Error estimates for moving least square approximations

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Abstract

In this paper we obtain error estimates for moving least square approximations in the one dimensional case. For the application of this method to the numerical solution of differential equations it is fundamental to have error estimates for the approximations of derivatives. We prove that, under appropriate hypothesis on the weight function and the distribution of points, the method produces optimal order approximations of the function and its first and second derivatives. As a consequence, we obtain optimal order error estimates for Galerkin approximations of coercive problems. Finally, as an application of the moving least square method we consider a convection-diffusion equation and propose a way of introducing up-wind by means of a non-symmetric weight function. We present several numerical results showing the good behavior of the method.

Key words. error estimates, moving least square, Galerkin approximations, convection-diffusion.

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1 Introduction

The moving least square (MLS) as approximation method has been introduced by Shepard [12] in the lowest order case and generalized to higher degree by Lancaster and Salkauskas [6]. The object of those works was to provide an alternative to classic interpolation useful to approximate a function from its values given at irregularly spaced points by using weighted least square approximations. However, those papers do not attempt to present an approximation theory. More recently, the moving least square method for the numerical solution of differential equations has been considered in several papers, especially in the engineering literature, ([13], [1], [2]) giving rise to a particular class of the so called mesh-less methods

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which provide numerical approximations starting from a set of points instead of a mesh or triangulation.

For this kind of application it is fundamental to analyze the order of approximation, not only for the function itself, but also for its derivates.

We will show that, for the approximation in L^∞ of the function, optimal error estimates in the one dimensional case follows from known results concerning the interpolation properties of best approximations ([9],[8],[10]) under rather general hypothesis. Also, in a recent paper Levin [7] analyzed the MLS method for a particular weight function obtaining error estimates in the uniform norm for the approximation of a regular function in higher dimensions.

The main object of this work is to prove error estimates for the approximation of the first and second order derivatives of a function by the moving least square method in the one dimensional case. We use compact support weight functions as is usually done in the application for numerical solution of differential equations. In order to obtain these estimates we will need to impose more restrictive assumptions on the set of points and on the weight function used.

Our results can be applied in two ways for the approximation to second order problems. The MLS method can be used to construct finite dimensional approximation spaces to define Galerkin approximations. In this case, our results for the first derivatives provide optimal order error estimates by using standard arguments (i.e the Cea' s Lemma). On the other hand, the error estimates for the second derivatives allows to control the consistency error of finite difference or collocation methods obtained from an arbitrary set of points by the MLS method.

One of the most interesting features of the MLS method is the possibility of constructing approximation spaces which take into account special properties of the differential equation considered. As an example of this fact we consider a convection-diffusion equation and show how the MLS method, with a particular weight function, can be used to construct non symmetric approximation functions introducing appropriate up-wind.

We present numerical computations showing the good behavior of this approach (although we remark that we do not attempt here to provide a theoretical analysis for the convection dominated problem, i.e, to obtain error estimates valid uniformly on the relation between convection and diffusion. This is an interesting problem and will be the object of our future research).

Some of the techniques introduced here for the error analysis can be extended to higher dimensions although this is not trivial and will be presented elsewhere.

The paper is organized as follows. First, in Section 2 we present the moving least square method. Section 3 deals with the error estimates for the function and its first and second derivatives. Finally, in Section 4 we use the error estimates to prove the convergence of Galerkin approximations based on MLS method for second order coercive problem and we conclude with some numerical examples for a convection-diffusion equation.

2 The Moving Least Square Method

Given $R > 0$ let $\Phi_R \geq 0$ be a function such that $supp\ \Phi_R \subset \overline{B_R(0)} = \{z/|z| \leq R\}$. and $X_R = \{x_1, x_2, \dots, x_n\}$, $n = n(R)$, a set of points in $\Omega \subset \mathbb{R}$ an open interval and let $u_1, u_2, \dots u_n$ be the values of the function u in those points, i.e, $u_j = u(x_j)$, $1 \leq j \leq n$.

Let $\{p_0, \cdots, p_m\}$ be a basis of the Polynomial Space $\mathcal{P}_m$. (we can choose, for example $p_0 = 1, p_1 = x, \dots, p_m = x^m$) with $m << n$.
For each $x \in \Omega$ (fix) we consider $P^*(x,y) = \sum_{k=0}^{m} p_k(y)\alpha_k(x)$ where $\{\alpha_0(x), \cdots, \alpha_m(x)\}$ are chosen such that

$$J_x(\alpha) = \sum_{j=1}^{n} \Phi_R(x - x_j)(u_j - \sum_{k=0}^{m} p_k(x_j)\alpha_k)^2 \,,$$

is minimized.
In order to simplify notation we will drop the subscript R from the weight function Φ_R.

Then, we define the approximation of u as

$$\hat{u}(x) = P^*(x,x) = \sum_{k=0}^{m} p_k(x)\alpha_k(x) \,. \tag{2.1}$$

In order to have this approximation well defined we need to guarantee that the minimization problem has a solution.
We define

$$< f, g >_x = \sum_{j=1}^{n} \Phi(x - x_j) f(x_j) g(x_j) \,.$$

Then,

$$\|f\|_x^2 = \sum_{j=1}^{n} \Phi(x - x_j) f(x_j)^2,$$

is a discrete norm on the polynomial space $\mathcal{P}_m$ if the weight function Φ satisfies the following property
Property P: For each $x \in \Omega$, $\Phi(x - x_j) > 0$ at least for $m + 1$ indices j.

Therefore (see for example [5])

Theorem 2.1 *Assume that the weight function satisfies property P. Then, for any $x \in \Omega$ there exists $P^*(x,\cdot) \in \mathcal{P}_m$ which satisfies $\|u - P^*(x,.)\|_x \leq \|u - P\|_x$ for all $P \in \mathcal{P}_m$.*

Observe that the polynomial $P^*(x,y)$ can be obtained solving the normal equations for the minimization problem. In fact, if we denote $\mathbf{p}(x) = (p_0(x) \cdots p_m(x))^t$,
$A(x) = \sum_{k=1}^{n} \Phi(x - x_k)\mathbf{p}(x_k)\mathbf{p}^t(x_k)$ and $\mathbf{c}_j(x) = \Phi(x - x_j)\mathbf{p}(x_j)$ then, $\alpha(x) = (\alpha_0(x), \cdots, \alpha_m(x))^t$ is the solution of the following system

$$A(x)\alpha(x) = \sum_{j=1}^{n} \mathbf{c}_j(x) u_j \,.$$

So, an easy calculation shows that $P^*(x,x)$ may be written as

$$P^*(x,x) = \sum_{j=1}^{n} \beta_j(x) u_j \,, \tag{2.2}$$

where $\beta_j(x) = \mathbf{p}^t(x) A^{-1}(x) \mathbf{c}_j(x)$ are functions with the same support and regularity than Φ ([6], section 2). We also note that if $u \in \mathcal{P}_k$ with $k \le m$, then $\hat{u} = u$. In particular $\{\beta_j\}_{1 \le j \le n}$ is a partition of unity, i.e, $\sum_{j=1}^{n} \beta_j(x) = 1$, $\forall x \in \Omega$. However, the β_j are not in general non negative functions and this fact make the error analysis more complicated.

3 Error Estimates

The object of this section is to obtain error estimates in terms of the parameter R which plays the role of the mesh size. The letter C will stand for a constant, independent of R, not necessarily the same in each occurrence. We will indicate in each case which parameters it depends on.

The approximation order of $\hat{u}$ to the function u can be obtained from known results on best least square approximations. Indeed, the following theorem is known (see Theorem 8 in [9])

Theorem 3.1 *Assume that Φ satisfies property P. Let $x \in \Omega$ and $u \in C^{m+1}(\overline{B_R(x) \cap \Omega})$. If $P^*(x,y)$ is the polynomial in y that minimizes $J_x(\alpha)$, then $u(y) - P^*(x,y)$ has at least $m+1$ zeros in $\overline{B_R(x) \cap \Omega}$.*

Therefore, from standard interpolation error estimates we have the following

Corollary 3.1 *If Φ satisfies property P and $u \in C^{m+1}(\overline{\Omega})$ then, there exists C, depending only on m, such that for each $x \in \Omega$ and any $y \in B_R(x) \cap \Omega$*

$$|u(y) - P^*(x,y)| \le C \|u^{(m+1)}\|_{L^\infty(\Omega)} R^{m+1} \,, \tag{3.1}$$

in particular taking $y = x$, we have

$$\|u - \hat{u}\|_{L^\infty(\Omega)} \le C \|u^{(m+1)}\|_{L^\infty(\Omega)} R^{m+1} \,. \tag{3.2}$$

For our subsequent error analysis we introduce the following properties about the weight function and the distribution of points. All the constants appearing below are independent of R.

1. Given $x \in \Omega$ there exist at least $m+1$ points $x_j \in X_R \cap B_{\frac{R}{2}}(x)$.

2. $\exists c_0 > 0$ such that $\Phi(z) \ge c_0 \;\; \forall z \in B_{\frac{R}{2}}(0)$.

3. $\Phi \in C^1(B_R(0)) \cap W^{1,\infty}(\mathbb{R})$ and $\exists\, c_1$ such that $\|\Phi'\|_{L^\infty(\mathbb{R})} \le \frac{c_1}{R}$.

4. $\exists c_p$ such that $\frac{R}{\sigma} \leq c_p$ where $\sigma = min|x_i - x_k|$ is the minimum over the $m+1$ points in condition 1).

5. $\exists c_{\#}$ such that for all $x \in \Omega$, $card\{X_R \cap B_{2R}(x)\} < c_{\#}$.

6. $\Phi \in C^2(B_R(0)) \cap W^{2,\infty}(\mathbb{R})$ and $\exists\, c_2$ such that $\|\Phi''\|_{L^\infty(\mathbb{R})} \leq \frac{c_2}{R^2}$.

Our next goal is to estimate the error in the approximation of u' by $\hat{u}'$. The following Lemma gives the fundamental tool to obtain this estimate.

Lemma 3.1 *Let $x \in \Omega$ such that $\frac{\partial P^*(x,y)}{\partial x}$ exists and $u \in C^{m+1}(\overline{\Omega})$. If properties 1) to 5) hold then, there exists $C = C(c_0, c_1, c_p, c_{\#}, m)$ such that $\forall y \in B_R(x) \cap \Omega$:*

$$|\frac{\partial P^*(x,y)}{\partial x}| \leq C R^m \|u^{(m+1)}\|_{L^\infty(\Omega)} \tag{3.3}$$

Proof. Given $x \in \Omega$, in view of property 1) there exist points $x_{j_1}, x_{j_2}, \cdots, x_{j_{m+1}} \in B_{\frac{R}{2}}(x)$. For any $h > 0$ we define

$$S(x) = \sum_{k \in \{j_i\}} |P^*(x+h, x_k) - P^*(x, x_k)|^2 \tag{3.4}$$

Then, by property 2)

$$\begin{aligned} S(x) &\leq \frac{1}{c_0} \sum_{k \in \{j_i\}} \Phi(x - x_k)(P^*(x+h, x_k) - P^*(x, x_k))^2 \\ &\leq \frac{1}{c_0} \sum_{k=1}^{n} \Phi(x - x_k)(P^*(x+h, x_k) - P^*(x, x_k))^2 \\ &= \frac{1}{c_0} \sum_{k=1}^{n} \Phi(x - x_k)(P^*(x+h, x_k) - P^*(x, x_k))(P^*(x+h, x_k) - u(x_k)) \\ &+ \frac{1}{c_0} \sum_{k=1}^{n} \Phi(x - x_k)(P^*(x+h, x_k) - P^*(x, x_k))(u(x_k) - P^*(x, x_k)) \,. \end{aligned} \tag{3.5}$$

Let Q be the polynomial of degree $\leq m$ defined by $Q(y) = P^*(x+h, y) - P^*(x, y)$, then since the minimum is attained at P^* we have

$$< u(y) - P^*(x,y), Q(y) >_x = \sum_{k=1}^{n} \Phi(x - x_k) Q(x_k)(u(x_k) - P^*(x, x_k)) = 0 \,. \tag{3.6}$$

Then,

$$S(x) \leq \frac{1}{c_0} \sum_{k=1}^{n} \Phi(x - x_k) Q(x_k)(P^*(x+h, x_k) - u(x_k)) \,. \tag{3.7}$$

Since Φ is in $C^1(B_R(0)) \cap W^{1,\infty}(\mathbb{R})$, for h small enough, $\exists \theta_k$ such that $\Phi(x-x_k) = \Phi(x+h-x_k) - \Phi'(\theta_k)h$ and so, replacing in (3.7) we obtain

$$\begin{aligned} S(x) &\leq \frac{1}{c_0}\sum_{k=1}^{n} \Phi(x+h-x_k)Q(x_k)(P^*(x+h,x_k)-u(x_k)) \\ &- \frac{h}{c_0}\sum_{k=1}^{n} \Phi'(\theta_k)Q(x_k)(P^*(x+h,x_k)-u(x_k)) \,. \end{aligned}$$

Since $< u(y) - P^*(x+h,y), Q(y) >_{x+h} = 0$ and using that Φ' has compact support together with property 3) we have

$$\begin{aligned} S(x) &\leq \frac{h}{c_0}\sum_{k=1}^{n} \Phi'(\theta_k)Q(x_k)(u(x_k) - P^*(x+h,x_k)) \\ &\leq \frac{h}{c_0}\sum_{k=1}^{n} |\Phi'(\theta_k)||Q(x_k)||(P^*(x+h,x_k)-u(x_k))| \\ &\leq \frac{c_1 h}{c_0 R}\sum_{x_k \in B_{2R}(x)} |Q(x_k)||(P^*(x+h,x_k)-u(x_k))| \,. \end{aligned}$$

By Corollary 3.1 we know that $|P^*(x+h,x_k) - u(x_k)| \leq C\|u^{(m+1)}\|_{L^\infty(\Omega)} R^{m+1}$ then,

$$S(x) \leq Ch\|u^{(m+1)}\|_{L^\infty(\Omega)} R^m \sum_{x_k \in B_{2R}(x)} |Q(x_k)| \tag{3.8}$$

Since Q is a polynomial of degree $\leq m$ it can be written as

$$Q(y) = \sum_{k \in \{j_i\}} Q(x_k) l_k(y) \,, \tag{3.9}$$

where $l_k(y)$ are the Lagrange's polynomial basis functions and therefore

$$\sum_{x_k \in B_{2R}(x)} |Q(x_k)| \leq \sum_{i \in \{j_k\}} |Q(x_i)| (\sum_{x_k \in B_{2R}(x)} |l_i(x_k)|)$$

From property 4) $|l_i(y)| \leq (\frac{2R}{\sigma})^{m+1} \leq c(c_p, m)$ $\forall i \in \{j_k, 1 \leq k \leq m+1\}$ and therefore, it follows from property 5) and (3.8) that

$$\begin{aligned} (\sum_{k \in \{j_i\}} |Q(x_k)|)^2 &\leq (m+1)\sum_{k \in \{j_i\}} |Q(x_k)|^2 = (m+1)S(x) \\ &\leq Ch\|u^{(m+1)}\|_{L^\infty(\Omega)} R^m c_{\#} \sum_{k \in \{j_i\}} |Q(x_k)| \end{aligned} \tag{3.10}$$

Now, we obtain

$$\sum_{k \in \{j_i\}} |Q(x_k)| \leq Ch\|u^{(m+1)}\|_{L^\infty(\Omega)} R^m \,. \tag{3.11}$$

Finally, using (3.11) and property 4) in (3.9) we get $\forall y \in B_R(x) \cap \Omega$

$$\frac{|Q(y)|}{h} = \frac{|P^*(x+h,y) - P^*(x,y)|}{h} \leq C\|u^{(m+1)}\|_{L^\infty(\Omega)} R^m \,, \tag{3.12}$$

and so, if x is a point such that $\frac{\partial P^*(x,y)}{\partial x}$ exists, the proof concludes by taking $h \to 0$. □

The following Theorem states the order which $\hat{u}'$ approximates u'.

Theorem 3.2 *If $u \in C^{m+1}(\overline{\Omega})$ and properties 1) to 5) hold then, there exists $C = C(c_0, c_1, c_p, c_\#, m)$ such that*

$$\|u' - \hat{u}'\|_{L^\infty(\Omega)} \leq C\|u^{(m+1)}\|_{L^\infty(\Omega)} R^m \,. \tag{3.13}$$

Proof. Observe that, since $\Phi \in C^1(B_R(0)) \cap W^{1,\infty}(\mathbb{R})$ then, $P^*(x,y)$ is continuous everywhere and differentiable for every x up to a finite set (this can be seen using again the argument given in [6], section 2). Therefore, $P^*(\cdot,y) \in W^{1,\infty}(\Omega)$. For any $x \in \Omega$ such that $\frac{\partial P^*(x,y)}{\partial x}$ exists, we want to estimate $|u'(x) - \hat{u}'(x)| = |u'(x) - \frac{d}{dx}P^*(x,x)|$. We note that

$$\frac{d}{dx}P^*(x,x) = \{\frac{\partial P^*(x,y)}{\partial x} + \frac{\partial P^*(x,y)}{\partial y}\frac{\partial y}{\partial x}\}|_{y=x} \tag{3.14}$$

So, we will estimate $|u'(y) - \frac{\partial P^*(x,y)}{\partial x} - \frac{\partial P^*(x,y)}{\partial y}|$, $\forall y \in B_R(x) \cap \Omega$. Since $P^*(x,y)$ is a polynomial of degree $\leq m$ that interpolates u at $m+1$ points (Theorem 3.1) then, $\frac{\partial P^*(x,y)}{\partial y}$ interpolates u' in m points and therefore,

$$|u'(y) - \frac{\partial P^*(x,y)}{\partial y}| \leq C\|u^{(m+1)}\|_{L^\infty(\Omega)} R^m \quad \forall y \in B_R(x) \cap \Omega \tag{3.15}$$

From Lemma 3.1 we know that $|\frac{\partial P^*(x,y)}{\partial x}| \leq C\|u^{(m+1)}\|_{L^\infty(\Omega)} R^m$ which together with (3.15) yields

$$|u'(y) - \frac{\partial P^*(x,y)}{\partial x} - \frac{\partial P^*(x,y)}{\partial y}| \leq |u'(y) - \frac{\partial P^*(x,y)}{\partial y}| + |\frac{\partial P^*(x,y)}{\partial x}| \leq C\|u^{(m+1)}\|_{L^\infty(\Omega)} R^m$$

In particular taking $y = x$ we conclude the proof. □

Our next goal is to find error estimates for the approximation of u'' by $\hat{u}''$. The idea of the proof is similar to that used for the first derivative. However, instead of using the orthogonality (3.6) we have to use a relation which is obtained in the next lemma by differentiating it. From now on we will assume that $\Phi \in C^1(\mathbb{R})$ so, in particular $P^*(\cdot,y) \in C^1(\Omega)$ using the same argument mentioned above.

Lemma 3.2 *For any $Q(x,y)$ polynomial in y of degree $\leq m$ and differentiable as a function of x we have*

$$\sum_{j=1}^{n} \Phi'(x - x_j) Q(x, x_j)(P^*(x, x_j) - u_j) + \sum_{j=1}^{n} \Phi(x - x_j) Q(x, x_j) \frac{\partial P^*(x, x_j)}{\partial x} = 0\,. \tag{3.16}$$

Proof. Since $P^*(x,y)$ is the minimum, for each $x \in \Omega$,

$$< u(y) - P^*(x,y), Q(x,y) >_x = 0 \quad \forall\ Q(x,\cdot) \in \mathcal{P}_m \tag{3.17}$$

then if we take derivative with respect to x we have

$$\begin{aligned}\sum_{j=1}^{n} \Phi'(x-x_j)Q(x,x_j)(P^*(x,x_j)-u_j) \quad &+ \quad \sum_{j=1}^{n} \Phi(x-x_j)\frac{\partial Q(x,x_j)}{\partial x}(P^*(x,x_j)-u_j) \\ &+ \quad \sum_{j=1}^{n} \Phi(x-x_j)Q(x,x_j)\frac{\partial P^*(x,x_j)}{\partial x} = 0\,.\end{aligned}$$

Since for each x, $\frac{\partial Q(x,y)}{\partial x}$ is in $\mathcal{P}_m$ and P^* satisfies (3.17) the second term is 0 and therefore we have the result. □

Proceeding as we have done for the error in the first derivative we will need to estimate the second derivatives $\frac{\partial^2 P^*(x,y)}{\partial x^2}$ and $\frac{\partial^2 P^*(x,y)}{\partial x \partial y}$ for each $x \in \Omega$ where they exist. This is done in the two following Lemmas.

Lemma 3.3 *Let $x \in \Omega$ such that $\frac{\partial^2 P^*(x,y)}{\partial x^2}$ exists , $m \geq 1$ and $u \in C^{m+1}(\overline{\Omega})$. If properties 1) to 6) hold then, there exists $C = C(c_0, c_1, c_2, c_p, c_\#, m)$ such that $\forall y \in B_R(x) \cap \Omega$*

$$|\frac{\partial^2 P^*(x,y)}{\partial x^2}| \leq C R^{m-1} \|u^{(m+1)}\|_{L^\infty(\Omega)} \tag{3.18}$$

Proof. Let $x \in \Omega$ and $x_{j_1}, x_{j_2}, \cdots, x_{j_{m+1}}$ be as in the proof of Lemma 3.1. Let $h > 0$ and $Q \in \mathcal{P}_m$ be defined by $Q(y) = \dfrac{\partial P^*(x+h,y)}{\partial x} - \dfrac{\partial P^*(x,y)}{\partial x}$ and

$$S(x) = \sum_{k \in \{j_i\}} |Q(x_j)|^2\,. \tag{3.19}$$

From property 2) we have

$$\begin{aligned} S(x) \quad &\leq \quad \frac{1}{c_0} \sum_{k \in \{j_i\}} \Phi(x-x_k)Q(x_k)^2 \leq \frac{1}{c_0}\sum_{k=1}^{n} \Phi(x-x_k)Q(x_k)^2 \\ &= \quad \frac{1}{c_0}\{\sum_{k=1}^{n} \Phi(x-x_k)Q(x_k)\frac{\partial P^*(x+h,x_k)}{\partial x} - \sum_{k=1}^{n} \Phi(x-x_k)Q(x_k)\frac{\partial P^*(x,x_k)}{\partial x}\}\,. \end{aligned} \tag{3.20}$$

Now we consider the first term on the right hand side of (3.20). Using that for h small enough $\Phi(x-x_k) = \Phi(x+h-x_k) - h\Phi'(\theta_k)$ and Lemma 3.2 we have

$$\begin{aligned} \sum_{k=1}^{n} \Phi(x-x_k)Q(x_k)\frac{\partial P^*(x+h,x_k)}{\partial x} \quad &= \quad \sum_{k=1}^{n} \Phi(x+h-x_k)Q(x_k)\frac{\partial P^*(x+h,x_k)}{\partial x} \\ &- \quad h\sum_{k=1}^{n} \Phi'(\theta_k)Q(x_k)\frac{\partial P^*(x+h,x_k)}{\partial x} \\ &= \quad \sum_{k=1}^{n} \Phi'(x+h-x_k)Q(x_k)(u_k - P^*(x+h,x_k)) \\ &- \quad h\sum_{k=1}^{n} \Phi'(\theta_k)Q(x_k)\frac{\partial P^*(x+h,x_k)}{\partial x}\,, \end{aligned}$$

and using Lemma 3.2 in the second term on the right hand side of (3.20) we have

$$\begin{aligned} S(x) &\leq \frac{1}{c_0}\{\sum_{k=1}^{n}\Phi'(x+h-x_k)Q(x_k)(u_k-P^*(x+h,x_k)) \\ &- h\sum_{k=1}^{n}\Phi'(\theta_k)Q(x_k)\frac{\partial P^*(x+h,x_k)}{\partial x} \\ &+ \sum_{k=1}^{n}\Phi'(x-x_k)Q(x_k)(P^*(x,x_k)-u_k)\}\,. \end{aligned}$$

Since Φ is in $C^2(B_R(0))\cap W^{2,\infty}(\mathbb{R})$, for h small enough, $\exists \eta_k$ such that $\Phi'(x+h-x_k)=\Phi'(x-x_k)+\Phi''(\eta_k)h$. Then, replacing in the first term we obtain

$$\begin{aligned} S(x) &\leq \frac{1}{c_0}\{\sum_{k=1}^{n}\Phi'(x-x_k)Q(x_k)(P^*(x,x_k)-P^*(x+h,x_k)) \\ &- h\sum_{k=1}^{n}\Phi'(\theta_k)Q(x_k)\frac{\partial P^*(x+h,x_k)}{\partial x}+h\sum_{k=1}^{n}x_k\in B_R(x)\Phi''(\eta_k)Q(x_k)(u_k-P^*(x,x_k))\} \\ &\leq \frac{1}{c_0}\{\sum_{k=1}^{n}|\Phi'(x-x_k)||Q(x_k)||P^*(x,x_k)-P^*(x+h,x_k)| \\ &+ h\sum_{k=1}^{n}|\Phi'(\theta_k)||Q(x_k)||\frac{\partial P^*(x+h,x_k)}{\partial x}|+h\sum_{k=1}^{n}|\Phi''(\eta_k)||Q(x_k)||u_k-P^*(x,x_k)|\}\,. \end{aligned}$$

From inequality (3.12), Lemma 3.1, Corollary 3.1 and properties 3) and 6) we have

$$S(x)\leq Ch\|u^{(m+1)}\|_{L^\infty(\Omega)}\{\sum_{x_k\in B_{2R}(x)}\frac{c_1}{R}|Q(x_k)|R^m+\sum_{x_k\in B_{2R}(x)}\frac{c_2}{R^2}|Q(x_k)|R^{m+1}\},$$

and therefore,

$$S(x)\leq Ch\|u^{(m+1)}\|_{L^\infty(\Omega)}R^{m-1}\sum_{x_k\in B_{2R}(x)}|Q(x_k)|.$$

Using the same argument as in the proof of Lemma 3.1 we have that

$$\frac{|Q(y)|}{h}\leq C\|u^{(m+1)}\|_{L^\infty(\Omega)}R^{m-1}, \tag{3.21}$$

and then, if x is such that $\frac{\partial^2 P^*(x,y)}{\partial x^2}$ exists, we can take $h\to 0$ to conclude the proof. □

Lemma 3.4 *Let $x\in\Omega$, $m\geq 1$ and $u\in C^{m+1}(\overline{\Omega})$. If properties 1) to 6) hold then, there exists $C=C(c_0,c_1,c_2,c_p,c_\#,m)$ such that $\forall y\in B_R(x)\cap\Omega$*

$$|\frac{\partial^2 P^*(x,y)}{\partial x\partial y}|\leq C\|u^{(m+1)}\|_{L^\infty(\Omega)}R^{m-1}$$

Proof. Let $x\in\Omega$ and let $x_{j_1},\cdots,x_{j_{m+1}}$ be as in the proof of Lemma 3.1. Since $P^*(x,\cdot)\in\mathcal{P}_m$ then, $P^*(x,y)=\sum_{k\in\{j_i\}}P^*(x,x_k)l_k(y)$ where $l_k(y)$ (with $k\in\{j_i,1\leq i\leq m+1\}$) are the Lagrange's polynomial basis functions associated with the points $x_{j_1}, \cdots, x_{j_{m+1}}$. Since property 6) holds P^* is C^1 in the first variable. Therefore, for any x we have

$$\frac{\partial^2 P^*(x,y)}{\partial x \partial y} = \sum_{k\in\{j_i\}} \frac{\partial P^*(x,x_k)}{\partial x} l'_k(y).$$

By Lemma 3.1 we know that $\forall y \in B_R(x)$: $|\frac{\partial P^*(x,y)}{\partial x}| \leq C\|u^{(m+1)}\|_{L^\infty(\Omega)} R^m$. Then

$$|\frac{\partial^2 P^*(x,y)}{\partial x \partial y}| \leq \sum_{k\in\{j_i\}} |\frac{\partial P^*(x,x_k)}{\partial x}||l'_k(y)| \leq C\|u^{(m+1)}\|_{L^\infty(\Omega)} R^m \sum_{k\in\{j_i\}} |l'_k(y)|. \tag{3.22}$$

An easy calculation shows that from property 5) it follows that $|l'_k(y)| \leq m\frac{c_p^{m-1}}{\sigma} \leq \frac{c}{\sigma}$ and using this estimate in (3.22) and property 5) again we obtain the result □

Theorem 3.3 *Let $m \geq 1$, if $u \in C^{m+1}(\overline{\Omega})$ and properties 1) to 6) hold then, there exists $C = C(c_0, c_1, c_2, c_p, c_\#, m)$ such that*

$$\|u'' - \hat{u}''\|_{L^\infty(\Omega)} \leq C\|u^{(m+1)}\|_{L^\infty(\Omega)} R^{m-1}. \tag{3.23}$$

Proof. Since $\Phi \in C^2(B_R(0)) \cap W^{2.\infty}(\mathbb{R})$ then, $\frac{\partial^2 P^*(x,y)}{\partial x \partial y}$ exists everywhere and $\frac{\partial^2 P^*(x,y)}{\partial x^2}$ exists for every x up to a finite set and $P^*(\cdot,y) \in W^{2,\infty}(\Omega)$. (this can be seen using again the argument given in [6] mentioned above).
Given any $x \in \Omega$, such that $\frac{\partial^2 P^*(x,y)}{\partial x^2}$ exists, we want to estimate $|u''(x) - \hat{u}''(x)| = |u''(x) - \frac{d^2}{dx^2}P^*(x,x)|$. We note that

$$\frac{d^2}{dx^2}P^*(x,x) = \{\frac{\partial^2 P^*(x,y)}{\partial x^2} + 2\frac{\partial^2 P^*(x,y)}{\partial x \partial y} + \frac{\partial^2 P^*(x,y)}{\partial y^2}\}|_{y=x} \tag{3.24}$$

So, $\forall y \in B_R(x) \cap \Omega$

$$|u''(y) - \frac{\partial^2 P^*(x,y)}{\partial y^2} - 2\frac{\partial^2 P^*(x,y)}{\partial x \partial y} - \frac{\partial^2 P^*(x,y)}{\partial x^2}| \leq |u''(y) - \frac{\partial^2 P^*(x,y)}{\partial y^2}| + 2|\frac{\partial^2 P^*(x,y)}{\partial x \partial y}| + |\frac{\partial^2 P^*(x,y)}{\partial x^2}| \tag{3.25}$$

Now observe that, $\frac{\partial^2 P^*(x,y)}{\partial y^2}$ is a polynomial of degree $\leq m-2$ which interpolates u'' in $m-1$ points and therefore

$$|u''(y) - \frac{\partial^2 P^*(x,y)}{\partial y^2}| \leq C\|u^{(m+1)}\|_{L^\infty(\Omega)} R^{m-1} \quad \forall y \in B_R(x) \cap \Omega \tag{3.26}$$

Then, the proof concludes by taking $y = x$ and using Lemma 3.3, Lemma 3.4 and (3.25).□

4 Numerical Examples

The object of this section is to present some examples on the numerical solution of differential equations by the MLS method.

In order to show the different possibilities of this method we apply it to solve the convection-diffusion equation

$$\begin{aligned} -u'' + bu' &= 0, \ in \ (0,1) \\ u(0) = 0, \ u(1) &= 1 \end{aligned} \tag{4.1}$$

with different values of the constant $b > 0$ and using Galerkin and Collocation methods. One of the interesting features of the MLS methods is the possibility of choosing appropriate weight function for the problem considered. For example, for the equation (4.1) one can choose Φ depending on b taking into account the non-symmetric character of the problem and thus, introducing some kind of upwinding when b is large in order to avoid oscillations.

In the first two examples we use Galerkin approximations. In general, given the following variational problem find $u \in V \subset H^1(\Omega)$ such that

$$a(u,v) = L(v), \ \forall v \in V$$

where a is a bilinear form, continuous and coercive on V and L is a linear and continuous operator, we can use the MLS method to define Galerkin approximation in the following way.

Let $V_R = span\{\beta_1, \ldots, \beta_n\}$, with β_j, $1 \le j \le n$ the basis functions defined in (2.2). Observe that, since the basis β_j have the same regularity as Φ, if $\Phi \in C^1(B_R(0)) \cap W^{1,\infty}(\mathbb{R})$ then $\beta_j \in H^1(\Omega)$ and therefore we can define the Galerkin approximation $u_R \in V_R$ as

$$u_R(x) = \sum_{j=1}^{n} \beta_j(x) u_j,$$

where $u_1, \ldots, u_n$ is the solution of the following system

$$\sum_{j=1}^{n} a(\beta_j, \beta_k) u_j = L(\beta_k), \ 1 \le k \le n$$

From <u>Céa's Lemma</u> ([3], [4]), Theorem 3.2 and Theorem 3.3 we have that :

$$\|u - u_R\|_V \le \frac{\gamma}{\delta} \min_{v \in V_h} \|u - v\|_V \le \frac{\gamma}{\delta} \|u - \hat{u}\|_V \le C \|u^{(m+1)}\|_{L^\infty(\Omega)} R^m \tag{4.2}$$

where γ is the continuity constant and δ is the coercivity constant of $a(\cdot,\cdot)$ on V. In particular we have that u_R converges to u when $R \to 0$.

Now, we write the convection-diffusion equation (4.1) in the following equivalent form. Let $\tilde{u}(x) = u(x) - x$ then, we have

$$\begin{aligned} -\tilde{u}'' + b\tilde{u}' &= -b \\ \tilde{u}(0) = \tilde{u}(1) &= 0 \end{aligned} \tag{4.3}$$

The variational problem is to find $\tilde{u} \in V = H_0^1(0,1)$, such that:

$$a(\tilde{u}, v) = L(v), \ \forall v \in V$$

where

$$
\begin{aligned}
a(\tilde{u}, v) &= \int_0^1 \tilde{u}'v' + \int_0^1 b\tilde{u}'v \\
L(v) &= -\int_0^1 bv
\end{aligned}
$$

The bilinear form $a(\cdot,\cdot)$ is coercive and continuous [11] therefore, the error estimate (4.2) holds (with the constant C depending on b).

For $R > 0$ we take the following weight function

$$
\Phi(z) = \begin{cases} \frac{e^{\delta(\frac{z}{R})^2} - e^{\delta}}{1-e^{\delta}} & \text{if } -R < z < 0 \\ \frac{e^{\gamma(\frac{z}{R})^2} - e^{\gamma}}{1-e^{\gamma}} & \text{if } 0 \leq z < R \\ 0 & \text{otherwise,} \end{cases} \tag{4.4}
$$

where δ and γ are constants to be chosen in each example. We consider the two cases $b = 1$ and $b = 20$. In order to compute the integrals involved in the Galerkin approximation we proceed as follows: Since the basis functions are smooth except in a finite number of points, we divide their supports in a finite number of intervals and compute each integral using the mid point quadrature rule.

In the first case the convection and the diffusion are of the same order and so, no upwinding is needed. We include this example to check the predicted convergence order. Therefore, we take $\delta = \gamma = 1$ to obtain a symmetric weight function (see Figure 1). We show in Figure 2 the approximate solution for equally spaced points with $n = 5$ and $m = 1$. The Figures 3 and 4 show the logarithm of the error in L^∞ norm as a function of the logarithm of R for the function and its first derivative respectively. It is observed that optimal order approximation is obtained in L^∞ both for u and u'. In particular, the predicted order of convergence in H^1 is obtained.

For the case $b = 20$ the problem becomes convection dominated and so we take a non-symmetric weight function by choosing $\delta = -30$ and $\gamma = 1$ (see Figure 5). We show the results obtained by using standard linear finite elements, usual up-wind finite difference and MLS for equaly spaced points. Figure 6, 7 and 8 show the results for n=5 and Figure 9,10 and 11 for n=10. As it is known, the first method produces oscillations. Instead the other two methods do not present oscillations but MLS produces a better approximation. Indeed, although we have not proved it, we expect that the MLS method allows to introduce up-wind but preserving the second order convergence when $R \to 0$.

Another way of applying the MLS to solve differential equations is by using collocation. Observe that, if $\Phi \in C^2(\mathbb{R})$ then the basis function β_j are in $C^2(\Omega)$. So, taking $m = 2$ it follows from Theorem 3.2 and Theorem 3.3 that the collocation method is consistent. Again we can introduce up-wind by means of a non-symmetric weight function. We modify the function (4.4) in order to obtain a C^2 function as follows (Figure 12)

$$
\Phi(z) = \begin{cases} \frac{e^{\delta(\frac{z}{R})^4} - e^{\delta}}{1-e^{\delta}}(1 - (\frac{z}{R})^2)^4 & \text{if } -R < z < 0 \\ \frac{e^{\gamma(\frac{z}{R})^4} - e^{\gamma}}{1-e^{\gamma}}(1 - (\frac{z}{R})^2)^4 & \text{if } 0 \leq z < R \\ 0 & \text{otherwise,} \end{cases}
$$

where $\delta = -30$ and $\gamma = 1$. We show the approximate solution for equally spaced points with $n = 8$ and $n = 15$ (Figure 13 and Figure 14). Again we observe that oscillations do not appear.

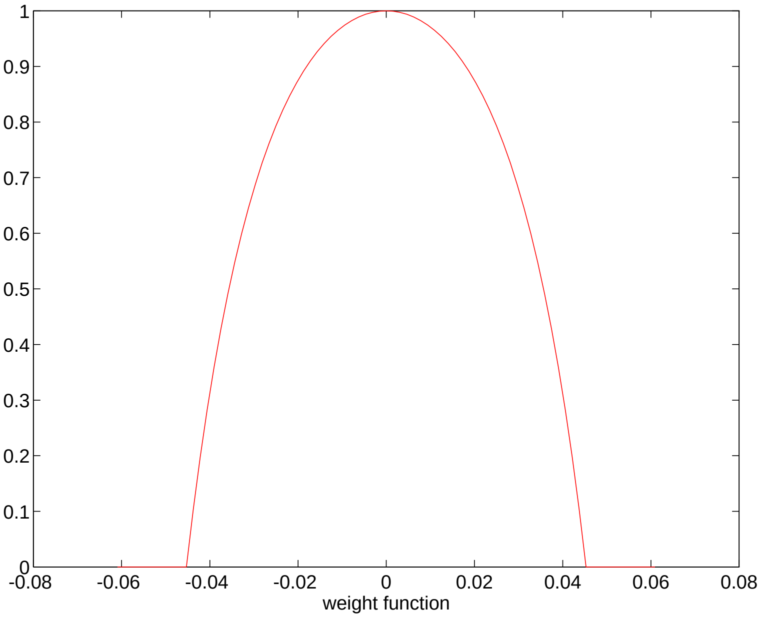

Figure 1: Symmetric weight function

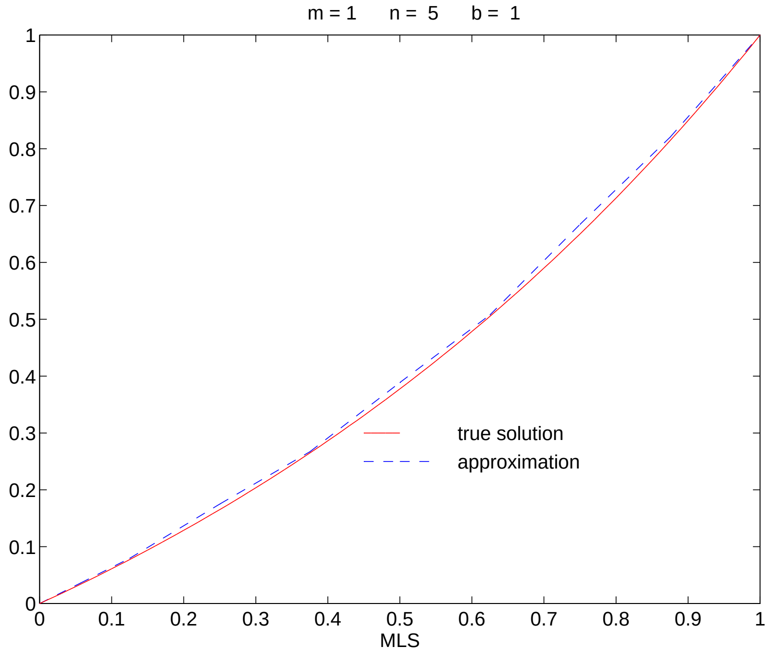

Figure 2: Exact solution vs. MLS approximation

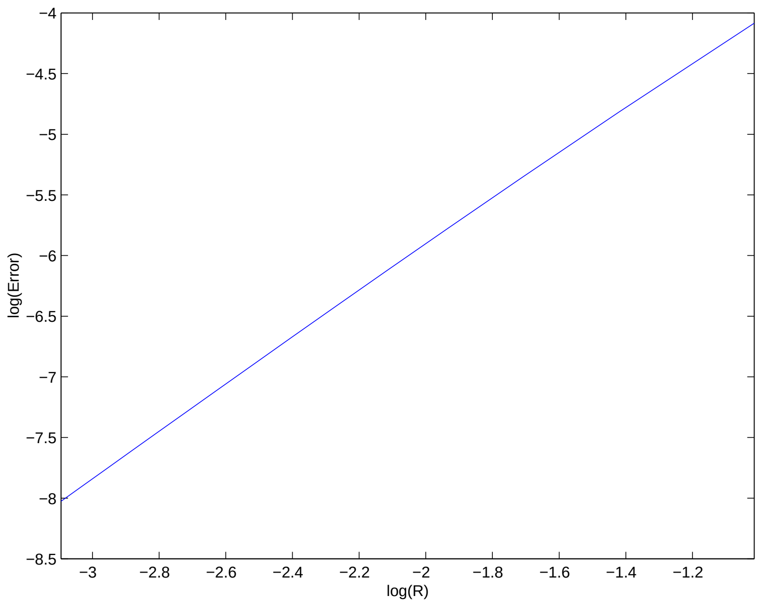

Figure 3: $\|u - u_R\|_{L^\infty(\Omega)} = O(R^2)$

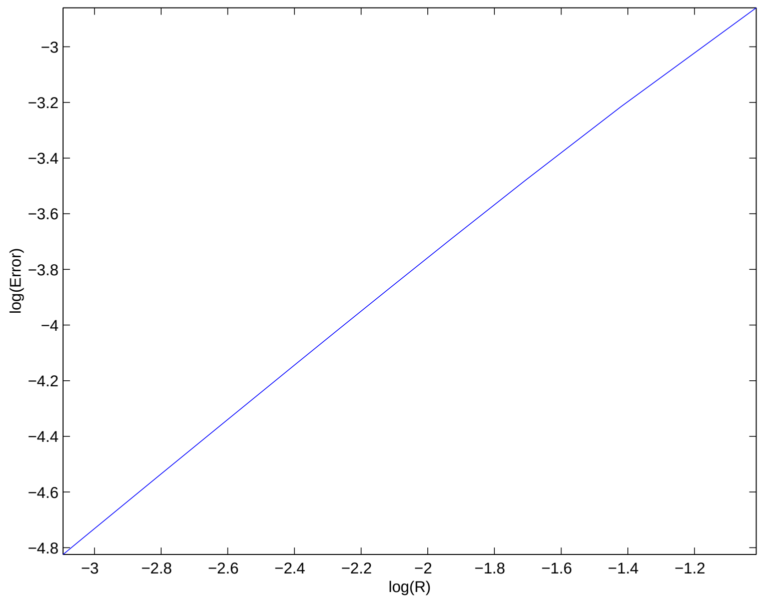

Figure 4: $\|u' - u'_R\|_{L^\infty(\Omega)} = O(R)$

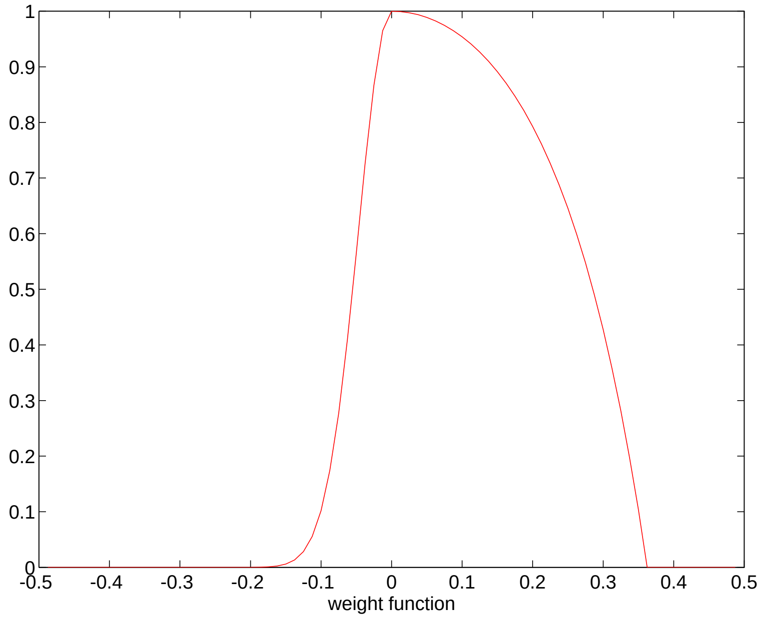

Figure 5: Nonsymmetric weight function

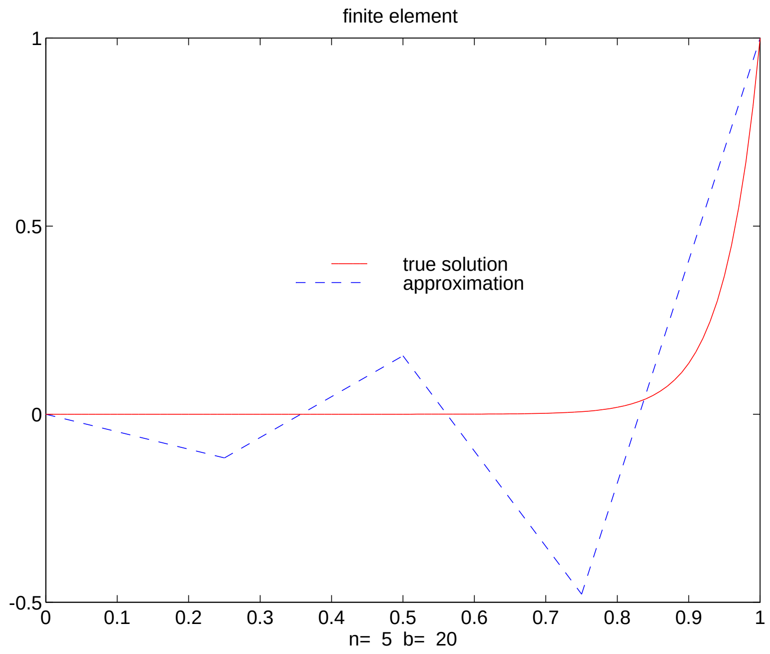

Figure 6: Exact Solution vs. Finite Element approximation

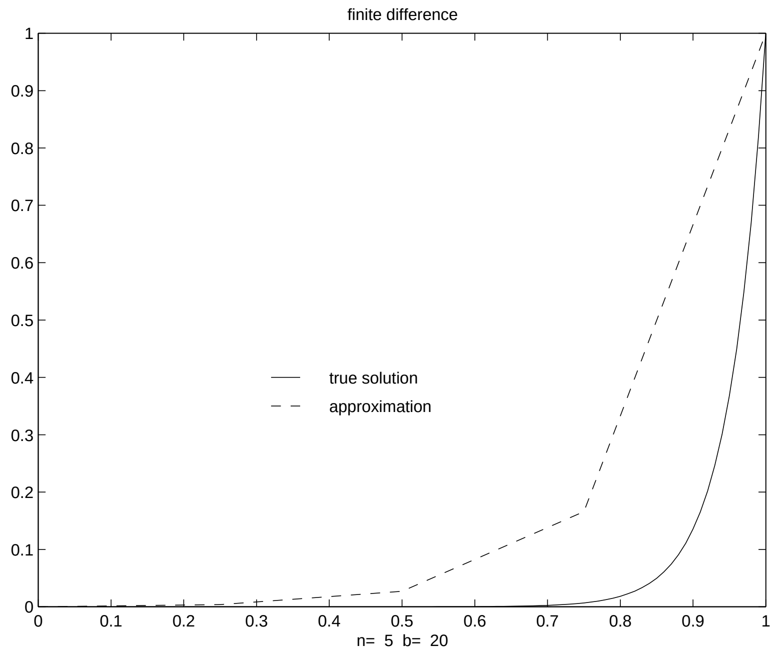

Figure 7: Exact solution vs. up-wind Finite Difference approximation

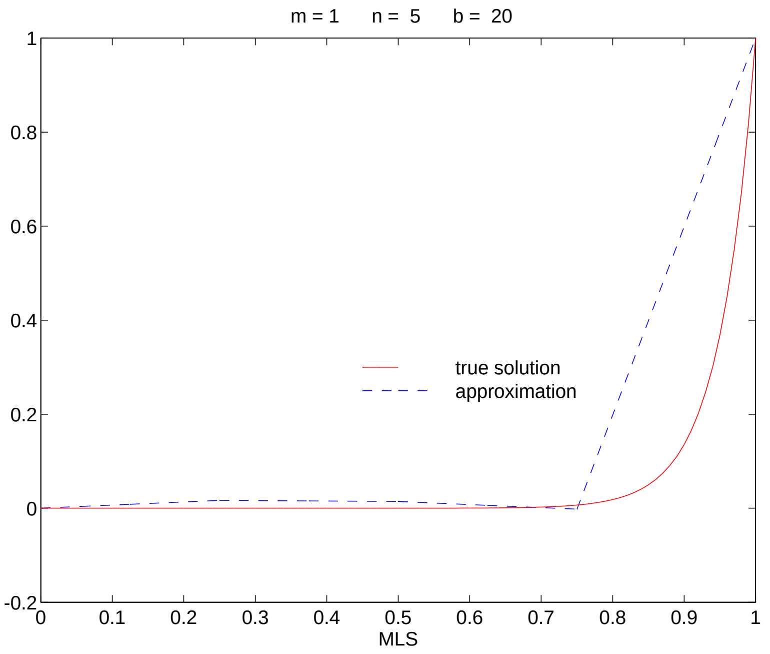

Figure 8: Exact solution vs. MLS approximation

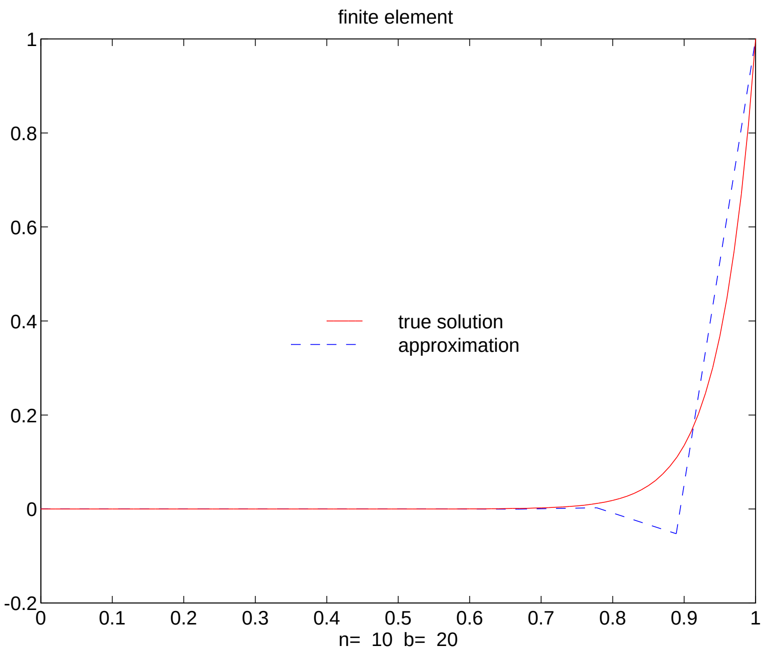

Figure 9: Exact Solution vs. Finite Element approximation

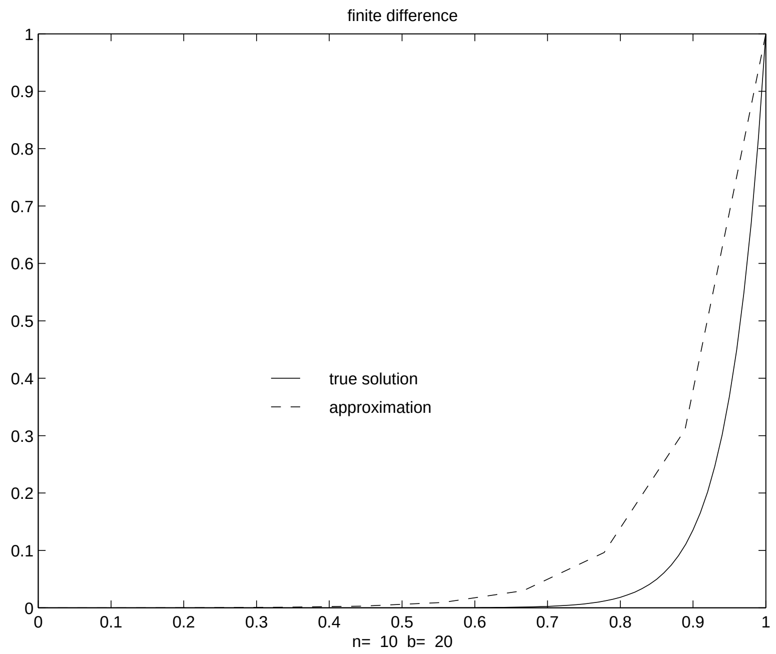

Figure 10: Exact solution vs. up-wind Finite Difference approximation

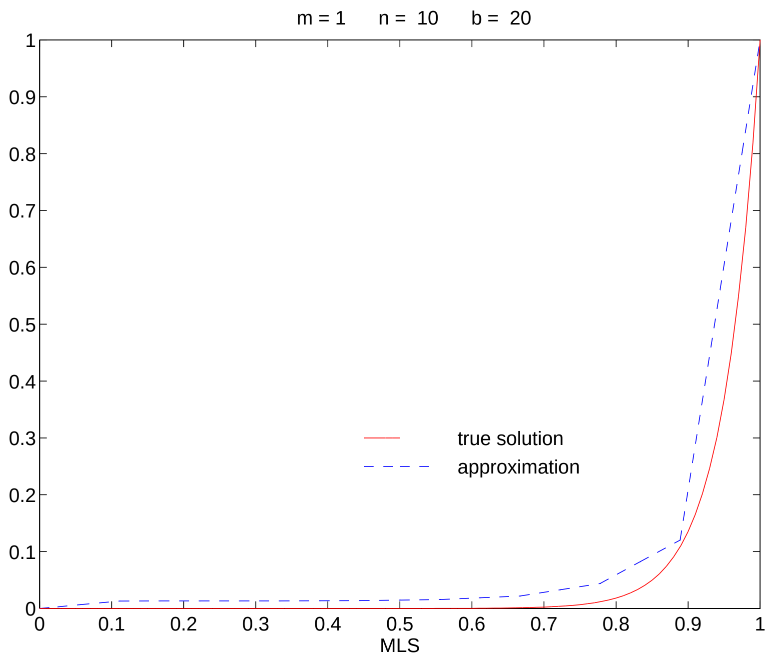

Figure 11: Exact solution vs. MLS approximation

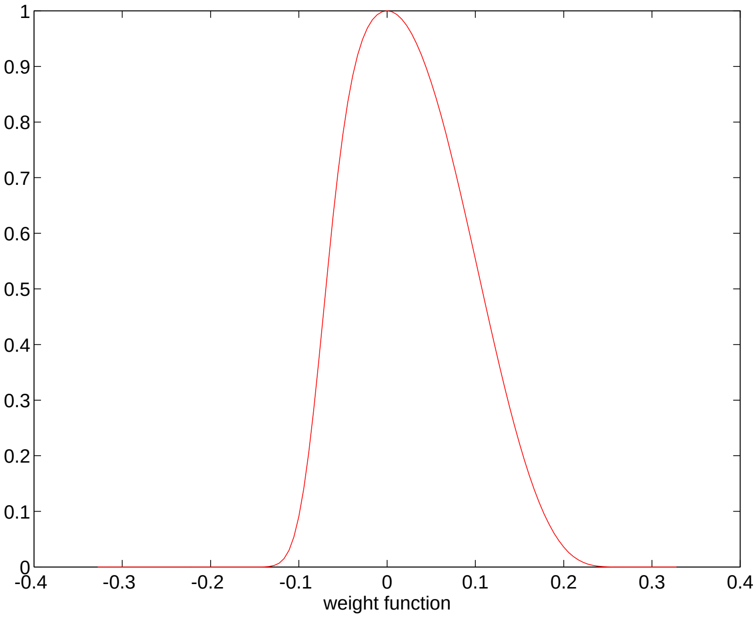

Figure 12: Smooth weight function

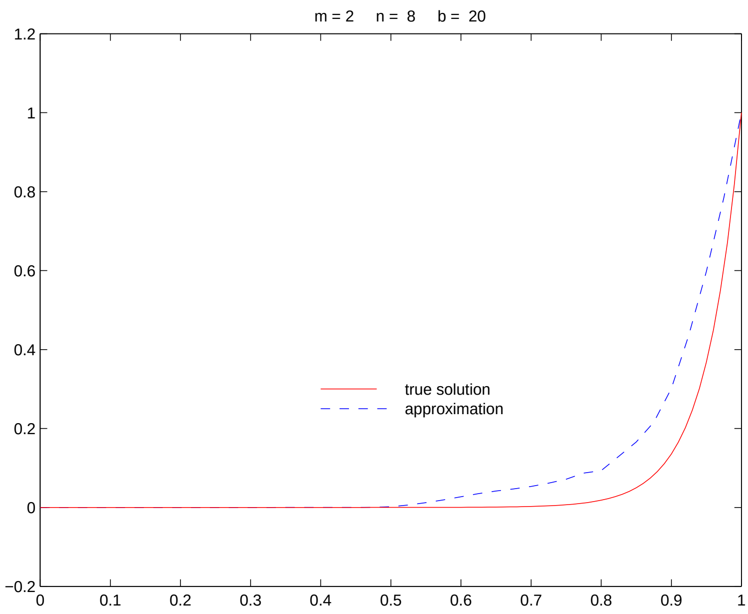

Figure 13: Exact solution vs. MLS approximation using Collocation

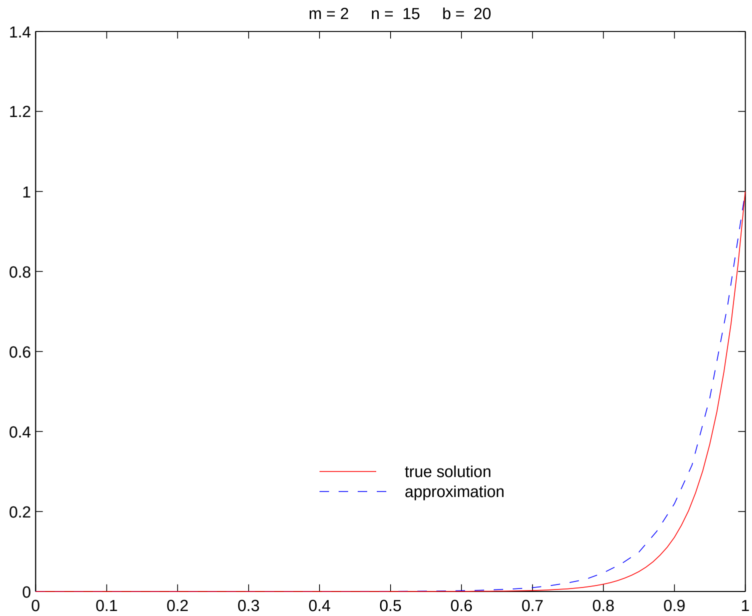

Figure 14: Exact solution vs. MLS approximation using Collocation

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