_2}\right) \circ J^{N_3+1}(f)
\end{aligned}
$$

Here the equalities follow from the definition of $g \star f$, Lemma 3.7.1 (i), Lemma 3.7.2 and Lemma 3.7.1 (ii) —in that order. We are also using Lemma 3.6.3 to handle the $(-1)^{N_2 N_3}$.

Secondly, we have:

$$
\begin{aligned}
(-1)^{N_2 N_3}\Lambda^{N_4}(g) \star f &= \\
&= (-1)^{N_2}\mu_C^{N_4+1,N_2} \circ \left(\mu_C^{N_4,1} \circ g^{\mathfrak{S}_1} \circ \lambda_{J^{N_3}B}\right)^{\mathfrak{S}_{N_2}} \circ \kappa_B^{N_3+1,N_2} \circ J^{N_3+1}(f) \\
&= (-1)^{N_2}\mu_C^{N_4+1,N_2} \circ \left(\mu_C^{N_4,1}\right)^{\mathfrak{S}_{N_2}} \circ \left(g^{\mathfrak{S}_1}\right)^{\mathfrak{S}_{N_2}} \circ (\lambda_{J^{N_3}B})^{\mathfrak{S}_{N_2}} \circ \kappa_B^{N_3+1,N_2} \circ J^{N_3+1}(f) \\
&= (-1)^{N_2}\mu_C^{N_4,1+N_2} \circ \mu_{C^{\mathfrak{S}_{N_4}}}^{1,N_2} \circ \left(g^{\mathfrak{S}_1}\right)^{\mathfrak{S}_{N_2}} \circ (\lambda_{J^{N_3}B})^{\mathfrak{S}_{N_2}} \circ \kappa_B^{N_3+1,N_2} \circ J^{N_3+1}(f) \\
&= (-1)^{N_2}\mu_C^{N_4,1+N_2} \circ g^{\mathfrak{S}_{1+N_2}} \circ \mu_{J^{N_3}B}^{1,N_2} \circ (\lambda_{J^{N_3}B})^{\mathfrak{S}_{N_2}} \circ \kappa_B^{N_3+1,N_2} \circ J^{N_3+1}(f)
\end{aligned}
$$

Here the equalities follow from the definition of $\star$ and Remark 3.6.2, the functoriality of $(?)^{\mathfrak{S}_{N_2}}$, the associativity of μ and the naturality of μ —in that order.

Finally, we have:

$$
\begin{aligned}
(-1)^{N_2 N_3}g \star \Lambda^{N_2}(f) &= \\
&= (-1)^{N_3}\mu_C^{N_4,N_2+1} \circ g^{\mathfrak{S}_{N_2+1}} \circ \kappa_B^{N_3,N_2+1} \circ J^{N_3}\Lambda^{N_2}(f) \\
&= (-1)^{N_3}\mu_C^{N_4,N_2+1} \circ g^{\mathfrak{S}_{N_2+1}} \circ \kappa_B^{N_3,N_2+1} \circ J^{N_3}\left(\Lambda^{N_2}(\mathrm{id}_{B^{\mathfrak{S}_{N_2}}}) \circ J(f)\right) \\
&= (-1)^{N_3}\mu_C^{N_4,N_2+1} \circ g^{\mathfrak{S}_{N_2+1}} \circ \kappa_B^{N_3,N_2+1} \circ J^{N_3}\Lambda^{N_2}(\mathrm{id}_{B^{\mathfrak{S}_{N_2}}}) \circ J^{N_3+1}(f)
\end{aligned}
$$

Here the equalities follow from the definition of $\star$, Remark 3.6.1 and the functoriality of J^{N_3} —in that order.

Thus, $\Lambda^{N_2+N_4}(g \star f) = g \star \Lambda^{N_2}(f)$ by Lemma 3.7.6 and to prove that $\Lambda^{N_2+N_4}(g \star f) = \Lambda^{N_4}(g) \star f$ it is enough to show that:

$$\Lambda^{N_2}\left(\kappa_B^{N_3,N_2}\right) = (-1)^{N_2}\mu_{J^{N_3}B}^{1,N_2} \circ (\lambda_{J^{N_3}B})^{\mathfrak{S}_{N_2}} \circ \kappa_B^{N_3+1,N_2}$$

We have:

$$
\begin{aligned}
\Lambda^{N_2}\left(\kappa_B^{N_3,N_2}\right) &= \Lambda^{N_2}\left(1_{(J^{N_3}B)^{\mathfrak{S}_{N_2}}} \circ \kappa_B^{N_3,N_2}\right)\\
&= \Lambda^{N_2}\left(1_{(J^{N_3}B)^{\mathfrak{S}_{N_2}}}\right) \circ J\left(\kappa_B^{N_3,N_2}\right)\\
&= (-1)^{N_2}\mu_{J^{N_3}B}^{1,N_2} \circ (\lambda_{J^{N_3}B})^{\mathfrak{S}_{N_2}} \circ \kappa_{J^{N_3}B}^{1,N_2} \circ J\left(\kappa_B^{N_3,N_2}\right)\\
&= (-1)^{N_2}\mu_{J^{N_3}B}^{1,N_2} \circ (\lambda_{J^{N_3}B})^{\mathfrak{S}_{N_2}} \circ \kappa_B^{N_3+1,N_2}
\end{aligned}
$$

The first equality is trivial and the others follow from Lemma 3.7.1 (i), Lemma 3.7.5 and Lemma 3.3.1 —in that order. □

Theorem 3.7.8. *The operation $\star$ defines an associative composition law that makes $\mathfrak{K}^C$ into a category, as described in Definition 3.6.4.*

Proof. The composition is well-defined by Lemma 3.7.7. The associativity is a straightforward but lengthy verification. □

Example 3.7.9. There is a functor $j : C \to \mathfrak{K}^C$ defined by $A \mapsto (A, 0)$ on objects, that sends a morphism $f : A \to B$ to its class in $\mathrm{Hom}_{\mathfrak{K}^C}((A, 0), (B, 0))$. It is easily verified that this functor factors through $C \to [C]$. We will often write f instead of $j(f)$ and A instead of $j(A)$ and $(A, 0)$. We will sometimes drop C from the notation and write $\mathfrak{K}$ instead of $\mathfrak{K}^C$.

3.8 Additivity

The hom-sets in $\mathfrak{K}^C$ are abelian groups; indeed, they are defined as the (filtered) colimit of a diagram of abelian groups. Next we show that composition is bilinear.

Lemma 3.8.1. *The composition in $\mathfrak{K}^C$ is bilinear.*

Proof. Let $g \in [J^{n+w}B, C_\bullet^{\mathfrak{S}_{k+w}}]$ represent an element $\beta \in \mathfrak{K}((B, n), (C, k))$. Let us show that $\beta_* : \mathfrak{K}((A, m), (B, n)) \to \mathfrak{K}((A, m), (C, k))$ is a group homomorphism. Represent elements $\alpha, \alpha' \in \mathfrak{K}((A, m), (B, n))$ by $f, f' \in [J^{m+v}A, B_\bullet^{\mathfrak{S}_{n+v}}]$ —we may assume that $n + v \geq 2$ by choosing v large enough. To alleviate notation, write

$$N_1 := m + v, \quad N_2 := n + v, \quad N_3 := n + w \quad \text{and} \quad N_4 := k + w,$$

as in Definition 3.6.4. By definition of the composition in $\mathfrak{K}$, the following diagram of sets commutes:

$$
\begin{array}{ccc}
[J^{N_1}A, B_\bullet^{\mathfrak{S}_{N_2}}] & \xrightarrow{\ g\star? \ } & [J^{N_1+N_3}A, C_\bullet^{\mathfrak{S}_{N_2+N_4}}]\\
\downarrow & & \downarrow\\
\mathfrak{K}((A, m), (B, n)) & \xrightarrow{\ \beta_* \ } & \mathfrak{K}((A, m), (C, k))
\end{array}
$$

Here, the vertical arrows are the structural morphisms into the colimits —hence they are group homomorphisms. Since $\beta_*(\alpha + \alpha')$ is represented by $g \star (f + f')$, to prove that $\beta_*(\alpha + \alpha') = \beta_*(\alpha) + \beta_*(\alpha')$ it suffices to show that $g \star ?$ is a group homomorphism. The function $g \star ?$ and the composite of the following three functions differ in the sign $(-1)^{N_2 N_3}$:

$$[J^{N_1}A, B_\bullet^{\mathfrak{S}_{N_2}}] \xrightarrow{\text{Lemma } 3.5.1} [J^{N_1+N_3}A, (J^{N_3}B)_\bullet^{\mathfrak{S}_{N_2}}]$$

$$[J^{N_1+N_3}A, (J^{N_3}B)_\bullet^{\mathfrak{S}_{N_2}}] \xrightarrow{\quad g_* \quad} [J^{N_1+N_3}A, (C_\bullet^{\mathfrak{S}_{N_4}})_\bullet^{\mathfrak{S}_{N_2}}]$$

$$[J^{N_1+N_3}A, (C_\bullet^{\mathfrak{S}_{N_4}})_\bullet^{\mathfrak{S}_{N_2}}] \xrightarrow{\left(\mu_C^{N_4,N_2}\right)_*} [J^{N_1+N_3}A, C_\bullet^{\mathfrak{S}_{N_2+N_4}}]$$

It follows that $g \star ?$ is a group homomorphism, since the three functions above are —recall Lemma 3.5.3.

Now let $f \in [J^{m+v}A, B_\bullet^{\mathfrak{S}_{n+v}}]$ represent an element $\alpha \in \mathfrak{K}((A, m), (B, n))$. Let us show that $\alpha^* : \mathfrak{K}((B, n), (C, k)) \to \mathfrak{K}((A, m), (C, k))$ is a group homomorphism. Represent elements $\beta, \beta' \in \mathfrak{K}((B, n), (C, k))$ by $g, g' \in [J^{n+w}B, C_\bullet^{\mathfrak{S}_{k+w}}]$ —we may assume that $k + w \geq 2$ by choosing w large enough. As before, write

$$N_1 := m + v, \quad N_2 := n + v, \quad N_3 := n + w \quad \text{and} \quad N_4 := k + w.$$

By definition of the composition in $\mathfrak{K}$, the following diagram of sets commutes:

$$\begin{array}{ccc}
[J^{N_3}B, C_\bullet^{\mathfrak{S}_{N_4}}] & \xrightarrow{\;? \star f\;} & [J^{N_1+N_3}A, C_\bullet^{\mathfrak{S}_{N_2+N_4}}] \\
\downarrow & & \downarrow \\
\mathfrak{K}((B, n), (C, k)) & \xrightarrow{\;\alpha^*\;} & \mathfrak{K}((A, m), (C, k))
\end{array}$$

Here, the vertical arrows are the structural morphisms into the colimits —hence they are group homomorphisms. Since $\alpha^*(\beta + \beta')$ is represented by $(g + g') \star f$, to prove that $\alpha^*(\beta + \beta') = \alpha^*(\beta) + \alpha^*(\beta')$ it suffices to show that $? \star f$ is a group homomorphism. The function $? \star f$ and the composite of the following two functions differ in the sign $(-1)^{N_2 N_3}$:

$$[J^{N_3}B, C_\bullet^{\mathfrak{S}_{N_4}}] \xrightarrow{\mu_C^{N_4,N_2} \circ (?)^{\mathfrak{S}_{N_2}}} [(J^{N_3}B)^{\mathfrak{S}_{N_2}}, C_\bullet^{\mathfrak{S}_{N_4+N_2}}] \tag{19}$$

$$[(J^{N_3}B)^{\mathfrak{S}_{N_2}}, C_\bullet^{\mathfrak{S}_{N_4+N_2}}] \xrightarrow{\left(\kappa_B^{N_3,N_2} \circ J^{N_3}f\right)^*} [J^{N_1+N_3}A, C_\bullet^{\mathfrak{S}_{N_4+N_2}}] \tag{20}$$

The function (19) is a group homomorphism by Lemma 3.5.4 and the function (20) is a group homomorphism by Remark 2.3.4. Thus, $? \star f$ is a group homomorphism. $\qquad\square$

Lemma 3.8.2. *The category $\mathfrak{K}^C$ has finite products.*

Proof. Let $B, C \in C$ and let $n \in \mathbb{Z}$. Let us first show that $(B \times C, n) = (B, n) \times (C, n)$ in $\mathfrak{K}$. For any $(A, m) \in \mathfrak{K}$, we have:

$$
\begin{aligned}
\mathfrak{K}((A, m), (B \times C, n)) &= \operatorname*{colim}_{v,r} [J^{m+v} A, (B \times C)_r^{\mathfrak{S}_{n+v}}] \\
&\cong \operatorname*{colim}_{v,r} [J^{m+v} A, B_r^{\mathfrak{S}_{n+v}} \times C_r^{\mathfrak{S}_{n+v}}] \\
&\cong \operatorname*{colim}_{v,r} \left\{ [J^{m+v} A, B_r^{\mathfrak{S}_{n+v}}] \times [J^{m+v} A, C_r^{\mathfrak{S}_{n+v}}] \right\} \\
&\cong \operatorname*{colim}_{v,r} [J^{m+v} A, B_r^{\mathfrak{S}_{n+v}}] \times \operatorname*{colim}_{v,r} [J^{m+v} A, C_r^{\mathfrak{S}_{n+v}}] \\
&= \mathfrak{K}((A, m), (B, n)) \times \mathfrak{K}((A, m), (C, n))
\end{aligned}
$$

Here we are using that the functors $(?)_r^{\mathfrak{S}_N} : C \to C$ and $C \to [C]$ commute with finite products, and that filtered colimits of sets commute with finite products.

In order to prove that $\mathfrak{K}$ has finite products, we can reduce to the special case above, as we proceed to explain. We claim that, for any $(B, n) \in \mathfrak{K}$ and any $p \geq 1$, we have an isomorphism $(B, n) \cong (J^p B, n - p)$. Using this, any pair of objects of $\mathfrak{K}$ can be replaced by a new pair of objects —each of them isomorphic to one of the original ones— with equal second coordinate. The claim follows easily from Lemma 3.10.1, and the proof of this lemma relies only on Lemma 3.7.3 and the definition of the composition in $\mathfrak{K}$. $\qquad\square$

Proposition 3.8.3. *The category $\mathfrak{K}^C$ is additive.*

Proof. It follows from Lemmas 3.8.1 and 3.8.2; see [3, Lemma 6.41] and [10, Section VIII.2]. $\qquad\square$

3.9 Excision

In this section we closely follow [2, Section 6.3]. Let $f : A \to B$ be a morphism in C. The *mapping path* $(P_f)_\bullet$ is the $\mathbb{Z}_{\geq 0}$-diagram in C defined by the following pullbacks:

$$
\begin{array}{ccc}
(P_f)_r & \xrightarrow{\;\pi_f\;} & A \\
\big\downarrow & & \big\downarrow{\scriptstyle f} \\
(PB)_r & \xrightarrow{\;d_1\;} & B
\end{array}
\qquad (21)
$$

Notice that π_f is a split surjection in $\mathcal{U}(C)$; indeed, it is the pullback of the split surjection d_1. Define morphisms $\iota_f : B_r^{\mathfrak{S}_1} \to (P_f)_r$ by the following diagrams in C:

$$
(22)
$$

Lemma 3.9.1 (cf. [2, Lemma 6.3.1])**.** *Let $f : A \to B$ be a morphism in C and let $C \in C$.*
Then the following sequence is exact:

$$\operatorname*{colim}_{r} \mathfrak{R}^C(C, (P_f)_r) \longrightarrow \mathfrak{R}^C(C, A) \xrightarrow{\;f_*\;} \mathfrak{R}^C(C, B) \qquad (23)$$

Here, the map on the left is induced by the morphisms π_f in (21).

Proof. Let $r \geq 0$ and note that $\mathfrak{R}(C, (PB)_r) = 0$ because $(PB)_r$ is contractible. Then the
following composite is zero, because it factors through $\mathfrak{R}(C, (PB)_r)$:

$$\mathfrak{R}(C, (P_f)_r) \xrightarrow{\;(\pi_f)_*\;} \mathfrak{R}(C, A) \xrightarrow{\;f_*\;} \mathfrak{R}(C, B)$$

This shows that the composite of the morphisms in (23) is zero.

Now let $g : J^m C \to A_r^{\mathfrak{S}_m}$ be a morphism in C that represents an element $\alpha \in \mathfrak{R}(C, A)$
such that $f_*(\alpha) = 0 \in \mathfrak{R}(C, B)$. Increasing m if necessary, we may assume without loss of
generality that the following composite in C is nullhomotopic:

$$J^m C \xrightarrow{\;g\;} A_r^{\mathfrak{S}_m} \xrightarrow{\;f^{\mathfrak{S}_m}\;} B_r^{\mathfrak{S}_m}$$

It follows that there is a commutative diagram in C:

$$\begin{array}{ccc}
J^m C & \xrightarrow{\;g\;} & A_r^{\mathfrak{S}_m} \\
\downarrow & & \downarrow{\scriptstyle f^{\mathfrak{S}_m}} \\
(B_r^{\mathfrak{S}_m})_s^{(I,\{1\})} & \xrightarrow{\;(d^1)^*\;} & B_r^{\mathfrak{S}_m}
\end{array} \qquad (24)$$

Since $(?)_r^{\mathfrak{S}_m} : C \to C$ commutes with finite limits, we get the following pullback:

$$\begin{array}{ccc}
((P_f)_s)_r^{\mathfrak{S}_m} & \longrightarrow & A_r^{\mathfrak{S}_m} \\
\downarrow & & \downarrow{\scriptstyle f^{\mathfrak{S}_m}} \\
(B_s^{(I,\{1\})})_r^{\mathfrak{S}_m} & \xrightarrow{\;(d^1)^*\;} & B_r^{\mathfrak{S}_m}
\end{array}$$

Notice that $(B_s^{(I,\{1\})})_r^{\mathfrak{S}_m} \cong (B_r^{\mathfrak{S}_m})_s^{(I,\{1\})}$. The diagram (24) determines a morphism:

$$J^m C \longrightarrow ((P_f)_s)_r^{\mathfrak{S}_m}$$

It is easily verified that this gives an element $\beta \in \mathfrak{R}(C, (P_f)_s)$ that maps to α.   □

Definition 3.9.2. Let $f : A \to B$ be a morphism in C. We call f a $\mathfrak{R}$-*equivalence* if it
becomes invertible upon applying $j : C \to \mathfrak{R}^C$.

Lemma 3.9.3 (c.f. [2, Lemma 6.3.2])**.** *Let $f : A \to B$ be a morphism in C that is a split*
surjection in $\mathcal{U}(C)$. Then the natural maps $\ker f \to (P_f)_r$ are $\mathfrak{R}$-equivalences for all r.

Proof. The proof is like that of [2, Lemma 6.3.2].   □
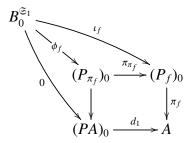
Let $f : A \to B$ be a morphism in C that is a split surjection in $\mathcal{U}(C)$. As explained in the discussion following [2, Lemma 6.3.2], Lemma 3.9.3 implies that the morphisms $(P_f)_r \to (P_f)_{r+1}$ are $\mathfrak{K}$-equivalences for all $r \geq 0$. Indeed, this follows from the 'two out of three' property of $\mathfrak{K}$-equivalences. Combining this fact with Lemma 3.9.1 we get:

Corollary 3.9.4. *Let $f : A \to B$ be a morphism in C that is a split surjection in $\mathcal{U}(C)$ and let $C \in C$. Then the following sequence is exact:*

$$\mathfrak{K}^C(C, (P_f)_0) \xrightarrow{(\pi_f)_*} \mathfrak{K}^C(C, A) \xrightarrow{f_*} \mathfrak{K}^C(C, B)$$

Corollary 3.9.5 ([2, Corollary 6.3.3]). *Let $f : A \to B$ be a morphism in C. Recall the definitions of π_f and ι_f from (21) and (22). Let $\phi_f : B_0^{\mathfrak{S}_1} \to (P_{\pi_f})_0$ be the morphism defined by the following diagram:*

Then ϕ_f is a $\mathfrak{K}$-equivalence.

Proof. The morphism π_f is a split surjection in $\mathcal{U}(C)$. The result follows from Lemma 3.9.3 if we show that $\iota_f : B_0^{\mathfrak{S}_1} \to (P_f)_0$ is a kernel of π_f, and this is easily verified. $\quad\square$

Corollary 3.9.6 ([2, Corollary 6.3.4]). *Let $D \in C$ and let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a split short exact sequence in C. Then the following sequence is exact:

$$0 \longrightarrow \mathfrak{K}^C(D, A) \xrightarrow{f_*} \mathfrak{K}^C(D, B) \xrightarrow{g_*} \mathfrak{K}^C(D, C) \longrightarrow 0 \qquad (25)$$

Proof. The proof of [2, Corollary 6.3.4] carries over verbatim in this setting; we include it here for completeness. The following sequence is exact at $\mathfrak{K}(D, B)$ by Corollary 3.9.4 and exact at $\mathfrak{K}(D, C)$ because g is a retraction:

$$0 \longrightarrow \mathfrak{K}(D, (P_g)_0) \xrightarrow{(\pi_g)_*} \mathfrak{K}(D, B) \xrightarrow{g_*} \mathfrak{K}(D, C) \longrightarrow 0 \qquad (26)$$

By Lemma 3.9.3, we have an isomorphism:

$$\mathfrak{K}(D, A) \xrightarrow{\cong} \mathfrak{K}(D, (P_g)_0)$$

Under this identification, (26) becomes (25). Thus, it suffices to show that $(\pi_g)_*$ in (26) is injective. In the following diagram, the sequence of solid arrows is exact by Corollary
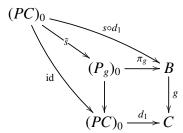
3.9.4 and $(\phi_g)_*$ is an isomorphism by Corollary 3.9.5. Thus, showing that $(\pi_g)_*$ is injective is equivalent to showing that $(\iota_g)_*$ is the zero morphism.

$$
\begin{array}{ccc}
\Re(D, (P_{\pi_g})_0) \xrightarrow{(\pi_{\pi_g})_*} \Re(D, (P_g)_0) \xrightarrow{(\pi_g)_*} \Re(D, B) \\
\big\uparrow {\scriptstyle(\phi_g)_*} \cong \qquad \nearrow \\
\Re(D, C_0^{\mathfrak{S}_1}) \qquad {\scriptstyle(\iota_g)_*}
\end{array}
\tag{27}
$$

Let $s : C \to B$ be a splitting of g and let $\tilde{s} : (PC)_0 \to (P_g)_0$ be the morphism defined by the following diagram:

$$
\begin{array}{ccc}
(PC)_0 & & \\
 & \searrow^{s \circ d_1} & \\
{\scriptstyle\tilde{s}} \searrow & (P_g)_0 \xrightarrow{\pi_g} B & \\
{\scriptstyle\mathrm{id}} \searrow & \big\downarrow & \big\downarrow g \\
 & (PC)_0 \xrightarrow{d_1} C &
\end{array}
$$

It is easily verified that ι_g equals the composite:

$$
C_0^{\mathfrak{S}_1} \longrightarrow (PC)_0 \xrightarrow{\tilde{s}} (P_g)_0
$$

This implies that $(\iota_g)_* = 0$ because $(PC)_0$ is contractible. $\square$

Lemma 3.9.7. *Let $f : A \to B$ be a morphism in C. Then the following diagram in $[C]$ commutes:*

$$
\begin{array}{ccccccc}
A_0^{\mathfrak{S}_1} & \xrightarrow{(f^{\mathfrak{S}_1})^{-1}} & B_0^{\mathfrak{S}_1} & \xrightarrow{\iota_f} & (P_f)_0 & \xrightarrow{\pi_f} & A \\
{\scriptstyle\mathrm{id}}\big\downarrow & & {\scriptstyle\phi_f}\big\downarrow & & {\scriptstyle\mathrm{id}}\big\downarrow & & {\scriptstyle\mathrm{id}}\big\downarrow \\
A_0^{\mathfrak{S}_1} & \xrightarrow{\iota_{\pi_f}} & (P_{\pi_f})_0 & \xrightarrow{\pi_{\pi_f}} & (P_f)_0 & \xrightarrow{\pi_f} & A
\end{array}
$$

Here, ϕ_f is the morphism defined in Corollary 3.9.5.

Proof. The square in the middle commutes by definition of ϕ_f. We still have to show that the square on the left commutes. In the whole proof, we will omit the subscript 0 and write $A^{\mathfrak{S}_1}$ instead of $A_0^{\mathfrak{S}_1}$, PB instead of $(PB)_0$, P_f instead of $(P_f)_0$, etc. For $C \in C$, consider the isomorphism $C^{\Delta^1} \cong C[t]$, $t_1 \leftrightarrow t$. Under this identification, we have:

$$
PB = (t-1)B[t]
$$

$$
B^{\mathfrak{S}_1} = (t^2 - t)B[t]
$$

$$
P_f = \{(p(t), a) \in PB \times A : p(0) = f(a)\}
$$

The map $\pi_f : P_f \to A$ is given by $(p(t), a) \mapsto a$. The map $\iota_f : B^{\mathfrak{S}_1} \to P_f$ is given by $p(t) \mapsto (p(t), 0)$. We also have:

$$
PA = (t-1)A[t]
$$

$$A^{\mathfrak{S}_1} = (t^2 - t)A[t]$$

$$P_{\pi_f} = \{(p(t), a, q(t)) \in PB \times A \times PA : p(0) = f(a), a = q(0)\}$$
$$= \{(p(t), q(t)) \in PB \times PA : p(0) = f(q(0))\}$$

The map $\phi_f : B^{\mathfrak{S}_1} \to P_{\pi_f}$ is given by $p(t) \mapsto (p(t), 0)$. The map $\iota_{\pi_f} : A^{\mathfrak{S}_1} \to P_{\pi_f}$ is given by $q(t) \mapsto (0, q(t))$. To prove the result, it suffices to show that the following morphisms $A^{\mathfrak{S}_1} \to P_{\pi_f}$ are homotopic:

$$\phi_f \circ \left(f^{\mathfrak{S}_1}\right)^{-1} : q(t) \mapsto (f(q(1-t)), 0)$$
$$\iota_{\pi_f} : q(t) \mapsto (0, q(t))$$

An easy verification shows that:

$$P_{\pi_f}[u] = \{(p(t, u), q(t, u)) \in (t-1)B[t, u] \times (t-1)A[t, u] : p(0, u) = f(q(0, u))\}$$

Let $H : A^{\mathfrak{S}_1} \to P_{\pi_f}[u]$ be the homotopy defined by:

$$H(q(t)) = (f(q((1-t)u)), q(1 - (1-t)(1-u)))$$

Let $\mathrm{ev}_i : P_{\pi_f}[u] \to P_{\pi_f}$ be the evaluation at i. We have:

$$(\mathrm{ev}_0 \circ H)(q(t)) = (0, q(t))$$

$$(\mathrm{ev}_1 \circ H)(q(t)) = (f(q(1-t)), 0)$$

The result follows. $\square$

Theorem 3.9.8 ([2, Theorem 6.3.6]). *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an extension in C. Then, for any $D \in C$, the following sequence is exact:*

$$\mathfrak{K}^C(D, B_0^{\mathfrak{S}_1}) \xrightarrow{(g^{\mathfrak{S}_1})_*} \mathfrak{K}^C(D, C_0^{\mathfrak{S}_1}) \xrightarrow{\partial} \mathfrak{K}^C(D, A) \xrightarrow{f_*} \mathfrak{K}^C(D, B) \xrightarrow{g_*} \mathfrak{K}^C(D, C)$$

Here, the morphism ∂ is the composite:

$$\mathfrak{K}^C(D, C_0^{\mathfrak{S}_1}) \xrightarrow{(\iota_g)_*} \mathfrak{K}^C(D, (P_g)_0) \xleftarrow{\cong} \mathfrak{K}^C(D, A)$$

Proof. Both g and π_g are split surjections in $\mathcal{U}(C)$; then, the following sequence is exact by Corollary 3.9.4:

$$\mathfrak{K}(D, (P_{\pi_g})_0) \xrightarrow{(\pi_{\pi_g})_*} \mathfrak{K}(D, (P_g)_0) \xrightarrow{(\pi_g)_*} \mathfrak{K}(D, B) \xrightarrow{g_*} \mathfrak{K}(D, C) \qquad (28)$$

We have a commutative diagram:

The morphism $A \to (P_g)_0$ is a $\Re$-equivalence by Lemma 3.9.3. Thus, we can replace $(\pi_g)_*$ in (28) by the composite

$$\Re(D, (P_g)_0) \xleftarrow{\cong} \Re(D, A) \xrightarrow{f_*} \Re(D, B)$$

and we get an exact sequence:

$$\Re(D, (P_{\pi_g})_0) \longrightarrow \Re(D, A) \xrightarrow{f_*} \Re(D, B) \xrightarrow{g_*} \Re(D, C) \tag{29}$$

By Corollary 3.9.5, we can identify $(\phi_g)_* : \Re(D, C_0^{\mathfrak{S}_1}) \xrightarrow{\cong} \Re(D, (P_{\pi_g})_0)$ and (29) becomes:

$$\Re(D, C_0^{\mathfrak{S}_1}) \longrightarrow \Re(D, A) \xrightarrow{f_*} \Re(D, B) \xrightarrow{g_*} \Re(D, C) \tag{30}$$

It is easily verified that ∂ is the leftmost morphism in (30); indeed, this follows from the equality $\pi_{\pi_g} \circ \phi_g = \iota_g$.

By Lemma 3.9.7, we have a commutative diagram as follows:

$$\begin{array}{ccccc}
\Re(D, B_0^{\mathfrak{S}_1}) & \xrightarrow{-(g^{\mathfrak{S}_1})_*} & \Re(D, C_0^{\mathfrak{S}_1}) & \xrightarrow{(\iota_g)_*} & \Re(D, (P_g)_0) \\
\text{id} \downarrow & & \downarrow (\phi_g)_* & & \downarrow \text{id} \\
\Re(D, B_0^{\mathfrak{S}_1}) & \xrightarrow{(\iota_{\pi_g})_*} & \Re(D, (P_{\pi_g})_0) & \xrightarrow{(\pi_{\pi_g})_*} & \Re(D, (P_g)_0) \\
(\phi_{\pi_g})_* \downarrow & & \downarrow \text{id} & & \downarrow \text{id} \\
\Re(D, (P_{\pi_g})_0) & \xrightarrow{(\pi_{\pi_g})_*} & \Re(D, (P_{\pi_g})_0) & \xrightarrow{(\pi_{\pi_g})_*} & \Re(D, (P_g)_0)
\end{array} \tag{31}$$

Notice that $\ker \partial = \ker\left((\iota_g)_* : \Re(D, C_0^{\mathfrak{S}_1}) \to \Re(D, (P_g)_0)\right)$. Thus, to finish the proof, it suffices to show that the top row in (31) is exact. Since $(\phi_g)_*$ and $(\phi_{\pi_g})_*$ are isomorphisms by Corollary 3.9.5, the top row in (31) is exact if and only if the bottom one is. But π_{π_g} is a split surjection in $\mathcal{U}(C)$, and so the bottom row in (31) is exact by Corollary 3.9.4. The result follows. $\qquad \square$

3.10 The translation functor

Define a functor $L : \Re^C \to \Re^C$ as follows: For $(A, m) \in \Re^C$ put $L(A, m) := (A, m + 1)$. The functor L on morphisms is defined by the following identification:

$$\begin{aligned}
\Re^C((A, m), (B, n)) &= \operatorname*{colim}_v [J^{m+v} A, B_\bullet^{\mathfrak{S}_{n+v}}] \\
&\cong \operatorname*{colim}_v [J^{m+1+v} A, B_\bullet^{\mathfrak{S}_{n+1+v}}] = \Re^C((A, m + 1), (B, n + 1))
\end{aligned}$$

It is clear that L is an automorphism of $\Re^C$.

Recall the definition of $\lambda_B : JB \to B_r^{\mathfrak{S}_1}$ from Example 3.2.2. For any $m \in \mathbb{Z}$, we can consider:

$$\begin{aligned}
[\lambda_B] &\in \Re^C((JB, m), (B, 1 + m)) \\
[\operatorname{id}_{JB}] &\in \Re^C((B, 1 + m), (JB, m))
\end{aligned} \tag{32}$$

These two morphisms are mutually inverses in $\mathfrak{K}^C$, as we prove below. From now on, each time we identify $(B, 1 + m) \cong (JB, m)$ it will be using these isomorphisms. Using this identification n times we get:

$$(B, n + m) \cong (J^n B, m) \tag{33}$$

It is easily verified that (33) is represented by $\mathrm{id}_{J^n B} : J^n B \to J^n B$.

Lemma 3.10.1. *The morphisms in* (32) *are mutually inverses in* $\mathfrak{K}^C$.

Proof. The composite $[\mathrm{id}_{JB}] \circ [\lambda_B]$ is represented by $\mathrm{id}_{JB} \star \lambda_B \in [J(JB), (JB)_\bullet^{\mathfrak{S}_1}]$. By definition of $\star$, $(\mathrm{id}_{JB} \star \lambda_B)^{-1}$ equals the following composite in $[C]^{\mathrm{ind}}$:

$$J(JB) \xrightarrow{\ J(\lambda_B)\ } J(B_\bullet^{\mathfrak{S}_1}) \xrightarrow{\ \kappa_B^{1,1}\ } (JB)_\bullet^{\mathfrak{S}_1}$$

It follows from Lemma 3.7.3 that $\mathrm{id}_{JB} \star \lambda_B = \lambda_{JB} \in [J(JB), (JB)_\bullet^{\mathfrak{S}_1}]$. This implies that $[\mathrm{id}_{JB}] \circ [\lambda_B] = \mathrm{id}_{(JB,m)}$.

The composite $[\lambda_B] \circ [\mathrm{id}_{JB}]$ is respresented by $\lambda_B \star \mathrm{id}_{JB} \in [J(B), (B)_\bullet^{\mathfrak{S}_1}]$. It is easily verified that $\lambda_B \star \mathrm{id}_{JB} = \lambda_B$. It follows that $[\lambda_B] \circ [\mathrm{id}_{JB}] = \mathrm{id}_{(B,1+m)}$. $\square$

We can also consider:

$$[\lambda_B] \in \mathfrak{K}^C((B, 1 + m), (B_0^{\mathfrak{S}_1}, m))$$
$$[\mathrm{id}_{B^{\mathfrak{S}_1}}] \in \mathfrak{K}^C((B_0^{\mathfrak{S}_1}, m), (B, 1 + m)) \tag{34}$$

As before, we will show that these morphisms are mutually inverses in $\mathfrak{K}^C$ and each time we identify $(B, 1 + m) \cong (B_0^{\mathfrak{S}_1}, m)$ it will be using these isomorphisms. We need the following result.

Lemma 3.10.2 ([2, Lemma 6.3.10]). *The morphism* $\lambda_B : JB \to B_0^{\mathfrak{S}_1}$ *is a* $\mathfrak{K}$-*equivalence.*

Proof. The proof of [2, Lemma 6.3.10] works verbatim, but we include it here for completeness. Consider the (unique) strong morphism of extensions:

$$
\begin{array}{ccccc}
JB & \longrightarrow & TB & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_B} \\
B_0^{\mathfrak{S}_1} & \longrightarrow & (PB)_0 & \longrightarrow & B
\end{array}
$$

We have $TB \cong 0 \cong (PB)_0$ in $\mathfrak{K}$, since TB and $(PB)_0$ are contractible. It follows from Theorem 3.9.8 that

$$(\lambda_B)_* : \mathfrak{K}((D, 0), (JB, 0)) \to \mathfrak{K}((D, 0), (B_0^{\mathfrak{S}_1}, 0))$$

is bijective for all $D \in C$. Then $\lambda_B : (JB, 0) \to (B_0^{\mathfrak{S}_1}, 0)$ is an isomorphism by Yoneda. $\square$

Lemma 3.10.3. *The morphisms in* (34) *are mutually inverses in* $\mathfrak{K}^C$.

Proof. By Lemma 3.10.2, the morphism $j(\lambda_B) : (JB, 0) \rightarrow (B_0^{\mathfrak{S}_1}, 0)$ is an isomorphism. Since L is an automorphism of $\mathfrak{K}$, $L^m(j(\lambda_B))$ is an isomorphism for all $m \in \mathbb{Z}$. Is is straightforward to check that the morphism $[\lambda_B]$ in (34) equals the composite:

$$(B, 1 + m) \xrightarrow{\;[\mathrm{id}_{JB}]\;} (JB, m) \xrightarrow{\;L^m(j(\lambda_B))\;} (B_0^{\mathfrak{S}_1}, m) \,,$$

Since $[\mathrm{id}_{JB}]$ is an isomorphism by Lemma 3.10.1, it follows that $[\lambda_B]$ in (34) is an isomorphism too.

To finish the proof it suffices to show that $[\mathrm{id}_{B^{\mathfrak{S}_1}}] \circ [\lambda_B] = \mathrm{id}_{(B,1+m)}$; this follows immediately from the definitions. $\qquad\square$

Remark 3.10.4. It follows from Lemma 3.10.3 that the functors $j \circ (?)_0^{\mathfrak{S}_1} : C \rightarrow \mathfrak{K}^C$ and $L \circ j : C \rightarrow \mathfrak{K}^C$ are naturally isomorphic. Indeed, it is easily verified that the following diagram commutes for every morphism $f : A \rightarrow B$ in C:

$$
\begin{array}{ccc}
(A_0^{\mathfrak{S}_1}, 0) & \xrightarrow{\;[\mathrm{id}_{A^{\mathfrak{S}_1}}]\;} & (A, 1) \\
{\scriptstyle j(f_0^{\mathfrak{S}_1})}\big\downarrow & & \big\downarrow{\scriptstyle L(j(f))} \\
(B_0^{\mathfrak{S}_1}, 0) & \xrightarrow{\;[\mathrm{id}_{B^{\mathfrak{S}_1}}]\;} & (B, 1)
\end{array}
$$

As a consecuence of this, if f is a $\mathfrak{K}$-equivalence then so is $f_0^{\mathfrak{S}_1}$.

The morphism $\mathrm{id}_{B^{\mathfrak{S}_n}}$ induces:

$$[\mathrm{id}_{B^{\mathfrak{S}_n}}] \in \mathfrak{K}((B_r^{\mathfrak{S}_n}, m), (B, n + m)) \tag{35}$$

We will show that $[\mathrm{id}_{B^{\mathfrak{S}_n}}]$ is an isomorphism. We need some preliminary results.

Lemma 3.10.5. *The morphisms $B_r^{\mathfrak{S}_n} \rightarrow B_{r+1}^{\mathfrak{S}_n}$ are $\mathfrak{K}$-equivalences.*

Proof. We will prove the assertion by induction on n. First notice that the result holds for $n = 0$; indeed, in this case $B_r^{\mathfrak{S}_0} \rightarrow B_{r+1}^{\mathfrak{S}_0}$ is the identity morphism of B. For the inductive step, consider the following morphism of extensions induced by the last vertex map:

$$
\begin{array}{ccc}
B_r^{\mathfrak{S}_{n+1}} \longrightarrow P(n, B)_r \longrightarrow B_r^{\mathfrak{S}_n} \\
\big\downarrow \qquad\qquad \big\downarrow \qquad\qquad \big\downarrow \\
B_{r+1}^{\mathfrak{S}_{n+1}} \longrightarrow P(n, B)_{r+1} \longrightarrow B_{r+1}^{\mathfrak{S}_n}
\end{array}
$$

The vertical morphism in the middle is a $\mathfrak{K}$-equivalence because both its source and its target are contractible by Lemma 3.2.5. The vertical morphism on the right is a $\mathfrak{K}$-equivalence by induction hypothesis. By Theorem 3.9.8, the vertical morphism on the left is a $\mathfrak{K}$-equivalence too. $\qquad\square$

Lemma 3.10.6. *The morphisms $\mu_B^{m,n} : (B_r^{\mathfrak{S}_m})_s^{\mathfrak{S}_n} \rightarrow B_{r+s}^{\mathfrak{S}_{m+n}} \in C$ are $\mathfrak{K}$-equivalences for all m and n.*

Proof. Let us start with the case $n = 1$. Consider the following morphism of extensions:

$$
\begin{array}{ccccc}
(B_r^{\mathfrak{S}_m})_s^{\mathfrak{S}_1} & \longrightarrow & P(B_r^{\mathfrak{S}_m})_s & \longrightarrow & B_r^{\mathfrak{S}_m} \\
\mu^{m,1} \downarrow & & \downarrow \mu^{(I^m, \partial I^m),(I,\{1\})} & & \downarrow (\gamma^s)^* \\
B_{r+s}^{\mathfrak{S}_{m+1}} & \longrightarrow & P(m, B)_{r+s} & \longrightarrow & B_{r+s}^{\mathfrak{S}_m}
\end{array}
$$

The result follows from Theorem 3.9.8 since $P(B_r^{\mathfrak{S}_m})_s$ and $P(m, B)_{r+s}$ are contractible by Lemma 3.2.5 and $(\gamma^s)^*$ is a $\mathfrak{K}$-equivalence by Lemma 3.10.5.

The general case will follow from the previous one by induction on n. Suppose that $\mu_B^{m,n}$ is a $\mathfrak{K}$-equivalence for every $B \in C$; we will show that $\mu_B^{m,n+1}$ is a $\mathfrak{K}$-equivalence too. By the associativity of μ discussed in Example 2.2.4, the following diagram commutes:

$$
\begin{array}{ccc}
\left(\left(B_r^{\mathfrak{S}_m}\right)_s^{\mathfrak{S}_n}\right)_0^{\mathfrak{S}_1} & \xrightarrow{\mu^{n,1}_{B_r^{\mathfrak{S}_m}}} & \left(B_r^{\mathfrak{S}_m}\right)_s^{\mathfrak{S}_{n+1}} \\
(\mu_B^{m,n})_0^{\mathfrak{S}_1} \downarrow & & \downarrow \mu_B^{m,n+1} \\
\left(B_{r+s}^{\mathfrak{S}_{m+n}}\right)_0^{\mathfrak{S}_1} & \xrightarrow{\mu_B^{m+n,1}} & B_{r+s}^{\mathfrak{S}_{m+n+1}}
\end{array}
$$

The horizontal morphisms are $\mathfrak{K}$-equivalences by the case $n = 1$. The morphism $(\mu_B^{m,n})_0^{\mathfrak{S}_1}$ is a $\mathfrak{K}$-equivalence because $\mu_B^{m,n}$ is; see Remark 3.10.4. Then $\mu_B^{m,n+1}$ is a $\mathfrak{K}$-equivalence. $\square$

Lemma 3.10.7. *The morphism* (35) *is an isomorphism.*

Proof. By Lemma 3.10.5 we may assume that $r = 0$. We will prove the result by induction on n. The case $n = 1$ holds by Lemma 3.10.3. Suppose now that the result holds for $n \geq 1$. It is easily verified that the following diagram in $\mathfrak{K}$ commutes:

$$
\begin{array}{ccc}
((B_0^{\mathfrak{S}_n})_0^{\mathfrak{S}_1}, m) & \xrightarrow{\cong} & (B_0^{\mathfrak{S}_n}, 1 + m) \\
L^m j(\mu_B^{n,1}) \downarrow \cong & & \downarrow \cong \\
(B_0^{\mathfrak{S}_{n+1}}, m) & \xrightarrow{[\mathrm{id}_{B^{\mathfrak{S}_{n+1}}}]} & (B, n + 1 + m)
\end{array}
$$

The horizontal top and vertical right morphisms are isomorphisms by induction hypothesis and $j(\mu_B^{n,1})$ is an isomorphism by Lemma 3.10.6. The result follows. $\square$

Remark 3.10.8. It follows from Lemma 3.10.7 that we have natural isomorphisms of functors $j \circ (?)_0^{\mathfrak{S}_n} \cong L^n \circ j$ and $j \circ ((?)_0^{\mathfrak{S}_n})_0^{\mathfrak{S}_1} \cong L^{n+1} \circ j$. Indeed, it is easily verified that the following diagrams commute for every morphism $f : A \to B$ in C:

$$
\begin{array}{ccc}
(A_0^{\mathfrak{S}_n}, 0) & \xrightarrow[\cong]{[\mathrm{id}_{A^{\mathfrak{S}_n}}]} & (A, n) \\
f_* \downarrow & & \downarrow f_* \\
(B_0^{\mathfrak{S}_n}, 0) & \xrightarrow[\cong]{[\mathrm{id}_{B^{\mathfrak{S}_n}}]} & (B, n)
\end{array}
\qquad
\begin{array}{ccc}
((A_0^{\mathfrak{S}_n})_0^{\mathfrak{S}_1}, 0) & \xrightarrow[\cong]{[\mu_A^{n,1}]} & (A, n + 1) \\
f_* \downarrow & & \downarrow f_* \\
((B_0^{\mathfrak{S}_n})_0^{\mathfrak{S}_1}, 0) & \xrightarrow[\cong]{[\mu_B^{n,1}]} & (B, n + 1)
\end{array}
$$

Lemma 3.10.9. *Let $\alpha \in \Re^C((A, m), (B, n))$ be represented by $f : J^{m+u}A \to B_r^{\mathfrak{S}_{n+u}}$. Then the following diagram commutes:*

$$
\begin{array}{ccc}
(A, m) & \xrightarrow{\quad \alpha \quad} & (B, n) \\
{\scriptstyle [\mathrm{id}_{J^{m+u}A}]} \downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \uparrow {\scriptstyle [\mathrm{id}_{B^{\mathfrak{S}_{n+u}}}]} \\
(J^{m+u}A, -u) & \xrightarrow{\ L^{-u}(j(f))\ } & (B_r^{\mathfrak{S}_{n+u}}, -u)
\end{array}
$$

Proof. The vertical morphisms are the isomorphisms in (33) and Lemma 3.10.7. The commutativity of the diagram follows from a straightforward computation.   □

3.11   Long exact sequences associated to extensions

Lemma 3.11.1. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an extension in C and let $D \in C$. Then there is a long exact sequence:*

$$
\dashrightarrow \Re^C(D, (A, n)) \xrightarrow{f_*} \Re^C(D, (B, n)) \xrightarrow{g_*} \Re^C(D, (C, n)) \xrightarrow{\partial} \Re^C(D, (A, n - 1)) \dashrightarrow
$$

Moreover, this sequence is natural with respect to morphisms of extensions.

Proof. It follows from Theorem 3.9.8, as we proceed to explain. The following diagram is an extension in C:

$$
A_0^{\mathfrak{S}_n} \xrightarrow{\ f^{\mathfrak{S}_n}\ } B_0^{\mathfrak{S}_n} \xrightarrow{\ g^{\mathfrak{S}_n}\ } C_0^{\mathfrak{S}_n} \tag{36}
$$

By Theorem 3.9.8 applied to (36), we have an exact sequence:

$$
\Re(D, (B_0^{\mathfrak{S}_n})_0^{\mathfrak{S}_1}) \xrightarrow{g_*} \Re(D, (C_0^{\mathfrak{S}_n})_0^{\mathfrak{S}_1}) \xrightarrow{\partial} \Re(D, A_0^{\mathfrak{S}_n}) \xrightarrow{f_*} \Re(D, B_0^{\mathfrak{S}_n}) \xrightarrow{g_*} \Re(D, C_0^{\mathfrak{S}_n})
$$

Under the natural identifications described in Remark 3.10.8, the latter becomes:

$$
\Re(D, (B, n + 1)) \xrightarrow{g_*} \Re(D, (C, n + 1)) \xrightarrow{\partial} \Re(D, (A, n)) \xrightarrow{f_*} \Re(D, (B, n)) \xrightarrow{g_*} \Re(D, (C, n))
$$

For varying $n \geq 0$, these sequences assemble into a long exact sequence, infinite to the left, ending in $\Re(D, (C, 0))$. It remains to show how to extend this sequence to the right. Upon applying $\Re(D_0^{\mathfrak{S}_n}, ?)$ to the extension $A \xrightarrow{f} B \xrightarrow{g} C$, we get the following exact sequence:

$$
\Re(D_0^{\mathfrak{S}_n}, (B, 1)) \xrightarrow{g_*} \Re(D_0^{\mathfrak{S}_n}, (C, 1)) \xrightarrow{\partial} \Re(D_0^{\mathfrak{S}_n}, A) \xrightarrow{f_*} \Re(D_0^{\mathfrak{S}_n}, B) \xrightarrow{g_*} \Re(D_0^{\mathfrak{S}_n}, C)
$$

After identifying $\Re(D_0^{\mathfrak{S}_n}, ?) \cong \Re((D, n), ?) \cong \Re(D, L^{-n}(?))$, this sequence becomes:

$$
\Re(D, (B, 1 - n)) \xrightarrow{g_*} \Re(D, (C, 1 - n)) \xrightarrow{\partial} \Re(D, (A, -n)) \xrightarrow{f_*} \Re(D, (B, -n)) \xrightarrow{g_*} \Re(D, (C, -n))
$$

Now glue these for varying $n \geq 0$ to extend the long exact sequence to the right.   □

Lemma 3.11.2. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an extension in C and let $C' \xrightarrow{c} C$ be a morphism in C. Let $B' \xrightarrow{g'} C'$ be the pullback of g along c. Then g' is a split surjection in $\mathcal{U}(C)$ with kernel A and c fits into the following morphism of extensions:*

$$
\begin{array}{ccccc}
A & \longrightarrow & B' & \xrightarrow{g'} & C' \\
\text{id} \downarrow & & \downarrow & & \downarrow c \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C
\end{array}
$$

Moreover, for any $D \in C$, there is a long exact Mayer-Vietoris sequence:

$$\cdots\rightarrow \mathfrak{R}(D, (B', n)) \rightarrow \mathfrak{R}(D, (B, n)) \oplus \mathfrak{R}(D, (C', n)) \rightarrow \mathfrak{R}(D, (C, n)) \xrightarrow{\partial} \mathfrak{R}(D, (B', n-1)) \cdots\rightarrow$$

Proof. The existence of the long exact Mayer-Vietoris sequence follows from Lemma 3.11.1 and from the argument explained in [3, Theorem 2.41]. □

Corollary 3.11.3. *Let $f : A \rightarrow B$ be any morphism in C. Then the last vertex map induces $\mathfrak{R}$-equivalences $(P_f)_r \rightarrow (P_f)_{r+1}$.*

Proof. Consider the path extension:

$$\mathscr{P}_{0,B} : B_r^{\mathfrak{S}_1} \longrightarrow (PB)_r \longrightarrow B$$

If we pullback $\mathscr{P}_{0,B}$ along $f : A \rightarrow B$, we get a long exact Mayer-Vietoris sequence as explained in Lemma 3.11.2. Since $(PB)_r$ is contractible, this sequence takes the form:

$$\cdots\rightarrow \mathfrak{R}(D, (A, 1)) \longrightarrow \mathfrak{R}(D, (B, 1)) \xrightarrow{\partial} \mathfrak{R}(D, (P_f)_r)) \longrightarrow \mathfrak{R}(D, A) \longrightarrow \mathfrak{R}(D, B) \cdots\rightarrow$$

It is easily verified that the sequence above is natural in r; then the result follows from the five lemma and Yoneda. □

3.12 Triangulated structure

Let $f : A \rightarrow B$ be a morphism in C. Recall the definitions of π_f and ι_f from (21) and (22).

Definition 3.12.1. We call *mapping path triangle* to a diagram in $\mathfrak{R}^C$ of the form

$$\triangle_{f,n} : L(B, n) \xrightarrow{\partial_{f,n}} ((P_f)_0, n) \xrightarrow{L^n j(\pi_f)} (A, n) \xrightarrow{L^n j(f)} (B, n) ,$$

where $f : A \rightarrow B$ is a morphism in C, $n \in \mathbb{Z}$ and $\partial_{f,n}$ equals the composite:

$$(B, n+1) \xleftarrow[\cong]{[\text{id}_B \mathfrak{S}_1]} (B_0^{\mathfrak{S}_1}, n) \xrightarrow{(-1)^{n+1} L^n j(\iota_f)} ((P_f)_0, n)$$

A *distinguished triangle* in $\mathfrak{R}^C$ is a triangle isomorphic (as a triangle) to some $\triangle_{f,n}$.

We are ready to verify that $\mathfrak{K}^C$ satisfies the axioms of a triangulated category with the translation functor L and the distinguished triangles defined above.

Axiom 3.12.2 (TR0). *Any triangle which is isomorphic to a distinguished triangle is itself distinguished. For any $B \in C$ and any $n \in \mathbb{Z}$, the following triangle is distinguished:*

$$L(B, n) \longrightarrow 0 \longrightarrow (B, n) \xrightarrow{\mathrm{id}_{(B,n)}} (B, n)$$

Proof. The first assertion is clear and the second one follows from the fact that the mapping path $(P_{\mathrm{id}_B})_0 \cong (PB)_0$ is contractible.   □

Axiom 3.12.3 (TR1). *Every morphism $\alpha : X \to Y$ in $\mathfrak{K}^C$ fits into a distinguished triangle of the form:*

$$L(Y) \longrightarrow Z \longrightarrow X \xrightarrow{\alpha} Y$$

Proof. By Lemma 3.10.9 we can assume that $X = (C, k)$, $Y = (B, k)$ and $\alpha = L^k j(f)$ with $f : C \to D$ a morphism in C. In this case α fits into the mapping path triangle $\triangle_{f,k}$.   □

Definition 3.12.4. Consider a triangle $\triangle$ in $\mathfrak{K}$:

$$\triangle : \ L(Z) \xrightarrow{\alpha} X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z \tag{37}$$

We define the *rotated triangle $R(\triangle)$* by:

$$R(\triangle) : \ L(Y) \xrightarrow{-L\gamma} L(Z) \xrightarrow{-\alpha} X \xrightarrow{-\beta} Y$$

Remark 3.12.5. As explained in [3, Definition 6.51], we have an isomorphism:

$$R(\triangle) \cong (\ L(Y) \xrightarrow{-L\gamma} L(Z) \xrightarrow{\alpha} X \xrightarrow{\beta} Y \ )$$

Axiom 3.12.6 (TR2). *A triangle $\triangle$ is distinguished if and only if $R(\triangle)$ is.*

Proof. Let us first show that if $\triangle$ is distinguished, then $R(\triangle)$ is distinguished as well. It suffices to prove that the rotation of a mapping path triangle is distinguished. Let $f : A \to B$ be a morphism in C and consider the following mapping path triangles:

$$\triangle_{f,n} : \ L(B, n) \xrightarrow{\partial_{f,n}} ((P_f)_0, n) \xrightarrow{L^n j(\pi_f)} (A, n) \xrightarrow{L^n j(f)} (B, n)$$

$$\triangle_{\pi_f,n} : \ L(A, n) \xrightarrow{\partial_{\pi_f,n}} ((P_{\pi_f})_0, n) \xrightarrow{L^n j(\pi_{\pi_f})} ((P_f)_0, n) \xrightarrow{L^n j(\pi_f)} (A, n)$$

Let $\varepsilon : L(B, n) \to ((P_{\pi_f})_0, n)$ be the following composite, where ϕ_f is the morphism defined in Corollary 3.9.5:

$$(B, n + 1) \xleftarrow[\cong]{[\mathrm{id}_B \mathfrak{S}_1]} (B_0^{\mathfrak{S}_1}, n) \xrightarrow{(-1)^{n+1} L^n j(\phi_f)} ((P_{\pi_f})_0, n)$$

Notice that ε is an isomorphism by Corollary 3.9.5. It follows from Lemma 3.9.7 that we have an isomorphism $R(\triangle_{f,n}) \cong \triangle_{\pi_f,n}$ as follows:

$$
\begin{array}{ccccccccc}
R(\triangle_{f,n}) & & L(A,n) & \xrightarrow{-L^{n+1}j(f)} & L(B,n) & \xrightarrow{-\partial_{f,n}} & ((P_f)_0,n) & \xrightarrow{-L^n j(\pi_f)} & (A,n) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle-\mathrm{id}} & & \downarrow{\scriptstyle\mathrm{id}} \\
\triangle_{\pi_f,n} & & L(A,n) & \xrightarrow{\partial_{\pi_f,n}} & ((P_{\pi_f})_0,n) & \xrightarrow{L^n j(\pi_{\pi_f})} & ((P_f)_0,n) & \xrightarrow{L^n j(\pi_f)} & (A,n)
\end{array}
$$

This shows that the rotation of a mapping path triangle is distinguished.

We still have to prove that if $R(\triangle)$ is distinguished, then $\triangle$ is distinguished. We claim that if $R^3(\triangle)$ is distinguished, then $\triangle$ is distinguished; suppose for a moment that this claim is proved. If $R(\triangle)$ is distinguished then $R^3(\triangle)$ is distinguished —because R preserves distinguished triangles— and so $\triangle$ is distinguished —by the claim. Thus, the proof will be finished if we prove the claim. Let $\triangle$ be the triangle in (37). Then:

$$
R^3(\triangle) \cong (\, L^2(Z) \xrightarrow{-L\alpha} L(X) \xrightarrow{L\beta} L(Y) \xrightarrow{L\gamma} L(Z) \,)
$$

Suppose that $R^3(\triangle)$ is distinguished. Then there exists a morphism $f : A \to B$ in C that fits into an isomorphism of triangles as follows:

$$
\begin{array}{ccccccc}
L^2(Z) & \xrightarrow{-L\alpha} & L(X) & \xrightarrow{L\beta} & L(Y) & \xrightarrow{L\gamma} & L(Z) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
L(B,n) & \xrightarrow{\partial_{f,n}} & ((P_f)_0,n) & \xrightarrow{L^n j(\pi_f)} & (A,n) & \xrightarrow{L^n j(f)} & (B,n)
\end{array} \qquad (38)
$$

Upon applying $L^{-1}(?)$ to (38) we get a commutative diagram as follows:

$$
\begin{array}{ccccccc}
L(Z) & \xrightarrow{-\alpha} & X & \xrightarrow{\beta} & Y & \xrightarrow{\gamma} & Z \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
(B,n) & \xrightarrow{-\partial_{f,n-1}} & ((P_f)_0,n-1) & \xrightarrow{L^{n-1}j(\pi_f)} & (A,n-1) & \xrightarrow{L^{n-1}j(f)} & (B,n-1)
\end{array}
$$

Thus, the vertical morphisms in the latter diagram assemble into an isomorphism of triangles $\triangle \cong \triangle_{f,n-1}$. Then $\triangle$ is distinguished. $\qquad\square$

Lemma 3.12.7. *Let $f : A \to B$ be a morphism in C, let $k \in \mathbb{Z}$ and let $n \geq 0$. Then there is a morphism of triangles as follows:*

$$
\begin{array}{ccccccc}
\triangle_{f,k+n} & & L(B,k+n) & \longrightarrow & ((P_f)_0,k+n) & \longrightarrow & (A,k+n) & \longrightarrow & (B,k+n) \\
\downarrow & & \downarrow{\scriptstyle\cong} & & \downarrow & & {\scriptstyle[\mathrm{id}_{J^n A}]}\downarrow{\scriptstyle\cong} & & {\scriptstyle[\mathrm{id}_{J^n B}]}\downarrow{\scriptstyle\cong} \\
\triangle_{J^n(f),k} & & L(J^n B,k) & \longrightarrow & ((P_{J^n(f)})_0,k) & \longrightarrow & (J^n A,k) & \longrightarrow & (J^n B,k)
\end{array}
$$

Proof. It is enough to construct a morphism $\triangle_{f,k+1} \to \triangle_{J(f),k}$ and then consider the composite:

$$
\triangle_{f,k+n} \longrightarrow \triangle_{J(f),k+n-1} \longrightarrow \triangle_{J^2(f),k+n-2} \longrightarrow \cdots \longrightarrow \triangle_{J^n(f),k}
$$

Let $c : J(P_f)_0 \to (P_{J(f)})_0$ be the morphism defined by the following diagram in C:

It is easily verified that the following diagram commutes, where the unlabelled vertical morphisms are induced by the natural isomorphism $j \circ J \cong L \circ j : C \to \mathfrak{K}$:

It is easily verified that the following diagram commutes:

To get the desired morphism of triangles $\triangle_{f,k+1} \to \triangle_{J(f),k}$, put together both diagrams above. $\qquad\square$

Lemma 3.12.8. *Let $f : A \to B$ be a morphism in C, let $k \in \mathbb{Z}$ and let $n \geq 0$. Then there is an isomorphism of triangles as follows:*
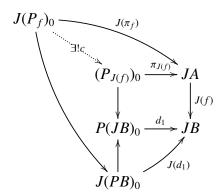
Proof. First notice that $((PB)_0)_r^{\mathfrak{S}_n} \cong P(B_r^{\mathfrak{S}_n})_0$ since there are natural isomorphisms $P(?)_0 \cong (?) \otimes P\mathbb{Z}_0$ and $(?)_r^{\mathfrak{S}_n} \cong (?) \otimes \mathbb{Z}_r^{\mathfrak{S}_n}$. It follows easily from this that $((P_f)_0)_r^{\mathfrak{S}_n} \cong (P_{f^{\mathfrak{S}_n}})_0$. We have the following commutative diagram, where the vertical morphisms from the second row to the first one are induced by the natural isomorphism $j \circ (?)_r^{\mathfrak{S}_n} \cong L^n \circ j : C \to \mathfrak{K}$:

$$
\begin{array}{ccccccc}
(B_0^{\mathfrak{S}_1}, n+k) & \xrightarrow{L^{n+k}j(\iota_f)} & ((P_f)_0, n+k) & \xrightarrow{L^{n+k}j(\pi_f)} & (A, n+k) & \xrightarrow{L^{n+k}j(f)} & (B, n+k) \\
\Big\uparrow{\scriptstyle\cong} & & \Big\uparrow{\scriptstyle\cong} & & \Big\uparrow{\scriptstyle\cong} & & \Big\uparrow{\scriptstyle\cong} \\
((B_0^{\mathfrak{S}_1})_r^{\mathfrak{S}_n}, k) & \xrightarrow{L^k j(\iota_f^{\mathfrak{S}_n})} & (((P_f)_0)_r^{\mathfrak{S}_n}, k) & \xrightarrow{L^k j(\pi_f^{\mathfrak{S}_n})} & (A_r^{\mathfrak{S}_n}, k) & \xrightarrow{L^k j(f^{\mathfrak{S}_n})} & (B_r^{\mathfrak{S}_n}, k) \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\text{id}} & & \Big\downarrow{\scriptstyle\text{id}} \\
((B_r^{\mathfrak{S}_n})_0^{\mathfrak{S}_1}, k) & \xrightarrow{L^k j(\iota_{f^{\mathfrak{S}_n}})} & ((P_{f^{\mathfrak{S}_n}})_0, k) & \xrightarrow{L^k j(\pi_{f^{\mathfrak{S}_n}})} & (A_r^{\mathfrak{S}_n}, k) & \xrightarrow{L^k j(f^{\mathfrak{S}_n})} & (B_r^{\mathfrak{S}_n}, k)
\end{array}
$$

It is easily verified that the following diagram commutes:

$$
\begin{array}{ccc}
(B, n+k+1) & \xleftarrow[\cong]{[\text{id}_{B^{\mathfrak{S}_1}}]} & (B_0^{\mathfrak{S}_1}, n+k) \\
& & \Big\uparrow{\scriptstyle\cong}{\scriptstyle[\text{id}_{(B^{\mathfrak{S}_1})^{\mathfrak{S}_n}}]} \\
{\scriptstyle[\text{id}_{B^{\mathfrak{S}_n}}]}\Big\uparrow{\scriptstyle\cong} & & ((B_0^{\mathfrak{S}_1})_r^{\mathfrak{S}_n}, k) \\
& & \Big\downarrow{\scriptstyle\cong} \\
(B_r^{\mathfrak{S}_n}, k+1) & \xleftarrow[\cong]{(-1)^n[\text{id}_{(B^{\mathfrak{S}_n})^{\mathfrak{S}_1}}]} & ((B_r^{\mathfrak{S}_n})_0^{\mathfrak{S}_1}, k)
\end{array}
$$

To get the desired isomorphism of triangles $\triangle_{f^{\mathfrak{S}_n}, k} \cong \triangle_{f, n+k}$, put together both diagrams above. $\qquad\square$

Axiom 3.12.9 (TR3). *For every diagram of solid arrows as follows, in which the rows are distinguished triangles, there exists a dotted arrow that makes the whole diagram commute.*

$$
\begin{array}{ccccccc}
L(Z') & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' \\
\Big\downarrow & & \Big\vdots & & \Big\downarrow & & \Big\downarrow \\
L(Z) & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z
\end{array}
\tag{39}
$$

Proof. Let us begin with a special case. Consider a commutative square in $[C]$:

$$
\begin{array}{ccc}
A' & \xrightarrow{f'} & B' \\
{\scriptstyle a}\Big\downarrow & & \Big\downarrow{\scriptstyle b} \\
A & \xrightarrow{f} & B
\end{array}
$$

Suppose that (39) takes the following form, where the rows are mapping path triangles:

$$
\begin{array}{ccccccc}
L(B', n) & \xrightarrow{\partial_{f', n}} & ((P_{f'})_0, n) & \xrightarrow{L^n j(\pi_{f'})} & (A', n) & \xrightarrow{L^n j(f')} & (B', n) \\
{\scriptstyle L^{n+1}j(b)}\Big\downarrow & & \Big\vdots & & {\scriptstyle L^n j(a)}\Big\downarrow & & {\scriptstyle L^n j(b)}\Big\downarrow \\
L(B, n) & \xrightarrow{\partial_{f, n}} & ((P_f)_0, n) & \xrightarrow{L^n j(\pi_f)} & (A, n) & \xrightarrow{L^n j(f)} & (B, n)
\end{array}
\tag{40}
$$

We want to show that a dotted arrow exists in this case. Let $H : A' \to B^{\mathrm{sd}^r I}$ be a homotopy such that $d_1 \circ H = f \circ a$ and $d_0 \circ H = b \circ f'$; we may assume $r \geq 1$. We have:

$$(P_{f'})_r = \{(x, y) \in A' \times (PB')_r \mid f'(x) = d_1(y)\}$$

$$(P_f)_{r+1} = \{(x, y) \in A \times (PB)_{r+1} \mid f(x) = d_1(y)\}$$

Define a morphism $c : (P_{f'})_r \to (P_f)_{r+1}$ by the formula:

$$c(x, y) = (a(x), H(x) \bullet P(b)(y)) \tag{41}$$

Here, the symbol $\bullet$ means *concatenation* of paths —we have $(PB)_{r+1} \cong B^{\mathrm{sd}^r I} \times_B (PB)_r$ so that we can *concatenate* an element of $B^{\mathrm{sd}^r I}$ with one of $(PB)_r$ to get an element of $(PB)_{r+1}$. Note that the concatenation in (41) makes sense since $d_0(H(x)) = b(f'(x)) = b(d_1(y)) = d_1(P(b)(y))$. Moreover, $c(x, y)$ is indeed an element of $(P_f)_{r+1}$ since we have:

$$d_1(H(x) \bullet P(b)(y)) = d_1(H(x)) = f(a(x))$$

Let $\chi : ((P_{f'})_0, n) \to ((P_f)_0, n)$ be the composite:

$$((P_{f'})_0, n) \xrightarrow{\ \cong\ } ((P_{f'})_r, n) \xrightarrow{\ L^n(j(c))\ } ((P_f)_{r+1}, n) \xleftarrow{\ \cong\ } ((P_f)_0, n)$$

We claim that taking the dotted arrow in (40) equal to χ makes the whole diagram commute. It is easily verified that the following square commutes in $\mathcal{C}$, and this implies that the middle square in (40) commutes:

$$
\begin{array}{ccc}
(P_{f'})_r & \xrightarrow{\ \pi_{f'}\ } & A' \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle a} \\
(P_f)_{r+1} & \xrightarrow{\ \pi_f\ } & A
\end{array}
$$

It is easily verified that the following diagram commutes in $[\mathcal{C}]$, and this implies that the left square in (40) commutes:

$$
\begin{array}{ccccc}
(B')_0^{\mathfrak{S}_1} & \longrightarrow & (B')_r^{\mathfrak{S}_1} & \xrightarrow{\ \iota_{f'}\ } & (P_{f'})_r \\
{\scriptstyle b^{\mathfrak{S}_1}}\downarrow & & & & \downarrow{\scriptstyle c} \\
B_0^{\mathfrak{S}_1} & \longrightarrow & B_{r+1}^{\mathfrak{S}_1} & \xrightarrow{\ \iota_f\ } & (P_f)_{r+1}
\end{array}
$$

This finishes the proof of the axiom in this special case.

In the general case, we may suppose that both triangles are mapping path triangles, so that (39) equals the following diagram, for some morphisms $f : A \to B$ and $f' : A' \to B'$:

$$
\begin{array}{ccccccc}
\triangle_{f',k'} & L(B', k') \longrightarrow ((P_{f'})_0, k') \longrightarrow (A', k') \longrightarrow (B', k') & \\
\vdots & \Big\downarrow \qquad\qquad \vdots \qquad\qquad \downarrow{\scriptstyle\alpha} \qquad\qquad \downarrow{\scriptstyle\beta} & \qquad (42) \\
\triangle_{f,k} & L(B, k) \longrightarrow ((P_f)_0, k) \longrightarrow (A, k) \longrightarrow (B, k) &
\end{array}
$$

We may choose l and r large enough so that α is represented by $a : J^{k'+l}A' \to A_r^{\mathfrak{S}k+l}$, β is represented by $b : J^{k'+l}B' \to B_r^{\mathfrak{S}k+l}$, and the following square in $[C]$ commutes:

$$
\begin{array}{ccc}
J^{k'+l}A' & \xrightarrow{\ J^{k'+l}(f')\ } & J^{k'+l}B' \\
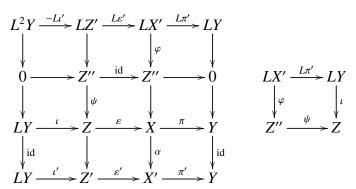{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
A_r^{\mathfrak{S}k+l} & \xrightarrow{\ f^{\mathfrak{S}k+l}\ } & B_r^{\mathfrak{S}k+l}
\end{array}
$$

By the special case we have already proven, we can extend a and b to a morphism of triangles $\triangle_{J^{k'+l}(f'),-l} \to \triangle_{f^{\mathfrak{S}k+l},-l}$. Then the composite

$$
\triangle_{f',k'} \xrightarrow{\text{Lemma 3.12.7}} \triangle_{J^{k'+l}(f'),-l} \xrightarrow{\hspace{3cm}} \triangle_{f^{\mathfrak{S}k+l},-l} \xrightarrow{\text{Lemma 3.12.8}} \triangle_{f,k}
$$

is a morphism of triangles that extends the diagram of solid arrows in (42). $\qquad\square$

Axiom 3.12.10 (TR4). *Let $\alpha : X \to X'$ and $\pi' : X' \to Y$ be composable morphisms in $\mathfrak{K}^C$ and put $\pi := \pi' \circ \alpha$. Then there exist commutative diagrams as follow, where the rows and columns of the big diagram are distinguished triangles.*

$$
\begin{array}{ccccccc}
L^2Y & \xrightarrow{-L\iota'} & LZ' & \xrightarrow{L\varepsilon'} & LX' & \xrightarrow{L\pi'} & LY \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow \\
0 & \longrightarrow & Z'' & \xrightarrow{\ \text{id}\ } & Z'' & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow & & \downarrow \\
LY & \xrightarrow{\ \iota\ } & Z & \xrightarrow{\ \varepsilon\ } & X & \xrightarrow{\ \pi\ } & Y \\
\downarrow{\scriptstyle\text{id}} & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\text{id}} \\
LY & \xrightarrow{\ \iota'\ } & Z' & \xrightarrow{\ \varepsilon'\ } & X' & \xrightarrow{\ \pi'\ } & Y
\end{array}
\qquad
\begin{array}{ccc}
LX' & \xrightarrow{L\pi'} & LY \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\iota} \\
Z'' & \xrightarrow{\ \psi\ } & Z
\end{array}
$$

Proof. Consider the following diagram in $\mathfrak{K}^C$:

$$
X \xrightarrow{\ \alpha\ } X' \xrightarrow{\ \pi'\ } Y \tag{43}
$$

We will say that this diagram *satisfies (TR4)* if the axiom holds for this particular pair of morphisms.

(i) It is straightforward to verify that if two of such diagrams are isomorphic, then one satisfies (TR4) if and only if the other does.

(ii) Consider the following triangle in $\mathfrak{K}$:

$$
\triangle : \ LW \xrightarrow{\ \rho\ } U \xrightarrow{\ \sigma\ } V \xrightarrow{\ \tau\ } W
$$

Recall from Definition 3.12.4 that we have:

$$
R^3(\triangle) \cong \left(L^2W \xrightarrow{-L\rho} LU \xrightarrow{L\sigma} LV \xrightarrow{L\tau} LW \right)
$$

By (TR2), $\triangle$ is distinguished if and only if $R^3(\triangle)$ is. Using this fact, it is easy to prove that (43) satisfies (TR4) if and only if the following diagram does:

$$
LX \xrightarrow{\ L\alpha\ } LX' \xrightarrow{\ L\pi'\ } LY
$$

We claim that any diagram like (43) is isomorphic to the diagram below, for some $n \in \mathbb{Z}$ and some morphisms $a : A \to B$ and $b : B \to C$ in $\mathcal{C}$:

$$(A, n) \xrightarrow{L^n j(a)} (B, n) \xrightarrow{L^n j(b)} (C, n)$$

Once this claim is proved, using (i) and (ii), the proof of the axiom (TR4) can be reduced to the special case when $\alpha = j(a) : (A, 0) \to (B, 0)$ and $\pi' = j(b) : (B, 0) \to (C, 0)$.

Let us prove the claim. Suppose that $X = (A, m)$, $X' = (B, n)$ and $Y = (C, k)$ and let $\alpha : (A, m) \to (B, n)$ be represented by $f : J^{m+u}A \to B_r^{\mathfrak{S}_{n+u}}$. Write $\tilde{A} := J^{m+u}A$ and $\tilde{B} := B_r^{\mathfrak{S}_{n+u}}$ to alleviate notation. By Lemma 3.10.9, we have an isomorphism of diagrams:

$$
\begin{array}{ccccc}
(A, m) & \xrightarrow{\alpha} & (B, n) & \xrightarrow{\pi'} & (C, k) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\text{id}} \\
(\tilde{A}, -u) & \xrightarrow{L^{-u} j(f)} & (\tilde{B}, -u) & \longrightarrow & (C, k)
\end{array}
$$

Hence we may assume that $m = n$ and that $\alpha = L^n j(f)$ for some morphism $f : A \to B$. Let $\pi' : (B, n) \to (C, k)$ be represented by $b : J^{n+v}B \to C_s^{\mathfrak{S}_{k+v}}$. By Lemma 3.10.9, we have an isomorphism of diagrams as follows, proving the claim:

$$
\begin{array}{ccccc}
(A, n) & \xrightarrow{L^n j(f)} & (B, n) & \xrightarrow{\pi'} & (C, k) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
(J^{n+v}A, -v) & \xrightarrow{L^{-v} j(J^{n+v}f)} & (J^{n+v}B, -v) & \xrightarrow{L^{-v} j(b)} & (C_s^{\mathfrak{S}_{k+v}}, -v)
\end{array}
$$

Let $a : A \to B$ and $b : B \to C$ be morphisms in $\mathcal{C}$. Let us prove that the following diagram satisfies (TR4):

$$(A, 0) \xrightarrow{j(a)} (B, 0) \xrightarrow{j(b)} (C, 0)$$

The argument is the one explained in [2, Axiom 6.5.7] but we give some more details. Put $c := b \circ a : A \to C$. We will use the identifications in Lemma 3.9.7 so that we have, for example:

$$(PC)_0 = (t - 1)C[t]$$

$$C_0^{\mathfrak{S}_1} = (t^2 - t)C[t]$$

$$(P_b)_0 = \{(p(t), y) \in C[t] \times B : p(0) = b(y) \text{ and } p(1) = 0\}$$

$$(P_c)_0 = \{(q(t), z) \in C[t] \times A : q(0) = c(z) \text{ and } q(1) = 0\}$$

Recall from (21) and (22) the definitions of $\pi_b : (P_b)_0 \to C$ and $\iota_b : C_0^{\mathfrak{S}_1} \to (P_b)_0$. For example, the morphism $\pi_b : (P_b)_0 \to B$ is defined by $\pi_b(p(t), y) = y$. The morphism $\eta : (P_c)_0 \to (P_b)_0$, $\eta(q(t), z) = (q(t), a(z))$, makes the following diagram in $\mathcal{C}$ commute:

$$
\begin{array}{ccccccc}
C_0^{\mathfrak{S}_1} & \xrightarrow{\iota_c} & (P_c)_0 & \xrightarrow{\pi_c} & A & \xrightarrow{c} & C \\
\downarrow{\scriptstyle\text{id}} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle\text{id}} \\
C_0^{\mathfrak{S}_1} & \xrightarrow{\iota_b} & (P_b)_0 & \xrightarrow{\pi_b} & B & \xrightarrow{b} & C
\end{array}
$$

By functoriality of the mapping path construction, there is a morphism θ making the following diagram commute:

$$
\begin{array}{ccc}
((P_b)_0)_0^{\mathfrak{S}_1} & \xrightarrow{(\pi_b)^{\mathfrak{S}_1}} & B_0^{\mathfrak{S}_1} \\
\downarrow{\iota_\eta} & & \downarrow{\iota_a} \\
(P_\eta)_0 & \xrightarrow{\theta} & (P_a)_0 \\
\downarrow{\pi_\eta} & & \downarrow{\pi_a} \\
C_0^{\mathfrak{S}_1} \xrightarrow{\iota_c} (P_c)_0 \xrightarrow{\pi_c} & A \xrightarrow{c} & C \\
\downarrow{\text{id}} \quad\quad \downarrow{\eta} & \downarrow{a} & \downarrow{\text{id}} \\
C_0^{\mathfrak{S}_1} \xrightarrow{\iota_b} (P_b)_0 \xrightarrow{\pi_b} & B \xrightarrow{b} & C
\end{array}
\tag{44}
$$

We claim that θ is a $\mathfrak{K}$-equivalence; indeed, it is a split surjection with contractible kernel, as we proceed to explain. We have:

$$
(P(P_b)_0)_0 = \left\{ (p(t, s), y(s)) \in C[t, s] \times B[s] : \begin{array}{l} p(0, s) = b(y)(s), \ p(1, s) = 0, \\ p(t, 1) = 0 \text{ and } y(1) = 0 \end{array} \right\}
$$

$$
(P_\eta)_0 = \{ (p(t, s), y(s), q(t), z) \in (P(P_b)_0)_0 \times (P_c)_0 : (p(t, 0), y(0)) = (q(t), a(z)) \}
$$

In the description of $(P_\eta)_0$ above, $(p(t, s), y(s), q(t), z)$ satisfies $q(t) = p(t, 0)$ so that we can get rid of q as long as we keep p. Hence, we have:

$$
(P_\eta)_0 = \left\{ (p(t, s), y(s), z) \in C[t, s] \times B[s] \times A : \begin{array}{l} p(0, s) = b(y)(s), \\ p(1, s) = 0, \ p(t, 1) = 0, \\ y(1) = 0, \ p(0, 0) = c(z) \\ \text{and } y(0) = a(z) \end{array} \right\}
$$

It is easily seen that, using this description of $(P_\eta)_0$, the morphism $\theta : (P_\eta)_0 \to (P_a)_0$ is given by $\theta(p(t, s), y(s), z) = (y(t), z)$. We have:

$$
\ker \theta = \{ (p(t, s), 0, 0) \in (P_\eta)_0 \}
$$
$$
\cong \{ p(t, s) \in C[t, s] : p(0, s) = 0, p(1, s) = 0 \text{ and } p(t, 1) = 0 \}
$$

It is easily verified that $\ker \theta$ is contractible. Moreover, θ is a split surjection with section:

$$
(P_a)_0 \ni (y(t), z) \mapsto (b(y)(1 - (1 - s)(1 - t)), y(s), z) \in (P_\eta)_0
$$

Upon applying j to (44) and identifying $(B_0^{\mathfrak{S}_1}, 0) \cong (B, 1)$, we get the following diagram in $\mathfrak{K}$ whose rows and columns are mapping path triangles; the diagram clearly

commutes, except maybe for the squares $*$ and $\star$:

$$
\begin{array}{ccccccc}
(C,2) & \xrightarrow{-\partial_{b,0}} & ((P_b)_0,1) & \xrightarrow{Lj(\pi_b)} & (B,1) & \xrightarrow{Lj(b)} & (C,1) \\
\downarrow & * & \downarrow{\scriptstyle\partial_{\eta,0}} & & \downarrow{\scriptstyle j(\theta)^{-1}\circ\partial_{a,0}} & & \downarrow \\
0 & \longrightarrow & ((P_\eta)_0,0) & \xrightarrow{\mathrm{id}} & ((P_\eta)_0,0) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle j(\pi_\eta)} & & \downarrow{\scriptstyle j(\pi_a)\circ j(\theta)} & \star & \downarrow \\
(C,1) & \xrightarrow{\partial_{c,0}} & ((P_c)_0,0) & \xrightarrow{j(\pi_c)} & (A,0) & \xrightarrow{j(c)} & (C,0) \\
\downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle j(\eta)} & & \downarrow{\scriptstyle j(a)} & & \downarrow{\scriptstyle\mathrm{id}} \\
(C,1) & \xrightarrow{\partial_{b,0}} & ((P_b)_0,0) & \xrightarrow{j(\pi_b)} & (B,0) & \xrightarrow{j(b)} & (C,0)
\end{array}
$$

The composite $c \circ \pi_a : (P_a)_0 \to C$ is easily seen to be nullhomotopic, so that the square $\star$ commutes. The composite

$$
(C_0^{\mathfrak{S}_1})_0^{\mathfrak{S}_1} \xrightarrow{(\iota_b)^{\mathfrak{S}_1}} ((P_b)_0)_0^{\mathfrak{S}_1} \xrightarrow{\iota_\eta} (P_\eta)_0
$$

is easily seen to factor through $\ker\theta$, which is contractible; this implies that the square $*$ commutes too.

We still have to show that the following square commutes:

$$
\begin{array}{ccc}
(B,1) & \xrightarrow{Lj(b)} & (C,1) \\
{\scriptstyle j(\theta)^{-1}\circ\partial_{a,0}}\downarrow & & \downarrow{\scriptstyle\partial_{c,0}} \\
((P_\eta)_0,0) & \xrightarrow{j(\pi_\eta)} & ((P_c)_0,0)
\end{array}
\tag{45}
$$

It is easily seen that the commutativity of (45) is implied by the commutativity of the following diagram in $[C]$:

$$
\begin{array}{ccc}
B_0^{\mathfrak{S}_1} & \xrightarrow{b^{\mathfrak{S}_1}} & C_0^{\mathfrak{S}_1} \\
{\scriptstyle\iota_a}\downarrow & & \downarrow{\scriptstyle\iota_c} \\
(P_a)_0 & \xrightarrow{\quad\xi\quad} & \\
{\scriptstyle\theta}\uparrow & & \downarrow \\
(P_\eta)_0 & \xrightarrow{\pi_\eta} & (P_c)_0
\end{array}
\tag{46}
$$

Here the morphism ξ is given by $\xi(y(t),z) = (b(y)(t),z)$. The square in (46) commutes on the nose. The triangle in (46) commutes in $[C]$, as we proceed to explain. Consider the following elementary homotopies $H_1, H_2 : (P_\eta)_0 \to (P_c)_0[u]$:

$$
H_1(p(t,s),y(s),z) = (p(tu,t),z)
$$

$$
H_2(p(t,s),y(s),z) = (p(t,tu),z)
$$

Then $\mathrm{ev}_{u=0} \circ H_1 = \xi \circ \theta$, $\mathrm{ev}_{u=1} \circ H_1 = \mathrm{ev}_{u=1} \circ H_2$ and $\mathrm{ev}_{u=0} \circ H_2 = \pi_\eta$, showing that $\xi \circ \theta = \pi_\eta$ in $[C]$. This finishes the proof of (TR4). $\qquad\square$

We have shown that $\mathfrak{K}^C$ is a triangulated category with the distinguished triangles being those triangles isomorphic to mapping path triangles. As in the topological setting [3, Section 6.6], the distinguished triangles could also be defined using extension triangles; we proceed to give the details of this.

Definition 3.12.11. Let $\mathscr{E} : A \xrightarrow{f} B \xrightarrow{g} C$ be an extension in C with classifying map $\xi : JC \to A$. Let $n \in \mathbb{Z}$ and let $\partial_{\mathscr{E},n}$ be the composite:

$$(C, n+1) \xrightarrow{\;[\mathrm{id}_{JC}]\;} (JC, n) \xrightarrow{\;(-1)^n L^n j(\xi)\;} (A, n)$$

We call *extension triangle* to a diagram in $\mathfrak{K}^C$ of the form:

$$\triangle_{\mathscr{E},n} : \; L(C,n) \xrightarrow{\;\partial_{\mathscr{E},n}\;} (A, n) \xrightarrow{\;L^n j(f)\;} (B, n) \xrightarrow{\;L^n j(g)\;} (C, n)$$

Proposition 3.12.12 ([3, Section 6.6]). *A triangle in $\mathfrak{K}^C$ is distinguished if and only if it is isomorphic to an extension triangle.*

Proof. Let us show first that every mapping path triangle is isomorphic to an extension triangle. Let $g : B \to C$ be any morphism in C. Consider the mapping cylinder:

$$Z_g := \{(p, b) \in C[t] \times B : p(0) = g(b)\}$$

Using the identifications in Lemma 3.9.7, we have:

$$(P_g)_0 := \{(p, b) \in (t-1)C[t] \times B : p(0) = g(b)\}$$

It is easily verified that the following diagram is an extension in C:

$$\mathscr{Z}_g : \; (P_g)_0 \xrightarrow{\;\mathrm{inc}\;} Z_g \xrightarrow{\;\varepsilon\;} C$$
$$(p, b) \longmapsto p(1)$$

Let $\mathrm{pr} : Z_g \to B$ be the natural projection; pr is easily seen to be a homotopy equivalence inverse to $b \mapsto (g(b), b)$. We claim that there is an isomorphism of triangles as follows:

$$
\begin{array}{ccccccc}
\triangle_{\mathscr{Z}_g,0} & (B,1) \xrightarrow{\partial_{\mathscr{Z}_g,0}} & ((P_g)_0, 0) \xrightarrow{j(\mathrm{inc})} & (Z_g, 0) \xrightarrow{j(\varepsilon)} & (C, 0) \\
\cong \downarrow & \mathrm{id}\downarrow & \mathrm{id}\downarrow & \cong\downarrow j(\mathrm{pr}) & \mathrm{id}\downarrow \\
\triangle_{g,0} & (B,1) \xrightarrow{\partial_{g,0}} & ((P_g)_0, 0) \xrightarrow{j(\pi_g)} & (B, 0) \xrightarrow{j(g)} & (C, 0)
\end{array}
$$

The middle and right squares clearly commute but we still have to show that $\partial_{\mathscr{Z}_g,0} = \partial_{g,0}$. Let $\omega : C_0^{\mathfrak{S}_1} \to C_0^{\mathfrak{S}_1}$ be the automorphism defined by $\omega(p(t)) = p(1-t)$. Consider the following morphism of extensions, where the vertical map in the middle is defined by $p(t) \mapsto (p(1-t), 0)$:

$$
\begin{array}{ccccc}
\mathscr{P}_{0,C} & C_0^{\mathfrak{S}_1} \longrightarrow & (PC)_0 \longrightarrow & C \\
\downarrow & \iota_g \circ \omega \downarrow & \downarrow & \mathrm{id}\downarrow \\
\mathscr{Z}_g & (P_g)_0 \longrightarrow & Z_g \longrightarrow & C
\end{array}
$$

By Proposition 3.1.3, the classifying map of $\mathcal{Z}_g$ equals $\iota_g \circ \omega \circ \lambda_C$; this is easily seen to imply that $\partial_{\mathcal{Z}_g,0} = \partial_{g,0}$.

Let us now show that every extension triangle is isomorphic to a mapping path triangle. Let $\mathcal{E} : A \xrightarrow{f} B \xrightarrow{g} C$ be an extension in C. Let $h : A \to (P_g)_0$ be the natural morphism that is a $\mathfrak{K}$-equivalence by Lemma 3.9.3. We claim that there is an isomorphism of triangles as follows:

$$
\begin{array}{ccccccccc}
\triangle_{\mathcal{E},0} & & (C,1) & \xrightarrow{\partial_{\mathcal{E},0}} & (A,0) & \xrightarrow{j(f)} & (B,0) & \xrightarrow{j(g)} & (C,0) \\
\cong \downarrow & & \mathrm{id}\downarrow & & \cong\downarrow j(h) & & \mathrm{id}\downarrow & & \mathrm{id}\downarrow \\
\triangle_{g,0} & & (C,1) & \xrightarrow{\partial_{g,0}} & ((P_g)_0,0) & \xrightarrow{j(\pi_g)} & (B,0) & \xrightarrow{j(g)} & (C,0)
\end{array}
$$

The middle and right squares clearly commute but we still have to show that $j(h) \circ \partial_{\mathcal{E},0} = \partial_{g,0}$. Above we proved that the classifying map of $\mathcal{Z}_g$ equals $\iota_g \circ \omega \circ \lambda_C$ in $[C]$. Now let $\xi : JC \to A$ be the classifying map of $\mathcal{E}$ and consider the following morphism of extensions, where the middle vertical map is $b \mapsto (g(b), b)$:

$$
\begin{array}{ccccccc}
\mathcal{E} : & & A & \longrightarrow & B & \longrightarrow & C \\
& & \downarrow & & \downarrow & & \downarrow \mathrm{id} \\
& & {}_h\downarrow & & \downarrow & & \\
\mathcal{Z}_g : & & (P_g)_0 & \xrightarrow{\mathrm{inc}} & Z_g & \xrightarrow{\varepsilon} & C
\end{array}
$$

By Proposition 3.1.3, the classifying map of $\mathcal{Z}_g$ equals $h \circ \xi$ in $[C]$. Then $\iota_g \circ \omega \circ \lambda_C = h \circ \xi$ in $[C]$, and this is easily seen to imply that $j(h) \circ \partial_{\mathcal{E},0} = \partial_{g,0}$. $\qquad\square$

3.13   Universal property

We recall from [2, Subsection 6.6] the definition of an excisive homology theory with values in a triangulated category.

Definition 3.13.1. Let $(\mathcal{T}, L)$ be a triangulated category. An *excisive homology theory* with values in $\mathcal{T}$ consists of the following data:

  (i) a functor $X : C \to \mathcal{T}$;

 (ii) a morphism $\delta_{\mathcal{E}} \in \mathrm{Hom}_{\mathcal{T}}(LX(C), X(A))$ for every extension $\mathcal{E}$ in C:

$$\mathcal{E} : A \xrightarrow{f} B \xrightarrow{g} C \tag{47}$$

These morphisms $\delta_{\mathcal{E}}$ are subjet to the following conditions:

  (a) For every extension (47), the following triangle is distinguished:

$$\triangle_{\mathcal{E}} : LX(C) \xrightarrow{\delta_{\mathcal{E}}} X(A) \xrightarrow{X(f)} X(B) \xrightarrow{X(g)} X(C)$$

  (b) The triangles $\triangle_{\mathcal{E}}$ are natural with respect to morphisms of extensions.

Example 3.13.2. Let $\mathscr{E} : A \xrightarrow{f} B \xrightarrow{g} C$ be an extension in C. Recall from Proposition 3.12.12 that we have a distinguished triangle in $\mathfrak{K}^C$:

$$\triangle_{\mathscr{E},0} : (C, 1) \xrightarrow{\partial_{\mathscr{E},0}} (A, 0) \xrightarrow{j(f)} (B, 0) \xrightarrow{j(g)} (C, 0)$$

Moreover, it follows from Proposition 3.1.3 that $\partial_{\mathscr{E},0}$ is natural with respect to morphisms of extensions. Then the functor $j : C \to \mathfrak{K}^C$ together with the morphisms $\partial_{\mathscr{E},0}$ is an excisive homology theory.

A *graded category* is a pair $(\mathscr{A}, L)$ where $\mathscr{A}$ is an additive category and L is an automorphism of $\mathscr{A}$. It $(\mathscr{A}, L)$ is a graded category and X is an object of $\mathscr{A}$, we will often write (X, n) instead of $L^n(X)$. A *graded functor* $F : (\mathscr{A}, L) \to (\mathscr{A}', L')$ is an additive functor $F : \mathscr{A} \to \mathscr{A}'$ such that $F \circ L = L' \circ F$. Let $F, G : (\mathscr{A}, L) \to (\mathscr{A}', L')$ be graded functors. A *graded natural transformation* $v : F \to G$ is a natural transformation v such that $L'(v_X) = v_{L(X)} : L'F(X) \to L'G(X)$ for all $X \in \mathscr{A}$.

Example 3.13.3. A triangulated category is a graded category.

Example 3.13.4. Let $(\mathscr{T}, L)$ be a triangulated category. Put $\mathscr{A} := \mathscr{T}^I$ where $I = \{0 \to 1\}$ is the interval category; it is easily verified that $\mathscr{A}$ is an additive category. The translation functor in $\mathscr{T}$ induces a translation functor in $\mathscr{A}$ that makes $\mathscr{A}$ into a graded category.

Example 3.13.5. Let GrAb be the category whose objects are $\mathbb{Z}$-graded abelian groups and whose morphisms graded morphisms of degree zero. Then $\mathscr{A} = $ GrAb is a graded category with the translation functor L defined by $L(M)_n = M_{n+1}$, $n \in \mathbb{Z}$, $M \in$ GrAb.

Definition 3.13.6. Let $(\mathscr{A}, L)$ be a graded category. A *δ-functor* with values in $\mathscr{A}$ consists of the following data:

(i) a functor $X : C \to \mathscr{A}$ that preserves finite products;

(ii) a morphism $\delta_{\mathscr{E}} \in \mathrm{Hom}_{\mathscr{A}}(LX(C), X(A))$ for every extension $\mathscr{E}$ in C:

$$\mathscr{E} : A \longrightarrow B \longrightarrow C$$

These morphisms $\delta_{\mathscr{E}}$ are subject to the following conditions:

(a) $\delta_{\mathscr{E}} : LX(C) \to X(A)$ is an isomorphism if $X(B) = 0$;

(b) The morphisms $\delta_{\mathscr{E}}$ are natural with respect to morphisms of extensions.

Example 3.13.7. An excisive homology theory $X : C \to \mathscr{T}$ is a δ-functor.

Example 3.13.8. Let $X, Y : C \to \mathscr{T}$ be excisive homology theories and let $v : X \to Y$ be a natural transformation such that, for every extension (47), the following diagram commutes:

$$
\begin{array}{ccc}
LX(C) & \xrightarrow{\delta_{\mathscr{E}}^X} & X(A) \\
{\scriptstyle L(v_C)}\downarrow & & \downarrow{\scriptstyle v_A} \\
LY(C) & \xrightarrow{\delta_{\mathscr{E}}^Y} & Y(A)
\end{array}
\tag{48}
$$

Let $\mathscr{A} = \mathscr{T}^I$ be the graded category of Example 3.13.4. Then the natural transformation ν induces a δ-functor $C \to \mathscr{A}$, that we still denote ν. Explicitely, the functor $\nu : C \to \mathscr{A}$ is defined as follows:

$$A \mapsto (\nu_A : X(A) \to Y(A))$$

$$f \in \mathrm{Hom}_C(A, B) \mapsto (X(f), Y(f)) \in \mathrm{Hom}_{\mathscr{A}}(\nu_A, \nu_B)$$

For an extension (47), the morphism $\delta_{\mathscr{E}} \in \mathrm{Hom}_{\mathscr{A}}(L(\nu_C), \nu_A)$ is defined by:

$$\delta_{\mathscr{E}} := (\delta_{\mathscr{E}}^X, \delta_{\mathscr{E}}^Y) \in \mathrm{Hom}_{\mathscr{A}}(L(\nu_C), \nu_A)$$

We want to show that $j : C \to \mathfrak{K}^C$ is the universal excisive and homotopy invariant homology theory, in the sense of [2, Section 6.6]. In order to deal with natural transformations, we will work in the slightly more general setting of δ-functors. From now on, fix a homotopy invariant δ-functor X with values in a graded category $(\mathscr{A}, L)$. A morphism in C will be called an *X-equivalence* if it becomes invertible upon applying X. For example, the following morphisms are X-equivalences:

1) The morphisms $B_r^{\mathfrak{S}_n} \to B_{r+1}^{\mathfrak{S}_n}$ for any $B \in C$, $n \in \mathbb{Z}_{\geq 0}$ and $r \geq 0$. Indeed, this follows by induction on n. For $n = 0$ there is nothing to prove since $B_r^{\mathfrak{S}_0} = B$. Now suppose that we know the result for $n \geq 0$ and consider the following morphism of extensions:

$$
\begin{array}{ccccc}
B_r^{\mathfrak{S}_{n+1}} & \longrightarrow & P(n, B)_r & \longrightarrow & B_r^{\mathfrak{S}_n} \\
\downarrow & & \downarrow & & \downarrow \\
B_{r+1}^{\mathfrak{S}_{n+1}} & \longrightarrow & P(n, B)_{r+1} & \longrightarrow & B_{r+1}^{\mathfrak{S}_n}
\end{array}
$$

Since X is a δ-functor, we have a commutative square as follows:

$$
\begin{array}{ccc}
LX(B_r^{\mathfrak{S}_n}) & \xrightarrow{\delta} & X(B_r^{\mathfrak{S}_{n+1}}) \\
\cong \downarrow & & \downarrow \\
LX(B_{r+1}^{\mathfrak{S}_n}) & \xrightarrow{\delta} & X(B_{r+1}^{\mathfrak{S}_{n+1}})
\end{array}
$$

The morphisms δ are isomorphisms since the middle terms of the extensions are contractible and X is homotopy invariant. This proves the result for $n + 1$.

2) The morphisms $\mu_B^{m,n} : (B_r^{\mathfrak{S}_m})_s^{\mathfrak{S}_n} \to B_{r+s}^{\mathfrak{S}_{m+n}}$ for $B \in C$ and $m, n, r, s \geq 0$. We will prove the assertion in the special case $n = 1$ and $r = s = 0$ since it is the only one we will use below. Consider the following morphism of extensions:

$$
\begin{array}{ccccc}
(B_0^{\mathfrak{S}_m})_0^{\mathfrak{S}_1} & \longrightarrow & P(B_0^{\mathfrak{S}_m})_0 & \longrightarrow & B_0^{\mathfrak{S}_m} \\
{\scriptstyle \mu_B^{m,1}} \downarrow & & \downarrow & & \downarrow {\scriptstyle \mathrm{id}} \\
B_0^{\mathfrak{S}_{m+1}} & \longrightarrow & P(m, B)_0 & \longrightarrow & B_0^{\mathfrak{S}_m}
\end{array}
$$

Since the middle terms of the extensions are contractible and X is a homotopy invariant δ-functor, we have a commutative square as follows:

$$
\begin{array}{ccc}
LX(B_0^{\mathfrak{S}_m}) & \xrightarrow[\cong]{\delta} & X((B_0^{\mathfrak{S}_m})_0^{\mathfrak{S}_1}) \\
{\scriptstyle \mathrm{id}}\Big\downarrow & & \Big\downarrow{\scriptstyle X(\mu_B^{m,1})} \\
LX(B_0^{\mathfrak{S}_m}) & \xrightarrow[\cong]{\delta} & X(B_0^{\mathfrak{S}_{m+1}})
\end{array}
$$

Since X is homotopy invariant, X induces a functor $[C] \to \mathscr{A}$ that we still denote X. Let $[C]_X^{\mathrm{ind}}$ be the full subcategory of $[C]^{\mathrm{ind}}$ whose objects are ind-objects (A, I) such that:

(i) I has an initial object i_0;

(ii) all the transition morphisms $A_i \to A_j$ are X-equivalences.

Notice that, for any $B \in C$, the ind-object $(B_\bullet^{\mathfrak{S}_n}, \mathbb{Z}_{\geq 0})$ is an object of $[C]_X^{\mathrm{ind}}$.

It is easily verified that $[C]$ has finite products and that the functor $C \to [C]$ commutes with finite products. Then $[C]^{\mathrm{ind}}$ has finite products too; explicitly, the product of $(A_\bullet, I)$ and $(B_\bullet, J)$ is the object $(A_\bullet \times B_\bullet, I \times J)$ with the obvious projections. Since X is a δ-functor, we have natural isomorphisms:

$$
X(A_i \times B_j) \cong X(A_i) \oplus X(B_j)
$$

Using this, it is easily seen that the product of two objects of $[C]_X^{\mathrm{ind}}$ is again an object of $[C]_X^{\mathrm{ind}}$. This shows that $[C]_X^{\mathrm{ind}}$ has finite products.

Let $(A, I), (B, J) \in [C]^{\mathrm{ind}}$. A morphism $f \in [(A, I), (B, J)]_C$ is a collection $\{[f_i]\}_{i \in I}$ of homotopy classes of morphisms $f_i : A_i \to B_{\theta(i)}$ subject to certain compatibility relations.

Lemma 3.13.9. *There is a functor $\tilde{X} : [C]_X^{\mathrm{ind}} \to \mathscr{A}$ such that $\tilde{X}(A, I) = X(A_{i_0})$ and such that $\tilde{X}(f)$ is the composite*

$$
X(A_{i_0}) \xrightarrow{X(f_{i_0})} X(B_{\theta(i_0)}) \xleftarrow{\cong} X(B_{j_0})
$$

for any $f \in [(A, I), (B, J)]_C$. Moreover, $\tilde{X}$ preserves finite products.

Proof. It is easily verified that $\tilde{X}$ is indeed a well-defined functor. The fact that $\tilde{X}$ preserves finite products follows from the fact that $X : [C] \to \mathscr{A}$ does. $\square$

Let $B \in C$ and $n \geq 1$. By Remark 2.3.4, $B_\bullet^{\mathfrak{S}_n}$ is a group object in $[C]_X^{\mathrm{ind}}$. It follows from Lemma 3.13.9 that $X(B_0^{\mathfrak{S}_n})$ has a group object structure induced by that of $B_\bullet^{\mathfrak{S}_n}$. Since $\mathscr{A}$ is an additive category, every object of $\mathscr{A}$ is an abelian group object. Thus, $X(B_0^{\mathfrak{S}_n})$ has two group object structures: the one coming from $B_\bullet^{\mathfrak{S}_n}$ and the other from being an object of $\mathscr{A}$. By the Eckmann-Hilton argument, both group structures coincide and the function

$$
\tilde{X} : [A_\bullet, B_\bullet^{\mathfrak{S}_n}]_C \longrightarrow \mathrm{Hom}_{\mathscr{A}}(X(A_{i_0}), X(B_0^{\mathfrak{S}_n}))
$$

is a group homomorphism for every (A, I) in $[C]_X^{\mathrm{ind}}$.

Let $A \in C$ and let $\mathcal{U}_A$ be the universal extension of A. Since TA is contractible, there is an isomorphism:

$$\delta_{\mathcal{U}_A} : (X(A), 1) \xrightarrow{\;\cong\;} (X(JA), 0)$$

Put $i_A^{J,1} := \delta_{\mathcal{U}_A}$ and define inductively $i_A^{J,n+1}$ as the composite:

$$(X(A), n+1) \xrightarrow{\;L(i_A^{J,n})\;} (X(J^n A), 1) \xrightarrow{\;i_{J^n A}^{J,1}\;} (X(J^{n+1}A), 0)$$

Let $i_A^{J,0}$ be the identity of $(X(A), 0)$. It is easily verified by induction on $n = p + q$ that the following equality holds for $p, q \geq 0$:

$$i_A^{J,p+q} = i_{J^p A}^{J,q} \circ L^q(i_A^{J,p})$$

The morphisms $i_?^{J,n}$ assemble into a natural isomorphism $L^n \circ X \cong X \circ J^n(?) : C \to \mathscr{A}$.

Let $A \in C$ and let $\mathscr{P}_{0,A}$ be the path extension of A. Since $(PA)_0$ is contractible, there is an isomorphism:

$$\delta_{\mathscr{P}_A} : (X(A), 1) \xrightarrow{\;\cong\;} (X(A_0^{\mathfrak{S}_1}), 0)$$

Put $i_A^{\mathfrak{S},1} := \delta_{\mathscr{P}_A}$ and define inductively $i_A^{\mathfrak{S},n+1}$ as the composite:

$$(X(A), n+1) \xrightarrow{\;L(i_A^{\mathfrak{S},n})\;} (X(A_0^{\mathfrak{S}_n}), 1) \xrightarrow{\;i_{A_0^{\mathfrak{S}_n}}^{\mathfrak{S},1}\;} (X((A_0^{\mathfrak{S}_n})_0^{\mathfrak{S}_1}), 0) \xrightarrow[\cong]{\;X(\mu_A^{n,1})\;} (X(A_0^{\mathfrak{S}_{n+1}}), 0)$$

Let $i_A^{\mathfrak{S},0}$ be the identity of $(X(A), 0)$. It is easily verified by induction on $n = p + q$ that the following equality holds for $p, q \geq 0$:

$$i_A^{\mathfrak{S},p+q} = X(\mu_A^{p,q}) \circ i_{A_0^{\mathfrak{S}_p}}^{\mathfrak{S},q} \circ L^q(i_A^{\mathfrak{S},p})$$

The morphisms $i_?^{\mathfrak{S},n}$ assemble into a natural isomorphism $L^n \circ X \cong X \circ (?)_0^{\mathfrak{S}_n} : C \to \mathscr{A}$.

Lemma 3.13.10. *Let $N_2, N_3 \geq 0$ and let $B \in C$. Then the following diagram in $\mathscr{A}$ commutes up to the sign $(-1)^{N_2 N_3}$:*

$$
\begin{array}{ccc}
(X(B), N_2 + N_3) & \xrightarrow{L^{N_3}(i_B^{\mathfrak{S},N_2})} (X(B_0^{\mathfrak{S}_{N_2}}), N_3) \xrightarrow{i_{B_0^{\mathfrak{S}_{N_2}}}^{J,N_3}} & (X(J^{N_3}(B_0^{\mathfrak{S}_{N_2}})), 0) \\
\downarrow{\scriptstyle L^{N_2}(i_B^{J,N_3})} & & \downarrow{\scriptstyle X(\kappa_B^{N_3,N_2})} \\
(X(J^{N_3}B), N_2) & \xrightarrow{\quad i_{J^{N_3}B}^{\mathfrak{S},N_2} \quad} & (X((J^{N_3}B)_0^{\mathfrak{S}_{N_2}}), 0)
\end{array}
$$

Proof. If $N_2 = 0$ or $N_3 = 0$ there is nothing to prove. Upon applying the functor $\tilde{X}$ to the diagram (18) we get that the followig diagram in $\mathscr{A}$ commutes up to the sign -1; the case $N_2 = N_3 = 1$ follows easily from this:

$$
\begin{array}{ccc}
X(J^2 B) & \xrightarrow{X(J(\lambda_B))} & X(J(B_0^{\mathfrak{S}_1})) \\
& \searrow{\scriptstyle X(\lambda_{JB})} & \downarrow{\scriptstyle X(\kappa_B^{1,1})} \\
& & X((JB)_0^{\mathfrak{S}_1})
\end{array}
$$

Once we know the case $N_2 = N_3 = 1$, the case $N_3 = 1$ with arbitrary N_2 follows by an easy induction on N_2. Once we know the result for $N_3 = 1$ and arbitrary N_2, the general case follows by an easy induction on N_3. $\qquad\square$

Theorem 3.13.11. *Let $(\mathscr{A}, L)$ be a graded category and let $X : C \to \mathscr{A}$ be a homotopy invariant δ-functor. Then there exists a unique graded functor $\bar{X} : \mathfrak{K}^C \to \mathscr{A}$ such that $\bar{X}(\partial_{\mathscr{E},0}) = \delta_{\mathscr{E}}$ for every extension $\mathscr{E}$ and such that the following diagram commutes:*

$$
\begin{array}{ccc}
C & \xrightarrow{\;\;j\;\;} & \mathfrak{K}^C \\
& \searrow{\scriptstyle X} & \big\downarrow{\scriptstyle \exists!\bar{X}} \\
& & \mathscr{A}
\end{array}
$$

Proof. Define $\bar{X}$ on objects by $\bar{X}(A, m) := (X(A), m)$. To define $\bar{X}$ on morphisms we must define, for every pair of objects of $\mathfrak{K}$, a group homomorphism:

$$
\bar{X}_{(A,m),(B,n)} : \; \mathfrak{K}((A, m), (B, n)) \longrightarrow \mathrm{Hom}_{\mathscr{A}}((X(A), m), (X(B), n)) \tag{49}
$$

Let $\bar{X}^v$ be the dotted composite:

$$
\begin{array}{ccc}
[J^{m+v}A, B_\bullet^{\mathfrak{S}_{n+v}}] & \xrightarrow{\;\tilde{X}\;} & \mathrm{Hom}_{\mathscr{A}}((X(J^{m+v}A), 0), (X(B_0^{\mathfrak{S}_{n+v}}), 0)) \\
& & \Big\downarrow{\scriptstyle \cong}\; {\scriptstyle (i_B^{\mathfrak{S},n+v})^{-1}\circ(?)\circ i_A^{J,m+v}} \\
& & \mathrm{Hom}_{\mathscr{A}}((X(A), m+v), (X(B), n+v)) \\
& & \Big\downarrow{\scriptstyle \cong}\; {\scriptstyle L^{-v}} \\
& \cdots\cdots\!\!\!\longrightarrow & \mathrm{Hom}_{\mathscr{A}}((X(A), m), (X(B), n))
\end{array}
$$

The function $\bar{X}^v$ is a group homomorphism by the discussion following Lemma 3.13.9. Moreover, it is easily verified that this diagram commutes:

$$
\begin{array}{ccc}
[J^{m+v}A, B_\bullet^{\mathfrak{S}_{n+v}}] & \xrightarrow{\;\bar{X}^v\;} & \mathrm{Hom}_{\mathscr{A}}((X(A), m), (X(B), n)) \\
\Big\downarrow{\scriptstyle \Lambda^{n+v}} & \nearrow{\scriptstyle \bar{X}^{v+1}} & \\
[J^{m+v+1}A, B_\bullet^{\mathfrak{S}_{n+v+1}}] & &
\end{array}
$$

Thus, the morphisms $\bar{X}^v$ induce the desired group homomorphism $\bar{X}_{(A,m),(B,n)}$ in (49); this defines $\bar{X}$ on morphisms. It is straightforward but tedious to verify that the definitions above indeed give rise to an additive functor $\bar{X} : \mathfrak{K}^C \to \mathscr{A}$. When verifying that $\tilde{X}$ preserves composition, Lemma 3.13.10 is needed to show that the signs in Definition 3.6.4 work out. We clearly have $X = \bar{X} \circ j$ and $L \circ \bar{X} = \bar{X} \circ L$.

Let us now show that $\bar{X}(\partial_{\mathscr{E},0}) = \delta_{\mathscr{E}}$ for every extension $\mathscr{E}$ in C. Consider an extension as follows, with classifying map $\xi : JC \to A$:

$$
\mathscr{E} : \; A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C
$$

Recall from Definition (3.12.11) that $\partial_{\mathscr{E},0}$ equals the composite:

$$(C,1) \xrightarrow{[\mathrm{id}_{JC}]} (JC,0) \xrightarrow{j(\xi)} (A,0)$$

Upon applying $\bar{X}$ we get:

$$(X(C),1) \xrightarrow{i_C^{J,1}} (X(JC),0) \xrightarrow{X(\xi)} (X(A),0)$$

By naturality of δ, we have:

$$\bar{X}(\partial_{\mathscr{E},0}) = X(\xi) \circ i_C^{J,1} = X(\xi) \circ \delta_{\mathscr{U}_C} = \delta_{\mathscr{E}}$$

It remains to check the uniqueness of $\bar{X}$. Let $\bar{X} : \mathfrak{K}^C \to \mathscr{A}$ be any graded functor with the properties described in the statement of this theorem. Let $\alpha \in \mathfrak{K}((A,m),(B,n))$ be represented by $f : J^{m+v}A \to B_r^{\mathfrak{S}_{n+v}}$ and let $\gamma^r : B_0^{\mathfrak{S}_{n+v}} \to B_r^{\mathfrak{S}_{n+v}}$ be the morphism induced by the iterated last vertex map. By Lemma 3.10.9, the following diagram in $\mathfrak{K}^C$ commutes:

$$
\begin{array}{ccc}
(A,m) & \xrightarrow{\quad\alpha\quad} & (B,n) \\
{\scriptstyle\cong}\downarrow & & \uparrow{\scriptstyle\cong} \quad\nwarrow{\scriptstyle\cong} \\
(J^{m+v}A,-v) & \xrightarrow{L^{-v}j(f)} (B_r^{\mathfrak{S}_{n+v}},-v) & \xleftarrow[\cong]{L^{-v}j(\gamma^r)} (B_0^{\mathfrak{S}_{n+v}},-v)
\end{array}
$$

Upon applying $\bar{X}$ we get the following commutative diagram in $\mathscr{A}$:

$$
\begin{array}{ccc}
(X(A),m) & \xrightarrow{\quad X(\alpha)\quad} & (X(B),n) \\
{\scriptstyle L^{-v}(i_A^{J,m+v})}\downarrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong}\quad \nwarrow{\scriptstyle L^{-v}(i_B^{\mathfrak{S},n+v})} \\
(X(J^{m+v}A),-v) & \xrightarrow{L^{-v}X(f)} (X(B_r^{\mathfrak{S}_{n+v}}),-v) & \xleftarrow[\cong]{L^{-v}X(\gamma^r)} (X(B_0^{\mathfrak{S}_{n+v}}),-v)
\end{array}
$$

It follows that $\bar{X}$ is the functor defined above.   □

As a corollary we get:

Theorem 3.13.12. *Let $(\mathscr{T},L)$ be a triangulated category and let $X : C \to \mathscr{T}$ be an excisive and homotopy invariant homology theory. Then there exists a unique triangulated functor $\bar{X} : \mathfrak{K}^C \to \mathscr{T}$ such that $\bar{X}(\partial_{\mathscr{E}}) = \delta_{\mathscr{E}}$ for every extension $\mathscr{E}$, and such that the following diagram commutes:*

$$
\begin{array}{ccc}
C & \xrightarrow{\quad j\quad} & \mathfrak{K}^C \\
 & \searrow{\scriptstyle X} & \downarrow{\scriptstyle \exists! \bar{X}} \\
 & & \mathscr{T}
\end{array}
$$

Proof. By Theorem 3.13.11, there exists a unique graded functor $\bar{X}$ making the diagram commute and such that $\bar{X}(\partial_{\mathscr{E},0}) = \delta_{\mathscr{E}}$ for every extension $\mathscr{E}$ in C. It remains to check that $\bar{X}$ sends distinguished triangles in $\mathfrak{K}$ to distinguished triangles in $\mathscr{T}$. By Proposition 3.12.12 it suffices to show that $\bar{X}$ sends extension triangles $\triangle_{\mathscr{E},0}$ to distinguished triangles in $\mathscr{T}$, but this follows immediately from the fact that $\bar{X}(\partial_{\mathscr{E},0}) = \delta_{\mathscr{E}}$.   □

Remark 3.13.13. One way to summarize Theorem 3.13.12 is to say that $j : C \to \mathfrak{K}^C$ is the universal excisive and homotopy invariant homology theory with values in a triangulated category. Such a universal homology theory was already constructed by Garkusha in [6, Theorem 2.6 (2)] using completely different methods. Both constructions are, of course, naturally isomorphic since they satisfy the same universal property.

Theorem 3.13.14. *Let $F : C \to C$ be a functor satisfying the following properties:*

1) *F preserves homotopic morphisms;*

2) *F preserves extensions.*

Then there exists a unique triangulated functor $\bar{F}$ making the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\ j\ } & \mathfrak{K}^C \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \exists!\bar{F}} \\
C & \xrightarrow[\ j\ ]{} & \mathfrak{K}^C
\end{array}
$$

Let $F_1, F_2 : C \to C$ be two morphisms with the properties above and let $\eta : F_1 \to F_2$ be a natural transformation. Then there exists a unique (graded) natural transformation $\bar{\eta} : \bar{F}_1 \to \bar{F}_2$ such that $\bar{\eta}_{j(A)} = j(\eta_A)$ for all $A \in C$.

Proof. The existence and uniqueness of $\bar{F}$ follow from Theorem 3.13.12 once we notice that $j \circ F : C \to \mathfrak{K}$ is an excisive and homotopy invariant homology theory. Let us show now the existence of $\bar{\eta}$. For every $A \in C$ put:

$$
\nu_A := j(\eta_A) \in \mathfrak{K}(F_1(A), F_2(A))
$$

The ν_A assemble into a natural transformation $\nu : j \circ F_1 \to j \circ F_2 : C \to \mathfrak{K}$. Let $\mathscr{A} := (\mathfrak{K}^C)^I$ where I is the interval category. Recall from Example 3.13.8 that ν induces a homotopy invariant δ-functor $C \to \mathscr{A}$ if we show that the diagram (48) commutes. Let

$$
\mathscr{E} : \quad A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C
$$

be an extension in C. Then we have a morphism of extensions in C:

$$
\begin{array}{ccc}
F_1(A) & \xrightarrow{F_1(f)} F_1(B) & \xrightarrow{F_1(g)} F_1(C) \\
{\scriptstyle \eta_A}\downarrow & \quad\downarrow{\scriptstyle \eta_B} & \quad\downarrow{\scriptstyle \eta_C} \\
F_2(A) & \xrightarrow{F_2(f)} F_2(B) & \xrightarrow{F_2(g)} F_2(C)
\end{array}
$$

Since j sends extensions to triangles in a natural way, the following diagram in $\mathfrak{K}^C$ commutes:

$$
\begin{array}{ccc}
(F_1(C), 1) & \xrightarrow{\delta^{j \circ F_1}} & (F_1(A), 0) \\
{\scriptstyle L(\nu_C)}\downarrow & & \downarrow{\scriptstyle \nu_A} \\
(F_2(C), 1) & \xrightarrow{\delta^{j \circ F_2}} & (F_2(A), 0)
\end{array}
$$

Thus, ν induces a homotopy invariant δ-functor $C \to \mathscr{A}$, which in turn induces a graded functor $\bar{\nu} : \mathfrak{K}^C \to \mathscr{A}$ by Theorem 3.13.11. It is easily verified that this graded functor $\bar{\nu}$ corresponds to the desired natural transformation $\bar{\eta} : \bar{F}_1 \to \bar{F}_2$. $\qquad\square$

Remark 3.13.15. Let $F, F', F'' : C \to C$ be functors satisfying the hypothesis of Theorem 3.13.14 and let $\eta : F \to F'$ and $\eta' : F' \to F''$ be natural transformations. Then $\overline{\eta' \circ \eta} = \bar{\eta}' \circ \bar{\eta}$.

Chapter 4

Bivariant K-theory spaces

Resumen del capítulo

Para cada par de álgebras A y B, Garkusha [5] construyó un espectro $\mathbb{K}(A, B)$ que representa a la teoría de homología universal, escisiva e invariante por homotopía [5, Comparison Theorem B]. En las secciones 4.2 y 4.3 recordamos la definición de $\mathbb{K}(A, B)$ y probamos un resultado análogo el teorema de representabilidad de Garkusha, usando la categoría $\mathfrak{K}^C$ definida en el capítulo 3. Más precisamente, en el Teorema 4.3.3, mostramos que hay un isomorfismo natural:

$$\pi_n \mathbb{K}(A, B) \cong \mathfrak{K}^C(A, (B, n))$$

Este resultado se deduce fácilmente del Teorema 2.3.3. Sea X un conjunto simplicial finito. En la sección 4.4 mostramos que el n-ésimo grupo de homología y el n-ésimo grupo de cohomología de X con coeficientes en $\mathbb{K}(A, B)$ [12] están dados, respectivamente, por $\mathfrak{K}^C(A^X, (B, n))$ y por $\mathfrak{K}^C(A, (B^X, -n))$. Más precisamente, en las proposiciones 4.4.1 y 4.4.5, construímos equivalencias débiles de espectros:

$$\widetilde{\mathbb{K}}(A, B) \wedge X_+ \xrightarrow{\ \sim\ } \mathbb{K}(A^X, B)$$

$$\mathbb{K}(A, B^X) \xrightarrow{\ \sim\ } \mathrm{Map}(X, \mathbb{K}(A, B))$$

Aquí, $\widetilde{\mathbb{K}}(A, B)$ es un reemplazo cofibrante de $\mathbb{K}(A, B)$ en la categoría de modelos estable. En el Corolario 4.4.2 probamos que una equivalencia débil $f : X \to Y$ entre conjuntos simpliciales finitos induce una $\mathfrak{K}$-equivalencia $f^* : A^Y \to A^X$. En el Corolario 4.4.3 mostramos que, para conjuntos simpliciales finitos X e Y, el morfismo de multiplicación $\mu^{X,Y} : (A^X)^Y \to A^{X \times Y}$ definido en la sección 2.2 es una $\mathfrak{K}$-equivalencia.

Chapter summary

For any pair of algebras (A, B), Garkusha [5] constructed a spectrum $\mathbb{K}(A, B)$ that represents the universal excisive and homotopy invariant homology theory [5, Comparison

Theorem B]. In sections 4.2 and 4.3 we recall the definition of $\mathbb{K}(A, B)$ and prove an analog of Garkusha's representability theorem, using the category $\mathfrak{K}^C$ defined in Chapter 3. More precisely, in Theorem 4.3.3, we show that there is a natural isomorphism:

$$\pi_n \mathbb{K}(A, B) \cong \mathfrak{K}^C(A, (B, n))$$

This result follows easily from Theorem 2.3.3. In section 4.4 we show that, for a finite simplicial set X, the n-th homology and n-th cohomology groups of X with coefficients in $\mathbb{K}(A, B)$ [12] equal $\mathfrak{K}^C(A^X, (B, n))$ and $\mathfrak{K}^C(A, (B^X, -n))$ respectively. More precisely, in Propositions 4.4.1 and 4.4.5, we construct weak equivalences of spectra:

$$\widetilde{\mathbb{K}}(A, B) \wedge X_+ \overset{\sim}{\longrightarrow} \mathbb{K}(A^X, B)$$

$$\mathbb{K}(A, B^X) \overset{\sim}{\longrightarrow} \mathrm{Map}(X, \mathbb{K}(A, B))$$

Here, $\widetilde{\mathbb{K}}(A, B)$ is a cofibrant replacement of $\mathbb{K}(A, B)$ in the stable model category of spectra. In Corollary 4.4.2 we prove that a weak equivalence $f : X \to Y$ of finite simplicial sets induces a $\mathfrak{K}$-equivalence $f^* : A^Y \to A^X$. In Corollary 4.4.3 we show that, for finite simplicial sets X and Y, the multiplication morphism $\mu^{X,Y} : (A^X)^Y \to A^{X \times Y}$ defined in section 2.2 is a $\mathfrak{K}$-equivalence.

4.1   Path extensions revisited

Let $B \in C$ and let $n, q \geq 0$. We will define a class of extensions $\mathscr{P}^q_{n,B}$ that generalize the path extensions $\mathscr{P}_{n,B}$ defined in Section 3.2. Put:

$$P(n, B)^q_\bullet := B_\bullet^{(I^{n+1} \times \Delta^q, (\partial I^n \times I \times \Delta^q) \cup (I^n \times \{1\} \times \Delta^q))}$$

On the one hand, the composite $I^n \times \Delta^q \cong I^n \times \{0\} \times \Delta^q \subseteq I^{n+1} \times \Delta^q$ induces a morphism of $\mathbb{Z}_{\geq 0}$-diagrams in C:

$$P(n, B)^q_\bullet \overset{p^q_{n,B}}{\longrightarrow} B_\bullet^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)}$$

On the other hand, we have inclusions $B_r^{(I^{n+1} \times \Delta^q, \partial I^{n+1} \times \Delta^q)} \subseteq P(n, B)^q_r$. Proceeding in analogy to what was done for (4) on page 36, one can show that there is an extension:

$$\mathscr{P}^q_{n,B} : B_r^{(I^{n+1} \times \Delta^q, \partial I^{n+1} \times \Delta^q)} \longrightarrow P(n, B)^q_r \overset{p^q_{n,B}}{\longrightarrow} B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)}$$

A splitting of $p_{n,B}^q$ in $\mathcal{U}(C)$ can be constructed as follows: Recall that we have an element $t_0 \in \mathbb{Z}_0^{(I,\{1\})}$. Let $s_{n,B}^q$ be the composite:

$$
\begin{array}{ccc}
B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)} & \xrightarrow{? \otimes t_0} & B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)} \otimes \mathbb{Z}_0^{(I,\{0\})} \\
 & & \downarrow \cong \\
 & & \left(B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)} \right)_0^{(I,\{0\})} \\
 & & \downarrow \mu \\
s_{n,B}^q & & B_r^{(I^n \times \Delta^q \times I, (I^n \times \Delta^q \times \{0\}) \cup (\partial I^n \times \Delta^q \times I))} \\
 & & \downarrow \cong \\
 & & B_r^{(I^n \times I \times \Delta^q, (I^n \times \{0\} \times \Delta^q) \cup (\partial I^n \times I \times \Delta^q))}
\end{array}
$$

It is straightforward to check that $s_{n,B}^q$ is a section to $p_{n,B}^q$ in $\mathcal{U}(C)$.

Remark 4.1.1. It is easily verified that the extensions $(\mathcal{P}_{n,B}^q, s_{n,B}^q)$ are:

 (i) natural in B with respect to morphisms in C;

 (ii) natural in r with respect to the last vertex map;

 (iii) and natural in q with respect to morphisms of ordinal numbers.

4.2   Bivariant K-theory space

Let $A, B \in C$ and let $n \geq 0$. From the proof of Theorem 2.3.3, it follows that there is a natural bijection:

$$
\left(\Omega^n Ex^\infty \mathrm{Hom}_C(J^n A, B^\Delta) \right)_q \cong \operatorname{colim}_r \mathrm{Hom}_C \left(J^n A, B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)} \right)
$$

Let $f \in \mathrm{Hom}_C(J^n A, B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)})$ and define $\zeta^n(f)$ as the classifying map of f with respect to the extension $\mathcal{P}_{n,B}^q$:

$$
\begin{array}{ccccc}
\mathcal{U}_{J^n A} & J^{n+1}A & \longrightarrow & T J^n A & \longrightarrow & J^n A \\
\exists! \downarrow & \zeta^n(f) \downarrow & & \downarrow & & \downarrow f \\
\mathcal{P}_{n,B}^q & B_r^{(I^{n+1} \times \Delta^q, \partial I^{n+1} \times \Delta^q)} & \longrightarrow & P(n,B)_r^q & \longrightarrow & B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)}
\end{array}
$$

It follows from Remark 4.1.1 that this defines a morphism of simplicial sets:

$$
\zeta^n : \Omega^n Ex^\infty \mathrm{Hom}_C(J^n A, B^\Delta) \longrightarrow \Omega^{n+1} Ex^\infty \mathrm{Hom}_C(J^{n+1}A, B^\Delta) \tag{1}
$$

Definition 4.2.1 (Garkusha). Let $A, B \in C$. The *bivariant K-theory space* of the pair (A, B) is the simplicial set defined by:

$$
\mathcal{K}^C(A, B) := \operatorname{colim}_n \Omega^n Ex^\infty \mathrm{Hom}_C(J^n A, B^\Delta)
$$

Here, the transition morphisms are the ζ^n defined in (1).

Note that $\mathscr{K}^C(A, B)$ is a fibrant simplicial set, since it is a filtering colimit of fibrant simplicial sets. This definition of $\mathscr{K}^C(A, B)$ is easily seen to be the same as the one given in [5, Section 4]. We will often drop C from the notation and write $\mathscr{K}$ instead of $\mathscr{K}^C$.

Theorem 4.2.2 (cf. [5, Corollary 7.1, Comparison Theorem B]). *Let $n \geq 0$. Then there is a natural isomorphism:*
$$\pi_n \mathscr{K}^C(A, B) \cong \mathfrak{K}^C(A, (B, n))$$

Proof. Since $\pi_n \cong \pi_0 \Omega^n$ commutes with filtered colimits, we have:

$$\pi_n \mathscr{K}(A, B) \cong \operatorname*{colim}_v \pi_0 \Omega^n \Omega^v Ex^\infty \operatorname{Hom}_C(J^v A, B^\Delta)$$
$$\cong \operatorname*{colim}_v \pi_0 \Omega^{v+n} Ex^\infty \operatorname{Hom}_C(J^v A, B^\Delta)$$
$$\cong \operatorname*{colim}_v [J^v A, B_\bullet^{\mathfrak{S}_{v+n}}]_C \qquad\qquad \text{(by Theorem 2.3.3)}$$

Notice that $\Omega^n \Omega^v \cong \Omega^{v+n}$ because of our conventions on iterated loop spaces; see section 1.3.3. Recall from Definition 3.6.4 that

$$\mathfrak{K}(A, (B, n)) = \operatorname*{colim}_v [J^v A, B_\bullet^{\mathfrak{S}_{n+v}}]_C$$

where the transition functions are the Λ^{n+v} of Lemma 3.6.3. Thus, we need to compare Λ^{n+v} with:

$$\pi_n \zeta^v : [J^v A, B_\bullet^{\mathfrak{S}_{n+v}}]_C \longrightarrow [J^{v+1} A, B_\bullet^{\mathfrak{S}_{n+v+1}}]_C$$

Let $c_{v,n} : I^v \times I^n \xrightarrow{\cong} I^n \times I^v$ be the commutativity isomorphism; $c_{v,n}$ induces an isomorphism $(c_{v,n})^* : B_r^{\mathfrak{S}_{n+v}} \to B_r^{\mathfrak{S}_{v+n}}$. It is straightforward to verify that the following square commutes for all v:

$$
\begin{array}{ccc}
[J^v A, B_\bullet^{\mathfrak{S}_{v+n}}]_C & \xrightarrow{\pi_n \zeta^v} & [J^{v+1} A, B_\bullet^{\mathfrak{S}_{v+1+n}}]_C \\
{\scriptstyle (c_{v,n})^*} \uparrow {\scriptstyle \cong} & & {\scriptstyle \cong} \uparrow {\scriptstyle (c_{v+1,n})^*} \\
[J^v A, B_\bullet^{\mathfrak{S}_{n+v}}]_C & \xrightarrow{\Lambda^{n+v}} & [J^{v+1} A, B_\bullet^{\mathfrak{S}_{n+v+1}}]_C
\end{array}
$$

These squares assemble into a morphism of diagrams that, upon taking colimit in v, induces an isomorphism:

$$\pi_n \mathscr{K}(A, B) \cong \mathfrak{K}(A, (B.n))$$

This finishes the proof. $\square$

4.3   Bivariant K-theory spectrum

The following result is [5, Theorem 5.1]; we sketch its proof here for future reference.

Theorem 4.3.1 (Garkusha). *There is a natural isomorphism of simplicial sets:*

$$\mathscr{K}^C(A, B) \cong \Omega \mathscr{K}^C(JA, B)$$

Proof. Let $c_{1,v} : I \times I^v \overset{\cong}{\to} I^v \times I$ be the commutativity isomorphism. For a simplicial set X, we have an induced isomorphism $(c_{1,v})^* : \Omega^{v+1}X \to \Omega^{1+v}X$. There is a commutative square:

$$
\begin{array}{ccc}
\Omega^{v+1}Ex^\infty \mathrm{Hom}_C(J^{1+v}A, B^\Delta) & \xrightarrow{\Omega\zeta^v} & \Omega^{v+2}Ex^\infty \mathrm{Hom}_C(J^{1+v+1}A, B^\Delta) \\
{\scriptstyle (c_{1,v})^*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle (c_{1,v+1})^*} \\
\Omega^{1+v}Ex^\infty \mathrm{Hom}_C(J^{1+v}A, B^\Delta) & \xrightarrow{\zeta^{1+v}} & \Omega^{1+v+1}Ex^\infty \mathrm{Hom}_C(J^{1+v+1}A, B^\Delta)
\end{array}
$$

The colimit of the top horizontal morphisms is $\Omega\mathcal{K}(JA, B)$ and the colimit of the bottom ones is $\mathcal{K}(A, B)$. The result follows. $\qquad\square$

Let $S^1 := \Delta^1/\partial\Delta^1$. A *spectrum* is a collection $\{X^n\}_{n \geq 0}$ of pointed simplicial sets, together with pointed morphisms $S^1 \wedge X^n \to X^{n+1}$, which we call *bonding maps*. A *morphism of spectra* $f : X \to Y$ is a sequence of pointed morphisms $f : X^n \to Y^n$ commuting with the bonding maps. Write Spt for the category of spectra and morphisms of spectra. We will consider in Spt the stable model structure, which is proper and simplicial; see, for example, [7, Theorem 4.15]. Let X be a spectrum and let $m \in \mathbb{Z}$. The m-th homotopy group of X is defined as the following colimit:

$$
\pi_m X := \operatorname*{colim}_v \pi_{m+v} X^v
$$

A morphism of spectra is a stable equivalence iff it induces an isomorphism upon taking π_m, for all m. A spectrum X is stably fibrant iff all the X^n are fibrant simplicial sets and all the adjoints $X^n \to \Omega X^{n+1}$ of the bonding maps are weak equivalences of simplicial sets.

Definition 4.3.2 (Garkusha). Let $A, B \in C$. The *bivariant K-theory spectrum* $\mathbb{K}^C(A, B)$ consists of the spaces

$$
\mathcal{K}^C(A, B), \mathcal{K}^C(JA, B), \mathcal{K}^C(J^2A, B), \ldots
$$

together with the bonding maps that are adjoint to the isomorphisms of Theorem 4.3.1.

Notice that $\mathbb{K}^C(A, B)$ is a fibrant spectrum. As usual, we may drop C from the notation.

Theorem 4.3.3 (cf. [5, Corollary 7.1, Comparison Theorem B]). *For every $n \in \mathbb{Z}$ there is a natural isomorphism:*
$$
\pi_n\mathbb{K}^C(A, B) \cong \mathfrak{K}^C(A, (B, n))
$$

Proof. For $n \geq 0$ this follows from Theorem 4.2.2 since $\pi_n\mathbb{K}(A, B) \cong \pi_n\mathcal{K}(A, B)$. For $n \leq 0$ we have:

$$
\begin{aligned}
\pi_n\mathbb{K}(A, B) &\cong \pi_0\mathcal{K}(J^{-n}A, B) \\
&\cong \operatorname*{colim}_v \pi_0\Omega^v Ex^\infty \mathrm{Hom}_C(J^{-n+v}A, B^\Delta) \\
&\cong \operatorname*{colim}_v [J^{-n+v}A, B_\bullet^{\mathfrak{S}_v}]_C \\
&\cong \mathfrak{K}((A, -n), (B, 0)) \\
&\cong \mathfrak{K}((A, 0), (B, n)) \qquad\qquad\square
\end{aligned}
$$

Remark 4.3.4. Let $X = \Omega^n Ex^\infty \mathrm{Hom}(A, B^\Delta)$. Let $\epsilon_X : S^1 \wedge \Omega X \to X$ be the counit of the adjunction between the suspension and loop functors. We want an explicit description of ϵ_X. More precisely, we have:

$$(\Omega X)_q \cong \operatorname*{colim}_r \mathrm{Hom}(A, B_r^{(I^{n+1} \times \Delta^q, \partial I^{n+1} \times \Delta^q)})$$
$$X_q \cong \operatorname*{colim}_r \mathrm{Hom}(A, B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)}) \tag{2}$$

We want a description of ϵ_X in terms of the right-hand side of (2). First of all, for any pointed simplicial set Y, the projection $\Delta^1 \to S^1$ induces an isomorphism:

$$\frac{\Delta^1 \times Y}{(\partial \Delta^1 \times Y) \cup (\Delta^1 \times \{*\})} \xrightarrow{\cong} S^1 \wedge Y$$

Hence, in order to describe ϵ_X it suffices to give a morphism $\Delta^1 \times \Omega X \to X$. Let $[f]$ denote the class of $f \in \mathrm{Hom}(A, B_r^{(I^{n+1} \times \Delta^q, \partial I^{n+1} \times \Delta^q)})$ in $(\Omega X)_q$. Let $x : \Delta^q \to \Delta^1$ be a q-simplex of Δ^1. It is easily verified that $\epsilon_X(x, [f])$ equals the class of the composite

$$A \xrightarrow{\ f\ } B_r^{(I^{n+1} \times \Delta^q, \partial I^{n+1} \times \Delta^q)} \xrightarrow{\ \star\ } B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)}$$

where the morphism $\star$ is induced by the following morphisms of simplicial sets:

$$I^n \times \Delta^q \xrightarrow{I^n \times \mathrm{diag}} I^n \times \Delta^q \times \Delta^q \xrightarrow{I^n \times x \times \Delta^q} I^n \times I \times \Delta^q$$

Example 4.3.5. Theorem 4.3.1 states the existence of isomorphisms

$$\mathcal{K}(A, B) \xrightarrow{\cong} \Omega \mathcal{K}(JA, B) \tag{3}$$

that are then used to construct the bonding maps of the spectrum $\mathbb{K}(A, B)$. It will be useful to have an explicit description of the adjoints of these morphisms:

$$S^1 \wedge \mathcal{K}(A, B) \longrightarrow \mathcal{K}(JA, B) \tag{4}$$

Recall that:

$$(\Omega^{1+n} Ex^\infty \mathrm{Hom}(J^{1+n}A, B^\Delta))_q \cong \operatorname*{colim}_r \mathrm{Hom}(J^{1+n}A, B_r^{(I^{1+n} \times \Delta^q, \partial I^{1+n} \times \Delta^q)}) \tag{5}$$

For every $n \geq 0$, we proceed to define a function:

$$S_n : (\Delta^1 \times \Omega^{1+n} Ex^\infty \mathrm{Hom}(J^{1+n}A, B^\Delta))_q \longrightarrow (\Omega^n Ex^\infty \mathrm{Hom}(J^{1+n}A, B^\Delta))_q$$

Let $[f]$ denote the class of $f \in \mathrm{Hom}(J^{1+n}A, B_r^{(I^{1+n} \times \Delta^q, \partial I^{1+n} \times \Delta^q)})$ in (5) and let $t : \Delta^q \to \Delta^1$ be a q-simplex of Δ^1. Define $S_n(t, [f])$ as the class of the composite

$$J^{1+n}A \xrightarrow{\ f\ } B_r^{(I^{1+n} \times \Delta^q, \partial I^{1+n} \times \Delta^q)} \xrightarrow{\ \star\ } B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)}$$

where the morphism $\star$ is induced by the following morphisms of simplicial sets:

$$I^n \times \Delta^q \xrightarrow{I^n \times \mathrm{diag}} I^n \times \Delta^q \times \Delta^q \cong \Delta^q \times I^n \times \Delta^q \xrightarrow{t \times I^n \times \Delta^q} I^{1+n} \times \Delta^q$$

Clearly, the S_n assemble into a morphism of simplicial sets:

$$S_n : \Delta^1 \times \Omega^{1+n} Ex^\infty \mathrm{Hom}(J^{1+n}A, B^\Delta) \longrightarrow \Omega^n Ex^\infty \mathrm{Hom}(J^{1+n}A, B^\Delta)$$

It is straightforward but tedious to verify that these morphisms are compatible with (1). Hence, upon taking colimit in n, we get a morphism:

$$S : \Delta^1 \times \mathscr{K}(A, B) \longrightarrow \mathscr{K}(JA, B)$$

We claim that the morphism above induces (4). Again, the verification of the latter is tedious but straightforward: first smash (3) with S^1 and then compose with the counit of Remark 4.3.4; the resulting morphism equals S.

4.4   The (co)homology theory with coefficients in $\mathbb{K}(A, B)$

Let X be a simplicial set and let $A, B \in C$. We proceed to construct a natural morphism:

$$\phi_{A,B,X} : \mathbb{K}(A, B) \wedge X_+ \longrightarrow \mathbb{K}(A^X, B) \tag{6}$$

To alleviate notation, we will write ϕ instead of $\phi_{A,B,X}$. We have:

$$\mathscr{K}(J^nA, B) \wedge X_+ \cong \frac{\mathscr{K}(J^nA, B) \times X}{\{*\} \times X}$$

Thus, to define ϕ at level n it suffices to give a morphism $\phi^n : \mathscr{K}(J^nA, B) \times X \to \mathscr{K}(J^n(A^X), B)$ that sends $\{*\} \times X$ to the basepoint —the zero morphism. Let us define ϕ^n in dimension q. Recall that:

$$\mathscr{K}(J^nA, B)_q = \operatorname*{colim}_v \left(\Omega^v Ex^\infty \mathrm{Hom}_C(J^v J^nA, B^\Delta) \right)_q$$
$$= \operatorname*{colim}_v \operatorname*{colim}_r \mathrm{Hom}_C \left(J^{n+v}A, B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)} \right)$$

Since cartesian products commute with colimits, to define ϕ^n_q it suffices to give compatible functions:

$$\mathrm{Hom}\left(J^{n+v}A, B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)} \right) \times X_q \xrightarrow{\phi_q} \mathrm{Hom}\left(J^{n+v}(A^X), B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)} \right)$$

Let $f \in \operatorname{Hom}\left(J^{n+v}A, B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)}\right)$ and let $x : \Delta^q \to X$ be a q-simplex of X. Define $\phi_q^n(f, x)$ as the following composite function:

$$
\begin{array}{ccc}
J^{n+v}(A^X) \xrightarrow{\ J^{n+v}(x^*)\ } J^{n+v}(A^{\Delta^q}) \xrightarrow{\ \text{Remark 3.3.2}\ } (J^{n+v}A)^{\Delta^q} \\[2mm]
\Big\downarrow{\scriptstyle f^{\Delta^q}} \\[2mm]
\left(B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)}\right)^{\Delta^q} \\[2mm]
\Big\downarrow{\scriptstyle \mu} \\[2mm]
B_r^{(I^v \times \Delta^q \times \Delta^q, \partial I^v \times \Delta^q \times \Delta^q)} \\[2mm]
\Big\downarrow{\scriptstyle \star} \\[2mm]
B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)}
\end{array}
$$

$$\phi_q^n(f,x)$$

Here, the morphism $\star$ is the one induced by the diagonal map $\Delta^q \to \Delta^q \times \Delta^q$. It is straightforward but tedious to verify that these functions ϕ_q^n are compatible with the colimit maps and induce a function:

$$\phi_q^n : \mathcal{K}(J^nA, B)_q \times X_q \longrightarrow \mathcal{K}(J^n(A^X), B)_q \tag{7}$$

It is easily seen that, for varying q, the functions (7) assemble into a morphism of simplicial sets. Moreover, the latter clearly sends $X \times \{*\}$ to the basepoint of $\mathcal{K}(A^X, B)$. So far, we have defined a morphism of simplicial sets:

$$\phi^n : \mathcal{K}(J^nA, B) \wedge X_+ \longrightarrow \mathcal{K}(J^n(A^X), B)$$

Finaly, a tedious but straightforward verification shows that these morphisms ϕ^n assemble into a morphism of spectra. This completes the construction of (6).

Notice that $\mathcal{K}(J^nA, B) \wedge \Delta_+^0 \cong \mathcal{K}(J^nA, B)$ and $A^{\Delta^0} \cong A$. It is easily verified that, under these identifications, the morphism ϕ_{A,B,Δ^0} equals the identity of $\mathbb{K}(A, B)$.

Let $c : \widetilde{\mathbb{K}}(A, B) \to \mathbb{K}(A, B)$ be a cofibrant replacement in the category of spectra.

Proposition 4.4.1. *Let X be a finite simplicial set. Then the following composite is a stable weak equivalence of spectra:*

$$\widetilde{\mathbb{K}}(A, B) \wedge X_+ \xrightarrow{\ c \wedge X_+\ } \mathbb{K}(A, B) \wedge X_+ \xrightarrow{\ \phi_X\ } \mathbb{K}(A^X, B) \tag{8}$$

Proof. Let $\widetilde{\phi}_X$ be the composite in (8); we have to prove that $\widetilde{\phi}_X$ is a weak equivalence for finite X. The assertion holds for $X = \emptyset$ and $X = \Delta^0$ —in these cases, ϕ_X is an isomorphism of spectra. Let us show now that $\widetilde{\phi}_{\Delta^p}$ is a weak equivalence for any p. The following diagram commutes:

$$
\begin{array}{ccc}
\widetilde{\mathbb{K}}(A, B) \wedge \Delta_+^p & \xrightarrow{\ \widetilde{\phi}_{\Delta^p}\ } & \mathbb{K}(A^{\Delta^p}, B) \\[2mm]
\Big\downarrow & & \Big\downarrow \\[2mm]
\widetilde{\mathbb{K}}(A, B) \wedge \Delta_+^0 & \xrightarrow[\ \cong\ ]{\ \widetilde{\phi}_{\Delta^0}\ } & \mathbb{K}(A^{\Delta^0}, B)
\end{array}
$$

Here, the vertical morphisms are induced by $\Delta^p \to \Delta^0$. The vertical morphism on the left is a weak equivalence because $\widetilde{\mathbb{K}}(A, B) \wedge ?$ is a left Quillen functor, and so it takes weak equivalences between cofibrant objects to weak equivalences [9, Lemma 1.1.12]. The vertical morphism on the right is a weak equivalence because $A^{\Delta^0} \to A^{\Delta^p}$ is a homotopy equivalence.

Consider a pushout diagram of finite simplicial sets as follows, where the horizontal morphisms are inclusions:

$$
\begin{array}{ccc}
Y' & \rightarrowtail & Y \\
\downarrow & & \downarrow \\
X' & \rightarrowtail & X
\end{array}
\tag{9}
$$

Upon applying $A^? : \mathbb{S}^{\mathrm{op}} \to C$ we get a morphism of extensions as follows:

$$
\begin{array}{ccc}
A^{(Y,Y')} \longrightarrow A^Y \longrightarrow A^{Y'} \\
\cong \uparrow \qquad\qquad \uparrow \qquad\quad \uparrow \\
A^{(X,X')} \longrightarrow A^X \longrightarrow A^{X'}
\end{array}
$$

The rows are indeed extensions by [2, Lemma 3.1.2] and the vertical map on the left is an isomorphism because the square on the right is cartesian. Upon applying $\mathbb{K}(?, B)$ we get the following diagram of spectra:

$$
\begin{array}{ccc}
\mathbb{K}(A^{Y'}, B) & \longrightarrow & \mathbb{K}(A^Y, B) \\
\downarrow & & \downarrow \\
\mathbb{K}(A^{X'}, B) & \longrightarrow & \mathbb{K}(A^X, B) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathbb{K}(A^{(X,X')}, B)
\end{array}
$$

The outer and bottom squares are homotopy cartesian by [5, Theorem 6.6]; hence the top square is homotopy cartesian too, by [11, Lemma 1.1.8]. Then the upper square is homotopy cocartesian as well because the model structure is stable. Now consider the following commutative diagram of spectra:

We already know that the front face is homotopy cocartesian. The back face is homotopy cocartesian too since $\widetilde{\mathbb{K}}(A, B)\wedge?$ preserves pushouts along cofibrations. If we know that $\widetilde{\phi}_{Y'}$, $\widetilde{\phi}_Y$ and $\widetilde{\phi}_{X'}$ are weak equivalences, then $\widetilde{\phi}_X$ is a weak equivalence too by [8, Proposition 13.5.10]. Using the latter, we will finish the proof by induction on $\dim X$.

Consider the following pushout diagram of simplicial sets:

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & \Delta^p \\
\downarrow & & \downarrow \\
\Delta^p & \longrightarrow & \Delta^p \coprod \Delta^p
\end{array}
$$

By the previous paragraph, it follows that $\widetilde{\phi}_{\Delta^p \coprod \Delta^p}$ is a weak equivalence. By induction, one proves that $\widetilde{\phi}_X$ is a weak equivalence when X is a finite disjoint union of Δ^p's. In particular, $\widetilde{\phi}_X$ is a weak equivalence for any finite X of dimension 0. Now suppose that $\widetilde{\phi}_X$ is a weak equivalence for every finite X of dimension $< p$, and let Y be a finite simplicial set of dimension p. We have a pushout of finite simplicial sets as follows:

$$
\begin{array}{ccc}
\coprod \partial \Delta^p & \longrightarrow & \coprod \Delta^p \\
\downarrow & & \downarrow \\
\mathrm{sk}_{p-1} X & \longrightarrow & X
\end{array}
$$

Once more, by the previous paragraph, $\widetilde{\phi}_X$ is a weak equivalence. □

Corollary 4.4.2. *Let $f : X \to Y$ be a weak equivalence between finite simplicial sets and let $A \in C$. Then $f^* : A^Y \to A^X$ is a $\Re$-equivalence.*

Proof. Let $B \in C$. Note that $\widetilde{\mathbb{K}}(A, B)\wedge?$ preserves weak equivalences [9, Lemma 1.1.12]. By Proposition 4.4.1, $f_* : \mathbb{K}(A^X, B) \to \mathbb{K}(A^Y, B)$ is a weak equivalence of spectra. Upon applying π_n and making the identifications in Theorem 4.3.3, we have an isomorphism:

$$
f_* : \Re(A^X, (B, n)) \longrightarrow \Re(A^Y, (B, n))
$$

Then $f^* : A^Y \to A^X$ is an isomorphism in $\Re$ by Yoneda. □

Corollary 4.4.3. *Let X and Y be two finite simplicial sets and let $A \in C$. Then the morphism $\mu^{X,Y} : (A^X)^Y \to A^{X \times Y}$ is a $\Re$-equivalence.*

Proof. Let $B \in C$. It is enough to show that $\mu^* : \mathbb{K}(A^{X \times Y}, B) \to \mathbb{K}((A^X)^Y, B)$ is a weak equivalence of spectra. A straightforward verification shows that the following diagram of spectra commutes:

$$
\begin{array}{ccc}
\mathbb{K}(A, B) \wedge X_+ \wedge Y_+ & \xrightarrow{\phi_X \wedge Y_+} & \mathbb{K}(A^X, B) \wedge Y_+ \\
\cong \downarrow & & \downarrow \phi_Y \\
\mathbb{K}(A, B) \wedge (X \times Y)_+ \xrightarrow{\phi_{X \times Y}} \mathbb{K}(A^{X \times Y}, B) & \xrightarrow{\mu^*} & \mathbb{K}((A^X)^Y, B)
\end{array}
$$

Upon taking cofibrant replacements $c : \widetilde{\mathbb{K}} \to \mathbb{K}$ of the bivariant K-theory spectra, we get the following commutative diagram of solid arrows:

$$
\begin{array}{c}
 \widetilde{\mathbb{K}}(A^X, B) \wedge Y_+ \\
\end{array}
\tag{10}
$$

The proof will be finished if we show there exists a dotted weak equivalence making the diagram commute in the homotopy category of spectra.

Recall that the K-theory spectra $\mathbb{K}$ are fibrant in the stable model category of spectra. Factoring $* \to \mathbb{K}$ into a cofibration followed by a trivial fibration, we can choose a cofibrant replacement $c : \widetilde{\mathbb{K}} \to \mathbb{K}$ so that $\widetilde{\mathbb{K}}$ is still fibrant. Since $\widetilde{\mathbb{K}}(A, B) \wedge X_+$ is cofibrant and $c : \widetilde{\mathbb{K}}(A^X, B) \to \mathbb{K}(A^X, B)$ is a weak equivalence between fibrant spectra, there exists a morphism of spectra ψ making the following diagram commute up to left homotopy [9, Proposition 1.2.5 (iv)]:

$$
\begin{array}{c}
\widetilde{\mathbb{K}}(A^X, B) \\
\psi \nearrow \qquad \downarrow c \\
\widetilde{\mathbb{K}}(A, B) \wedge X_+ \xrightarrow{\widetilde{\phi}_X} \mathbb{K}(A^X, B)
\end{array}
\tag{11}
$$

Moreover, ψ is a weak equivalence since both $\widetilde{\phi}_X$ and c are. Since ψ is a weak equivalence between cofibrant spectra and $? \wedge Y_+$ is a left Quillen functor, then $\psi \wedge Y_+$ is a weak equivalence of spectra. We claim that $\psi \wedge Y_+$ is a dotted arrow that fits (10) making the diagram commute in the homotopy category. Every left Quillen functor preserves left homotopies between morphisms with cofibrant domain. Thus, upon applying $? \wedge Y_+$ to (11) we get a diagram that commutes up to left homotopy —hence, it commutes in the homotopy category of spectra by [9, Theorem 1.2.10 (iii)]. $\qquad \square$

Remark 4.4.4. Let X be a finite simplicial set. By Proposition 4.4.1, $\mathfrak{K}(A^X, (B, n))$ equals the n-th homology group of X with coefficients in $\mathbb{K}(A, B)$ —in the sense of [12].

We now proceed to construct a natural morphism:

$$
\vartheta_{A,B,X} : \mathbb{K}(A, B^X) \longrightarrow \mathbb{K}(A, B)^X
\tag{12}
$$

In order to define ϑ at level n, recall that:

$$
\mathscr{K}(J^n A, B^X)_p \cong \operatorname*{colim}_{v,r} \left(\Omega^v Ex^r \operatorname{Hom}(J^{n+v} A, (B^X)^{\Delta}) \right)_p
$$

$$
\cong \operatorname*{colim}_{v,r} \operatorname{Hom}\left(J^{n+v} A, (B^X)_r^{(I^v \times \Delta^p, \partial I^v \times \Delta^p)} \right)
\tag{13}
$$

There is a natural morphism of simplicial sets as follows —this need not be an isomorphism unless X is finite:

$$\operatorname*{colim}_{v,r} \operatorname{Map}\left(X, \Omega^v Ex^r \operatorname{Hom}(J^{n+v}A, B^\Delta)\right) \longrightarrow \mathscr{K}(J^n A, B)^X \tag{14}$$

We have:

$$\left[\operatorname*{colim}_{v,r} \operatorname{Map}\left(X, \Omega^v Ex^r \operatorname{Hom}(J^{n+v}A, B^\Delta)\right)\right]_p \cong \operatorname*{colim}_{v,r} \operatorname{Hom}\left(J^{n+v}A, B_r^{(I^v \times X \times \Delta^p, \partial I^v \times X \times \Delta^p)}\right) \tag{15}$$

Let $f \in \operatorname{Hom}\left(J^{n+v}A, (B^X)_r^{(I^v \times \Delta^p, \partial I^v \times \Delta^p)}\right)$ and define $\vartheta^n(f)$ to be the following composite, where the vertical isomorphism is induced by the commutativity of the product:

$$
\begin{array}{ccc}
J^{n+v}A \xrightarrow{\;f\;} (B^X)^{(I^v \times \Delta^p, \partial I^v \times \Delta^p)_r} \xrightarrow{\;\mu\;} & B_r^{(X \times I^v \times \Delta^p, X \times \partial I^v \times \Delta^p)} \\
& \downarrow \cong \\
\xdashrightarrow{\quad \vartheta^n(f) \quad} & B_r^{(I^v \times X \times \Delta^p, \partial I^v \times X \times \Delta^p)}
\end{array}
$$

It is easily verified that these functions ϑ^n are compatible with the transition maps of the colimits in (13) and (15), giving a morphism of simplicial sets:

$$\vartheta^n : \mathscr{K}(J^n A, B^X) \longrightarrow \operatorname*{colim}_{v,r} \operatorname{Map}\left(X, \Omega^v Ex^r \operatorname{Hom}(J^{n+v}A, B^\Delta)\right)$$

Composing the latter with (14) we get:

$$\vartheta^n : \mathscr{K}(J^n A, B^X) \longrightarrow \mathscr{K}(J^n A, B)^X$$

A straightforward verification shows that these ϑ^n assemble into the desired morphism of spectra (12).

Proposition 4.4.5. *Let X be a finite simplicial set. Then the morphism $\vartheta_{A,B,X}$ constructed above is a weak equivalence of spectra.*

Proof. This is very similar to the proof of Proposition 4.4.1 but there is no need to take fibrant replacement because the bivariant K-theory spectrum $\mathbb{K}$ is already fibrant. First of all, notice that ϑ_X is an isomorphism in the cases $X = \emptyset$ and $X = \Delta^0$. Proceeding as in the proof of Proposition 4.4.1 one shows that ϑ_{Δ^p} is a weak equivalence for all p. Now consider a pushout like (9) where the horizontal morphism are inclusions; we want to show that if $\vartheta_{Y'}$, ϑ_Y and $\vartheta_{X'}$ are weak equivalences, then ϑ_X is a weak equivalence too. The latter follows from a reasoning like in the proof of Proposition 4.4.1, using [5, Theorem 5.3] and the fact that $\mathbb{K}(A, B)^? : \mathbb{S}^{\mathrm{op}} \to \mathrm{Spt}$ is a right Quillen functor. $\square$

Remark 4.4.6. Let X be a finite simplicial set. By Proposition 4.4.5, $\mathfrak{K}(A, (B^X, -n))$ equals the n-th cohomology group of X with coefficients in $\mathbb{K}(A, B)$ —in the sense of [12].

Chapter 5

Matrix-stable bivariant K-theories

Resumen del capítulo

En este capítulo estudiamos diferentes estabilizaciones de $\mathfrak{K}$ con respecto a álgebras de matrices. En la sección 5.1 construímos una teoría de homología universal, escisiva, invariante por homotopía y M_n-estable, que denotamos por $j_{\mathrm{f}} : C \to \mathfrak{K}_{\mathrm{f}}$. La condición de M_n-estabilidad significa que, para todo $A \in C$ y todo $n \geq 1$, el morfismo $A \to M_n A$, $a \mapsto e_{11} \otimes a$, se vuelve inversible luego de aplicar j_{f}. Una teoría con estas características ya había sido construída por Garkusha [5, Theorem 6.5 (2)]. En la sección 5.2 construímos, para cualquier conjunto infinito X, una teoría de homología universal, escisiva, invariante por homotopía y M_X-estable, que denotamos por $j_X : C \to \mathfrak{K}_X$. Aquí, M_X es la ℓ-álgebra de matrices finitas con coeficientes en ℓ indexadas por $X \times X$. En el caso $X = \mathbb{N}$, $\mathfrak{K}_X$ coincide con construcciones previas de Cortiñas-Thom [2, Theorem 6.6.2] y Garkusha [5, Theorem 9.3.2]. Probamos que, para todo conjunto infinito X y toda ℓ-álgebra A, hay un isomorfismo natural $\mathfrak{K}_X(\ell, A) \cong KH_0 A$; esto extiende [2, Theorem 8.2.1]. En la sección 5.3 definimos, para cualquier grupo G, una teoría de homología universal, escisiva, invariante por homotopía y G-estable, que denotamos por $j^G : G\mathrm{Alg}_\ell \to \mathfrak{K}^G$. Aquí seguimos de cerca la construcción hecha por Ellis [4], pero nuestra definición es un poco más general ya que no imponemos restricciones sobre el cardinal de G. En el Teorema 5.3.11 probamos que hay un espectro $\mathbb{K}^G(A, B)$ que representa a $\mathfrak{K}^G$. Finalmente, en los teoremas 5.3.15 y 5.3.18, mostramos que el teorema de Green-Julg [4, Theorem 5.2.1] y la adjunción entre inducción y restricción [4, Theorem 6.14] se levantan a equivalencias débiles de espectros.

Chapter summary

In this chapter we discuss different ways of stabilizing $\mathfrak{K}$ with respect to matrix algebras. In section 5.1 we construct a universal excisive, homotopy invariant and M_n-stable homology theory $j_{\mathrm{f}} : C \to \mathfrak{K}_{\mathrm{f}}$. The M_n-stability condition means that, for any $A \in C$ and any $n \geq 1$, the morphism $A \to M_n A$, $a \mapsto e_{11} \otimes a$, becomes invertible upon applying j_{f}. Such a theory was already constructed by Garkusha [5, Theorem 6.5 (2)]. In section 5.2 we construct, for any infinite set X, a universal excisive, homotopy invariant and M_X-stable

homology theory $j_\mathcal{X} : C \to \mathfrak{K}_\mathcal{X}$. Here, $M_\mathcal{X}$ is the ℓ-algebra of finite matrices with coefficients in ℓ indexed by $\mathcal{X} \times \mathcal{X}$. In the case $\mathcal{X} = \mathbb{N}$, we recover previous constructions by Cortiñas-Thom [2, Theorem 6.6.2] and Garkusha [5, Theorem 9.3.2]. We prove that, for any (infinite) set $\mathcal{X}$ and any ℓ-algebra A, there is a natural isomorphism $\mathfrak{K}_\mathcal{X}(\ell, A) \cong KH_0 A$; this extends [2, Theorem 8.2.1]. In section 5.3 we define, for any group G, a universal excisive, homotopy invariant and G-stable homology theory $j^G : G\mathrm{Alg}_\ell \to \mathfrak{K}^G$. Here, we closely follow Ellis [4], but our definition is slightly more general since we do not impose any restriction on the cardinality of G. In Theorem 5.3.11 we prove that there is a spectrum $\mathbb{K}^G(A, B)$ representing $\mathfrak{K}^G$. Finally, in Theorems 5.3.15 and 5.3.18, we show that the Green-Julg theorem [4, Theorem 5.2.1] and the adjunction between induction and restriction [4, Theorem 6.14] lift to weak equivalences of spectra.

5.1   Stabilization by finite matrices

For $n \geq 0$, let M_n be the algebra of $n \times n$-matrices with coefficients in ℓ. In the G-equivariant setting, we will consider M_n as a G-ℓ-algebra with the trivial action of G. For $A \in C$, we will write $M_n A$ instead of $M_n \otimes_\ell A$; this is, of course, the algebra of $n \times n$-matrices with coefficients in A. In the case $C = G\mathrm{Alg}_\ell$, we will consider $M_n A$ as a G-ℓ-algebra with the diagonal G-action. We have a natural morphism $s_n : A \to M_n A$ defined by $s_n(a) = e_{11} \otimes a$. We also have a morphism $\iota_{n,N} : M_n A \to M_N A$ that sends $M_n A$ into the upper left corner of $M_N A$ for $n \leq N$. We can regard M_n as a functor $M_n : C \to C$. It is easily verified that this functor preserves extensions and homotopies, and thus induces a triangulated functor $M_n : \mathfrak{K}^C \to \mathfrak{K}^C$ by Theorem 3.13.14. Moreover, we can regard $s_n : \mathrm{id} \to M_n$ and $\iota_{n,N} : M_n \to M_N$ as natural transformations; by Theorem 3.13.14 they induce graded natural transformations between the corresponding triangulated functors.

Remark 5.1.1. Let $p, q \geq 0$. Any bijection $\theta : \{1, \ldots, p\} \times \{1, \ldots, q\} \to \{1, \ldots, pq\}$ induces a natural isomorphism $\theta : M_p M_q \cong M_{pq}$ determined by $e_{ij} \otimes e_{kl} \leftrightarrow e_{\theta(i,k)\theta(j,l)}$. By Theorem 3.13.14, the latter induces a graded natural isomorphism $\theta : M_p M_q \cong M_{pq}$ of triangulated functors $\mathfrak{K}^C \to \mathfrak{K}^C$.

Example 5.1.2. Let $\Theta_{p,q} : \{1, \ldots, p\} \times \{1, \ldots, q\} \to \{1, \ldots, pq\}$ be the bijection defined by $\Theta_{p,q}(i, j) := j + (i - 1)q$. The corresponding isomorphism $\Theta_{p,q} : M_p M_q \cong M_{pq}$ can be described as follows: An element of $M_p M_q$ is a matrix $A = (A_{ij})$ of size $p \times p$ whose coefficients are matrices A_{ij} of size $q \times q$ with coefficients in ℓ; $\Theta_{p,q}(A)$ is the $pq \times pq$-matrix obtained by drawing A and then deleting the parentheses of the A_{ij}'s.

Definition 5.1.3 (cf. [6, Section 6]). We proceed to define a category $\mathfrak{K}_\mathrm{f}^C$ —the subscript f standing for *finite matrices*. The objects of $\mathfrak{K}_\mathrm{f}^C$ are the objects of $\mathfrak{K}^C$ and the hom-sets are defined by:

$$\mathrm{Hom}_{\mathfrak{K}_\mathrm{f}^C}((A, m), (B, n)) := \operatorname*{colim}_p \mathfrak{K}^C((A, m), M_p(B, n))$$

Here, the transition maps are induced by $\iota_{p,p+1} : M_p(B, n) \to M_{p+1}(B, n)$. It is clear that the hom-sets in $\mathfrak{K}_\mathrm{f}^C$ are abelian groups. As usual, we may drop C from the notation and

write $\mathfrak{K}_{\mathrm{f}}$ instead of $\mathfrak{K}_{\mathrm{f}}^C$. We proceed to describe a composition:

$$\mathfrak{K}_{\mathrm{f}}((B,n),(C,k)) \times \mathfrak{K}_{\mathrm{f}}((A,m),(B,n)) \longrightarrow \mathfrak{K}_{\mathrm{f}}((A,m),(C,k))$$

Suppose we have morphisms in $\mathfrak{K}_{\mathrm{f}}$ represented by $\alpha \in \mathfrak{K}((A,m), M_p(B,n))$ and $\beta \in \mathfrak{K}((B,n), M_q(C,k))$; their composition in $\mathfrak{K}_{\mathrm{f}}$ is, by definition, represented by the following composite in $\mathfrak{K}$:

$$(A,m) \xrightarrow{\ \alpha\ } M_p(B,n) \xrightarrow{\ M_p\beta\ } M_pM_q(C,k) \xrightarrow[\cong]{\ \theta\ } M_{pq}(C,k) \qquad (1)$$

Here the isomorphism θ on the right is induced by a bijection $\theta : \{1,\ldots,p\} \times \{1,\ldots,q\} \to \{1,\ldots,pq\}$ as explained in Remark 5.1.1. This composition rule is well defined and independent from θ, as we prove below.

Lemma 5.1.4. *The definitions above make $\mathfrak{K}_{\mathrm{f}}$ into a category.*

Proof. Suppose η is a natural transformation between functors $C \to C$ that preserve extensions and homotopy. By Theorem 3.13.14, η induces a graded natural transformation $\bar{\eta}$ between the corresponding triangulated functors $\mathfrak{K} \to \mathfrak{K}$. Any commuting diagram involving such natural transformations induces a commuting diagram involving the corresponding graded natural transformations; see Remark 3.13.15. This will be used without further mention.

Fix p and q and choose a bijection $\theta : \{1,\ldots,p\} \times \{1,\ldots,q\} \to \{1,\ldots,pq\}$. Let us first show that the formula (1) defines a function

$$\mathfrak{K}((B,n), M_q(C,k)) \times \mathfrak{K}((A,m), M_p(B,n)) \longrightarrow \mathfrak{K}_{\mathrm{f}}((A,m),(C,k)) \qquad (2)$$

which is independent from θ. Let $\theta' : \{1,\ldots,p\} \times \{1,\ldots,q\} \to \{1,\ldots,pq\}$ be another bijection. It is easily seen that there is a permutation matrix $P \in M_{pq}$ such that the following diagram in C commutes:

$$
\begin{array}{ccc}
M_pM_q & \xrightarrow{\ \theta\ } & M_{pq} \\
& \theta' \searrow & \downarrow{\scriptstyle P\cdot(?)\cdot P^{-1}} \\
& & M_{pq}
\end{array}
$$

By [6, Proposition 3.1] (cf. [2, Lemma 4.1.1]), the following diagram commutes in $[C]$:

$$
\begin{array}{ccccc}
M_pM_q & \xrightarrow{\quad \theta \quad} & M_{pq} & & \\
& \theta' \searrow & \downarrow{\scriptstyle P\cdot(?)\cdot P^{-1}} & \searrow^{s_2} & \\
& & M_{pq} & \xrightarrow{\ s_2\ } & M_2 M_{pq}
\end{array}
$$

It is possible to choose an isomorphism $M_2M_{pq} \cong M_{2pq}$ such that the composite $M_{pq} \xrightarrow{s_2} M_2M_{pq} \cong M_{2pq}$ equals $\iota_{pq,2pq}$. It follows that the diagram below commutes in $\mathfrak{K}$, proving

the independence of (2) from θ.

$$
\begin{array}{ccc}
M_p M_q(C,k) & \xrightarrow{\ \theta\ } & M_{pq}(C,k) \\
& \searrow^{\theta'} \quad \downarrow{\scriptstyle P\cdot(?)\cdot P^{-1}} & \searrow^{\iota_{pq,2pq}} \\
& M_{pq}(C,k) \xrightarrow{\ \iota_{pq,2pq}\ } & M_{2pq}(C,k)
\end{array}
$$

Let us show now that the functions (2) are compatible with the transition maps in the first variable, $\iota_{q,q+1} : M_q(C,k) \to M_{q+1}(C,k)$. It is enough to find bijections $\theta : \{1,\ldots,p\}\times\{1,\ldots,q\} \to \{1,\ldots,pq\}$ and $\theta' : \{1,\ldots,p\}\times\{1,\ldots,q+1\} \to \{1,\ldots,p(q+1)\}$ such that the following diagram commutes —and this is easily done:

$$
\begin{array}{ccc}
M_p M_q & \xrightarrow{\ \theta\ } & M_{pq} \\
{\scriptstyle M_p\iota_{q,q+1}}\downarrow & & \downarrow{\scriptstyle \iota_{pq,p(q+1)}} \\
M_p M_{q+1} & \xrightarrow{\ \theta'\ } & M_{p(q+1)}
\end{array}
$$

Let us show now that the functions (2) are compatible with the transition maps in the second variable, $\iota_{p,p+1} : M_p(B,n) \to M_{p+1}(B,n)$. It is enough to find bijections $\theta : \{1,\ldots,p\}\times\{1,\ldots,q\} \to \{1,\ldots,pq\}$ and $\theta' : \{1,\ldots,p+1\}\times\{1,\ldots,q\} \to \{1,\ldots,(p+1)q)\}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
M_p M_q & \xrightarrow{\ \theta\ } & M_{pq} \\
{\scriptstyle \iota_{p,p+1}}\downarrow & & \downarrow{\scriptstyle \iota_{pq,(p+1)q}} \\
M_{(p+1)} M_q & \xrightarrow{\ \theta'\ } & M_{(p+1)q}
\end{array}
$$

Again, it is easy to find such θ and θ'. We can take, for example, $\theta = \Theta_{p,q}$ and $\theta' = \Theta_{p+1,q}$ as defined in Example 5.1.2.

We have proved that the formula (1) gives a well defined composition:

$$
\mathfrak{K}_{\mathrm{f}}((B,n),(C,k)) \times \mathfrak{K}_{\mathrm{f}}((A,m),(B,n)) \longrightarrow \mathfrak{K}_{\mathrm{f}}((A,m),(C,k))
$$

This composition is easily seen to be associative. The identity morphism of (A,m) in $\mathfrak{K}_{\mathrm{f}}$ is the class of $\mathrm{id}_{(A,m)} \in \mathfrak{K}((A,m),(A,m)) = \mathfrak{K}((A,m),M_1(A,m))$. This finishes the proof. $\qquad\square$

Remark 5.1.5. It is easily verified that the translation functor $L : \mathfrak{K}^C \to \mathfrak{K}^C$ induces a translation functor $L : \mathfrak{K}^C_{\mathrm{f}} \to \mathfrak{K}^C_{\mathrm{f}}$. Thus, $(\mathfrak{K}^C_{\mathrm{f}}, L)$ is a graded category.

There is a graded functor $t_{\mathrm{f}} : \mathfrak{K}^C \to \mathfrak{K}^C_{\mathrm{f}}$ that is the identity on objects and that sends a morphism $\alpha \in \mathfrak{K}((A,m),(B,n))$ to its class in:

$$
\mathfrak{K}_{\mathrm{f}}((A,m),(B,n)) = \operatorname*{colim}_{p} \mathfrak{K}((A,m),M_p(B,n))
$$

Precomposing t_{f} with $j : C \to \mathfrak{K}^C_{\mathrm{f}}$, we get a functor $j_{\mathrm{f}} : C \to \mathfrak{K}^C_{\mathrm{f}}$. For $A \in C$, we will usually write A instead of $j_{\mathrm{f}}(A)$ and $(A,0)$.

Lemma 5.1.6. *The functor* $j_f : C \to \mathfrak{K}_f^C$ *sends* $s_q : B \to M_q B$ *to an isomorphism for all* $q \geq 1$ *and all* $B \in C$.

Proof. It is enough to show that the following function is bijective for all (A, m):

$$(s_q)_* : \mathfrak{K}_f((A, m), B) \to \mathfrak{K}_f((A, m), M_q B) \tag{3}$$

Recall the definition of $\Theta_{p,q}$ from Example 5.1.2. The following diagram in C commutes for all p:

$$
\begin{array}{ccc}
M_p M_q & \xrightarrow{\iota_{p,p+1}} & M_{p+1} M_q \\
\Theta_{p,q} \downarrow & & \downarrow \Theta_{p+1,q} \\
M_{pq} & \xrightarrow{\iota_{pq,(p+1)q}} & M_{(p+1)q}
\end{array}
$$

It follows that the diagram below commutes, where the unlabeled arrows are the structural morphisms into the colimit:

$$
\begin{array}{ccc}
\mathfrak{K}((A, m), M_p M_q B) & \xrightarrow{\iota_{p,p+1}} & \mathfrak{K}((A, m), M_{p+1} M_q B) \\
\Theta_{p,q} \downarrow & & \downarrow \Theta_{p+1,q} \\
\mathfrak{K}((A, m), M_{pq} B) & \xrightarrow{\iota_{pq,(p+1)q}} & \mathfrak{K}((A, m), M_{(p+1)q} B) \\
& & \downarrow \\
& & \mathfrak{K}_f((A, m), B)
\end{array}
$$

This diagram induces a function $\rho : \mathfrak{K}_f((A, m), M_q B) \to \mathfrak{K}_f((A, m), B)$. Its is tedious but straightforward to verify that this function ρ is the inverse of (3). As in the proof of Lemma 5.1.4, it is used that two morphisms $M_q \to M_q$ which are conjugate to each other by a permutation matrix become homotopic when composed with $s_2 : M_q \to M_2 M_q$; of course, these composites become equal in $\mathfrak{K}$ by homotopy invariance. See also [6, Section 3] and [2, Section 4.1]. □

Remark 5.1.7. Suppose $a \in \mathfrak{K}_f((A, m), (B, n))$ is represented by $\alpha \in \mathfrak{K}((A, m), M_q(B, n))$. It is easily verified that the following diagram in $\mathfrak{K}_f$ commutes:

$$
\begin{array}{ccc}
(A, m) & \xrightarrow{t_f(\alpha)} & M_q(B, n) \\
& \searrow_{a} & \cong \uparrow t_f(s_q) \\
& & (B, n)
\end{array}
$$

Define a triangulation in $\mathfrak{K}_f^C$ as follows: a triangle in $\mathfrak{K}_f^C$ is distinguished if and only if it is isomorphic to the image by $t_f : \mathfrak{K}^C \to \mathfrak{K}_f^C$ of a distinguished triangle in $\mathfrak{K}^C$.

Lemma 5.1.8. *The triangulation above makes* $\mathfrak{K}_f^C$ *into a triangulated category.*

Proof. The verification of axioms (TR0) and (TR2) is straightforward.

Let us verify (TR1): Let $a \in \mathfrak{K}_f((A, m), (B, n))$ be represented by a morphism $\alpha \in \mathfrak{K}((A, m), M_q(B, n))$. Since $\mathfrak{K}$ is triangulated, α fits into a distinguished triangle in $\mathfrak{K}$. Then $t_f(\alpha)$ fits into a distinguished triangle in $\mathfrak{K}_f$ and so does a, by Remark 5.1.7.

Let us verify (TR3): Consider a diagram like the diagram of solid arrows (39) on page 71, where the rows are distinguished triangles; we have to show that the dotted arrow exists. We may assume that the rows are images by t_f of distinguished triangles in $\mathfrak{K}$:

$$
\begin{array}{ccccccc}
(C, k+1) & \xrightarrow{t_f(\gamma)} & (A, m) & \xrightarrow{t_f(\alpha)} & (B, n) & \xrightarrow{t_f(\beta)} & (C, k) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
(C', k'+1) & \xrightarrow{t_f(\gamma')} & (A', m') & \xrightarrow{t_f(\alpha')} & (B', n') & \xrightarrow{t_f(\beta')} & (C', k')
\end{array}
\qquad (4)
$$

Pick q large enough so that f and g are represented by $\phi \in \mathfrak{K}((B, n), M_q(B', n'))$ and $\psi \in \mathfrak{K}((C, k), M_q(C', k'))$ respectively, and such that the following diagram in $\mathfrak{K}$ commutes:

$$
\begin{array}{ccc}
(B, n) & \xrightarrow{\beta} & (C, k) \\
\phi \downarrow & & \downarrow \psi \\
M_q(B', n') & \xrightarrow{M_q\beta'} & M_q(C', k')
\end{array}
$$

It follows from Remark 5.1.7 that the diagram (4) is isomorphic in $\mathfrak{K}_f$ to:

$$
\begin{array}{ccccccc}
(C, k+1) & \xrightarrow{t_f(\gamma)} & (A, m) & \xrightarrow{t_f(\alpha)} & (B, n) & \xrightarrow{t_f(\beta)} & (C, k) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle t_f(\phi)} & & \downarrow{\scriptstyle t_f(\psi)} \\
M_q(C', k'+1) & \xrightarrow{t_f(M_q\gamma')} & M_q(A', m') & \xrightarrow{t_f(M_q\alpha')} & M_q(B', n') & \xrightarrow{t_f(M_q\beta')} & M_q(C', k')
\end{array}
$$

This diagram is the image by t_f of a commutative diagram in $\mathfrak{K}$ whose rows are distinguished triangles; thus, it can be completed to a morphism of triangles.

Recall the statement of (TR4) from Axiom 3.12.10. The axiom clearly holds if α and π' are in the image of t_f; we can easily reduce to this case using Remark 5.1.7. $\qquad \square$

Remark 5.1.9. The functor $t_f : \mathfrak{K}^C \to \mathfrak{K}_f^C$ is triangulated. Consider an extension in C:

$$
\mathscr{E} : \quad A \longrightarrow B \longrightarrow C
$$

Recall that we have morphisms $\partial_{\mathscr{E}} \in \mathfrak{K}^C((C, 1), A)$ such that the following diagram is a distinguished triangle in $\mathfrak{K}^C$:

$$
(C, 1) \xrightarrow{\partial_{\mathscr{E}}} A \longrightarrow B \longrightarrow C
$$

Upon applying t_f, we get a distinguished triangle in $\mathfrak{K}_f^C$. It is easily verified that the functor $j_f : C \to \mathfrak{K}_f^C$ together with the morphisms $\{t_f(\partial_{\mathscr{E}})\}_{\mathscr{E}}$ is an excisive homology theory.

Definition 5.1.10. Let $X : C \to \mathfrak{C}$ be a functor. We say that X is M_n-*stable* if $X(s_n : A \to M_nA)$ is an isomorphism for all $n \in \mathbb{N}$ and all $A \in C$.

Theorem 5.1.11. *The functor $j_f : C \to \mathfrak{K}_f^C$ is the universal homotopy invariant and M_n-stable δ-functor with values in a graded category.*

Proof. Let $X : C \to \mathscr{A}$ be a homotopy invariant and M_n-stable δ-functor with values in a graded category $(\mathscr{A}, L)$. By Theorem 3.13.11, there exists a unique graded functor $\bar{X}$ making the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\;j\;} & \mathfrak{K} \\
 & \searrow{\scriptstyle X} & \downarrow{\scriptstyle \bar{X}} \\
 & & \mathscr{A}
\end{array}
$$

We proceed to define a functor $\tilde{X} : \mathfrak{K}_f \to \mathscr{A}$. Put $\tilde{X}(A, m) := (X(A), m)$ for every $(A, m) \in \mathfrak{K}_f$. Let $\tilde{X} : \mathfrak{K}_f((A, m), (B, n)) \to \mathrm{Hom}_{\mathscr{A}}((X(A), m), (X(B), n))$ be the function determined by the commutative diagram:

$$
\begin{array}{ccc}
\mathfrak{K}((A, m), M_q(B, n)) & \xrightarrow{\;(\iota_{q,q+1})_*\;} & \mathfrak{K}((A, m), M_{q+1}(B, n)) \\
\bar{X} \downarrow & & \downarrow \bar{X} \\
\mathscr{A}((X(A), m), (X(M_q B), n)) & \xrightarrow{L^n X(\iota_{q,q+1})_*} & \mathscr{A}((X(A), m), (X(M_{q+1} B), n)) \\
 & \underset{L^n X(s_q)_*}{\overset{\cong}{\longleftarrow}} & \uparrow{\scriptstyle L^n X(s_{q+1})_*} \; \cong \\
 & & \mathscr{A}((X(A), m), (X(B), n))
\end{array}
$$

It is easy to see that these definitions give rise to a functor $\tilde{X}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
C & \xrightarrow{\;j_f\;} & \mathfrak{K}_f \\
 & \searrow{\scriptstyle X} & \downarrow{\scriptstyle \tilde{X}} \\
 & & \mathscr{A}
\end{array}
$$

Moreover, $\tilde{X} \circ t_f = \bar{X} : \mathfrak{K} \to \mathscr{A}$. Using that $\bar{X}$ is a graded functor, it is easily seen that $\tilde{X}$ is graded too. The uniqueness of $\tilde{X}$ follows from the uniqueness of $\bar{X}$ and Remark 5.1.7. $\qquad \square$

Theorem 5.1.12 (cf. [6, Theorem 6.5 (2)]). *The functor $j_f : C \to \mathfrak{K}_f^C$ is the universal excisive, homotopy invariant and M_n-stable homology theory.*

Proof. Let $X : C \to \mathscr{T}$ be an excisive, homotopy invariant and M_n-stable homology theory. By Theorem 5.1.11, there is a unique graded functor $\tilde{X}$ making the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\;j_f\;} & \mathfrak{K}_f \\
 & \searrow{\scriptstyle X} & \downarrow{\scriptstyle \tilde{X}} \\
 & & \mathscr{T}
\end{array}
$$

We claim that $\tilde{X}$ is triangulated, as we proceed to explain. Recall from the proof of Theorem 5.1.11 that $\tilde{X} \circ t_f = \bar{X}$, where $\bar{X}$ is the unique graded functor making the following

diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\ j\ } & \Re \\
& {}_{X}\searrow & \downarrow{}^{\bar{X}} \\
& & \mathscr{T}
\end{array}
$$

Since X is an excisive and homotopy invariant homology theory, it follows from Theorem 3.13.12 that $\bar{X}$ is triangulated. Any distinguished triangle in $\Re_{\mathrm{f}}$ is isomorphic to the image by t_{f} of a distinguished triangle in $\Re$; since $\tilde{X} \circ t_{\mathrm{f}} = \bar{X}$ and $\bar{X}$ is triangulated, it follows that $\tilde{X}$ is triangulated too. $\qquad\square$

Remark 5.1.13. A universal excisive, homotopy invariant and M_n-stable homology theory of algebras was already constructed by Garkusha [6, Theorem 6.5 (2)]. Of course, $\Re_{\mathrm{f}}$ and the theory developed in op. cit. are naturally isomorphic, since they both satisfy the same universal property. What we here call M_n-*stability* is referred to in [6] and [5] as *Morita invariance*.

Theorem 5.1.14. *Let $B \in C$. Then there is a unique triangulated functor $(B\tilde{\otimes}_\ell?) : \Re_{\mathrm{f}}^C \to \Re_{\mathrm{f}}^C$ making the following diagram commute:*

$$
\begin{array}{ccc}
C & \xrightarrow{\ j_{\mathrm{f}}\ } & \Re_{\mathrm{f}}^C \\
{}_{B\otimes_\ell?}\downarrow & & \downarrow{}^{B\tilde{\otimes}_\ell?} \\
C & \xrightarrow{\ j_{\mathrm{f}}\ } & \Re_{\mathrm{f}}^C
\end{array}
$$

Moreover, any morphism $f : B \to C$ in C induces a unique (graded) natural transformation $(f\tilde{\otimes}_\ell?) : (B\tilde{\otimes}_\ell?) \to (C\tilde{\otimes}_\ell?)$ such that:

$$
(f\tilde{\otimes}_\ell?)_{j_{\mathrm{f}}(A)} = j_{\mathrm{f}}(f\otimes_\ell A) : (B\otimes_\ell A, 0) \to (C\otimes_\ell A, 0)
$$

Proof. Let us prove the existence of $(B\tilde{\otimes}_\ell?)$. Consider the functor $j_{\mathrm{f}} \circ (B\otimes_\ell?) : C \to \Re_{\mathrm{f}}$. This functor is an excisive and homotopy invariant homology theory in a natural way since tensoring with B preserves extensions and homotopies. Moreover, we claim it is M_n-stable. Indeed, $j_{\mathrm{f}}(B\otimes_\ell s_n : B\otimes_\ell A \to B\otimes_\ell M_n A)$ is an isomorphism because there is an algebra isomorphism $B\otimes_\ell M_n A \cong M_n(B \otimes_\ell A)$ making the following diagram commute:

$$
\begin{array}{ccc}
B\otimes_\ell A & \xrightarrow{\ B\otimes_\ell s_n\ } & B\otimes_\ell M_n A \\
& {}_{s_n}\searrow & \downarrow{}^{\cong} \\
& & M_n(B\otimes_\ell A)
\end{array}
$$

Then the existence and uniqueness of $(B\tilde{\otimes}_\ell?)$ follow from Theorem 5.1.12.

Let $f : B \to C$ be a morphism in C. Then f induces a natural transformation $j_{\mathrm{f}}(f\otimes_\ell?) : j_{\mathrm{f}} \circ (B\otimes_\ell?) \to j_{\mathrm{f}} \circ (C\otimes_\ell?)$ of functors $C \to \Re_{\mathrm{f}}$. This natural transformation can be thought of as a homotopy invariant and M_n-stable δ-functor $X : C \to (\Re_{\mathrm{f}})^I$. By Theorem 5.1.11,

there exists a unique graded functor $\tilde{X}$ making the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{j_{\mathrm{f}}} & \mathfrak{K}_{\mathrm{f}} \\ & {}_{X}\searrow & \downarrow{\tilde{X}} \\ & & (\mathfrak{K}_{\mathrm{f}})^{I} \end{array}$$

This graded functor $\tilde{X}$ corresponds to the desired natural transformation:

$$(f\tilde{\otimes}_{\ell}?) : (B\tilde{\otimes}_{\ell}?) \to (C\tilde{\otimes}_{\ell}?) : \mathfrak{K}_{\mathrm{f}} \to \mathfrak{K}_{\mathrm{f}} \qquad\qquad \square$$

Definition 5.1.15 ([5, Section 9])**.** Let (A, B) be a pair of objects of C. The M_n-stable *bivariant K-theory spectrum* $\mathbb{K}_{\mathrm{f}}^{C}(A, B)$ is defined by:

$$\mathbb{K}_{\mathrm{f}}^{C}(A, B) := \operatorname*{colim}_{p\in\mathbb{N}} \mathbb{K}^{C}(A, M_p B)$$

Here, the transition maps are induced by $\iota_{p,p+1} : M_p B \to M_{p+1} B$.

Remark 5.1.16. The spectrum $\mathbb{K}_{\mathrm{f}}(A, B)$ was defined in [5, Section 9], where it was called the *Morita-stable Kasparov K-theory spectrum*. This spectrum represents the universal excisive, homotopy invariant and M_n-stable homology theory [5, Theorem 9.8], as we prove below.

Theorem 5.1.17 (cf. [5, Theorems 9.6 and 9.8])**.** *For every $m \in \mathbb{Z}$ there is a natural isomorphism:*

$$\pi_m \mathbb{K}_{\mathrm{f}}^{C}(A, B) \cong \mathfrak{K}_{\mathrm{f}}^{C}((A, 0), (B, m))$$

Proof. Since homotopy groups commute with filtered colimits, we have:

$$\begin{aligned} \pi_m \mathbb{K}_{\mathrm{f}}(A, B) &\cong \operatorname*{colim}_{p} \pi_m \mathbb{K}(A, M_p B) \\ &\cong \operatorname*{colim}_{p} \mathfrak{K}((A, 0), M_p(B, m)) \qquad\qquad \text{(by Theorem 4.3.3)} \\ &\cong \mathfrak{K}_{\mathrm{f}}((A, 0), (B, m)) \qquad\qquad\qquad\qquad\qquad \square \end{aligned}$$

5.2 Stabilization by finite matrices indexed on an arbitrary infinite set

Let $\mathcal{X}$ be an infinite set. In this section we construct a universal excisive, homotopy invariant and $M_{\mathcal{X}}$-stable homology theory $j_{\mathcal{X}} : C \to \mathfrak{K}_{\mathcal{X}}^{C}$. In the case $\mathcal{X} = \mathbb{N}$, we recover the universal excisive, homotopy invariant and M_{∞}-stable homology theory constructed by Cortiñas-Thom in [2, Theorem 6.6.2] and by Garkusha in [6, Theorem 9.3 (2)]. Moreover, we generalize [2, Theorem 8.2.1] proving that, in the case $C = \mathrm{Alg}_{\ell}$, $\mathfrak{K}_{\mathcal{X}}(\ell, A) \cong KH_0(A)$.

Put $M_{\mathcal{X}} := \{a : \mathcal{X} \times \mathcal{X} \to \ell : \mathrm{supp}(a) < \infty\}$. The set $M_{\mathcal{X}}$ is an ℓ-algebra with the usual sum and product of matrices. More precisely, for matrices $a, b \in M_{\mathcal{X}}$ we have:

$$(a + b)_{ij} = a_{ij} + b_{ij}$$

$$(ab)_{ij} = \sum_{k\in\mathcal{X}} a_{ik} b_{kj}$$

In the case $C = G\mathrm{Alg}_\ell$, we will consider M_X as a G-ℓ-algebra with trivial action. We will write $e_{ij} \in M_X$ ($i, j \in X$) for the matrix with a 1 in the (i, j)-place and 0 elsewhere. Fix $i_0 \in X$; the choice of i_0 is not important because of homotopy invariance. We have an ℓ-algebra homomorphism $s_X : \ell \to M_X$ defined by $s_X(1) = e_{i_0 i_0}$. For $A \in C$, we will write $M_X A$ instead of $M_X \otimes_\ell A$. In the case $C = G\mathrm{Alg}_\ell$, $M_X A$ will be considered as a G-ℓ-algebra with the diagonal G-action. By Theorem 5.1.14 we have a triangulated functor $M_X : \mathfrak{K}_\mathrm{f}^C \to \mathfrak{K}_\mathrm{f}^C$ such that $M_X(A, m) = (M_X A, m)$. Moreover, the morphism $s_X : \ell \to M_X$ induces a graded natural transformation $s_X : \mathrm{id} \to M_X : \mathfrak{K}_\mathrm{f} \to \mathfrak{K}_\mathrm{f}$.

Example 5.2.1. If $F \subset X$ is a subset, there is a natural inclusion $M_F \to M_X$. It is easily verified that $M_X = \mathrm{colim}\, M_F$, where the colimit runs over all the finite subsets $i_0 \in F \subset X$.

Notation 5.2.2. In the special case $X = \mathbb{N}$, $i_0 = 1$, we will write M_∞ instead of M_X and s_∞ instead of s_X.

For $A \in C$, let $\widetilde{A} = A \oplus \ell$ be the unitalization of A. Note that in the case $C = G\mathrm{Alg}$, $\widetilde{A}$ is a G-algebra in a natural way. Let $O_X A$ be the set of those $X \times X$-matrices with coefficients in $\widetilde{A}$ having finitely many nonzero coefficientes in each row and column. More precisely, define $O_X A$ as the set:

$$\left\{ (a_{ij}) \in \widetilde{A}^{X \times X} : \text{ for all } i \in X, |\{ j : a_{ij} \neq 0 \}| < \infty \text{ and } |\{ j : a_{ji} \neq 0 \}| < \infty \right\}$$

It is easily verified that $O_X A$ is a unital object of C and that $M_X A \subseteq O_X A$ is a two sided ideal. Every invertible matrix $P \in O_X A$ induces a morphism $\phi^P : M_X A \to M_X A$ by the formula $\phi^P(a) = PaP^{-1}$. Since $j_\mathrm{f} : C \to \mathfrak{K}_\mathrm{f}$ is M_2-stable, it follows from [1, Proposition 2.2.6] that $j_\mathrm{f}(\phi^P) : j_\mathrm{f}(M_X A) \to j_\mathrm{f}(M_X A)$ equals the identity of $j_\mathrm{f}(M_X A)$ in $\mathfrak{K}_\mathrm{f}$.

Remark 5.2.3. We will usually consider a special kind of invertible $P \in O_X A$ that arises in the following manner: Let $\varphi : X \to X$ be a bijection. Define $P \in O_X A$ by $P_{ij} = \delta_{i,\varphi(j)}$ —here δ is Kronecker's delta. Then $P \in O_X A$, P is invertible and $(P^{-1})_{ij} = \delta_{i,\varphi^{-1}(j)}$. The morphism $\phi^P : M_X A \to M_X A$ is determined by $\phi^P(e_{ij} \otimes_\ell a) = e_{\varphi(i)\varphi(j)} \otimes_\ell a$. We call P a *permutation matrix*.

Example 5.2.4. Any bijection $\theta : X \times X \to X$ induces an isomorphism $\theta : M_X M_X \xrightarrow{\cong} M_X$ by the formula $e_{ij} \otimes e_{kl} \mapsto e_{\theta(i,k)\theta(j,l)}$. By Theorem 5.1.14, the latter induces a natural isomorphism $\tilde{\theta} : M_X M_X \xrightarrow{\cong} M_X : \mathfrak{K}_\mathrm{f} \to \mathfrak{K}_\mathrm{f}$. We claim that $\tilde{\theta}$ is independent from θ. Let $\theta' : X \times X \to X$ be another bijection. It is easily verified that there is a permutation matrix P making the following diagram commute:

$$\begin{array}{ccc} M_X M_X & \xrightarrow{\ \theta\ } & M_X \\ & \searrow{\scriptstyle \theta'} & \downarrow{\scriptstyle \phi^P} \\ & & M_X \end{array} \qquad (5)$$

From the discussion above, it follows that $\tilde{\theta} = \tilde{\theta}'$.

Example 5.2.5. Any bijection $\theta : \mathbb{N} \times X \to X$ induces a natural isomorphism $M_\infty M_X \cong M_X$ in $\mathfrak{K}_\mathrm{f}$. Proceeding as in Example 5.2.4, it is easily verified that this isomorphism is independent from θ.

Example 5.2.6. Suppose that $|\mathcal{Y}| > |\mathcal{X}|$. Any injective function $\iota : \mathcal{X} \to \mathcal{Y}$ induces a morphism $\iota : M_\mathcal{X} \to M_\mathcal{Y}$ by the formula $e_{ij} \mapsto e_{\iota(i)\iota(j)}$. By Theorem 5.1.14, ι induces a natural transformation $\tilde{\iota} : M_\mathcal{X} \to M_\mathcal{Y}$ of functors $\mathfrak{K}_\mathrm{f} \to \mathfrak{K}_\mathrm{f}$. We claim that $\tilde{\iota}$ is independent from ι. Let $\iota' : \mathcal{X} \to \mathcal{Y}$ be another injective function. By our assumption on the cardinalities of $\mathcal{X}$ and $\mathcal{Y}$, any bijective function $\iota(\mathcal{X}) \to \iota'(\mathcal{X})$ can be extended to a bijection $\mathcal{Y} \to \mathcal{Y}$. This implies that there exists a permutation matrix P making the following diagram commute:

$$
\begin{array}{ccc}
M_\mathcal{X} & \xrightarrow{\ \iota\ } & M_\mathcal{Y} \\
& \searrow^{\iota'} & \downarrow{\phi^P} \\
& & M_\mathcal{Y}
\end{array}
$$

Then $\tilde{\iota} = \tilde{\iota}'$ because ϕ^P induces the identity in $\mathfrak{K}_\mathrm{f}$.

Definition 5.2.7. Define a category $\mathfrak{K}_\mathcal{X}^C$ as follows: The objects of $\mathfrak{K}_\mathcal{X}^C$ are the objects of $\mathfrak{K}_\mathrm{f}^C$ and the hom-sets are defined by:

$$
\mathrm{Hom}_{\mathfrak{K}_\mathcal{X}^C}((A, m), (B, n)) := \mathfrak{K}_\mathrm{f}^C((A, m), M_\mathcal{X}(B, n))
$$

Let $a \in \mathfrak{K}_\mathrm{f}^C((A, m), M_\mathcal{X}(B, n))$. We will often consider a both as a morphism in $\mathfrak{K}_\mathrm{f}$ and as a morphism $(A, m) \to (B, n)$ in $\mathfrak{K}_\mathcal{X}$. To avoid ambiguity, we will write $[a]$ instead of a when considering a as a morphism in $\mathfrak{K}_\mathcal{X}$ and we will say that $[a]$ is *represented* by a. We proceed to describe a composition:

$$
\mathfrak{K}_\mathcal{X}^C((B, n), (C, k)) \times \mathfrak{K}_\mathcal{X}^C((A, m), (B, n)) \longrightarrow \mathfrak{K}_\mathcal{X}^C((A, m), (C, k))
$$

Let $a \in \mathfrak{K}_\mathrm{f}((A, m), M_\mathcal{X}(B, n))$ and $b \in \mathfrak{K}_\mathrm{f}((B, n), M_\mathcal{X}(C, k))$; then we set $[b] \circ [a] := [c]$, where c is the following composite in $\mathfrak{K}_\mathrm{f}$:

$$
(A, m) \xrightarrow{\ a\ } M_\mathcal{X}(B, n) \xrightarrow{\ M_\mathcal{X}b\ } M_\mathcal{X}M_\mathcal{X}(C, k) \xrightarrow[\cong]{\ \theta\ } M_\mathcal{X}(C, k) \tag{6}
$$

Here the isomorphism on the right is induced by any bijection $\theta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ as explained in Example 5.2.4.

Lemma 5.2.8. *The definitions above make $\mathfrak{K}_\mathcal{X}^C$ into a category. There is a functor $t_\mathcal{X} : \mathfrak{K}_\mathrm{f}^C \to \mathfrak{K}_\mathcal{X}^C$ that is the identity on objects and that sends $a \in \mathfrak{K}_\mathrm{f}^C((A, m), (B, n))$ to the morphism in $\mathfrak{K}_\mathcal{X}^C$ represented by the following composite in $\mathfrak{K}_\mathrm{f}^C$:*

$$
(A, m) \xrightarrow{\ a\ } (B, n) \xrightarrow{\ s_{\mathcal{X},(B,n)}\ } M_\mathcal{X}(B, n)
$$

Proof. The composition rule in $\mathfrak{K}_\mathcal{X}$ is independent from θ by Example 5.2.4. The rest of the proof is tedious but straightforward, and relies on the fact that $j_\mathrm{f} : C \to \mathfrak{K}_\mathrm{f}$ sends ϕ^P to the identity of $j_\mathrm{f}(M_\mathcal{X}A)$ for every $A \in C$ and every invertible matrix $P \in O_\mathcal{X}A$. For example: To prove the associativity of the composition one has to show that certain diagram in $\mathfrak{K}_\mathrm{f}$ commutes. This diagram does not come from a commutative diagram in C, but it does come from a diagram that commutes up to conjungation by a permutation matrix $P \in O_\mathcal{X}A$. The result follows from the fact that conjugating by P induces the identity in $\mathfrak{K}_\mathrm{f}$. $\qquad\square$

The translation functor in $\mathfrak{K}_{\mathrm{f}}^C$ induces a translation functor in $\mathfrak{K}_{\mathcal{X}}^C$ that makes it into a graded category. The functor $t_{\mathcal{X}} : \mathfrak{K}_{\mathrm{f}}^C \to \mathfrak{K}_{\mathcal{X}}^C$ is clearly graded. Precomposing $t_{\mathcal{X}}$ with $j_{\mathrm{f}} : C \to \mathfrak{K}_{\mathrm{f}}^C$ we get a functor $j_{\mathcal{X}} : C \to \mathfrak{K}_{\mathcal{X}}^C$.

Example 5.2.9. Suppose that $|\mathcal{Y}| > |\mathcal{X}|$ and let $\iota : \mathcal{X} \to \mathcal{Y}$ be an injective function. By Example 5.2.6, ι induces a natural transformation $\tilde{\iota} : M_{\mathcal{X}} \to M_{\mathcal{Y}} : \mathfrak{K}_{\mathrm{f}} \to \mathfrak{K}_{\mathrm{f}}$ that is independent from ι. We proceed to define a functor $T : \mathfrak{K}_{\mathcal{X}} \to \mathfrak{K}_{\mathcal{Y}}$: Let T be the identity on objects. Define T on morphisms by:

$$\mathfrak{K}_{\mathcal{X}}((A, m), (B, n)) = \mathfrak{K}_{\mathrm{f}}((A, m), M_{\mathcal{X}}(B, n)) \xrightarrow{\tilde{\iota}_*} \mathfrak{K}_{\mathrm{f}}((A, m), M_{\mathcal{Y}}(B, n)) = \mathfrak{K}_{\mathcal{Y}}((A, m), (B, n))$$

By a cardinality argument, any bijection $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ can be extended to a bijection $\mathcal{Y} \times \mathcal{Y} \to \mathcal{Y}$. Using the latter, it is straightforward to verify that T is indeed a functor and that the following diagram commutes:

$$
\begin{array}{ccc}
\mathfrak{K}_{\mathrm{f}} & \xrightarrow{\ t_{\mathcal{X}}\ } & \mathfrak{K}_{\mathcal{X}} \\
& {\scriptstyle t_{\mathcal{Y}}} \searrow & \ \downarrow {\scriptstyle T} \\
& & \mathfrak{K}_{\mathcal{Y}}
\end{array}
$$

Definition 5.2.10. Let $X : C \to \mathfrak{C}$ be a functor. We say that X is $M_{\mathcal{X}}$-*stable* if $X(s_{\mathcal{X}} : A \to M_{\mathcal{X}}A)$ is an isomorphism for all $A \in C$.

Lemma 5.2.11. *The functor* $j_{\mathcal{X}} : C \to \mathfrak{K}_{\mathcal{X}}^C$ *is* $M_{\mathcal{X}}$-*stable.*

Proof. To alleviate notation, if B is an object of C we will still write B for its images under the functors $j_{\mathrm{f}} : C \to \mathfrak{K}_{\mathrm{f}}$ and $j_{\mathcal{X}} : C \to \mathfrak{K}_{\mathcal{X}}$. Let $A \in C$. The identity of $M_{\mathcal{X}}A$ in $\mathfrak{K}_{\mathrm{f}}$ represents a morphism $\varphi \in \mathfrak{K}_{\mathcal{X}}(M_{\mathcal{X}}A, A)$; we claim that φ is the inverse of $j_{\mathcal{X}}(s_{\mathcal{X},A})$.

Let us show that $j_{\mathcal{X}}(s_{\mathcal{X},A}) \circ \varphi$ is the identity of $M_{\mathcal{X}}A$ in $\mathfrak{K}_{\mathcal{X}}$. Fix a bijection $\theta : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$. Using the definition of the composition rule in $\mathfrak{K}_{\mathcal{X}}$, we are easily reduced to showing that the following diagram in $\mathfrak{K}_{\mathrm{f}}$ commutes:

$$
\begin{array}{ccc}
M_{\mathcal{X}}A & \xrightarrow{\ M_{\mathcal{X}}(s_{\mathcal{X},A})\ } & M_{\mathcal{X}}M_{\mathcal{X}}A \\
{\scriptstyle s_{\mathcal{X},M_{\mathcal{X}}A}} \downarrow & & \downarrow {\scriptstyle M_{\mathcal{X}}(s_{\mathcal{X},M_{\mathcal{X}}A})} \\
M_{\mathcal{X}}M_{\mathcal{X}}A & \xleftarrow{\ \theta \otimes M_{\mathcal{X}}A\ } & M_{\mathcal{X}}M_{\mathcal{X}}M_{\mathcal{X}}A
\end{array}
\qquad (7)
$$

To prove that (7) commutes we have to show that two morphisms $M_{\mathcal{X}}A \to M_{\mathcal{X}}M_{\mathcal{X}}A$ are equal; this is equivalent to showing that both morphism are equal when composed with the isomorphism $\theta : M_{\mathcal{X}}M_{\mathcal{X}}A \to M_{\mathcal{X}}A$. These two composites are not equal in C but they are easily seen to be conjugate by a permutation matrix $P \in O_{\mathcal{X}}A$. Again, the result follows from the fact that conjugating by P induces the identity in $\mathfrak{K}_{\mathrm{f}}$.

Showing that $\varphi \circ j_{\mathcal{X}}(s_{\mathcal{X},A})$ is the identity of A in $\mathfrak{K}_{\mathcal{X}}$ is easily reduced to showing that the following diagram in $\mathfrak{K}_{\mathrm{f}}$ commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ s_{\mathcal{X},A}\ } & M_{\mathcal{X}}A \\
{\scriptstyle s_{\mathcal{X},A}} \downarrow & & \downarrow {\scriptstyle s_{\mathcal{X},M_{\mathcal{X}}A}} \\
M_{\mathcal{X}}A & \xleftarrow{\ \theta \otimes A\ } & M_{\mathcal{X}}M_{\mathcal{X}}A
\end{array}
$$

This diagram commutes in C if we pick the bijection θ so that $\theta(i_0, i_0) = i_0$. $\square$

Remark 5.2.12. Let $a \in \mathfrak{R}^C_f((A, m), M_\chi(B, n))$. It is easily verified that the following diagram in $\mathfrak{R}^C_\chi$ commutes:

$$
\begin{array}{ccc}
(A, m) & \xrightarrow{\ t_\chi(a)\ } & (M_\chi B, n) \\
& \searrow^{[a]} & \big\uparrow{\cong}\ {t_\chi(s_\chi)} \\
& & (B, n)
\end{array}
$$

Define a triangulation in $\mathfrak{R}^C_\chi$ as follows: a triangle in $\mathfrak{R}^C_\chi$ is distinguished if and only if it is isomorphic (in $\mathfrak{R}^C_\chi$) to the image by t_χ of a distinguished triangle in $\mathfrak{R}^C_f$.

Lemma 5.2.13. *The triangulation above makes $\mathfrak{R}^C_\chi$ into a triangulated category.*

Proof. The verification of axioms (TR0) and (TR2) is straightforward.

Let us verify (TR1): Let $a \in \mathfrak{R}_f((A, m), M_\chi(B, n))$. Since $\mathfrak{R}_f$ is triangulated, a fits into a distinguished triangle in $\mathfrak{R}_f$. Then $t_\chi(a)$ fits into a distinguished triangle in $\mathfrak{R}_\chi$ and so does $[a]$, by Remark 5.2.12

Let us verify (TR3): Consider a diagram like the diagram of solid arrows (39) on page 71, where the rows are distinguished triangles; we have to show that the dotted arrow exists. We may assume that the rows are images by t_χ of distinguished triangles in $\mathfrak{R}_f$:

$$
\begin{array}{ccccccc}
LZ' & \xrightarrow{t_\chi(c)} & X' & \xrightarrow{t_\chi(a)} & Y' & \xrightarrow{t_\chi(b)} & Z' \\
\downarrow & & \downarrow & & \downarrow{[f]} & & \downarrow{[g]} \\
LZ & \xrightarrow{t_\chi(c')} & X & \xrightarrow{t_\chi(a')} & Y & \xrightarrow{t_\chi(b')} & Z
\end{array}
\tag{8}
$$

Here $f \in \mathfrak{R}_f(Y', M_\chi Y)$ and $g \in \mathfrak{R}_f(Z', M_\chi Z)$. We claim that the following square in $\mathfrak{R}_f$ commutes; indeed, this follows easily from the commutativity of the rightmost square in (8):

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ b\ } & Z' \\
{f}\downarrow & & \downarrow{g} \\
M_\chi Y & & M_\chi Z \\
{s_{\chi, M_\chi Y}}\downarrow & & \downarrow{s_{\chi, M_\chi Z}} \\
M_\chi M_\chi Y & \xrightarrow{M_\chi M_\chi b'} & M_\chi M_\chi Z
\end{array}
$$

It follows from Remark 5.2.12 that the diagram (8) is isomorphic in $\mathfrak{R}_\chi$ to:

$$
\begin{array}{ccccccc}
LZ' & \xrightarrow{t_\chi(c)} & X' & \xrightarrow{t_\chi(a)} & Y' & \xrightarrow{t_\chi(b)} & Z' \\
\downarrow & & & & \downarrow{t_\chi(s_\chi \circ f)} & & \downarrow{t_\chi(s_\chi \circ g)} \\
LM_\chi M_\chi Z & \xrightarrow{t_\chi(M_\chi M_\chi c')} & M_\chi M_\chi X & \xrightarrow{t_\chi(M_\chi M_\chi a')} & M_\chi M_\chi Y & \xrightarrow{t_\chi(M_\chi M_\chi b')} & M_\chi M_\chi Z
\end{array}
$$

This diagram is the image by t_χ of a commutative diagram in $\mathfrak{R}_f$ whose rows are distinguished triangles; thus, it can be completed to a morphism of triangles.

Recall the statement of (TR4) from Axiom 3.12.10. The axiom clearly holds if α and π' are in the image of t_χ; we can easily reduce to this case using Remark 5.2.12. $\qquad\square$

Remark 5.2.14. The functor $t_X : \mathfrak{K}_{\mathrm{f}}^C \to \mathfrak{K}_X^C$ is obviously triangulated and the functor $j_X : C \to \mathfrak{K}_X^C$ is an excisive, homotopy invariant and M_X-stable homology theory.

Theorem 5.2.15. *The functor $j_X : C \to \mathfrak{K}_X^C$ is the universal homotopy invariant and M_X-stable δ-functor with values in a graded category.*

Proof. Let $X : C \to \mathscr{A}$ be a homotopy invariant and M_X-stable δ-functor with values in a graded category $(\mathscr{A}, L)$. Since X is M_X-stable, then it is M_n-stable. By Theorem 5.1.11, there exists a unique graded functor $\tilde{X}$ making the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\;j_{\mathrm{f}}\;} & \mathfrak{K}_{\mathrm{f}} \\
 & {\scriptstyle X}\searrow & \downarrow{\scriptstyle \tilde{X}} \\
 & & \mathscr{A}
\end{array}
$$

We proceed to define a functor $\hat{X} : \mathfrak{K}_X \to \mathscr{A}$. Put $\hat{X}(A, m) := (X(A), m)$ for every $(A, m) \in \mathfrak{K}_X$. For $f \in \mathfrak{K}_{\mathrm{f}}((A, m), M_X(B, n))$ put:

$$\hat{X}([f]) := \tilde{X}(s_{X,(B,n)})^{-1} \circ \tilde{X}(f) \in \mathrm{Hom}_{\mathscr{A}}((X(A), m), (X(B), n))$$

This defines a function $\hat{X} : \mathfrak{K}_X((A, m), (B, n)) \to \mathrm{Hom}_{\mathscr{A}}((X(A), m), (X(B), n))$. Let us show that these definitions give rise to a functor $\hat{X} : \mathfrak{K}_X \to \mathscr{A}$. Let $f \in \mathfrak{K}_{\mathrm{f}}((A, m), M_X(B, n))$ and $g \in \mathfrak{K}_{\mathrm{f}}((B, n), M_X(C, k))$. Choose a bijection $\theta : X \times X \to X$. We have:

$$[g] \circ [f] = [\theta_{(C,k)} \circ M_X g \circ f] \in \mathfrak{K}_X((A, m), (C, k))$$

Here $\theta_{(C,k)} : M_X M_X(C, k) \to M_X(C, k)$ is the isomorphism induced by θ in $\mathfrak{K}_{\mathrm{f}}$; see Example 5.2.4. Then:

$$\hat{X}([g]) \circ \hat{X}([f]) = \tilde{X}(s_{X,(C,k)})^{-1} \circ \tilde{X}(g) \circ \tilde{X}(s_{X,(B,n)})^{-1} \circ \tilde{X}(f)$$
$$\hat{X}([g] \circ [f]) = \tilde{X}(s_{X,(C,k)})^{-1} \circ \tilde{X}(\theta_{(C,k)}) \circ \tilde{X}(M_X g) \circ \tilde{X}(f)$$

Thus, to prove that $\hat{X}([g] \circ [f]) = \hat{X}([g]) \circ \hat{X}([f])$ it is enough to show that $\tilde{X}(g) \circ \tilde{X}(s_{X,(B,n)})^{-1} = \tilde{X}(\theta_{(C,k)}) \circ \tilde{X}(M_X g)$. We have:

$$\tilde{X}(\theta_{(C,k)}) \circ \tilde{X}(M_X g) = \tilde{X}(\theta_{(C,k)}) \circ \tilde{X}(M_X g) \circ \tilde{X}(s_{X,(B,n)}) \circ \tilde{X}(s_{X,(B,n)})^{-1}$$
$$= \tilde{X}(\theta_{(C,k)}) \circ \tilde{X}(s_{X,M_X(C,k)}) \circ \tilde{X}(g) \circ \tilde{X}(s_{X,(B,n)})^{-1}$$

Thus, it suffices to show that:

$$\tilde{X}(\theta_{(C,k)}) \circ \tilde{X}(s_{X,M_X(C,k)}) = \mathrm{id}_{(X(M_X C), k)} \tag{9}$$

If we choose θ such that $\theta(i_0, i_0) = i_0$ then the following diagram in C commutes:

$$
\begin{array}{ccc}
C & \xrightarrow{\;s_{X,C}\;} & M_X C \\
{\scriptstyle s_{X,C}}\downarrow & & \downarrow{\scriptstyle s_{X,M_X C}} \\
M_X C & \xleftarrow{\;\theta\;} & M_X M_X C
\end{array}
$$

The equality (9) follows from this and from the fact that $\tilde{X}(s_{X,(C,k)})$ is invertible.

It straightforward to verify that $\hat{X} \circ t_X = \tilde{X} : \mathfrak{R}_{\mathrm{f}} \to \mathscr{A}$ and that the following diagram commutes:

$$
\begin{array}{ccc}
C & \xrightarrow{j_X} & \mathfrak{R}_X \\
 & {}_{X}\searrow & \downarrow{}^{\hat{X}} \\
 & & \mathscr{A}
\end{array}
$$

Using that $\tilde{X}$ is a graded functor, it is easily seen that $\hat{X}$ is graded too. The uniqueness of $\hat{X}$ follows from the uniqueness of $\tilde{X}$ and Remark 5.2.12. $\qquad\square$

Theorem 5.2.16 (cf. [2, Theorem 6.6.2] [6, Theorem 9.3 (2)]). *The functor $j_X : C \to \mathfrak{R}_X^C$ is the universal excisive, homotopy invariant and M_X-stable homology theory.*

Proof. The proof is like that of Theorem 5.1.12, with minor changes. $\qquad\square$

Remark 5.2.17. A universal excisive, homotopy invariant and M_∞-stable homology theory of algebras was already constructed by Cortiñas-Thom [2, Theorem 6.6.2] and by Garkusha [6, Theorem 9.3 (2)]. Of course, these constructions are naturally isomorphic to $\mathfrak{R}_\infty$ since they all satisfy the same universal property.

Theorem 5.2.18. *Let B be an object of C. Then there is a unique triangulated functor $(B\tilde{\otimes}_\ell?) : \mathfrak{R}_X^C \to \mathfrak{R}_X^C$ making the following diagram commute:*

$$
\begin{array}{ccc}
C & \xrightarrow{j_X} & \mathfrak{R}_X^C \\
{}_{B\otimes_\ell?}\downarrow & & \downarrow{}^{B\tilde{\otimes}_\ell?} \\
C & \xrightarrow{j_X} & \mathfrak{R}_X^C
\end{array}
$$

Moreover, any morphism $f : B \to C$ in C induces a unique (graded) natural transformation $(f\tilde{\otimes}_\ell?) : (B\tilde{\otimes}_\ell?) \to (C\tilde{\otimes}_\ell?)$ such that:

$$
(f\tilde{\otimes}_\ell?)_{j_X(A)} = j_X(f \otimes_\ell A) : (B \otimes_\ell A, 0) \to (C \otimes_\ell A, 0)
$$

Proof. The proof is like that of Theorem 5.1.14, with minor changes. $\qquad\square$

Example 5.2.19. In this example, $C = \mathrm{Alg}_\ell$. For $A \in \mathrm{Alg}_\ell$, let $KH_n(A)$ be Weibel's *homotopy K-theory* groups; see [1, Section 5]. The functors KH_n are homotopy invariant, M_∞-stable, commute with filtering colimits and satisfy excision [1, Theorem 5.1.1]. The latter means that to every extension of ℓ-algebras

$$
\mathscr{E} : A \longrightarrow B \longrightarrow C \tag{10}
$$

there corresponds a long exact sequence:

$$
\cdots \xrightarrow{\partial_{\mathscr{E}}} KH_{n+1}(C) \longrightarrow KH_n(A) \longrightarrow KH_n(B) \longrightarrow KH_n(C) \longrightarrow \cdots
$$

This sequence is, moreover, natural with respect to morphisms of extensions. Notice that KH_n is M_X-stable for any X because of M_∞-stability and the fact that it commutes with filtering colimits; see Example 5.2.1.

Let $\mathscr{A} = \mathrm{GrAb}$ be the category of graded abelian groups described in Example 3.13.5. There is a functor $KH : \mathrm{Alg}_\ell \to \mathrm{GrAb}$ that sends an algebra A to the graded abelian group $\oplus_n KH_n(A)$. For an extension $\mathscr{E}$ like (10), the morphisms $\partial_\mathscr{E}$ assemble into a morphism $\partial \in \mathscr{A}(L(KH(C)), KH(A))$. Thus, $KH : \mathrm{Alg}_\ell \to \mathscr{A}$ is a homotopy invariant and M_X-stable δ-functor. By Theorem 5.2.15 there exists a unique graded functor $\widehat{KH}$ making the following diagram commute:

$$
\begin{array}{ccc}
\mathrm{Alg}_\ell & \xrightarrow{\ j_X\ } & \mathfrak{K}_X \\
& {\scriptstyle KH}\searrow & \downarrow{\scriptstyle \widehat{KH}} \\
& & \mathrm{GrAb}
\end{array}
$$

Let $u : \mathbb{Z} \to \ell$ be the unique unital ring homomorphism. For any algebra A, we have a group homomorphism:

$$
\mathfrak{K}_X(\ell, A) \xrightarrow{\ \widehat{KH}\ } \mathrm{GrAb}(KH(\ell), KH(A)) \longrightarrow \mathrm{Ab}(KH_0\ell, KH_0A) \xrightarrow{\ u^*\ } \mathrm{Ab}(KH_0\mathbb{Z}, KH_0A)
$$

Since $KH_0\mathbb{Z} \cong \mathbb{Z}$, this gives a group homomorphism $\varphi_{X,A} : \mathfrak{K}_X(\ell, A) \to KH_0(A)$. Cortiñas-Thom showed that, in the case $X = \mathbb{N}$, $\varphi_{X,A}$ is an isomorphism [2, Example 6.6.8 and Theorem 8.2.1].

Theorem 5.2.20 (cf. [2, Theorem 8.2.1]). *Let $C = \mathrm{Alg}_\ell$ and let A be an ℓ-algebra. Then the morphism $\varphi_{X,A} : \mathfrak{K}_X(\ell, A) \to KH_0A$ defined in Example 5.2.19 is an isomophism.*

Proof. The case $X = \mathbb{N}$ is [2, Theorem 8.2.1] and the general case will follow easily from this. Suppose that $|X| > |\mathbb{N}|$. It is straightforward to verify that $\varphi_{\mathbb{N},A}$ equals the following composite:

$$
\mathfrak{K}_\infty(\ell, A) \xrightarrow[\ T\ ]{\text{Example 5.2.9}} \mathfrak{K}_X(\ell, A) \xrightarrow{\ \varphi_{X,A}\ } KH_0A
$$

Thus, it suffices to show that the morphism T of Example 5.2.9 is an isomorphism. Let $\theta : \mathbb{N} \times X \to X$ be any bijection. By Example 5.2.5, θ induces an isomorphism $\tilde{\theta} : M_\infty M_X A \cong M_X A$ in $\mathfrak{K}_\mathrm{f}$, which is independent from θ. A straightforward verification shows that T equals the following composite:

$$
\mathfrak{K}_\infty(\ell, A) \xrightarrow{\ (s_X)_*\ } \mathfrak{K}_\infty(\ell, M_X A) = \mathfrak{K}_\mathrm{f}(\ell, M_\infty M_X A) \xrightarrow[\cong]{\ \tilde{\theta}_*\ } \mathfrak{K}_\mathrm{f}(\ell, M_X A) = \mathfrak{K}_X(\ell, A)
$$

But $(s_X)_*$ is an isomorphism because $\mathfrak{K}_\infty(\ell, ?) \cong KH_0(?)$ is M_X-stable. This finishes the proof. $\qquad\square$

Definition 5.2.21. Let (A, B) be a pair of objects of C. The *M_X-stable bivariant K-theory spectrum* $\mathbb{K}_X^C(A, B)$ is defined by:

$$
\mathbb{K}_X^C(A, B) := \mathbb{K}_\mathrm{f}^C(A, M_X B)
$$

Theorem 5.2.22 (cf. [5, Theorem 9.7])**.** *For every $m \in \mathbb{Z}$ there is a natural isomorphism:*

$$\pi_m \mathbb{K}_{\chi}^C(A, B) \cong \mathfrak{R}_{\chi}^C((A, 0), (B, m))$$

Proof. It follows immediately from Theorem 5.1.17 and the definition of $\mathfrak{R}_{\chi}$. $\qquad\square$

Remark 5.2.23. A spectrum representing the universal excisive, homotopy invariant and M_{∞}-stable homology theory was already constructed by Garkusha [5, Theorem 9.8]. However, the spectrum $\mathbb{K}_{\infty}(A, B)$ is slightly different from the *stable Kasparov K-theory spectrum* defined in [5, Section 9].

5.3 Equivariant bivariant algebraic K-theory

Let G be a group. We briefly recall the definition of G-stability from [4, Section 3].

A *G-ℓ-module* is an ℓ-module with an action of G. A *G-ℓ-module with basis* is a pair $(\mathcal{W}, B)$ where $\mathcal{W}$ is a G-ℓ-module that is a free ℓ-module with basis B. If $(\mathcal{W}, B)$ is a G-ℓ-module with basis, we define:

$$\mathcal{L}(\mathcal{W}, B) := \{\psi : B \times B \to \ell : \{v : \psi(v, w) \neq 0\} \text{ is finite for all } w\}$$

Notice that $\mathcal{L}(\mathcal{W}, B)$ is an ℓ-algebra with the usual matrix product. We have an ℓ-algebra isomorphism:

$$\mathrm{End}_{\ell}(\mathcal{W}) \to \mathcal{L}(\mathcal{W}, B), \quad f \mapsto \psi_f, \quad \psi_f(v, w) = p_v(f(w)) \tag{11}$$

where $p_v : \mathcal{W} \to \ell$ is the projection into the submodule of W generated by v. Put:

$$C(\mathcal{W}, B) = \{\psi \in \mathcal{L}(\mathcal{W}, B) : \{w : \psi(v, w) \neq 0\} \text{ is finite for all } v\}$$
$$\mathcal{F}(\mathcal{W}, B) = \{\psi \in \mathcal{L}(\mathcal{W}, B) : \{(v, w) : \psi(v, w) \neq 0\} \text{ is finite}\}$$

Note that $C(\mathcal{W}, B)$ is a unital subalgebra of $\mathcal{L}(\mathcal{W}; B)$ and $\mathcal{F}(\mathcal{W}, B)$ is an ideal of $C(\mathcal{W}, B)$. Put:

$$\mathrm{End}_{\ell}^C(\mathcal{W}, B) = \{f \in \mathrm{End}_{\ell}(\mathcal{W}) : \psi_f \in C(\mathcal{W}, B)\}$$
$$\mathrm{End}_{\ell}^F(\mathcal{W}, B) = \{f \in \mathrm{End}_{\ell}(\mathcal{W}) : \psi_f \in \mathcal{F}(\mathcal{W}, B)\}$$

The isomorphism (11) identifies $\mathrm{End}_{\ell}^C(\mathcal{W}, B) \cong C(\mathcal{W}, B)$ and $\mathrm{End}_{\ell}^F(\mathcal{W}, B) \cong \mathcal{F}(\mathcal{W}, B)$. Clearly, $\mathrm{End}_{\ell}^C(\mathcal{W}, B) \subseteq \mathrm{End}_{\ell}(\mathcal{W})$ is a subalgebra and $\mathrm{End}_{\ell}^F(\mathcal{W}, B) \subseteq \mathrm{End}_{\ell}^C(\mathcal{W}, B)$ is an ideal.

Example 5.3.1. Let M_G be the ℓ-algebra of matrices with coefficients in ℓ indexed by $G \times G$. We consider M_G as a G-ℓ-algebra with the action:

$$g \cdot e_{s,t} = e_{gs,gt}$$

As usual, for $A \in \mathrm{GAlg}_{\ell}$, we will write $M_G A$ instead of $M_G \otimes_{\ell} A$ —with the diagonal G-action. As explained in [4, Example 3.1.3], $M_G = \mathcal{F}(\mathcal{W}, B) \cong \mathrm{End}_{\ell}^F(\mathcal{W}, B)$ where $\mathcal{W} = \ell[G]$ is the group algebra with the regular representation.

Let $(\mathcal{W}; B)$ be a G-ℓ-module with basis and consider the representation $\rho : G \to \text{End}_\ell(\mathcal{W})$, $\rho_g(w) = g \cdot w$. We say that $(\mathcal{W}, B)$ is a *G-ℓ-module by locally finite automorphisms* if $\rho(G) \subseteq \text{End}_\ell^C(\mathcal{W}, B)$. In this case, $\text{End}_\ell^F(\mathcal{W}, B)$ and $\text{End}_\ell^C(\mathcal{W}, B)$ are G-ℓ-algebras with the action:

$$g \cdot f = \rho_g \circ f \circ \rho_g^{-1}$$

Definition 5.3.2. Let $(\mathcal{W}_1, B_1)$ and $(\mathcal{W}_2, B_2)$ be G-ℓ-modules by locally finite automorphisms such that $|B_1|, |B_2| \leq |G \times \mathbb{N}|$. The inclusion $\mathcal{W}_1 \subseteq \mathcal{W}_1 \oplus \mathcal{W}_2$ induces a morphism of G-ℓ-algebras:

$$\iota : \text{End}_\ell^F(\mathcal{W}_1) \to \text{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2), \quad f \mapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}.$$

A functor $X : G\text{Alg}_\ell \to \mathfrak{C}$ is *G-stable* if, for any $(\mathcal{W}_1, B_1)$ and $(\mathcal{W}_2, B_2)$ as above and any G-ℓ-algebra A, the following morphism in $\mathfrak{C}$ is an isomorphism:

$$X(\iota \otimes_\ell A) : X(\text{End}_\ell^F(\mathcal{W}_1) \otimes_\ell A) \to X(\text{End}_\ell^F(\mathcal{W}_1 \oplus \mathcal{W}_2) \otimes_\ell A)$$

Proposition 5.3.3 (cf. [4, Proposition 3.1.9]). *Let G be a group and let $\mathcal{X}$ be a set with $|\mathcal{X}| = |G \times \mathbb{N}|$. Let $X : G\text{Alg}_\ell \to \mathfrak{C}$ be an $M_\mathcal{X}$-stable functor. Then the following composite functor is G-stable:*

$$G\text{Alg}_\ell \xrightarrow{\ M_G \otimes_\ell ?\ } G\text{Alg}_\ell \xrightarrow{\ X\ } \mathfrak{C}$$

Proof. The proof is like that of [4, Proposition 3.1.9], replacing M_∞ with $M_\mathcal{X}$.   □

Definition 5.3.4 (cf. [4, Section 4.1]). We proceed to define the *G-equivariant bivariant algebraic K-theory category* $\mathfrak{K}^G$. From now on, fix a set $\mathcal{X}$ with $|\mathcal{X}| = |G \times \mathbb{N}|$. It is easily seen that the functor $j_\mathcal{X} \circ (M_G \otimes_\ell ?) : G\text{Alg}_\ell \to \mathfrak{K}_\mathcal{X}^{G\text{Alg}_\ell}$ is an excisive, homotopy invariant and $M_\mathcal{X}$-stable homology theory. By Theorem 5.2.16 there exists a unique triangulated functor M_G making the following diagram commute:

$$
\begin{array}{ccc}
G\text{Alg}_\ell & \xrightarrow{\ j_\mathcal{X}\ } & \mathfrak{K}_\mathcal{X}^{G\text{Alg}_\ell} \\
{\scriptstyle M_G \otimes_\ell ?}\Big\downarrow & & \Big\downarrow{\scriptstyle M_G} \\
G\text{Alg}_\ell & \xrightarrow{\ j_\mathcal{X}\ } & \mathfrak{K}_\mathcal{X}^{G\text{Alg}_\ell}
\end{array}
$$

Explicitely, we have $M_G(A, m) = (M_G \otimes_\ell A, m)$. Define $\mathfrak{K}^G$ as follows: The objects of $\mathfrak{K}^G$ are the objects of $\mathfrak{K}_\mathcal{X}^{G\text{Alg}_\ell}$. The hom-sets are defined by:

$$\text{Hom}_{\mathfrak{K}^G}((A, m), (B, n)) := \mathfrak{K}_\mathcal{X}^{G\text{Alg}_\ell}(M_G(A, m), M_G(B, n))$$

Let $a \in \mathfrak{K}_\mathcal{X}^{G\text{Alg}_\ell}(M_G(A, m), M_G(B, n))$. We will often consider a both as a morphism in $\mathfrak{K}_\mathcal{X}^{G\text{Alg}_\ell}$ and as a morphism $(A, m) \to (B, n)$ in $\mathfrak{K}^G$. To avoid ambiguity, we will write $[a]$ instead of a when considering a as a morphism in $\mathfrak{K}^G$ and we will say that $[a]$ is *represented* by a. The composition in $\mathfrak{K}^G$ is the one induced by the composition in $\mathfrak{K}_\mathcal{X}^{G\text{Alg}_\ell}$;

this means that for $a \in \mathfrak{K}_X^{GAlg_\ell}(M_G(A, m), M_G(B, n))$ and $b \in \mathfrak{K}_X^{GAlg_\ell}(M_G(B, n), M_G(C, r))$ we have:

$$[b] \circ [a] = [b \circ a]$$

It is easily seen that the translation functor in $\mathfrak{K}_X^{GAlg_\ell}$ induces a translation functor in $\mathfrak{K}^G$ that makes it into a graded category.

There is a functor $j^G : GAlg_\ell \to \mathfrak{K}^G$ that sends a G-ℓ-algebra A to $(A, 0)$ and a morphism of G-ℓ-algebras $f : A \to B$ to:

$$[j_X(M_G \otimes_\ell f)] = [M_G j_X(f)] \in \mathfrak{K}^G((A, 0), (B, 0))$$

There is a functor $t^G : \mathfrak{K}_X^{GAlg_\ell} \to \mathfrak{K}^G$ that is the identity on objects and that sends a morphism $a \in \mathfrak{K}_X^{GAlg_\ell}((A, m), (B, n))$ to:

$$[M_G(a)] \in \mathfrak{K}^G((A, m), (B, n))$$

It is easily seen that t^G is a graded functor and that the following diagram commutes:

$$
\begin{array}{ccc}
GAlg_\ell & \xrightarrow{\;j_X\;} & \mathfrak{K}_X^{GAlg_\ell} \\
& \searrow^{j^G} & \downarrow^{t^G} \\
& & \mathfrak{K}^G
\end{array}
$$

Lemma 5.3.5. *The functor* $j^G : GAlg_\ell \to \mathfrak{K}^G$ *is G-stable.*

Proof. Let $X : GAlg_\ell \to \mathfrak{C}$ be a functor. By Yoneda, X is G-stable is and only if $\mathrm{Hom}_\mathfrak{C}(c, X(?)) : GAlg_\ell \to \mathrm{Set}$ is G-stable for every object c of $\mathfrak{C}$. Thus, it is enough to show that $\mathfrak{K}^G((A, m), j^G(?)) : GAlg_\ell \to \mathrm{Set}$ is G-stable for any (A, m). It is easy to see that $\mathfrak{K}^G((A, m), j^G(?))$ equals the following composite:

$$GAlg_\ell \xrightarrow{\;M_G \otimes_\ell ?\;} GAlg_\ell \xrightarrow{\;j_X\;} \mathfrak{K}_X^{GAlg_\ell} \xrightarrow{\;\mathfrak{K}_X^{GAlg_\ell}(M_G(A,m),?)\;} \mathrm{Set}$$

This composite is M_G-stable by Proposition 5.3.3. $\qquad\qquad\square$

Let $G \sqcup \{*\}$ be the disjoint union of the G-sets G and the singleton $\{*\}$; here we consider G as a G-set with the action by left translation. The inclusions of $\{*\}$ and G into $G \sqcup \{*\}$ induce morphisms of G-ℓ-algebras $\iota : \ell \to M_{G \sqcup \{*\}}$ and $\iota' : M_G \to M_{G \sqcup \{*\}}$ respectively. By Theorem 5.2.18, upon tensoring with ι and ι' we get graded natural transformations:

$$\iota_{(A, m)} \in \mathfrak{K}_X^{GAlg_\ell}((A, m), M_{G \sqcup \{*\}}(A, m))$$

$$\iota'_{(A, m)} \in \mathfrak{K}_X^{GAlg_\ell}(M_G(A, m), M_{G \sqcup \{*\}}(A, m))$$

It follows from the G-stability of $j^G : GAlg_\ell \to \mathfrak{K}^G$ that we have the following identifications in $\mathfrak{K}^G$; see [4, Example 3.1.8] for details:

$$(A, m) \xrightarrow[\cong]{\;t^G(\iota_{(A,m)})\;} M_{G \sqcup \{*\}}(A, m) \xleftarrow[\cong]{\;t^G(\iota'_{(A,m)})\;} M_G(A, m)$$

Lemma 5.3.6. *Let $f \in \mathfrak{K}_X^{\mathrm{GAlg}_\ell}(M_G(A, m), M_G(B, n))$. Then the following diagram in $\mathfrak{K}^G$ commutes:*

$$
\begin{array}{ccc}
M_G(A, m) & \xrightarrow{\;\; t^G(f) \;\;} & M_G(B, n) \\[4pt]
{\scriptstyle t^G(\iota'_{(A,m)})} \downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \downarrow {\scriptstyle t^G(\iota'_{(B,n)})} \\[4pt]
M_{G \sqcup \{*\}}(A, m) & & M_{G \sqcup \{*\}}(B, n) \\[4pt]
{\scriptstyle t^G(\iota_{(A,m)})} \uparrow {\scriptstyle \cong} & & {\scriptstyle \cong} \uparrow {\scriptstyle t^G(\iota_{(B,n)})} \\[4pt]
(A, m) & \xrightarrow{\;\; [f] \;\;} & (B, n)
\end{array}
\tag{12}
$$

Proof. To alleviate notation we will write $G_+ = G \sqcup \{*\}$, $X = (A, m)$ and $Y = (B, n)$. Let $S : M_G M_G \to M_G M_G$ and $S' : M_G M_{G_+} \to M_{G_+} M_G$ be the isomorphisms defined by the formula $x \otimes y \mapsto y \otimes x$. As usual, by Theorem 5.2.18, tensoring with S and S' induces natural transformations in $\mathfrak{K}_X^{\mathrm{GAlg}_\ell}$:

$$
S_{(A,m)} : M_G M_G(A, m) \longrightarrow M_G M_G(A, m)
$$
$$
S'_{(A,m)} : M_G M_{G_+}(A, m) \longrightarrow M_{G_+} M_G(A, m)
$$

It is easily verified that the following diagram in $\mathfrak{K}_X^{\mathrm{GAlg}_\ell}$ commutes:

$$
\begin{array}{ccccccc}
M_G M_G X & \xrightarrow[\cong]{S_X} & M_G M_G X & \xrightarrow{M_G f} & M_G M_G Y & \xleftarrow[\cong]{S_Y} & M_G M_G Y \\[4pt]
{\scriptstyle M_G(\iota'_X)} \downarrow & & {\scriptstyle \iota'_{M_G X}} \downarrow {\scriptstyle \cong} & & {\scriptstyle \cong} \downarrow {\scriptstyle \iota'_{M_G Y}} & & \downarrow {\scriptstyle M_G(\iota'_Y)} \\[4pt]
M_G M_{G_+} X & \xrightarrow[\cong]{S'_X} & {}' M_{G_+} M_G X & \xrightarrow{M_{G_+} f} & M_{G_+} M_G Y & \xleftarrow[\cong]{S'_Y} & M_G M_{G^+} Y \\[4pt]
& {\scriptstyle M_G(\iota_X)} \searrow & {\scriptstyle \iota_{M_G X}} \uparrow {\scriptstyle \cong} & & {\scriptstyle \cong} \uparrow {\scriptstyle \iota_{M_G Y}} & {\scriptstyle M_G(\iota_Y)} \nearrow & \\[4pt]
& & M_G X & \xrightarrow{\;\; f \;\;} & M_G Y & &
\end{array}
$$

Thus, to show that (12) commutes it is enough to show that S_X equals the identity of $M_G M_G X$. Suppose for a moment that the G-ℓ-algebra homomorphism $d : M_G \to M_G M_G$, $d(e_{g,h}) = e_{g,h} \otimes e_{g,h}$, induces an isomorphism in $\mathfrak{K}_X^{\mathrm{GAlg}_\ell}$:

$$
d_X : M_G X \xrightarrow{\;\cong\;} M_G M_G X
\tag{13}
$$

It is easily verified that $S_X \circ d_X = d_X$. Since d_X is an isomorphism, it follows that $S_X = \mathrm{id}_{M_G M_G X}$. Let $M_{|G|}$ be the ℓ-algebra M_G considered as a G-ℓ-algebra with the trivial action of G. Let $T : M_G M_{|G|} \to M_G M_G$ be the G-ℓ-algebra isomorphism defined by $T(e_{g,h} \otimes e_{s,t}) = e_{g,h} \otimes e_{gs,ht}$. It is easily verified that $d : M_G \to M_G M_G$ equals the following composite:

$$
M_G \xrightarrow{\;\; ? \otimes e_{1,1} \;\;} M_G M_{|G|} \xrightarrow[\cong]{\;\; T \;\;} M_G M_G
$$

The map on the left induces an isomorphism in $\mathfrak{K}_X^{\mathrm{GAlg}_\ell}$ by M_X-stability; this implies that (13) is an isomorphism. $\qquad\square$

Define a triangulation in $\mathfrak{K}^G$ as follows: a triangle in $\mathfrak{K}^G$ is distinguished if and only if it is isomorphic (in $\mathfrak{K}^G$) to the image by $t^G : \mathfrak{K}_X^{\mathrm{GAlg}_\ell} \to \mathfrak{K}^G$ of a distinguished triangle in $\mathfrak{K}_X^{\mathrm{GAlg}_\ell}$.

Lemma 5.3.7. *The definition above makes $\mathfrak{K}^G$ into a triangulated category.*

Proof. Axioms (TR0) and (TR2) are obvious. Axiom (TR1) follows immediately from Lemma 5.3.6, since every morphism in $\mathfrak{K}^G$ is isomorphic —in the arrow category of $\mathfrak{K}^G$— to a morphism lying in the image of t^G.

Let us show that (TR3) holds: Consider a diagram in $\mathfrak{K}^G$ like the diagram of solid arrows (39) on page 71; we may assume that the rows are images by t^G of distinguished triangles in $\mathfrak{K}_\chi^{G\mathrm{Alg}_\ell}$:

$$
\begin{array}{ccccccc}
LZ' & \xrightarrow{t^G(c')} & X' & \xrightarrow{t^G(a')} & Y' & \xrightarrow{t^G(b')} & Z' \\
& & & & \downarrow{[f]} & & \downarrow{[g]} \\
LZ & \xrightarrow{t^G(c)} & X & \xrightarrow{t^G(a)} & Y & \xrightarrow{t^G(b)} & Z
\end{array}
$$

By Lemma 5.3.6, the diagram above is isomorphic to the following one:

$$
\begin{array}{ccccccc}
LM_GZ' & \xrightarrow{t^G M_G(c')} & M_GX' & \xrightarrow{t^G M_G(a')} & M_GY' & \xrightarrow{t^G M_G(b')} & M_GZ' \\
& & & & \downarrow{t^G(f)} & & \downarrow{t^G(g)} \\
LM_GZ & \xrightarrow{t^G M_G(c)} & M_GX & \xrightarrow{t^G M_G(a)} & M_GY & \xrightarrow{t^G M_G(b)} & M_GZ
\end{array} \tag{14}
$$

This diagram is the image by t^G of a diagram in $\mathfrak{K}_\chi^{G\mathrm{Alg}_\ell}$ whose rows are distinguished triangles, but the square in the latter may not commute. Nevertheless, the following square in $\mathfrak{K}_\chi^{G\mathrm{Alg}_\ell}$ does commute:

$$
\begin{array}{ccc}
M_GM_GY' & \xrightarrow{M_GM_G(b')} & M_GM_GZ' \\
\downarrow{M_G(f)} & & \downarrow{M_G(g)} \\
M_GM_GY & \xrightarrow{M_GM_G(b)} & M_GM_GZ
\end{array}
$$

Thus, we can complete the following diagram to a morphism of triangles in $\mathfrak{K}_\chi^{G\mathrm{Alg}_\ell}$:

$$
\begin{array}{ccccccc}
LM_GM_GZ' & \xrightarrow{M_GM_G(c')} & M_GM_GX' & \xrightarrow{M_GM_G(a')} & M_GM_GY' & \xrightarrow{M_GM_G(b')} & M_GM_GZ' \\
& & & & \downarrow{M_G(f)} & & \downarrow{M_G(g)} \\
LM_GM_GZ & \xrightarrow{M_GM_G(c)} & M_GM_GX & \xrightarrow{M_GM_G(a)} & M_GM_GY & \xrightarrow{M_GM_G(b)} & M_GM_GZ
\end{array} \tag{15}
$$

Using Lemma 5.3.6 once more, diagram (14) is isomorphic to the one obtained upon applying t^G to (15); this proves axiom (TR3).

To prove Axiom (TR4) note that, by Lemma 5.3.6, every diagram in $\mathfrak{K}^G$ like (43) on page 73 is isomorphic to one lying in the image of t^G. □

The functor $t^G : \mathfrak{K}_\chi^{G\mathrm{Alg}_\ell} \to \mathfrak{K}^G$ is obviously triangulated. Let $\mathscr{E} : A \to B \to C$ be an extension in $G\mathrm{Alg}_\ell$. Since $j : G\mathrm{Alg}_\ell \to \mathfrak{K}_\chi^{G\mathrm{Alg}_\ell}$ is an excisive homology theory, we have a morphism:

$$
\partial_{\mathscr{E}} \in \mathfrak{K}_\chi^{G\mathrm{Alg}_\ell}(LC, A)
$$

Define $\partial_{\mathscr{E}}^G := t^G(\partial_{\mathscr{E}}) \in \mathfrak{K}^G(LC, A)$.

Theorem 5.3.8 ([4, Theorem 4.1.1])**.** *The functor $j^G : GAlg_\ell \to \mathfrak{K}^G$ together with the collection $\{\partial_{\mathcal{E}}^G\}_{\mathcal{E}}$ is the universal excisive, homotopy invariant and G-stable homology theory.*

Proof. Let $X : GAlg_\ell \to \mathcal{T}$ be an excisive, homotopy invariant and G-stable homology theory. Since X is G-stable then it is M_χ-stable; see [4, Section 3]. By Theorem 5.2.16 there exists a unique triangulated functor $\bar{X}$ making the following diagram commute:

$$
\begin{array}{ccc}
GAlg_\ell & \xrightarrow{\;j_\chi\;} & \mathfrak{K}_\chi^{GAlg_\ell} \\
& \searrow{\scriptstyle X} & \downarrow{\scriptstyle \bar{X}} \\
& & \mathcal{T}
\end{array}
$$

Suppose that there is a triangulated functor $X' : \mathfrak{K}^G \to \mathcal{T}$ such that $X' \circ j^G = X$. Then the following diagram commutes:

$$
\begin{array}{ccc}
GAlg_\ell & \xrightarrow{\;j_\chi\;} & \mathfrak{K}_\chi^{GAlg_\ell} \\
& \searrow{\scriptstyle X} & \downarrow{\scriptstyle X' \circ t^G} \\
& & \mathcal{T}
\end{array}
$$

By the uniqueness of $\bar{X}$ we have $\bar{X} = X' \circ t^G$. Then, by Lemma 5.3.6, we must have

$$
X'([f]) = \bar{X}(\iota_{(B,n)})^{-1} \circ \bar{X}(\iota'_{(B,n)}) \circ \bar{X}(f) \circ \bar{X}(\iota'_{(A,m)})^{-1} \circ \bar{X}(\iota_{(A,m)})
$$

for any $f \in \mathfrak{K}_\chi^{GAlg_\ell}(M_G(A, m), M_G(B, n))$. It is straightforward to verify that the equation above defines a triangulated functor $X' : GAlg_\ell \to \mathcal{T}$ with the desired properties. □

Remark 5.3.9. For a countable group G, a universal excisive, homotopy invariant and G-stable homology theory was already constructed by Ellis [4, Theorem 4.1.1]. The results above extend the construction in op. cit. to arbitrary groups.

A spectrum for G-equivariant bivariant algebraic K-theory

Definition 5.3.10. Let (A, B) be a pair of G-ℓ-algebras. The *G-equivariant bivariant K-theory spectrum* $\mathbb{K}^G(A, B)$ is defined as:

$$
\mathbb{K}^G(A, B) := \mathbb{K}_\chi^{GAlg_\ell}(M_G A, M_G B)
$$

Theorem 5.3.11. *For every $m \in \mathbb{Z}$ there is a natural isomorphism:*

$$
\pi_m \mathbb{K}^G(A, B) \cong \mathfrak{K}^G(A, (B, m))
$$

Proof. It follows immediately from Theorem 5.2.22 and the definition of $\mathfrak{K}^G$. □

Crossed product

From now on, we assume G to be a countable group. Let $\rtimes G : \mathrm{GAlg}_\ell \to \mathrm{Alg}_\ell$ be the crossed product functor. It can be shown that this functor induces a unique triangulated functor making the following diagram commute; see [4, Proposition 5.1.2]:

$$
\begin{array}{ccc}
\mathrm{GAlg}_\ell & \xrightarrow{\ j^G\ } & \mathfrak{K}^G \\
{\scriptstyle \rtimes G}\Big\downarrow & & \Big\downarrow{\scriptstyle \exists!\ \rtimes G} \\
\mathrm{Alg}_\ell & \xrightarrow{\ j_\infty\ } & \mathfrak{K}_\infty
\end{array}
$$

Example 5.3.12. Let A be a G-ℓ-algebra. Recall from Lemma 5.3.6 that we have isomorphisms in $\mathfrak{K}^G$:

$$
A \xrightarrow[\cong]{\ t^G(\iota_A)\ } M_{G_+}A \xleftarrow[\cong]{\ t^G(\iota'_A)\ } M_G A
$$

Upon applying $\rtimes G$ to the zig-zag above, we get the following diagram in $\mathfrak{K}_\infty$:

$$
A \rtimes G \xrightarrow[\cong]{\ j_\infty(\iota_A \rtimes G)\ } (M_{G_+}A) \rtimes G \xleftarrow[\cong]{\ j_\infty(\iota'_A \rtimes G)\ } (M_G A) \rtimes G \tag{16}
$$

We have ℓ-algebra isomorphisms $(M_{G_+}A)\rtimes G \cong M_{|G_+|}(A\rtimes G)$ and $(M_G A)\rtimes G \cong M_{|G|}(A\rtimes G)$ given by the formula $(e_{s,t}\otimes a)\rtimes g \leftrightarrow e_{s,g^{-1}\cdot t}\otimes(a\rtimes g)$; see [4, Proposition 5.1.1] for details. Using these identifications and the M_∞-stability, it is easily verified that the composite (16) equals the following composite in $\mathfrak{K}_\infty$:

$$
A \rtimes G \xrightarrow[\cong]{\ j_\infty(e_{1,1}\otimes?)\ } M_{|G|}(A \rtimes G) \cong (M_G A) \rtimes G \tag{17}
$$

Let $e : G \to \mathbb{N}$ be any injective function, so that e induces an ℓ-algebra homomorphism $e : M_{|G|} \to M_\infty$. It is also easy to see that the inverse of (17) is the morphism represented —in the sense explained in Section 5.2— by the following composite in $\mathfrak{K}_f$:

$$
(M_G A) \rtimes G \cong M_{|G|}(A \rtimes G) \xrightarrow{\ e\ } M_\infty(A \rtimes G)
$$

Let A and B be two G-ℓ-algebras. We aim to construct a morphism of spectra $\rtimes G : \mathbb{K}^G(A,B) \to \mathbb{K}_\infty(A \rtimes G, B \rtimes G)$ that induces $\rtimes G : \mathfrak{K}^G \to \mathfrak{K}_\infty$ upon taking π_n. We start by defining a morphism:

$$
\mathbb{K}(M_G A, M_\infty M_G B) \xrightarrow{\ \psi\ } \mathbb{K}((M_G A) \rtimes G, (M_\infty M_G B) \rtimes G) \tag{18}
$$

In order to describe (18) at level n, recall that:

$$
\mathscr{K}^C(J^n C, D)_q \cong \operatorname*{colim}_v \operatorname*{colim}_r \operatorname{Hom}(J^{n+v}C, D_r^{(I^v\times\Delta^q, \partial I^v\times\Delta^q)}) \tag{19}
$$

Here, C and D may be either two ℓ-algebras or two G-ℓ-algebras. Represent an element of $\mathscr{K}(J^n(M_G A), M_\infty M_G B)_q$ by $f \in \operatorname{Hom}_{\mathrm{GAlg}_\ell}(J^{n+v}(M_G A), (M_\infty M_G B)_r^{(I^v\times\Delta^q, \partial I^v\times\Delta^q)})$, and define $\psi^n(f)$ as the class of the following composite:

$$
\begin{array}{ccc}
J^{n+v}((M_G A) \rtimes G) \longrightarrow J^{n+v}(M_G A) \rtimes G \xrightarrow{\ f\rtimes G\ } (M_\infty M_G B)_r^{(I^v\times\Delta^q, \partial I^v\times\Delta^q)} \rtimes G \\
\Big\downarrow{\scriptstyle \cong} \\
\xdashrightarrow{\ \psi^n(f)\ } ((M_\infty M_G B) \rtimes G)_r^{(I^v\times\Delta^q, \partial I^v\times\Delta^q)}
\end{array}
$$

Here, the vertical isomorphism is the obvious one; if G acts trivially on a ring C, then $? \otimes C$ and $? \rtimes G$ commute. We have defined a morphism:

$$\mathscr{K}^{\mathrm{GAlg}_\ell}(J^n(M_G A), M_\infty M_G B) \xrightarrow{\psi^n} \mathscr{K}^{\mathrm{Alg}_\ell}(J^n((M_G A) \rtimes G), (M_\infty M_G B) \rtimes G) \qquad (20)$$

One has to check that the given formula for ψ^n is compatible with the transition maps of the colimits in (19); this is a straightforward verification. Another verification shows that, for varying n, the morphisms (20) assemble into a morphism of spectra like (18). Now fix an injection $e : G \to \mathbb{N}$ and a bijection $\theta : \mathbb{N} \times G \to \mathbb{N}$, and consider the following morphisms of ℓ-algebras:

$$A \rtimes G \xrightarrow{e_{1,1} \otimes ?} M_{|G|}(A \rtimes G) \cong (M_G A) \rtimes G$$

$$(M_\infty M_G B) \rtimes G \cong M_\infty M_{|G|}(B \rtimes G) \xrightarrow{e} M_\infty M_\infty(B \rtimes G) \xrightarrow{\theta} M_\infty(B \rtimes G) \qquad (21)$$

Let $\rtimes G$ be the following composite, where the vertical morphism is induced by (21):

$$\mathbb{K}^{\mathrm{GAlg}_\ell}(M_G A, M_\infty M_G B) \xrightarrow{\psi} \mathbb{K}^{\mathrm{Alg}_\ell}((M_G A) \rtimes G, (M_\infty M_G B) \rtimes G)$$

$$\xrightarrow{\rtimes G} \mathbb{K}^{\mathrm{Alg}_\ell}(A \rtimes G, M_\infty(B \rtimes G))$$

It is clear that $\rtimes G : \mathbb{K}^{\mathrm{GAlg}_\ell}(M_G A, M_\infty M_G B) \to \mathbb{K}^{\mathrm{Alg}_\ell}(A \rtimes G, M_\infty(B \rtimes G))$ is functorial in A and B with respect to morphisms of G-ℓ-algebras. Taking the colimit of $\rtimes G$ along the system

$$B = M_1 B \xrightarrow{\iota_{1,2}} M_2 B \xrightarrow{\iota_{2,3}} M_3 B \longrightarrow \cdots$$

we get a morphism:

$$\rtimes G : \mathbb{K}^G(A, B) \longrightarrow \mathbb{K}_\infty(A \rtimes G, B \rtimes G) \qquad (22)$$

Proposition 5.3.13. *Let A and B be two G-ℓ-algebras. Upon taking π_n and making the identifications in Theorems 5.2.22 and 5.3.11, the morphism (22) induces:*

$$\rtimes G : \mathfrak{K}^G(A, (B, n)) \longrightarrow \mathfrak{K}_\infty(A \rtimes G, (B, n) \rtimes G)$$

Proof. Represent $f' \in \mathfrak{K}^G(A, (B, n))$ by $f \in \mathrm{Hom}_{\mathrm{GAlg}_\ell}(J^\nu(M_G A), (M_m M_\infty M_G B)_r^{\mathfrak{S}_{n+\nu}})$. It is straightforward to verify that, under the identifications in Theorems 5.2.22 and 5.3.11, $\pi_n(\rtimes G)(f')$ is represented by the composite of the solid ℓ-algebra homomorphisms in the following diagram:

$$
\begin{array}{ccc}
J^\nu(A \rtimes G) \xrightarrow{(21)} J^\nu((M_G A) \rtimes G) \longrightarrow & J^\nu(M_G A) \rtimes G \\
& & \downarrow{\scriptstyle f \rtimes G} \\
& & (M_m M_\infty M_G B)_r^{\mathfrak{S}_{n+\nu}} \rtimes G \\
& & \downarrow{\scriptstyle \cong} \qquad\qquad (23) \\
& & (M_m M_\infty(M_G B) \rtimes G)_r^{\mathfrak{S}_{n+\nu}} \\
& & \downarrow{\scriptstyle (21)} \\
& & M_m M_\infty(B \rtimes G)_r^{\mathfrak{S}_{n+\nu}}
\end{array}
$$

The equality $\pi_n(\rtimes G)(f') = f' \rtimes G$ follows from a lengthy verification, of which we proceed to sketch the main steps. It is easily seen that the crossed product with G induces a triangulated functor $\rtimes G : \Re_\infty^{G\mathrm{Alg}_\ell} \to \Re_\infty^{\mathrm{Alg}_\ell}$ making the following diagram commute:

$$
\begin{array}{ccccc}
G\mathrm{Alg}_\ell & \xrightarrow{\ j_\infty\ } & \Re_\infty^{G\mathrm{Alg}_\ell} & \xrightarrow{\ t^G\ } & \Re^G \\
& \searrow{\scriptstyle j_\infty(?\rtimes G)} & \downarrow{\scriptstyle \rtimes G} & \swarrow{\scriptstyle \rtimes G} & \\
& & \Re_\infty^{\mathrm{Alg}_\ell} & &
\end{array}
$$

Note that f represents a morphism $f'' \in \Re_\infty^{G\mathrm{Alg}_\ell}(M_G A, (M_G B, n))$; in the notation of Section 5.3 we have $f' = [f'']$. By the proofs of Theorems 3.13.11, 5.1.11 and 5.2.15, $f'' \rtimes G$ is represented by the dotted morphism in (23). By Lemma 5.3.6, the following diagram in $\Re_\infty^{\mathrm{Alg}_\ell}$ commutes:

$$
\begin{array}{ccc}
(M_G A) \rtimes G & \xrightarrow{\ f'' \rtimes G\ } & ((M_G B) \rtimes G, n) \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
(M_{G_+} A) \rtimes G & & ((M_{G_+} B) \rtimes G, n) \\
{\scriptstyle \cong}\uparrow & & \uparrow{\scriptstyle \cong} \\
A \rtimes G & \xrightarrow[\ f' \rtimes G\ ]{} & (B, n)
\end{array}
$$

The diagram above —together with Example 5.3.12— implies that $f' \rtimes G$ is represented by the composite of the solid morphisms in (23). This finishes the proof. $\qquad\square$

The Green-Julg Theorem

Let $\tau : \mathrm{Alg}_\ell \to G\mathrm{Alg}_\ell$ be the functor that sends an ℓ-algebra A to the same algebra, considered as a G-ℓ-algebra with the trivial G-action. It can be shown that this functor induces a unique triangulated functor making the following diagram commute; see [4, Section 5.1]:

$$
\begin{array}{ccc}
\mathrm{Alg}_\ell & \xrightarrow{\ j_\infty\ } & \Re_\infty \\
{\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \exists!\,\tau} \\
G\mathrm{Alg}_\ell & \xrightarrow[\ j^G\ ]{} & \Re^G
\end{array}
$$

We recall the Green-Julg Theorem for bivariant algebraic K-theory.

Theorem 5.3.14 ([4, Theorem 5.2.1])**.** *Let G be a finite group of n elements and suppose that n is invertible in ℓ. Then there is an adjunction:*

$$
\Re^G(A^\tau, B) \cong \Re_\infty(A, B \rtimes G)
$$

Proof. Let A be an ℓ-algebra and define $\alpha_A : A \to A^\tau \rtimes G$ by $\alpha_A(a) = a \otimes \frac{1}{n}\sum_{g \in G} g$; it is easily verified that α_A is an ℓ-algebra homomorphism. Put:

$$
\bar{\alpha}_A := j(\alpha_A) \in \Re_\infty(A, A^\tau \rtimes G)
$$

Let B be a G-ℓ-algebra and let $\iota_B : B \to M_G B$ be the G-equivariant morphism given by:

$$\iota_B(b) = \frac{1}{n} \sum_{p,g \in G} e_{p,q} \otimes b$$

Note that $j^G(\iota_B)$ is an isomorphism by [4, Remark 3.1.11]. Let $\beta_B : (B \rtimes G)^\tau \to M_G B$ be the G-equivariant morphism defined by:

$$\beta_B(b \otimes g) = \sum_{s \in G} e_{s,sg} \otimes s(b)$$

Finally, let $\bar{\beta}_B \in \mathfrak{R}^G((B \rtimes G)^\tau, B)$ be the following composite:

$$(B \rtimes G)^\tau \xrightarrow{\ j^G(\beta_B)\ } M_G B \xleftarrow[\cong]{\ j^G(\iota_B)\ } B$$

One checks that $\bar{\alpha}_A$ and $\bar{\beta}_B$ are respectively the unit and counit of an adjunction; see the proof of [4, Theorem 5.2.1]. $\qquad\square$

Theorem 5.3.15 (cf. [4, Theorem 5.2.1])**.** *Let G be a finite group of n elements and suppose that n is invertible in ℓ. Let $A \in \mathrm{Alg}_\ell$ and let $B \in \mathrm{GAlg}_\ell$. Let $\alpha_A : A \to A^\tau \rtimes G$ be the ℓ-algebra homomorphism defined in the proof of Theorem 5.3.14. Then the following composite is a weak equivalence of spectra:*

$$\mathbb{K}^G(A^\tau, B) \xrightarrow[(22)]{\rtimes G} \mathbb{K}_\infty(A^\tau \rtimes G, B \rtimes G) \xrightarrow{(\alpha_A)^*} \mathbb{K}_\infty(A, B \rtimes G) \tag{24}$$

Proof. Let ψ denote the composite in (24). By Proposition 5.3.13, upon making the identifications in Theorems 5.2.22 and 5.3.11, $\pi_n(\psi)$ equals the following composite:

$$\mathfrak{R}^G(A^\tau, (B, n)) \xrightarrow{\rtimes G} \mathfrak{R}_\infty(A^\tau \rtimes G, (B \rtimes G, n)) \xrightarrow{(\alpha_A)^*} \mathfrak{R}_\infty(A, (B \rtimes G, n))$$

Then $\pi_n(\psi)$ is an isomorphism by Theorem 5.3.14. $\qquad\square$

The restriction functor

Let G be a countable group and let $H \subseteq G$ be a subgroup. If B is a G-ℓ-algebra, we can restrict the action of G to obtain an H-ℓ-algebra $\mathrm{Res}_G^H(B)$; thus, we have a functor $\mathrm{Res}_G^H : \mathrm{GAlg}_\ell \to \mathrm{HAlg}_\ell$. By Theorem 5.3.8, there exists a unique triangulated functor $\mathrm{Res}_G^H : \mathfrak{R}^G \to \mathfrak{R}^H$ making the following diagram commute:

$$
\begin{array}{ccc}
\mathrm{GAlg}_\ell & \xrightarrow{\ \mathrm{Res}_G^H\ } & \mathrm{HAlg}_\ell \\
{\scriptstyle j^G}\downarrow & & \downarrow{\scriptstyle j^H} \\
\mathfrak{R}^G & \xrightarrow{\ \mathrm{Res}_G^H\ } & \mathfrak{R}^H
\end{array}
$$

Let A and B be two G-ℓ-algebras. We proceed to construct a morphism of spectra

$$\mathrm{Res} : \mathbb{K}^G(A, B) \longrightarrow \mathbb{K}^H(\mathrm{Res}_G^H(A), \mathrm{Res}_G^H(B)) \tag{25}$$

that induces $\operatorname{Res}_G^H : \mathfrak{K}^G(A, (B, n)) \to \mathfrak{K}^H(\operatorname{Res}_G^H(A), \operatorname{Res}_G^H(B, n))$ upon taking π_n and making the identifications in Theorem 5.3.11. We start by defining a morphism:

$$\mathbb{K}^{G\mathrm{Alg}_\ell}(M_G A, M_\infty M_G B) \xrightarrow{\;\psi\;} \mathbb{K}^{H\mathrm{Alg}_\ell}(M_H A, M_\infty M_H B) \tag{26}$$

First of all, fix a system of representatives $\{g_i\}$ of the cosets G/H. Each choice of representatives induces a bijection $G/H \times H \cong G$, $(g_i H, h) \mapsto h g_i^{-1}$, that is H-equivariant if we consider G/H as an H-set with trivial action. Moreover, this bijection induces an H-ℓ-algebra isomorphism $M_{|G/H|} M_H \cong M_G$. Now choose an injection $e : G/B \to \mathbb{N}$ and a bijection $\theta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. These choices determine a morphism of H-algebras:

$$M_\infty M_G \cong M_\infty M_{|G/H|} M_H \xrightarrow{M_\infty \otimes e \otimes M_H} M_\infty M_\infty M_H \xrightarrow{\theta \otimes M_H} M_\infty M_H \tag{27}$$

In order to describe (26) at level n, recall that:

$$\mathscr{K}(J^n C, D)_q \cong \operatorname*{colim}_v \operatorname*{colim}_r \operatorname{Hom}(J^{n+v} C, D_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)}) \tag{28}$$

Here, C and D may be either two G-algebras or two H-algebras. Represent an element of $\mathscr{K}(J^n(M_G A), M_\infty M_G B)_q$ by $f \in \operatorname{Hom}_{G\mathrm{Alg}_\ell}(J^{n+v}(M_G A), (M_\infty M_G B)_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)})$ and define $\psi^n(f)$ as the class of the following composite morphism of H-ℓ-algebras:

$$
\begin{array}{ccccc}
J^{n+v}(M_H A) & \longrightarrow & J^{n+v}(M_G A) & \xrightarrow{\;f\;} & M_\infty M_G B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)} \\
& & & & \Big\downarrow{\scriptstyle (27)} \\
& \xrightarrow{\quad\psi^n(f)\quad} & & & M_\infty M_H B_r^{(I^v \times \Delta^q, \partial I^v \times \Delta^q)}
\end{array}
$$

It is easily verified that the given formula for ψ^n is compatible with the transition maps of the colimits in (28); thus, it defines a morphism of simplicial sets:

$$\mathscr{K}(J^n(M_G A), M_G B) \xrightarrow{\;\psi^n\;} \mathscr{K}(J^n(M_H A), M_H B) \tag{29}$$

Another verification shows that, for varying n, the morphisms (29) assemble into a morphism of spectra like (26). It is clear that (26) is functorial in A and B with respect to morphisms of G-algebras. Taking the colimit of (26) along the system

$$B = M_1 B \xrightarrow{\;\iota_{1,2}\;} M_2 B \xrightarrow{\;\iota_{2,3}\;} M_3 B \longrightarrow \cdots$$

we get the desired morphism (25).

Theorem 5.3.16. *The morphism* $\operatorname{Res} : \mathbb{K}^G(A, B) \to \mathbb{K}^H(\operatorname{Res}_G^H(A), \operatorname{Res}_G^H(B))$ *constructed above induces* $\operatorname{Res}_G^H : \mathfrak{K}^G(A, (B, n)) \to \mathfrak{K}^H(\operatorname{Res}_G^H(A), \operatorname{Res}_G^H(B, n))$ *upon taking* π_n *and making the identifications in Theorem 5.3.11.*

Proof. It is a tedious but straightforward verification. $\qquad\square$

Adjuntion between induction and restriction

Let G be a countable group and let $H \subseteq G$ be a subgroup. Let $\pi : G \to G/H$ be the natural projection and let A be an H-ℓ-algebra. Put:

$$\mathrm{Ind}_H^G(A) := \left\{ f \in A^G : \begin{array}{l} f(s) = h \cdot f(sh) \ \forall s \in G, \forall h \in H \\ |\pi(\mathrm{supp}(f))| < \infty \end{array} \right\}$$

It is easily verified that $\mathrm{Ind}_H^G(A)$ is a G-ℓ-algebra with pointwise sum and multiplication, and the following G-action:

$$(g \cdot f)(s) = f(g^{-1}s) \quad (f \in \mathrm{Ind}_H^G(A), \ g, s \in G.)$$

Moreover, an H-equivariant morphism $A \to A'$ induces a morphism of G-ℓ-algebras $\mathrm{Ind}_H^G(A) \to \mathrm{Ind}_H^G(A')$ in the obvious way.

It can be shown that there exists a unique triangulated functor $\overline{\mathrm{Ind}}_H^G : \mathfrak{K}^H \to \mathfrak{K}^G$ making the following diagram commute; see [4, Proposition 6.9]:

$$
\begin{array}{ccc}
H\mathrm{Alg}_\ell & \xrightarrow{\mathrm{Ind}_H^G(M_H \otimes ?)} & G\mathrm{Alg}_\ell \\
{\scriptstyle j^H} \downarrow & & \downarrow {\scriptstyle j^G} \\
\mathfrak{K}^H & \xrightarrow{\overline{\mathrm{Ind}}_H^G} & \mathfrak{K}^G
\end{array}
$$

We recall the adjunction between induction and restriction.

Theorem 5.3.17 ([4, Theorem 6.14]). *Let G be a countable group and let $H \subseteq G$ be a subgroup. Then there is an adjunction:*

$$\mathfrak{K}^G(\overline{\mathrm{Ind}}_H^G(A), B) \cong \mathfrak{K}^H(A, \mathrm{Res}_G^H(B))$$

Proof. We recall some details that we will use below; for a full proof see [4, Theorem 6.14]. For $a \in A$ and $g \in G$, define $\xi_H(g, a) : G \to A$ by the formula:

$$\xi_H(g, a)(s) := \left\{ \begin{array}{cl} (g^{-1}s) \cdot a & \text{if } s \in gH, \\ 0 & \text{if } s \notin gH. \end{array} \right.$$

It is easily verified that $\xi_H(g, a) \in \mathrm{Ind}_H^G(A)$ and that these elements generate $\mathrm{Ind}_H^G(A)$ as an abelian group. See [4, Section 6] for a list of relations among the $\xi_H(g, a)$ for different g and a.

Let $B \in H\mathrm{Alg}_\ell$. Let $\psi_B : M_H B \to \mathrm{Res}_G^H \mathrm{Ind}_H^G(M_H B)$ be the H-equivariant morphism defined by:

$$\psi_B(e_{ij} \otimes b) = \xi_H(1, e_{ij} \otimes B)$$

Let $\bar{\psi}_B \in \mathfrak{K}^H(B, \mathrm{Res}_G^H \overline{\mathrm{Ind}}_H^G(B))$ be the following composite in $\mathfrak{K}^H$, where the isomorphism on the left is given by the functorial zig-zag of Lemma 5.3.6:

$$B \cong M_H B \xrightarrow{j^H(\psi_B)} \mathrm{Res}_H^G \overline{\mathrm{Ind}}_H^G(B)$$

It can be shown that $\bar{\psi}_B$ is the unit of the desired adjunction between Res_G^H and $\overline{\mathrm{Ind}}_H^G$; see the proof of [4, Theorem 6.14] for details. $\square$

Theorem 5.3.18 (cf. [4, Theorem 6.14])**.** *Let G be a countable group and let $H \subseteq G$ be a subgroup. Let $B \in H\mathrm{Alg}_\ell$ and let $C \in G\mathrm{Alg}_\ell$. Let $\psi_B : M_H B \to \mathrm{Res}_G^H \mathrm{Ind}_H^G (M_H B)$ be the H-equivariant morphism defined in the proof of Theorem 5.3.17. Then the following composite is a weak equivalence of spectra:*

$$\mathbb{K}^G(\overline{\mathrm{Ind}}_H^G B, C) \xrightarrow{\mathrm{Res}} \mathbb{K}^H(\mathrm{Res}_G^H \overline{\mathrm{Ind}}_H^G(B), \mathrm{Res}_G^H C) \xrightarrow{\psi_B^*} \mathbb{K}^H(M_H B, \mathrm{Res}_G^H C)$$

Proof. It is straightforward from Theorem 5.3.16 and the proof of Theorem 5.3.17. □

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