A POSTERIORI ERROR ESTIMATES FOR THE STEKLOV EIGENVALUE PROBLEM

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Abstract. In this paper we introduce and analyze an a posteriori error estimator for the linear finite element approximations of the Steklov eigenvalue problem. We define an error estimator of the residual type which can be computed locally from the approximate eigenpair and we prove that, up to higher order terms, the estimator is equivalent to the energy norm of the error. Finally, we prove that the volumetric part of the residual term is dominated by a constant times the edge residuals, again up to higher order terms.

1. Introduction

The aim of this paper is to propose and analyze an a posteriori error estimator, of the residual type, for the linear finite element approximations of the Steklov eigenvalue problem.

In recent years, numerical approximation of spectral problems arising in fluid mechanics have received increasing attention (see [8, 9, 12, 14, 17] and their references). Some of these spectral problems lead to a Steklov eigenvalue problem similar to the one considered here, for instance, in the study of surface waves [7], in the analysis of stability of mechanical oscillators immersed in a viscous fluid ([12] and the references therein) and in the study of the vibration modes of a structure in contact with an incompressible fluid (see, for example, [9]). In [4] optimal error estimates for the piecewise linear finite element approximation of the Steklov eigenvalue problem have been obtained.

In the numerical approximation of partial differential equations, the adaptive procedures based on a posteriori error estimators have gained an enormous importance. Several approaches have been considered to construct estimators based on the residual equations (see [3, 20] and their references). In particular, for second order elliptic eigenvalue problems, some results based on a general analysis for non-linear equations can be obtained (see [20, 21]) assuming that the numerical solution is close enough to the exact one. Using different approaches, similar results to those given in [21], have been obtained in [13, 16] for standard eigenvalue problem. However, in view of the boundary conditions in the Steklov eigenvalue problem, the standard arguments must be improved to obtain the equivalence between the estimator and the error. In this work we introduce an a posteriori error estimator for the Steklov eigenvalue problem and we prove that the estimator is equivalent to the error up to higher order terms. The constants involved in the higher order terms depend on the eigenvalue being approximated and the smallest angle in the triangulation.

Finally, we show that the volumetric part of the residual term is dominated by a constant times the edge residuals, up to a multiplicative constant depending only on the minimum angle, and so we obtain a simpler error estimator which turns out to be equivalent to the error up

1991 *Mathematics Subject Classification.* 65N25, 65N30, 65N15.

Key words and phrases. Steklov eigenvalue problem, finite elements, a posteriori error estimates.

Supported by ANPCyT under grant PICT 03-05009 and Fundación Antorchas. The authors are members of CONICET, Argentina.

to higher order terms. So, we extend to the Steklov eigenvalue problem a result that is well known for source problems (see, for instance, [5, 10, 18, 19]), and proved for standard eigenvalue problem in [13].

The rest of the paper is organized as follows. In Section 2 we introduce the model problem and we recall some known a priori error estimates for the linear finite element approximation. In Section 3 we introduce the error estimator and prove its equivalence with the error. Finally, in Section 4 we prove that the edge residuals dominate the volumetric part of the estimator and in view of this result, we propose a simpler error estimator which is also equivalent to the error.

2. The Steklov Eigenvalue Problem and Finite Element Approximation

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. We consider the following Steklov eigenvalue problem [[4], [6]]:

$$\begin{cases} -\triangle u + u &= 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= \lambda u \quad \text{on } \Gamma = \partial\Omega \end{cases} \tag{2.1}$$

The variational problem associated with (2.1) is given by:

Find λ and $u \in V = H^1(\Omega)$, $u \neq 0$ satisfying

$$\begin{cases} a(u,v) &= \lambda \int_\Gamma uv \qquad \forall\, v \in V \\ \|u\|_{L^2(\Gamma)} &= 1 \end{cases} \tag{2.2}$$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv$, which is continuous and coercive on V.

From the classical theory of abstract elliptic eigenvalue problems (see, for example, [1, 2, 4, 22]) we may infer that the problem (2.2) has a sequence of pairs (λ_j, u_j), with positive eigenvalues λ_j diverging to $+\infty$. We assume the eigenvalues to be increasingly ordered: $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$. The associated eigenfunctions u_j belong to the Sobolev space $H^{1+r}(\Omega)$, where $r = 1$ if Ω is convex and $r < \frac{\pi}{\omega}$ (with ω being the largest inner angle of Ω) otherwise (see, for instance, [15]).

The approximation of the eigenvalue λ and its associated eigenfunction u is obtained as follows:

Let $\{\mathcal{T}_h\}$ be a family of triangulations of Ω such that any two triangles in $\mathcal{T}_h$ share at most a vertex or an edge. Let h stand for the mesh-size; namely $h = \max_{T\in\mathcal{T}_h} h_T$, with h_T being the diameter of the triangle T. We assume that the family of triangulations $\{\mathcal{T}_h\}$ satisfies a minimal angle condition, i.e., there exists a constant $\sigma > 0$ such that $\frac{h_T}{\rho_T} \leq \sigma$, where ρ_T is the diameter of the largest circle contained in T.

Let $V_h = \{v \in V : \ v|_T \in \mathcal{P}_1 \quad \forall T \in \mathcal{T}_h\}$ where $\mathcal{P}_1$ denotes the space of linear polynomials. Then, the standard finite element approximation problem is the following:

Find λ_h and $u_h \in V_h$, $u_h \neq 0$ such that

$$\begin{cases} a(u_h,v) &= \lambda_h \int_\Gamma u_h v \qquad \forall v \in V_h \\ \|u_h\|_{L^2(\Gamma)} &= 1 \end{cases} \tag{2.3}$$

Let $\mathcal{N}$ be the set of vertices of the triangulation $\mathcal{T}_h$, we split $\mathcal{N}$ as $\mathcal{N} = \mathcal{N}_\Omega \cup \mathcal{N}_\Gamma$ where $\mathcal{N}_\Omega$ is the set of interior vertices and $\mathcal{N}_\Gamma$ is the set of vertices which lie in Γ. Let β_j, $1 \leq j \leq \mathcal{N}$ be the Lagrange basis of degree one, i.e., $\beta_j \in \mathcal{P}_1$ such that $\beta_j(P_i) = \delta_{i,j}$ where P_i denotes the node i.

The problem (2.3) reduces to a generalized eigenvalue problem given by: Find $u_h = \sum_{j=1}^{\mathcal{N}} z_j \beta_j$ such that

$$Az = \lambda_h Bz$$

with

$$A_{i,j} = \int_\Omega \nabla\beta_i \nabla\beta_j + \int_\Omega \beta_i\beta_j \qquad 1 \le i,j, \le \mathcal{N}$$

and

$$B_{i,j} = \int_\Gamma \beta_i\beta_j \qquad 1 \le i,j, \le \mathcal{N}$$

The matrix A is positive definite and symmetric, and the matrix B is non-negative definite, symmetric, and it has rank $\mathcal{N}_\Gamma$. So, this generalized eigenvalue problem attains a finite number of eigenpairs $(\lambda_{j,h}, u_{j,h})$, $1 \le j \le \mathcal{N}_\Gamma$, with positive eigenvalues which we assume increasingly ordered: $0 < \lambda_{1,h} \le \cdots \le \lambda_{\mathcal{N}_\Gamma,h}$.

In [4], the following a priori error estimates have been obtained: For any j, $1 \le j \le \mathcal{N}$,

$$\|u_j - u_{h,j}\|_{H^1(\Omega)} \le Ch^r, \tag{2.4}$$

$$\|u_j - u_{h,j}\|_{L^2(\Gamma)} \le Ch^{3r/2}, \tag{2.5}$$

$$|\lambda_j - \lambda_{h,j}| \le Ch^{2r}, \tag{2.6}$$

where C denotes a generic positive constant which depends only on the smallest angle in the triangulation.

In order to simplify notation from now on we will drop the subindex j in $\lambda_j, \lambda_{h,j}, u_j, u_{h,j}$.

3. Error Estimator

In this section we introduce the error estimator and prove its equivalence, up to higher order terms, with the energy norm of the error .

First we introduce some notation that we will need in the definition and analysis of the error estimator:

For any $T \in \mathcal{T}_h$ we denote by $\mathcal{E}_T$ the set of edges of T,

$$\mathcal{E} = \bigcup_{T\in\mathcal{T}_h} \mathcal{E}_T$$

and we decompose $\mathcal{E} = \mathcal{E}_\Omega \cup \mathcal{E}_\Gamma$ where $\mathcal{E}_\Gamma := \{\ell \in \mathcal{E} : \ell \subset \Gamma\}$ and $\mathcal{E}_\Omega = \mathcal{E} \setminus \mathcal{E}_\Gamma$.

For each $\ell \in \mathcal{E}_\Omega$ we choose an arbitrary unit normal vector n_ℓ and denote the two triangles sharing this edge T_{in} and T_{out}, where n_ℓ points outwards T_{in}. For $v_h \in V_h$ we set

$$\left[\!\!\left[\frac{\partial v_h}{\partial n_\ell}\right]\!\!\right]_\ell = \nabla\left(v_h|_{T_{\text{out}}}\right)\cdot n_\ell - \nabla\left(v_h|_{T_{\text{in}}}\right)\cdot n_\ell,$$

which corresponds to the jump of the normal derivatives of v_h across the edge ℓ. Notice that these values are independent of the choice of n_ℓ.

Let (λ, u) be an eigenpair and its corresponding finite element approximation (λ_h, u_h). From (2.2) and (2.3) we know that for any $v \in V_h$ the error $e = u - u_h$ satisfies

$$\int_\Omega \nabla e\cdot\nabla v + \int_\Omega ev = \int_\Gamma \lambda uv - \int_\Gamma \lambda_h u_h v \tag{3.7}$$

On the other hand, for any $v \in V$ using (2.2), integrating by parts and using that $\triangle u_h = 0$ on T, we have

$$\int_\Omega \nabla e \cdot \nabla v + \int_\Omega ev = a(u,v) - a(u_h,v) = \int_\Gamma \lambda uv - \sum_T \left\{ \int_{\partial T} \frac{\partial u_h}{\partial n} v + \int_T u_h v \right\}$$

and so

$$\begin{aligned}\int_\Omega \nabla e \cdot \nabla v + \int_\Omega ev &= \sum_T \left\{ -\int_T u_h v + \sum_{\ell \in \mathcal{E}_T \cap \mathcal{E}_\Gamma} \int_\ell (\lambda_h u_h - \frac{\partial u_h}{\partial n}) v + \frac{1}{2} \sum_{\ell \in \mathcal{E}_T \cap \mathcal{E}_\Omega} \int_\ell \left[\!\left[\frac{\partial u_h}{\partial n_\ell} \right]\!\right]_\ell v \right\} \\ &+ \int_\Gamma \lambda uv - \int_\Gamma \lambda_h u_h v \end{aligned} \tag{3.8}$$

For each $\ell \in \mathcal{E}$ we define J_ℓ by

$$J_\ell = \begin{cases} \frac{1}{2} \left[\!\left[\frac{\partial v_h}{\partial n_\ell} \right]\!\right]_\ell & \ell \in \mathcal{E}_\Omega \\ \lambda_h u_h - \frac{\partial u_h}{\partial n_\ell} & \ell \in \mathcal{E}_\Gamma \end{cases}$$

Now, the local error indicator η_T is defined as follows

$$\eta_T = \left(h_T^2 \|u_h\|_{L^2(T)}^2 + \sum_{\ell \in \mathcal{E}_T} |\ell| \|J_\ell\|_{L^2(\ell)}^2 \right)^{1/2} \tag{3.9}$$

where $|\ell|$ denotes the length of the edge ℓ. Then, the global error estimator is given by

$$\eta_\Omega = \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2}.$$

From now on we denote by C a generic positive constant, not necessarily the same at each occurrence, which depends only on the smallest angle in the triangulation.

In order to obtain the relation between the error and the estimator we will use the following well-known error estimates for the interpolation operator of Clement $I_h : V \to V_h$

$$\|v - I_h v\|_{L^2(T)} \leq C h_T \|v\|_{H^1(\tilde{T})} \tag{3.10}$$

$$\|v - I_h v\|_{L^2(\ell)} \leq C |\ell|^{\frac{1}{2}} \|v\|_{H^1(\tilde{\ell})} \tag{3.11}$$

where $\tilde{T}$ is the union of all the elements sharing a vertex with T and $\tilde{\ell}$ is the union of all the elements sharing a vertex with ℓ (see [11], [20]).

The following theorem provides the upper bound on the error.

Theorem 3.1. *There exists a constant C such that*

$$\|e\|_{H^1(\Omega)} \leq C \left\{ \eta_\Omega + \left(\frac{\lambda + \lambda_h}{2} \right) \|e\|_{L^2(\Gamma)} \right\}$$

Proof. We denote by e^I the Clement interpolant of e. By using (3.7) and (3.8) we obtain

$$\begin{aligned}\int_\Omega |\nabla e|^2 + |e|^2 &= \int_\Omega \nabla e \cdot (\nabla e - \nabla e^I) + \int_\Omega e(e - e^I) + \int_\Gamma \lambda u e^I - \int_\Gamma \lambda_h u_h e^I \\ &= \sum_T \left\{ -\int_T u_h (e - e^I) + \sum_{\ell \in \mathcal{E}_T} \int_\ell J_\ell (e - e^I) \right\} + \int_\Gamma (\lambda u - \lambda_h u_h) e\end{aligned}$$

Then by using Cauchy-Schwartz, (3.10), (3.11) and that the triangulation satisfies the minimum angle condition we obtain

$$\begin{aligned}
\|e\|^2_{H^1(\Omega)} &\leq \sum_T \|u_h\|_{L^2(T)}\|e-e^I\|_{L^2(T)} + \sum_T \sum_{\ell\in\mathcal{E}_T} \|J_\ell\|_{L^2(\ell)}\|e-e^I\|_{L^2(\ell)} + \int_\Gamma (\lambda u - \lambda_h u_h)e \\
&\leq C\sum_T h_T\|u_h\|_{L^2(T)}\|e\|_{H^1(\tilde{T})} + \sum_T\sum_{\ell\in\mathcal{E}_T} |\ell|^{\frac{1}{2}}\|J_\ell\|_{L^2(\ell)}\|e\|_{H^1(\tilde{\ell})} + \int_\Gamma(\lambda u - \lambda_h u_h)e \\
&\leq C\left\{\sum_T\left(\|u_h\|^2_{L^2(T)}h_T^2 + \sum_{\ell\in\mathcal{E}_T}|\ell|\|J_\ell\|^2_{L^2(\ell)}\right)\right\}^{\frac{1}{2}}\|e\|_{H^1(\Omega)} + \int_\Gamma(\lambda u - \lambda_h u_h)e \qquad (3.12)
\end{aligned}$$

Since $\|u\|_{L^2(\Gamma)} = 1$ and $\|u_h\|_{L^2(\Gamma)} = 1$ we have that

$$\int_\Gamma(\lambda u - \lambda_h u_h)e = \lambda + \lambda_h - (\lambda+\lambda_h)\int_\Gamma u u_h = \frac{\lambda+\lambda_h}{2}\|e\|^2_{L^2(\Gamma)}$$

and by using it in (3.12) and the trace theorem we have

$$\|e\|_{H^1(\Omega)} \quad\leq\quad C\left\{\sum_T h_T^2\|u_h\|^2_{L^2(T)} + \sum_{\ell\in\mathcal{E}_T}|\ell|\|J_\ell\|^2_{L^2(\ell)}\right\}^{\frac{1}{2}} + C\left(\frac{\lambda+\lambda_h}{2}\right)\|e\|_{L^2(\Gamma)}$$

and the proof concludes. □

Then, from (2.5) the global estimator provides an upper bound of the H^1 error up to a higher order term.

In order to guarantee that our estimator is efficient for practical adaptive refinement our next goal is to prove that the local estimator η_T is bounded by the H^1 error and higher order terms.

For $T \in \mathcal{T}_h$, let b_T be the standard cubic bubble given by:

$$b_T = \begin{cases} \delta_{1,T}\delta_{2,T}\delta_{3,T} & \text{in } T \\ 0 & \text{in } \Omega\setminus T\end{cases}$$

where $\delta_{1,T}, \delta_{2,T}$ and $\delta_{3,T}$ denote the barycentric coordinates of $T\in\mathcal{T}_h$.

For $\ell \in \mathcal{E}_T \cap \mathcal{E}_\Omega$, we denote by T_1 and T_2 the two triangles sharing ℓ and we enumerate the vertices of T_1 and T_2 such that the vertices of ℓ are numbered first. Then we consider the edge-bubble function b_ℓ

$$b_\ell = \begin{cases} \delta_{1,T_i}\delta_{2,T_i} & \text{in } T_i \\ 0 & \text{in } \Omega\setminus T_1\cup T_2\end{cases}$$

First, we obtain the upper estimate for the first term of η_T.

Lemma 3.1. *There exists a constant C such that*

$$h_T\|u_h\|_{L^2(T)} \leq C(\|\nabla e\|_{L^2(T)} + h_T\|e\|_{L^2(T)})$$

Proof. Let $v_T \in \mathcal{P}_4(T)\cap H^1_0(T)$ be such that

$$\int_T v_T w = -h_T^2\int_T u_h w \quad \forall w\in\mathcal{P}_1(T)$$

Such v_T exists and is unique. In fact, let $v_T = \sum_{i=1}^3 \alpha_i\varphi_i$ with $\{\varphi_i = \delta_{i,T}b_T\}_{1\leq i\leq 3}$ a basis of $\mathcal{P}_4(T)\cap H^1_0(T)$; using that $\int_T \delta_{1,T}^{n_1}\delta_{2,T}^{n_2}\delta_{3,T}^{n_3}dx = \frac{n_1!n_2!n_3!2!}{(n_1+n_2+n_3+2)!}|T|$ we can compute $\alpha_1,\alpha_2,\alpha_3$ by solving a non singular system.

An easy calculation shows that

$$|\alpha_i| \leq \frac{C}{|T|} \max_{1\leq i\leq 3} \left| \int_T h_T^2 u_h \delta_{i,T} \right|$$

and thus,

$$\|v_T\|_{L^2(T)} \leq C|T|^{1/2} \max_{1\leq i\leq 3} |\alpha_i| \leq C h_T^2 \|u_h\|_{L^2(T)}$$

Then, there exists a positive constant C, depending only on the regularity condition, such that

$$\|v_T\|_{L^2(T)} + h_T\|\nabla v_T\|_{L^2(T)} \leq C\|v_T\|_{L^2(T)} \leq C h_T^2 \ \|u_h\|_{L^2(T)} \tag{3.13}$$

Since $h_T^2\|u_h\|^2_{L^2(T)} = -\int_T v_T u_h$, using the residual equation (3.8) we obtain

$$h_T^2\|u_h\|^2_{L^2(T)} = \int_T \nabla e \cdot \nabla v_T + \int_T e v_T$$

Therefore, using Cauchy-Schwartz and (3.13) we get

$$h_T^2\|u_h\|^2_{L^2(T)} \leq C(\|\nabla e\|_{L^2(T)} + h_T\|e\|_{L^2(T)}) h_T\|u_h\|_{L^2(T)}$$

and the proof concludes from this estimate.□

For the second term of η_T we have:

Lemma 3.2. *a) For $\ell \in \mathcal{E}_T \cap \mathcal{E}_\Gamma$, there exists a constant C such that*

$$|\ell|^{1/2} \, \|J_\ell\|_{L^2(\ell)} \leq C(1+h_T)\|e\|^2_{H^1(T)} + |\ell|^{1/2}\|\lambda u - \lambda_h u_h\|_{L^2(\ell)}$$

b) For $\ell \in \mathcal{E}_T \cap \mathcal{E}_\Omega$, let $T_1, T_2 \in \mathcal{T}_h$ be the two triangles sharing ℓ. Then, there exists a constant C such that

$$|\ell|^{1/2} \, \|J_\ell\|_{L^2(\ell)} \leq C\|e\|_{H^1(T^1_\ell \cup T^2_\ell)}$$

Proof. a) For $\ell \in \mathcal{E}_T \cap \mathcal{E}_\Gamma$, let $v_\ell \in \mathcal{P}_3(T)$ be the function such that $v_\ell|_{\ell'} = 0$ for $\ell' \in \mathcal{E}_T$ $\ell' \neq \ell$ and

$$\begin{cases} \int_\ell v_\ell w = |\ell| \int_\ell J_\ell w & \forall w \in \mathcal{P}_1(T) \\ \|v_\ell\|_{L^2(\ell)} \leq C|\ell| \ \|J_\ell\|_{L^2(\ell)} \end{cases}$$

Such v_ℓ exists and is unique. In fact, let $v_\ell = \sum_{i=1}^2 \beta_i \psi_i$ where $\psi_i = \delta_{i,T} b_\ell$, and $\delta_{1,T}, \delta_{2,T}$ are the barycentric coordinates associated to the vertices of ℓ. Then, using that $\int_\ell \delta_{1,T}^{n_1}\delta_{2,T}^{n_2} = \frac{n_1! n_2!}{(n_1+n_2+1)!}|\ell|$ we can obtain β_1 and β_2 by solving a non singular system.

It is easy to see that

$$|\beta_i| \leq \frac{C}{|\ell|} \max \left| \int_\ell |\ell| J_\ell \delta_{i,T} \right|$$

and therefore,

$$\|v_\ell\|_{L^2(\ell)} \leq C|\ell|^{1/2} \max |\beta_i| \leq C|\ell| \|J_\ell\|_{L^2(\ell)}$$

and

$$\|v_\ell\|_{L^2(T)} \leq C h_T \max_{1\leq i\leq 2} |\beta_i| \leq C|\ell| \|J_\ell\|_{L^2(\ell)} \max_{1\leq i\leq 2} \|\delta_{i,T}\|_{L^2(\ell)} \leq C|\ell|^{3/2}\|J_\ell\|_{L^2(\ell)}$$

Then, $\|v_\ell\|_{L^2(T)} + h_T\|\nabla v_\ell\|_{L^2(T)} \leq C|\ell|^{3/2}\|J_\ell\|_{L^2(\ell)}$.

Since $|\ell| \int_\ell J_\ell^2 = \int_\ell v_\ell J_\ell$, using the residual equation (3.8) we get

$$|\ell| \|J_\ell\|^2_{L^2(\ell)} = \int_T \nabla e \cdot \nabla v_\ell + \int_T e v_\ell - \int_\ell (\lambda u - \lambda_h u_h) v_\ell + \int_T u_h v_\ell$$

Thus,

$$|\ell| \|J_\ell\|^2_{L^2(\ell)} \leq C \left(\|\nabla e\|_{L^2(T)} + h_T \|e\|_{L^2(T)} + |\ell|^{1/2} \|\lambda u - \lambda_h u_h\|_{L^2(\ell)} + h_T \|u_h\|_{L^2(T)} \right) |\ell| \|J_\ell\|_{L^2(\ell)}$$

This estimate and the previous lemma allows us to conclude the proof of a).

b) For $\ell \in \mathcal{E}_T \cap \mathcal{E}_\Omega$, let $T_1, T_2 \in \mathcal{T}_h$ be the two triangles sharing ℓ. Let $v_\ell \in H^1_0(T_1 \cup T_2)$ be a edge-bubble function such that $v_\ell|_{T_j} \in \mathcal{P}_2\,, j = 1, 2$, and

$$\begin{cases} \int_\ell v_\ell w = |\ell| \int_\ell J_\ell w & \forall w \in \mathcal{P}_0(T) \\ \|v_\ell\|_{L^2(\ell)} \leq C|\ell| \ \|J_\ell\|_{L^2(\ell)} \end{cases}$$

It is easy to see that $\|v_\ell\|_{L^2(T)} + h_T \|\nabla v_\ell\|_{L^2(T)} \leq C|\ell|^{3/2} \|J_\ell\|_{L^2(\ell)}$. Since $|\ell| \int_\ell J_\ell^2 = \int_\ell v_\ell J_\ell$, from the residual equation we get

$$|\ell| \|J_\ell\|^2_{L^2(\ell)} = \int_{T_1 \cup T_2} \nabla e \cdot \nabla v_\ell + \int_{T_1 \cup T_2} e v_\ell + \int_{T_1 \cup T_2} u_h v_\ell$$

Using Cauchy-Schwartz we obtain

$$|\ell| \|J_\ell\|^2_{L^2(\ell)} \leq C \left(\|\nabla e\|_{L^2(T_1 \cup T_2)} + h_T \|e\|_{L^2(T_1 \cup T_2)} + h_T \|u_h\|_{L^2(T_1 \cup T_2)} \right) |\ell| \|J_\ell\|_{L^2(\ell)}$$

From this estimate and the previous lemma we conclude the proof.□

Now, we are in condition to prove the efficiency of our estimator.

Theorem 3.2. *There exists a constant C such that*

a) For $T \in \mathcal{T}_h$, if $\partial T \cap \Gamma = \emptyset$ then

$$\eta_T \leq C \|e\|_{H^1(T^*)}$$

where T^* denote the union of T and the triangles sharing an edge with T.

b)For $T \in \mathcal{T}_h$, if $\partial T \cap \Gamma \neq \emptyset$ then

$$\eta_T \leq C \{ \|e\|_{H^1(T)} + \sum_{\ell \in \mathcal{E}_T \cap \mathcal{E}_\Gamma} |\ell| \, \|\lambda u - \lambda_h u_h\|_{L^2(\ell)} \}$$

Proof. It follows immediately from Lemmas 3.1 and 3.2. □.

Remark 3.1. *The term $|\ell| \|\lambda u - \lambda_h u_h\|^2_{L^2(\ell)}$ is a higher order term. In fact, for any $v \in V$ we have*

$$\begin{aligned} \int_\Gamma (\lambda u - \lambda_h u_h) v &= \int_\Gamma \lambda (u - u_h) v + \int_\Gamma (\lambda - \lambda_h) u_h v \\ &\leq \left(\lambda \|u - u_h\|_{L^2(\Gamma)} + (\lambda - \lambda_h) \|u_h\|_{L^2(\Gamma)} \right) \|v\|_{L^2(\Gamma)} \end{aligned}$$

So, taking $v = \lambda u - \lambda_h u_h$ and using the apriori estimates (2.5) and (2.6) we obtain

$$\|\lambda u - \lambda_h u_h\|_{L^2(\Gamma)} \leq C \{ h^{\frac{3}{2} r} + h^{2r} \}.$$

4. Edge Residual Error Estimator

In this section, we prove that the edge residuals dominate the other part of the estimator defined in the previous section and then, we propose a simpler error estimator which is also equivalent to the error up to higher order terms.

For any $P \in \mathcal{N}_\Omega$ we define $\Omega_P = \bigcup \{ T \in \mathcal{T}_h : P \in T \}$.

Lemma 4.1. *For any $P \in \mathcal{N}_\Omega$ we have that*

$$\sum_{T\subset\Omega_P} h_T^2 \|u_h\|^2_{L^2(T)} \le C \left(\sum_{\ell\subset\Omega_P} |\ell| \|J_\ell\|^2_{L^2(\ell)} + |\Omega_P|^2 \|\nabla u_h\|^2_{L^2(\Omega_P)} \right)$$

Proof. Let $\Pi_0(u_h)$ be the $L^2(\Omega_P)$ projection of u_h onto the constants, i.e,

$$\int_{\Omega_P} u_h v = \int_{\Omega_P} \Pi_0(u_h) v \quad \forall v \text{ constant in } \quad \Omega_P$$

Then,

$$\sum_{T\subset\Omega_P} h_T^2 \|u_h\|^2_{L^2(T)} \le |\Omega_P| \int_{\Omega_P} |u_h|^2 = |\Omega_P| \|u_h - \Pi_0(u_h)\|^2_{L^2(\Omega_P)} + |\Omega_P| \|\Pi_0(u_h)\|^2_{L^2(\Omega_P)}$$

Let ϕ_P be the corresponding Lagrange basis function with $\operatorname{supp} \phi_P = \Omega_P$

$$\begin{aligned}
\|\Pi_0(u_h)\|^2_{L^2(\Omega_P)} &= |\Pi_0(u_h)|^2 |\Omega_P| = \frac{9}{|\Omega_P|} \left(\int_{\Omega_P} \phi_P \Pi_0(u_h) \right)^2 \\
&\le \frac{18}{|\Omega_P|} \left(\int_{\Omega_P} \phi_P (\Pi_0(u_h) - u_h) \right)^2 + \frac{18}{|\Omega_P|} \left(\int_{\Omega_P} \phi_P u_h \right)^2 \qquad (4.14)
\end{aligned}$$

Since $P \in \mathcal{N}_\Omega$ by using (2.3) we have

$$-\int_{\Omega_P} \phi_P u_h = \int_{\Omega_P} \nabla u_h \cdot \nabla \phi_P = \frac{1}{2} \sum_{\ell\in\mathcal{E}_\Omega\cap\Omega_P} |\ell| \left[\!\!\left[\frac{\partial u_h}{\partial n_\ell} \right]\!\!\right]_\ell = \frac{1}{2} \sum_{\ell\in\mathcal{E}_\Omega\cap\Omega_P} \int_\ell \left[\!\!\left[\frac{\partial u_h}{\partial n_\ell} \right]\!\!\right]_\ell$$

and so,

$$\begin{aligned}
|\Omega_P| \|\Pi_0(u_h)\|^2_{L^2(\Omega_P)} &\le C\{ |\Omega_P| \|\Pi_0(u_h) - u_h\|^2_{L^2(\Omega_P)} + \frac{1}{2} \sum_{\ell\in\mathcal{E}_\Omega\cap\Omega_P} |\ell| \| \left[\!\!\left[\frac{\partial u_h}{\partial n_\ell} \right]\!\!\right]_\ell \|^2_{L^2(\ell)} \} \\
&\le C\{ |\Omega_P| \|\Pi_0(u_h) - u_h\|^2_{L^2(\Omega_P)} + \frac{1}{2} \sum_{\ell\subset\Omega_P} |\ell| \|J_\ell\|^2_{L^2(\ell)} \}
\end{aligned}$$

therefore

$$\sum_{T\subset\Omega_P} h_T^2 \|u_h\|^2_{L^2(T)} \le C\{ |\Omega_P| \|\Pi_0(u_h) - u_h\|^2_{L^2(\Omega_P)} + \frac{1}{2} \sum_{\ell\subset\Omega_P} |\ell| \|J_\ell\|^2_{L^2(\ell)} \}$$

The proof concludes by using the standard estimate for the L^2 projection. □

Now, we introduce a simplified indicator by omitting the volumetric part in the residual error estimator given in (3.9)

$$\tilde{\eta}_T = \left(\sum_{\ell\in\mathcal{E}_T} |\ell| \|J_\ell\|^2_{L^2(\ell)} \right)^{1/2} \qquad (4.15)$$

and the corresponding global error estimator

$$\tilde{\eta}_\Omega = \left\{ \sum \tilde{\eta}_T^2 \right\}^{1/2}$$

In the following Theorem we prove that this estimator is globally reliable and locally efficient up to higher order terms.

Theorem 4.1. *There exists a constant C such that*

$$\|e\|_{H^1(\Omega)} \leq C\left\{\tilde{\eta}_\Omega + \left(\frac{\lambda+\lambda_h}{2}\right)\|e\|_{L^2(\Gamma)} + h^2\right\}$$

and

$$\tilde{\eta}_T \leq C\begin{cases} \|e\|_{H^1(T^*)} & \textit{if } \partial T\cap\Gamma=\emptyset \\ \|e\|_{H^1(T)} + \sum_{\ell\in\mathcal{E}_T\cap\mathcal{E}_\Gamma}|\ell|\,\|\lambda u-\lambda_h u_h\|_{L^2(\ell)} & \textit{if } \partial T\cap\Gamma\neq\emptyset \end{cases}$$

Proof. It follows immediately from Theorem 3.1, Theorem 3.2 and Lemma 4.1.

Remark 4.1. *We have considered the case in which Ω is a polygonal domain only for simplicity, there is no difficulty extending all of the above to the 3d case with Ω a polyhedral domain. In such a case we have to replace, in the definition of the error indicator (3.9), the length of the edge by the diameter of the face and by following our ideas the equivalence between the error estimator and the energy norm can be obtained, up to higher order terms.*

Acknowledgments: We thank Prof. Rodolfo Rodriguez for his useful suggestions.

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