FINITE ELEMENT APPROXIMATIONS IN A NON-LIPSCHITZ DOMAIN: PART II

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Abstract. In [2] the finite element method was applied to a non-homogeneous Neumann problem on a cuspidal domain $\Omega \subset \mathbb{R}^2$, and quasi-optimal order error estimates in the energy norm were obtained for certain graded meshes. In this paper, we study the error in the L^2 norm obtaining similar results by using graded meshes of the type considered in [2]. Since many classical results in the theory Sobolev spaces do not apply to the domain under consideration, our estimates require a particular duality treatment working on appropriate weighted spaces.

On the other hand, since the discrete domain Ω_h verifies $\Omega \subset \Omega_h$, in [2] the source term of the Poisson problem was taken equal to 0 outside Ω in the variational discrete formulation. In this article we also consider the case in which this condition does not hold and obtain more general estimates, which can be useful in different problems, for instance in the study of the effect of numerical integration, or in eigenvalue approximations.

1. INTRODUCTION

The finite element method has been widely studied in several contexts involving different kinds of differential equations, however, the domains under consideration are in general polygons or smooth domains. In the recent paper [2], the piecewise linear finite element method was applied to a non-homogeneous Poisson problem in a domain with an external cusp. Despite its simplicity, this problem provides an interesting starting point for the finite element analysis of more general equations in non-Lipschitz domains. This kind of problems have interesting applications in fluid mechanics. For instance, the motion of rigid bodies immersed in a fluid can lead to the presence of cusps as a result of collisions between bodies or between a body and the boundary [18, 22].

An interesting feature related with problems in this kind of domains is that even regular solutions may require some type of mesh adaptivity. Indeed, as it was proved in [1], the solution of the proposed problem belongs to H^2 and, despite of this fact, uniform meshes show a poor convergence rate. The reason for this behavior is related with the fact that, in this context, classical extension theorems do not apply [20, 23]. A solution for this drawback was also given in [2] and (quasi) optimal convergence order error in the energy norm was recovered by using appropriate mesh adaptivity. Let us notice that, for problems in polygonal domains with solutions having corner like singularities, the use of graded meshes has been widely studied (see [8, 11, 12, 15] and the references therein) and optimal or quasi-optimal convergence rates for numerical approximations are usually obtained by using arguments based on weighted Sobolev spaces.

In this paper we continue the analysis of the finite element method for the same problem considered in [2], focusing on L^2 convergence results. These estimates require a particular treatment, making it necessary to take into account the regularity of the extended functions outside the non-Lipschitz domain under consideration. We introduce two different kinds of

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auxiliary problems: one in the original domain Ω, and the other in the discrete domain Ω_h. The first one leads us to standard estimates of the error in Ω, and the second one to estimates of the error between a certain extension of the original solution and the discrete solution in Ω_h. The latter case is more general than the former, however, it is also much more technical and relies on certain extra assumptions. In both cases quasi-optimal order of convergence with respect to the number of nodes is obtained by using appropriate graded meshes of the type considered in [2]. We present some numerical examples supporting this analytical result, and in particular we show that uniform meshes lead to poor L^2 convergence order (similar conclusions for the H^1 norm were obtained in [2]) .

Let $\Omega \subset \mathbb{R}^2$ be defined by:

$$\Omega = \{(x, y) : 0 < x < 1,\, 0 < y < x^\alpha\},$$

where $\alpha > 1$. We denote the boundary of Ω by $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 = \{0 \le x \le 1, y = 0\}, \quad \Gamma_2 = \{x = 1, 0 \le y \le 1\} \quad \text{and} \quad \Gamma_3 = \{0 \le x \le 1, y = x^\alpha\}$$

(see Figure 1).

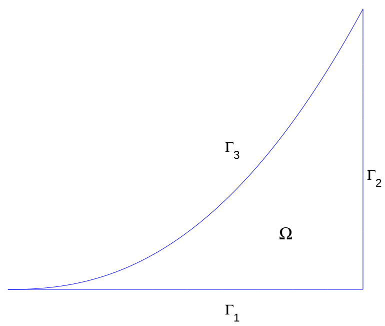

FIGURE 1. Cuspidal domain

Our model problem is: find u such that

$$\begin{cases} -\Delta u = f\,, & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = g\,, & \text{on } \Gamma_3 \\ \dfrac{\partial u}{\partial \nu} = 0\,, & \text{on } \Gamma_1 \\ u = 0\,, & \text{on } \Gamma_2 \end{cases} \tag{1.1}$$

where ν denotes the outside normal.

We will work along this paper with $g = 0$. This assumption partially simplifies the treatment of the error and can be justified by recalling that H^2 regularity results for (1.1) rely on the smoothness of g, and its fast decay to zero, i.e., $g \sim 0$ near the tip of the cusp [1].

Let $V = \{v \in H^1(\Omega) : v|_{\Gamma_2} = 0\}$. Then, the variational formulation of our model problem (1.1) is given by: find $u \in V$, such that

$$a(u, v) \;=\; L(v) \quad \forall v \in V \tag{1.2}$$

where $a(u, v) = \int_\Omega \nabla u \cdot \nabla v$ and $L(v) = \int_\Omega fv$. It is known that this problems has a unique solution in $H^2(\Omega)$ and that there exists a constant C such that (see [1, 15, 19])

$$\|u\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{1.3}$$

The natural way to approximate the solution of (1.2) is to replace Ω by a polygonal domain Ω_h and then use the standard finite element method. We will construct Ω_h in such a way that $\Omega \subset \Omega_h$ and the nodes of Γ_h, the boundary of Ω_h, are on Γ.

Let $\{\mathcal{T}_h\}$ be a family of triangulations of Ω_h verifying the maximum angle condition. We can associate to $\{\mathcal{T}_h\}$ the finite element space

$$V_h = \{v \in H^1(\Omega_h) : v|_{\Gamma_2} = 0 \,\text{and}\; v|_T \in \mathcal{P}_1 \quad \forall T \in \mathcal{T}_h\}$$

where $\mathcal{P}_1$ denotes the space of linear polynomials.

Then, the finite element approximation problem of (1.2) is: find $u_h \in V_h$, such that

$$a_h(u_h, v_h) \;\; = \;\; L_h(u_h) \quad \forall v_h \in V_h \tag{1.4}$$

where $a_h(u,v) = \int_{\Omega_h} \nabla u \cdot \nabla v$ and $L_h(v) = \int_{\Omega_h} fv$. Observe that the discrete problem corresponds to a boundary problem on Ω_h.

Since the solution of problem (1.1) depends on the values of f in Ω only, it seems natural to assume that f vanishes outside Ω, in which case we have $L_h(v) = \int_\Omega fv$ and so (1.4) agrees with the discrete problem from [2]. In this paper we also consider the case in which this assumption is dropped and obtain more general error estimates for finite element approximations. This approach introduces an extra difficulty that was not addressed in [2], however, it provides more information in different scenarios. Indeed, the contribution of terms like $\int_{\Omega_h \setminus \Omega} \tilde{f} v_h$, with $\tilde{f}$ being a certain approximation of f defined on Ω_h, can be useful to evaluate the effect of numerical integration (see, for example, [24]). Moreover, since $\Omega \neq \Omega_h$, the standard theory for eigenvalue approximations [9] does not apply straightforwardly. In fact, the study of convergence for this problems leads to problems like (1.4) with f not necessarily equal to zero outside Ω [17, 24]. On the other hand, the study of the error between a certain extension of the solution u and u_h, analyzed in Section 5, is also of interest in the context of eigenvalue approximations [17, 24].

Let us mention that, even when Ω is not regular enough, certain extension operators can be constructed. More precisely, the solution of (1.2) can be extended to a function in a weighted Sobolev space with the weight being a power of the distance to the cuspidal point (see [2, 20]). In fact, there exists a function $u^E \in H^2_\alpha(\mathbb{R}^2)$ such that $u^E|_\Omega = u$, and

$$\|u^E\|_{H^2_\alpha(\mathbb{R}^2)} \leq C\|u\|_{H^2(\Omega)}, \tag{1.5}$$

where the weighted Sobolev space H^2_α is defined, for any domain $\mathcal{D} \subset \mathbb{R}^2$, as follows:

$$H^2_\alpha(\mathcal{D}) = \left\{ v : r^{\frac{\alpha-1}{2}} D^\delta v \in L^2(\mathcal{D}) \;\; \forall\, \delta\,, |\delta| \leq 2 \right\} \tag{1.6}$$

with $r = \sqrt{x^2+y^2}$, and

$$\|v\|^2_{H^2_\alpha(\mathcal{D})} = \sum_{|\delta|\leq 2} \|r^{\frac{\alpha-1}{2}} D^\delta v\|^2_{L^2(\mathcal{D})}.$$

This extension result will be useful to bound the approximation error in L^2.

The rest of the paper is organized as follows. In Section 2 we present some results involving the graded meshes that we will use in the remainder of the paper. In Section 3, we obtain L^2 error estimates in Ω when $f \equiv 0$ outside Ω. Section 4 is devoted to obtain H^1 and L^2 error estimates when f is not necessarily equal to zero outside Ω in the discrete variational formulation. In Section 5 we introduce and analyze an auxiliary problem on Ω_h, which is the main tool to obtain L^2 error estimates in Ω_h between u_h and a certain extension of u. Finally, in Section 6, we explain how the graded meshes can be constructed and we present numerical approximations in which the error behaves according to our analytical results.

2. Graded Meshes

We will assume that the family of meshes $\{\mathcal{T}_h\}$ satisfies the same properties as those considered in [2]. More precisely, we take $1 < \alpha < 3$ and define $\gamma = (\alpha - 1)/2$. Let $\mathcal{T}_h$ be a triangulation of Ω_h, where Ω_h is an approximate polygon of Ω with all its vertices belonging to Γ, and $h > 0$ be a parameter that goes to 0. If for each $T \in \mathcal{T}_h$ we denote by h_T its diameter and by θ_T its maximum angle, we assume that there exist positive constants σ and $\theta_M < \pi$, independent of h, such that

(1) $\theta_T < \theta_M$, $\forall T \in \mathcal{T}_h$ *(the maximum angle condition).*
(2) $h_T \sim \sigma\, h^{\frac{1}{1-\gamma}}$, if $(0,0) \in T$.
(3) $h_T \leq \sigma\, h\, \inf_T x^\gamma$, if $(0,0) \notin T$.

We denote by $\Gamma_{3,h}^j$, $1 \leq j \leq n$, the edges on the boundary of Ω_h, by $P_{j-1} = (x_{j-1}, x_{j-1}^\alpha)$ and $P_j = (x_j, x_j^\alpha)$ their endpoints with $x_0 = 0$ and $x_n = 1$, and by Γ_3^j the part on Γ_3 with the same endpoints (see Figure 2). By Ω_h^j we denote the region bounded by Γ_3^j and $\Gamma_{3,h}^j$.

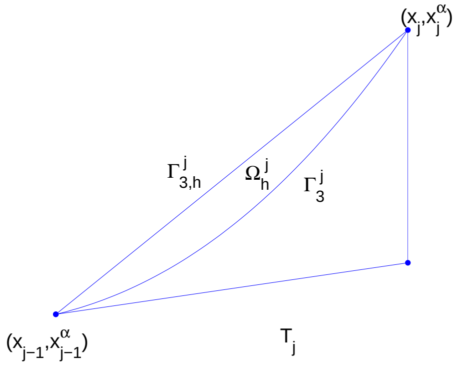

Figure 2

In addition to the assumptions (1), (2) and (3) we will need for our error analysis the following hypothesis on the meshes:

(Ha) For $1 \leq j \leq n$ the region Ω_h^j is strictly contained in only one triangle denoted by T_j. We denote the diameter of T_j by h_j.

Let us also notice that, for $2 \leq j \leq n$,

$$x_j \leq C x_{j-1}$$

where C can be taken independent of h. Indeed, from (Ha) we have $x_j - x_{j-1} \leq C|\Gamma_{3,h}^j|$ for some constant depending only on α. Then, $x_j - x_{j-1} \leq C h_j$, and hence, from assumption (3) we get

$$x_j \leq x_{j-1}\left(1 + C h x_{j-1}^{\gamma-1}\right).$$

Therefore, we have proved the following useful result

Lemma 2.1. *For $2 \leq j \leq n$,*

$$x_{j-1} \leq x_j \leq C x_{j-1}$$

with C depending only on α and σ.

Remark 2.1. *We will show in Section 6 that meshes verifying conditions (1),(2),(3) and (Ha), can indeed be constructed.*

We will assume that our family of triangulations satisfies conditions (1), (2) and (3), and hypothesis (Ha). The following result is obtained in the proof of Lemma 4.1 in [2], we reproduce it here as a separate result for the sake of completeness.

Lemma 2.2. *Let* $\gamma = \frac{\alpha-1}{2}$*, with* $1 < \alpha < 3$ *and choose* $0 < \beta$ *and* $q > 1$ *such that*

$$\beta q < \min\{2\gamma, 1\}.$$

Then

$$\int_{\Omega_h \setminus \Omega} x^{-2\beta q} \leq Ch^2$$

Proof.

$$\int_{\Omega_h \setminus \Omega} x^{-2\beta q} = \sum_{j=1}^{N} \int_{\Omega_h^j} x^{-2\beta q}. \tag{2.7}$$

Let us estimate each term in the right hand side of (2.7). Since

$$\Omega_h^1 \subset T = \{(x, y) : 0 \leq x \leq x_1,\, 0 \leq y \leq x_1^{\alpha-1} x\},$$

we have

$$\int_{\Omega_h^1} x^{-2\beta q} \leq \int_T x^{-2\beta q}.$$

Hence, using now that $h_1 \leq \sigma\, h^{\frac{1}{1-\gamma}}$ and $\beta q < 1$, we obtain

$$\int_T x^{-2\beta q} \leq C h_1^{2(\gamma+1-\beta q)} \leq C h^{2\frac{\gamma+1-\beta q}{1-\gamma}}$$

and, therefore,

$$\int_T x^{-2\beta q} \leq Ch^2$$

because $\beta q < 2\gamma$.

On the other hand, we have

$$\sum_{j>1} \int_{\Omega_h^j} x^{-2\beta q} \leq \sum_{j>1} x_{j-1}^{-2\beta q} |\Omega_h^j|,$$

but, by using the well known error formula for the trapezoidal rule, we obtain

$$|\Omega_h^j| \leq C h_j^3 x_{j-1}^{\alpha-2} = C h_j^3 x_{j-1}^{2\gamma-1}$$

where in the case when $\alpha > 2$ we have used $x_j \leq C x_{j-1}$. Therefore, since $h_j \leq \sigma h\, x_{j-1}^{\gamma}$ we have

$$\begin{aligned} \sum_{j>1} \int_{\Omega_h^j} x^{-2\beta q} &\leq C \sum_{j>1} x_{j-1}^{-2\beta q+2\gamma-1} h_j^3 \leq Ch^2 \sum_{j>1} x_{j-1}^{-2\beta q+4\gamma-1} h_j \\ &\leq Ch^2 \int_0^1 x^{-2\beta q+4\gamma-1} \end{aligned}$$

where we have used again that $x_j \leq C x_{j-1}$. But the last integral is finite because $\beta q < 2\gamma$. □

We will also need bounds for the measure of the set $\Omega_h \setminus \Omega$ in terms of the parameter h.

Lemma 2.3. *It holds*

$$|\Omega_h \setminus \Omega| \leq Ch^2.$$

Proof. It follows by using similar arguments as those in the proof of Lemma 2.2, or as a Corollary, taking into account that $1 \le x^{-2\beta q}$, where β and q are as in the previous Lemma. □

In order to obtain L^2 error estimates in the polygonal domains Ω_h we will need a careful estimate for the inner angles of Ω_h. This computation is carried out in the following Lemma.

Lemma 2.4. *Let ω_h be the maximum inner angle of Ω_h, then*

i) *If $\alpha < 2$,* $\quad \omega_h \le \pi + C\alpha(\alpha-1)h^{\frac{\alpha-1}{3-\alpha}}$

ii) *If $2 \le \alpha < 3$,* $\quad \omega_h \le \pi + C\alpha(\alpha-1)h$,

where C is independent of α and h.

Proof. Assume first $j \ge 2$ (the case $j = 1$ is treated below). If we denote by $\omega_{h,j}$ the inner angle between $\Gamma^j_{3,h}$ and $\Gamma^{j+1}_{3,h}$, we obviously have

$$\omega_{h,j} = \pi + \arctan\left(\frac{x^\alpha_{j+1} - x^\alpha_j}{x_{j+1} - x_j}\right) - \arctan\left(\frac{x^\alpha_j - x^\alpha_{j-1}}{x_j - x_{j-1}}\right) \tag{2.8}$$

But

$$\frac{x^\alpha_{j+1} - x^\alpha_j}{x_{j+1} - x_j} = \alpha\tilde{x}^{\alpha-1}_{j+1} \qquad \frac{x^\alpha_j - x^\alpha_{j-1}}{x_j - x_{j-1}} = \alpha\tilde{x}^{\alpha-1}_j$$

for some $\tilde{x}_{j+1} \in [x_j, x_{j+1}]$ $\tilde{x}_j \in [x_{j-1}, x_j]$, and using the fact that the function $\arctan(\alpha t^{\alpha-1})$ is increasing we get

$$\omega_{h,j} \le \pi + \arctan(\alpha x^{\alpha-1}_{j+1}) - \arctan(\alpha x^{\alpha-1}_{j-1}).$$

By the mean value theorem and Lemma 2.1,

$$\begin{aligned}
\arctan(\alpha x^{\alpha-1}_{j+1}) - \arctan(\alpha x^{\alpha-1}_{j-1}) &= C\frac{\alpha(\alpha-1)x^{\alpha-2}_j}{1+\alpha^2 x_j^{2(\alpha-1)}}(x_{j+1} - x_{j-1}) \\
&\le C\alpha(\alpha-1)x^{\alpha-2}_j(x_{j+1} - x_{j-1}) \\
&\le Ch\alpha(\alpha-1)x^{\alpha-2}_j x_j^{\frac{\alpha-1}{2}}
\end{aligned}$$

where in the last inequality we have used condition (3).

Now, if $\alpha \ge 2$ the result follows immediately. For the case $\alpha < 2$ we use that $x_j \ge x_1$ and that, by condition (2), $x_1 \sim h^{\frac{2}{3-\alpha}} = h^{\frac{1}{1-\gamma}}$, so

$$\arctan(\alpha x^{\alpha-1}_{j+1}) - \arctan(\alpha x^{\alpha-1}_{j-1}) \le C\alpha(\alpha-1)h^{\frac{2(\alpha-2)}{3-\alpha}+1} = C\alpha(\alpha-1)h^{\frac{\alpha-1}{3-\alpha}}$$

obtaining the desired result.

Let us focus now on the case $j = 1$. In this case (2.8) takes the form

$$\omega_{h,1} = \pi + \arctan(\frac{x^\alpha_2 - x^\alpha_1}{x_2 - x_1}) - \arctan(x^{\alpha-1}_1) = \pi + \arctan(\alpha\tilde{x}^{\alpha-1}_2) - \arctan(x^{\alpha-1}_1)$$

with $\tilde{x}_2 \in [x_1, x_2]$, but

$$x^{\alpha-1}_1 = \alpha\left(\frac{x_1}{\alpha^{1/(\alpha-1)}}\right)^{\alpha-1} = \alpha\tilde{x}^{\alpha-1}_1$$

so

$$\tilde{x}_1 \le x_1 \le C\tilde{x}_1 \tag{2.9}$$

with C depending only on α. Then

$$\omega_{h,1} \le \pi + \arctan(\alpha\tilde{x}^{\alpha-1}_2) - \arctan(\alpha\tilde{x}^{\alpha-1}_1)$$

and the result follows now as in the case $j \ge 2$ using (2.9).

□

In Theorem 2.4 from [5, page 63] T. Apel obtained interpolation error estimates for functions in weighted Sobolev spaces on tetrahedral elements under the maximum angle condition. However, we were unable to find analogous results for the two dimensional case. The reason for this seems to be that corner singularities, which lead to the kind of spaces considered in this work, do not require anisotropic elements in the case of polygonal domains (see, for instance, [15]). In our case, the external cusp enforces the occurrence of flat elements and, hence, we need to obtain the required error estimates for functions in H^2_α in dimension 2 under the maximal angle condition. In order to do that, we prove the following Poincaré type inequality for functions with zero average on a side of a triangle.

Lemma 2.5. *Let $\hat{T}$ be the following "reference" triangle, $\hat{T} = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le x\}$, and w such that $\|w\|_{L^2(\hat{T})} + \|\nabla w x^s\|_{L^2(\hat{T})} < \infty$ for some $0 \le s < 1$. If $\int_\ell w = 0$ where ℓ is a side of $\hat{T}$, then, there exists a positive constant C, depending only on s and $\hat{T}$, such that*

$$\|w\|_{L^2(\hat{T})} \le C\|\nabla w x^s\|_{L^2(\hat{T})}.$$

Proof. We observe that for any $0 \le s < 1$, there exists $p > 1$ such that

$$\|v\|_{L^p(\hat{T})} \le C\|v x^s\|_{L^2(\hat{T})}. \tag{2.10}$$

Indeed, since

$$\int_{\hat{T}} |v|^p = \int_{\hat{T}} |v|^p x^{ps} x^{-ps},$$

applying Hölder's inequality with exponent $\frac{2}{p}$ and its dual exponent $\frac{2}{2-p}$ we obtain (2.10) for any p such that $p < \frac{2}{s+1}$. On the other hand, it is easy to see by standard compactness arguments (see Lemma 2.2 in [3] for the case $p = 2$) that functions with zero average on one side of $\hat{T}$ verify

$$\|w\|_{L^p(\hat{T})} \le C\|\nabla w\|_{L^p(\hat{T})} \tag{2.11}$$

with C depending only on p and $\hat{T}$. Therefore,

$$\|w\|_{L^2(\hat{T})} \le C\|w\|_{W^{1,p}(\hat{T})} \le C\|\nabla w\|_{L^p(\hat{T})} \le C\|\nabla w x^s\|_{L^2(\hat{T})}.$$

Indeed, the first inequality follows by the classical embedding theorem, while the second and third inequalities are consequences of (2.11) and (2.10) with $v = \nabla w$. □

Theorem 2.1. *Let T be a triangle with a maximum interior angle θ_T, and let $\mathbf{v}_m$ be the vertex corresponding to the minimum interior angle of T. We denote by $d_{\mathbf{v}_m}(x, y)$ the distance from $(x, y) \in T$ to $\mathbf{v}_m$. Let v be such that $\|v\|_{L^2(T)} + \|\nabla v\|_{L^2(T)} + \sum_{|\delta|=2} \|D^\delta v d^s_{\mathbf{v}_m}\|_{L^2(T)} < \infty$ for some $0 \le s < 1$. Then, there exists a positive constant C, depending only on θ_T, such that*

$$\|\nabla(v - \Pi v)\|_{L^2(T)} \le C h_T^{1-s} \sum_{|\delta|=2} \|D^\delta v d^s_{\mathbf{v}_m}\|_{L^2(T)}, \tag{2.12}$$

$$\|v - \Pi v\|_{L^2(T)} \le C h_T^{2-s} \sum_{|\delta|=2} \|D^\delta v d^s_{\mathbf{v}_m}\|_{L^2(T)}. \tag{2.13}$$

where $\Pi v \in V_h$ denotes the piecewise linear Lagrange interpolation of v.

Proof. It is clear that it is enough to show (2.12) and (2.13) for a triangle obtained from T after a rigid movement. Hence, we can assume that T is a triangle with $\mathbf{v}_m = (0, 0)$ and with remaining vertices of the form $\mathbf{v}_2 = (h_1, 0)$ $\mathbf{v}_3 = (x_1, h_2)$ with $h_1, h_2 > 0$ and $h_1 \ge \sqrt{x_1^2 + h_2^2} \ge h_2$. Therefore, the angle θ_2 at $\mathbf{v}_2 = (h_1, 0)$ verifies $\theta_2 \le \pi/2$ and, since it is not the minimum angle of T, $\theta_2 \ge \frac{\pi - \theta_T}{2}$, i.e,

$$\frac{\pi - \theta_T}{2} \le \theta_2 \le \pi/2. \tag{2.14}$$

Let us introduce a further linear transformation L given by the matrix

$$A = \begin{pmatrix} 1 & \frac{x_1-h_1}{h_2} \\ 0 & 1 \end{pmatrix}.$$

It is clear that L transforms the right triangle T_R with vertices $(0,0),(h_1,0),(h_1,h_2)$ into T. From (2.14), it is easy to see that $\frac{x_1-h_1}{h_2} \le C$ for some $C = C(\theta_T)$ and, as a consequence, $\|A\| \le C$ and $\|A^{-1}\| \le C$. Since in both triangles T and T_R the minimum angle is placed at the origin, the inequalities $\|(x,y)\| \le \|A^{-1}\|\|L(x,y)\| \le \|A\|\|A^{-1}\|\|(x,y)\|$ imply the equivalence between the distance $d_{\mathbf{v}_m}(x,y)$ and the norm of $L(x,y)$. Therefore, changing variables, we have that it is enough to prove (2.12) and (2.13) for T_R with $h_1 \ge h_2$. On the other hand, in T_R it is clear that $d_{\mathbf{v}_m} = \|(x,y)\| \sim x$ and so in order to show that (2.12) holds it is enough to prove

$$\|\nabla(v - \Pi v)\|_{L^2(T_R)} \le C h_{T_R}^{1-s} \sum_{|\delta|=2} \|D^\delta v x^s\|_{L^2(T_R)}. \tag{2.15}$$

We prove the previous inequality for $\frac{\partial(v-\Pi v)}{\partial x}$; the other derivative can be treated in the same way. Taking $w = \frac{\partial(v-\Pi v)}{\partial x}$ we have that $\int_{\ell_1} w = 0$, ℓ_1 being the side joining the vertices $(0,0)$ and $(h_1,0)$. Changing variables to the reference element defined in Lemma 2.5 we get, taking $\hat{L}(\hat{x},\hat{y}) = (\hat{x}h_1, \hat{y}h_2)$, that the function $\hat{w} = w \circ \hat{L}$ has zero average on the side of $\hat{T}$ joining the vertices $(0,0)$ and $(1,0)$. Then, by Lemma 2.5 applied to $\hat{w}$ we get

$$\|w\|^2_{L^2(T_R)} = h_1 h_2 \|\hat{w}\|^2_{L^2(\hat{T})} \le h_1 h_2 C \|\nabla \hat{w} \hat{x}^s\|^2_{L^2(\hat{T})}$$

and changing variables back to the original T_R

$$\|w\|_{L^2(T_R)} \le C \left(h_1 \|\frac{\partial w}{\partial x}\left(\frac{x}{h_1}\right)^s\|_{L^2(\hat{T})} + h_2 \|\frac{\partial w}{\partial y}\left(\frac{x}{h_1}\right)^s\|_{L^2(\hat{T})} \right)$$

since $h_1 \ge h_2$,

$$\|w\|_{L^2(T_R)} \le C h_{T_R}^{1-s} \|\nabla w x^s\|_{L^2(T_R)}$$

and (2.15) follows. As it is usual when considering anisotropic elements, inequality (2.13) is easier to prove than (2.12), since its left hand side does not involve derivatives. The estimate for $\hat{T}$

$$\|\hat{v} - \Pi\hat{v}\|_{L^2(\hat{T})} \le C \sum_{|\delta|=2} \|D^\delta \hat{v} \hat{x}^s\|_{L^2(\hat{T})} \tag{2.16}$$

follows by using embedding results and standard Lagrange interpolation error estimates in L^p together with (2.10). In fact,

$$\|\hat{v} - \Pi\hat{v}\|_{L^2(\hat{T})} \le C \|\hat{v} - \Pi\hat{v}\|_{W^{1,p}(\hat{T})} \le C \sum_{|\delta|=2} \|D^\delta \hat{v}\|_{L^p(\hat{T})} \le \sum_{|\delta|=2} \|D^\delta \hat{v} \hat{x}^s\|_{L^2(\hat{T})}.$$

Now, (2.13) follows on T_R from (2.16) by using the change of variables $\hat{L}(\hat{x},\hat{y}) = (\hat{x}h_1, \hat{y}h_2)$ and taking into account that $\Pi\hat{v} = \Pi v \circ \hat{L}$. □

We define a fixed (i.e., independent of h) domain T_U, which contains our discrete domain Ω_h, in the following way:

$$T_U = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le x\}. \tag{2.17}$$

Although T_U agrees with $\hat{T}$, we use a different notation for both triangles for the sake of clarity, since $\hat{T}$ plays the standard role of the reference element in interpolation error estimates, and T_U is the domain where the extension u^E of u will be studied.

Lemma 2.6. *If $1 < \alpha < 3$, there exists a constant C, which depends only on α, such that*

$$\|u^E\|_{W^{2,p}(T_U)} \le C\|u^E\|_{H^2_\alpha(T_U)}$$

for all $1 \le p < \frac{4}{\alpha+1}$.

Proof. The proof follows by using (2.10), with $s = \frac{\alpha-1}{2} < 1$, for u^E and its derivatives. □

Remark 2.2. *Conditions (1), (2) and (Ha), together with the fact that $\alpha > 1$, imply that there exists only one triangle T in the mesh such that $(0,0) \in T$. Moreover, its vertices are necessarily of the form $(0,0)$, $(0,h_1)$, and (x_1, x_1^α). Furthermore, if $h_T \to 0$, the angle θ_0 placed at $(0,0)$ tends to zero since $\frac{x_1^\alpha}{x_1} = x_1^{\alpha-1} \le h_T^{\alpha-1} \to 0$ and, hence, condition (1) implies that for h_T small enough θ_0 is in fact the minimum interior angle.*

Now we can prove the following "global" version of the interpolation error estimates,

Theorem 2.2. *There exists a constant C depending only on θ_M, σ and α such that*

$$\|\nabla(u^E - \Pi u^E)\|_{L^2(\Omega_h)} \le Ch|u^E|_{H^2_\alpha(\Omega_h)}, \tag{2.18}$$

and

$$\|u^E - \Pi u^E\|_{L^2(\Omega_h)} \le Ch^2|u^E|_{H^2_\alpha(\Omega_h)}, \tag{2.19}$$

where $\Pi u^E \in V_h$ denotes the piecewise linear Lagrange interpolation of u^E and $|u^E|_{H^2_\alpha(\Omega_h)}$ denotes the usual semi-norm on H^2_α,

Proof. We will only sketch the proof because it is standard (see [15]). For (2.18) we write

$$\|\nabla(u^E - \Pi u^E)\|^2_{L^2(\Omega_h)} = \|\nabla(u^E - \Pi u^E)\|^2_{L^2(T_1)} + \sum_{T\in\mathcal{T}_h, T\neq T_1} \|\nabla(u^E - \Pi u^E)\|^2_{L^2(T)}$$

We observe that, in view of Remark 2.2, the triangle T_1 defined in (Ha) is the unique triangle which contains $(0,0)$. The first term can be bounded using condition (1), Theorem 2.1 with $s = \frac{\alpha-1}{2} < 1$ (recall that $\alpha < 3$), and noticing that Lemma 2.6 gives the necessary regularity for u^E (use embedding results on T_U). Finally, condition (2) allows us to replace h_{T_1} by h. The second term can be handled using error estimates for Lagrange interpolation for classical unweighted Sobolev spaces under the maximal angle condition (see, for example, [6]) together with condition (3). Indeed, since $(0,0)$ is not in T, we have

$$\|\nabla(u^E - \Pi u^E)\|_{L^2(T)} \le Ch_T|u^E|_{H^2(T)} \le Ch \inf_T x^\gamma |u^E|_{H^2(T)} \le Ch|u^E|_{H^2_\alpha(T)}.$$

The estimate (2.19) is handled in the same way. □

We finish this Section recalling the following estimate that will be useful later on.

Lemma 2.7. *If $1 < \alpha < 3$, then there exists a constant C, which depends only on α, θ_M and σ, such that*

$$\|\nabla u^E\|_{L^2(\Omega_h\setminus\Omega)} \le Ch\sqrt{log(1/h)}\|u\|_{H^2(\Omega)}.$$

Proof. See Lemma 4.1 in [2]. □

3. L^2 ERROR ESTIMATES IN Ω WHEN $f \equiv 0$ OUTSIDE Ω

In this section we obtain error estimates in $L^2(\Omega)$ of quasi-optimal order (i.e., optimal up to a logarithmic factor) with respect to the number of nodes using appropriate graded meshes, when f vanishes outside Ω.

The following error estimate in $H^1(\Omega)$ for the finite element approximation of the Poisson problem (1.2) with $f \equiv 0$ outside Ω was obtained in [2]

Theorem 3.1. *Let u be the solution of (1.2) and $u_h \in V_h$ be the solution of (1.4). Assume that $1 < \alpha < 3$ and that $f \in L^2(\Omega)$ is extended by zero outside Ω.*

If the family of meshes satisfies (1), (2), (3) and (Ha), then there exists a constant C depending only on α, θ_M and σ such that

$$\|u - u_h\|_{H^1(\Omega)} \leq \|u^E - u_h\|_{H^1(\Omega_h)} \leq Ch\sqrt{\log(1/h)}\,\|f\|_{L^2(\Omega)}.$$

Our next goal is to obtain error estimates in L^2 norm. In order to use the Aubin-Nitsche-duality arguments we introduce the following auxiliary problem: Let $\Phi \in H^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta\Phi = u - u_h\,, & \text{in } \Omega \\ \dfrac{\partial \Phi}{\partial \nu} = 0\,, & \text{on } \Gamma_1 \cup \Gamma_3 \\ \Phi = 0\,, & \text{on } \Gamma_2 \end{cases} \tag{3.20}$$

where ν denotes the outside normal. Applying the a priori estimate (1.3) to Φ, we have that $\Phi \in H^2(\Omega)$ and there exists a constant C such that

$$\|\Phi\|_{H^2(\Omega)} \leq C\|u - u_h\|_{L^2(\Omega)}. \tag{3.21}$$

Moreover, solutions of (3.20) can be extended to $\mathbb{R}^2$. Indeed, the analogous of (1.5) applied to Φ shows that there exists a function $\Phi^E \in H^2_\alpha(\mathbb{R}^2)$ such that $\Phi^E|_\Omega = \Phi$, and

$$\|\Phi^E\|_{H^2_\alpha(\mathbb{R}^2)} \leq C\|\Phi\|_{H^2(\Omega)}. \tag{3.22}$$

On the other hand, applying Lemma 2.6 to Φ^E, we get

$$\|\Phi^E\|_{W^{2,p}(T_U)} \leq C\|\Phi^E\|_{H^2_\alpha(T_U)} \tag{3.23}$$

for $1 \leq p < \frac{4}{\alpha+1}$.

Theorem 3.2. *Let u be the solution of (1.2) and u_h be the solution of (1.4). Assume that $1 < \alpha < 3$ and that $f \in L^2(\Omega)$ is extended by zero outside Ω. Then,*

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2 \log(1/h)\|f\|_{L^2(\Omega)}.$$

Proof. Let $e = u - u_h$ and Φ be the solution of (3.20) we have that

$$\int_\Omega e^2 = \int_\Omega (-\Delta\Phi)e = \int_\Omega \nabla\Phi\nabla e = \int_\Omega \nabla(\Phi - \Pi\Phi)\nabla e + \int_\Omega \nabla(\Pi\Phi)\nabla e \tag{3.24}$$

From (1.2) and (1.4) we get

$$\int_\Omega \nabla e\nabla v = \int_{\Omega_h\setminus\Omega} \nabla u_h \nabla v \qquad \forall v \in V_h$$

Hence,

$$\int_\Omega e^2 = \int_\Omega \nabla(\Phi - \Pi\Phi)\nabla e + \int_{\Omega_h\setminus\Omega} \nabla u_h \nabla(\Pi\Phi^E) \tag{3.25}$$

From Theorem 2.2 applied to Φ^E and (3.22) we have that

$$\|\nabla(\Phi - \Pi\Phi)\|_{L^2(\Omega)} \leq \|\nabla(\Phi^E - \Pi\Phi^E)\|_{L^2(\Omega_h)} \leq Ch\|\Phi^E\|_{H^2_\alpha(\Omega_h)} \leq Ch\|\Phi\|_{H^2(\Omega)}. \tag{3.26}$$

Then, the first term of (3.25) can be bounded by means of Theorem 3.1 and (3.21). Indeed,

$$\int_\Omega \nabla(\Phi - \Pi\Phi)\nabla e \leq Ch^2 \log(1/h)\|f\|_{L^2(\Omega)}\|e\|_{L^2(\Omega)}.$$

For the second term in (3.25) we have that

$$\int_{\Omega_h\setminus\Omega} \nabla u_h \nabla(\Pi\Phi^E) = \int_{\Omega_h\setminus\Omega} \nabla(u_h - u^E)\nabla(\Pi\Phi^E) + \int_{\Omega_h\setminus\Omega} \nabla u^E \nabla(\Pi\Phi^E). \tag{3.27}$$

The first term can be bounded using Theorem 3.1 by

$$\int_{\Omega_h\setminus\Omega} \nabla(u_h-u^E)\nabla(\Pi\Phi^E) \le Ch\sqrt{\log(1/h)}\|f\|_{L^2(\Omega)}\{\|\nabla(\Pi\Phi^E-\Phi^E)\|_{L^2(\Omega_h\setminus\Omega)}+\|\nabla\Phi^E\|_{L^2(\Omega_h\setminus\Omega)}\}$$

while the second term can be bounded using Lemma 2.7 and (1.3) by

$$\begin{aligned}\int_{\Omega_h\setminus\Omega} \nabla u^E\nabla(\Pi\Phi^E) &\le Ch\sqrt{\log(1/h)}\|f\|_{L^2(\Omega)}\|\nabla\Pi\Phi^E\|_{L^2(\Omega_h\setminus\Omega)}\\ &\le Ch\sqrt{\log(1/h)}\|f\|_{L^2(\Omega)}\{\|\nabla(\Pi\Phi^E-\Phi^E)\|_{L^2(\Omega_h\setminus\Omega)}+\|\nabla\Phi^E\|_{L^2(\Omega_h\setminus\Omega)}\}.\end{aligned}$$

Therefore, from (3.27) we get

$$\int_{\Omega_h\setminus\Omega} \nabla u_h\nabla(\Pi\Phi^E) \le Ch\sqrt{\log(1/h)}\|f\|_{L^2(\Omega)}\{\|\nabla(\Pi\Phi^E-\Phi^E)\|_{L^2(\Omega_h\setminus\Omega)}+\|\nabla\Phi^E\|_{L^2(\Omega_h\setminus\Omega)}\}$$

and using (3.26), Lemma 2.7 applied to Φ^E, and (3.21), we obtain

$$\int_{\Omega_h\setminus\Omega} \nabla u_h\nabla(\Pi\Phi^E) \le Ch^2\log(1/h)\|f\|_{L^2(\Omega)}\|e\|_{L^2(\Omega)}$$

and the theorem follows. ☐

Remark 3.1. *Whether or not the logarithmic factor* $\log h$ *in Theorem 3.1 and Theorem 3.2 can be removed is an open problem, and it is not easy to obtain information about either possibility from numerical experiments.*

4. Error estimates in the case in which $f \not\equiv 0$ outside Ω

In this section we will obtain error estimates in H^1 norm and L^2 norm when f does not necessarily vanish outside Ω. This kind of estimates can be useful in several situations. For example, even for simple sources like $f\equiv 1$ in Ω, the term $\int_{\Omega_h}\chi_\Omega v$ in (1.4) is usually replaced by $\int_{\Omega_h} 1v$ (i.e. as if f were defined as 1 over Ω_h) in numerical computations when any standard quadrature rule is applied. In general, the contribution of the terms like $\int_{\Omega_h\setminus\Omega}\tilde f v_h$, $\tilde f$ being a certain approximation of f defined on Ω_h, may be useful in order to evaluate the effect of numerical integration. On the other hand, in eigenvalue approximations the usual approach (see [9]) is based on the convergence of appropriate operators T_h to the limit operator T, T being the inverse of the Laplacian. Since $\Omega_h\ne\Omega$, the operators T_h are mesh dependent and the analysis leads to the study of problems like (1.4) with f not necessarily equal to zero outside Ω [17].

In order to analyze the contribution of the consistency term arising from the integral $\int_{\Omega_h\setminus\Omega} f v_h$ in equation (1.4) we will need, in addition to assumptions (1),(2),(3) and (Ha), the following hypothesis about the mesh:

(Hb) For each triangle T_j with vertices P_{j-1}, P_j, R_j, and for h_j small enough, the triangle $\tilde T_j$ with vertices $\frac{P_{j-1}+R_j}{2}, \frac{P_j+R_j}{2}, R_j$ (see Figure 3) does not intersect Ω_h^j.

Remark 4.1. *It can easily be deduced that (Hb) holds for meshes with only regular elements, and for domains with smooth boundaries. Meshes for the domains under consideration in this paper involve necessarily anisotropic elements (consider, for instance, any element with a vertex at* $(0,0)$*) and this kind of elements may fail to verify condition (Hb). In fact, an easy example is given by taking* $\alpha=5$ *and the triangle* T *defined by the vertices* $(0,0),(h,0),(h,h^5)$*. In Section 6 we will show that meshes verifying conditions (1),(2),(3),(Ha) and (Hb) can be constructed (we recall that* $1<\alpha<3$*).*

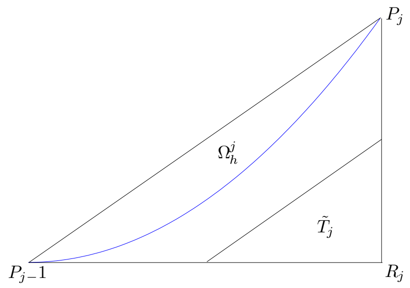

FIGURE 3

In what follows we will assume that the family of triangulations under consideration verifies (1),(2),(3), (Ha) and (Hb).

Our first goal is to obtain the H^1 error estimates for the solutions of (1.2) and (1.4). In order to do that we will use the following result (let us recall that Ω_h is not uniformly Lipschitz in h):

Lemma 4.1. *For any $v_h \in V_h$ there exists a constant C such that*

$$\|v_h\|_{L^p(\Omega_h)} \leq C\|v_h\|_{H^1(\Omega_h)}$$

for $1 \leq p \leq \frac{2(\alpha+1)}{\alpha-1}$.

Proof. Since $\Omega \subset \Omega_h$ and $v_h \in V_h$, we see that $v_h|_\Omega \in H^1(\Omega)$ and then, using the imbedding result for cusps given in Theorem 5.35 of [4] (with $\nu = \alpha - 1$), we get

$$\|v_h\|_{L^p(\Omega)} \leq C\|v_h\|_{H^1(\Omega)}$$

for $1 \leq p \leq \frac{2(\alpha+1)}{\alpha-1}$. We now need to show that v_h can also be bounded on $\Omega_h \setminus \Omega$. More precisely, since v_h is a piecewise linear function, we claim that

$$\|v_h\|_{L^p(\Omega_h)} \leq C\|v_h\|_{L^p(\Omega)} \tag{4.28}$$

from which we can easily obtain the desired result. Inequalities like (4.28) for Lipschitz domains have been obtained in different works (see, for example, [16]).

Let us introduce the notation $M^j = T_j \setminus \Omega_h^j$ (i.e. M^j stands for $\Omega \cap T_j$). All we need is to show that the local estimates

$$\|v_h\|_{L^p(T_j)} \leq C\|v_h\|_{L^p(M^j)} \tag{4.29}$$

hold with C depending only on α. From (Hb) we have that $\tilde{T}_j \cap \Omega_h^j = \emptyset$, so $\tilde{T}_j \subset M^j \subset T_j$. On the other hand, calling $\hat{T}$ and $\hat{T}_{\frac{1}{2}}$ the triangles of vertices $\{(0,0),(1,0),(0,1)\}$ and $\{(0,0),(\frac{1}{2},0),(0,\frac{1}{2})\}$ respectively, we have that there exists an affine mapping $\hat{F}$ such that $\hat{F}(\hat{T}) = T_j$ and $\hat{F}(\hat{T}_{\frac{1}{2}}) = \tilde{T}_j$. Now, since the space of linear functions has finite dimension, we have

$$\|\hat{v}\|_{L^p(\hat{T})} \leq C\|\hat{v}\|_{L^p(\hat{T}_{\frac{1}{2}})}$$

for any linear function $\hat{v}$ (the constant C depends only on $\hat{T}$ and $\hat{T}_{\frac{1}{2}}$). Changing variables we get

$$\|v_h\|_{L^p(T_j)} \leq C\|v_h\|_{L^p(\tilde{T}_j)}$$

from which (4.29) follows, since $\tilde{T}_j \subset M^j$. □

Remark 4.2. *Note that, since functions in V_h vanish at Γ_2, the previous Lemma implies, together with Poincaré inequality, that*

$$\|v_h\|_{L^p(\Omega_h)} \le C|v_h|_{H^1(\Omega_h)} \tag{4.30}$$

for $1 \le p \le \frac{2(\alpha+1)}{\alpha-1}$.

Theorem 4.1 below is a generalization of Theorem 3.1 and essentially says that the contribution to the error of the consistency type term due to fact that $f \not\equiv 0$ outside Ω is at most $Ch^{\frac{2}{\alpha+1}}\|f\|_{L^2(\Omega_h\setminus\Omega)}$.

Theorem 4.1. *Let u be the solution of (1.2) and u_h be the solution of (1.4). If we assume that $\alpha < 3$ and $f \in L^2(\mathbb{R}^2)$, then there exists a positive constant C, depending only on α, θ_M, and σ, such that*

$$\|u - u_h\|_{H^1(\Omega)} \le \|u^E - u_h\|_{H^1(\Omega_h)} \le Ch\sqrt{\log 1/h}\|f\|_{L^2(\Omega)} + Ch^{\frac{2}{\alpha+1}}\|f\|_{L^2(\Omega_h\setminus\Omega)}.$$

Proof. Since $\Omega \subset \Omega_h$, by Poincaré inequality and (1.3) we observe that it is enough to prove that

$$|u^E - u_h|_{H^1(\Omega_h)} \le Ch\sqrt{\log 1/h}\|u\|_{H^2(\Omega)} + Ch^{\frac{2}{\alpha+1}}\|f\|_{L^2(\Omega_h\setminus\Omega)}. \tag{4.31}$$

Now,

$$|u^E - u_h|^2_{H^1(\Omega_h)} = \int_{\Omega_h} \nabla(u^E - u_h)\cdot\nabla(u^E - \Pi u^E) + \int_{\Omega_h} \nabla(u^E - u_h)\cdot\nabla(\Pi u^E - u_h), \tag{4.32}$$

but we know from (1.5) and (2.18) that

$$|u^E - \Pi u^E|_{H^1(\Omega_h)} \le Ch\|u^E\|_{H^2_\alpha(\Omega_h)} \le Ch\|u\|_{H^2(\Omega)}.$$

Thus, for the first term in (4.32), by Young's inequality, we have

$$\int_{\Omega_h} \nabla(u^E - u_h)\cdot\nabla(u^E - \Pi u^E) \le \varepsilon|u^E - u_h|^2_{H^1(\Omega_h)} + C_\varepsilon h^2\|u\|^2_{H^2(\Omega)} \tag{4.33}$$

with ε to be chosen below.

For the second term of (4.32) we proceed as follows. Let us introduce the notation $w_h := \Pi u^E - u_h$. From (1.2) and (1.4) we have

$$\begin{aligned}\int_{\Omega_h} \nabla(u^E - u_h)\cdot\nabla w_h &= \int_{\Omega_h} \nabla u^E\cdot\nabla w_h - \int_{\Omega_h} \nabla u_h\cdot\nabla w_h \\ &= \int_{\Omega_h\setminus\Omega} \nabla u^E\cdot\nabla w_h - \int_{\Omega_h\setminus\Omega} f w_h. \end{aligned} \tag{4.34}$$

From Lemma 2.7 using Young's inequality again we obtain

$$\left|\int_{\Omega_h\setminus\Omega} \nabla u^E\cdot\nabla w_h\right| \le C_\varepsilon h^2\log(1/h)\|u\|^2_{H^2(\Omega)} + \varepsilon|w_h|^2_{H^1(\Omega_h)}, \tag{4.35}$$

while for the second term in (4.34), if we take $\frac{1}{p} + \frac{1}{q} = 1$ as

$$q = 2\frac{\alpha+1}{\alpha+3} < 2 \qquad p = 2\frac{\alpha+1}{\alpha-1},$$

we can write

$$|\int_{\Omega_h\setminus\Omega} f w_h| \le \left(\int_{\Omega_h\setminus\Omega} f^q\right)^{\frac{1}{q}}\left(\int_{\Omega_h\setminus\Omega} w_h^p\right)^{\frac{1}{p}} \le \left(\int_{\Omega_h\setminus\Omega} f^q\right)^{\frac{1}{q}}\left(\int_{\Omega_h} w_h^p\right)^{\frac{1}{p}}.$$

Applying again Hölder's inequality and Lemma 4.1 to the limit case $p = 2\frac{\alpha+1}{\alpha-1}$ ($w_h \in V_h$, see also (4.30)), we get

$$\left|\int_{\Omega_h\setminus\Omega} f w_h\right| \le C|\Omega_h \setminus \Omega|^{\frac{2-q}{2q}} \|f\|_{L^2(\Omega_h\setminus\Omega)} |w_h|_{H^1(\Omega_h)}$$

and by Young's inequality, Lemma 2.3, and replacing $q = 2\frac{\alpha+1}{\alpha+3}$ we obtain

$$\left|\int_{\Omega_h\setminus\Omega} f w_h\right| \le C_\varepsilon h^{\frac{4}{\alpha+1}} \|f\|^2_{L^2(\Omega_h\setminus\Omega)} + \varepsilon |w_h|^2_{H^1(\Omega_h)}.$$

This inequality together with (4.34) and (4.35) gives

$$\left|\int_{\Omega_h} \nabla(u^E - u_h)\cdot\nabla w_h\right| \le C_\varepsilon h^2 \log(1/h)\|u\|^2_{H^2(\Omega)} + C_\varepsilon h^{\frac{4}{\alpha+1}} \|f\|^2_{L^2(\Omega_h\setminus\Omega)} + 2\varepsilon|w_h|^2_{H^1(\Omega_h)}. \quad (4.36)$$

By (2.18)

$$|w_h|^2_{H^1(\Omega_h)} \le 2(|\Pi u^E - u^E|^2_{H^1(\Omega_h)} + |u^E - u_h|^2_{H^1(\Omega_h)}) \le Ch^2\|u\|^2_{H^2(\Omega)} + 2|u^E - u_h|^2_{H^1(\Omega_h)} \quad (4.37)$$

and replacing (4.37) in (4.36) we get (C_ε may change from line to line)

$$\begin{aligned}\left|\int_{\Omega_h} \nabla(u^E - u_h)\cdot\nabla w_h\right| &\le C_\varepsilon h^2 \log(1/h)\|u\|^2_{H^2(\Omega)} + C_\varepsilon h^{\frac{4}{\alpha+1}} \|f\|^2_{L^2(\Omega_h\setminus\Omega)} \\ &+ 4\varepsilon|\tilde{u} - u_h|^2_{H^1(\Omega_h)}. \end{aligned} \quad (4.38)$$

Finally, taking ε small enough, by (4.32), (4.33) and (4.38) we obtain (4.31), and the theorem follows. □

Our next goal is to obtain error estimates in $L^2(\Omega)$.

Theorem 4.2. *Let u be the solution of (1.2) and u_h be the solution of (1.4). Assume $\alpha < 3$, and $f \in L^2(\mathbb{R}^2)$. Then,*

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^2 \log(1/h)\|f\|_{L^2(\Omega)} + Ch\|f\|_{L^2(\Omega_h\setminus\Omega)}.$$

Proof. Let $e = u - u_h$ and Φ be the solution of (3.20). Then,

$$\int_\Omega e^2 = \int_\Omega(-\Delta\Phi)e = \int_\Omega \nabla\Phi\nabla e = \int_\Omega \nabla(\Phi - \Pi\Phi)\nabla e + \int_\Omega \nabla(\Pi\Phi)\nabla e.$$

From (1.2) and (1.4) we get

$$\int_\Omega \nabla e \nabla v = \int_{\Omega_h\setminus\Omega} \nabla u_h \nabla v - \int_{\Omega_h\setminus\Omega} f v \quad \forall v \in V_h,$$

hence,

$$\int_\Omega e^2 = \int_\Omega \nabla(\Phi - \Pi\Phi)\nabla e + \int_{\Omega_h\setminus\Omega} \nabla u_h \nabla(\Pi\Phi^E) - \int_{\Omega_h\setminus\Omega} f(\Pi\Phi^E). \quad (4.39)$$

The first term of (4.39) can be bounded by means of Theorem 4.1, (3.26) and the a priori estimates (3.21). Indeed,

$$\int_\Omega \nabla(\Phi - \Pi\Phi)\nabla e \le Ch\left\{h\sqrt{\log(1/h)}\|f\|_{L^2(\Omega)} + h^{\frac{2}{\alpha+1}}\|f\|_{L^2(\Omega_h\setminus\Omega)}\right\}\|e\|_{L^2(\Omega)}.$$

For the second term in (4.39), using Lemma 2.7 and Theorem 4.1 we know that

$$\begin{aligned}\int_{\Omega_h\setminus\Omega} \nabla u_h \nabla(\Pi\Phi^E) &= \int_{\Omega_h\setminus\Omega} \nabla(u_h - u^E)\nabla(\Pi\Phi^E) + \int_{\Omega_h\setminus\Omega} \nabla u^E \nabla(\Pi\Phi^E) \\ &\leq Ch\sqrt{\log(1/h)}\|f\|_{L^2(\Omega)}\{\|\nabla(\Pi\Phi^E - \Phi^E)\|_{L^2(\Omega_h\setminus\Omega)} + \|\nabla\Phi^E\|_{L^2(\Omega_h\setminus\Omega)}\}\end{aligned}$$

Then, using (3.26), Lemma 2.7 applied to Φ^E, and (3.21), we get

$$\int_{\Omega_h\setminus\Omega} \nabla u_h \nabla(\Pi\Phi^E) \leq Ch^2 \log(1/h)\|f\|_{L^2(\Omega)}\|e\|_{L^2(\Omega)}.$$

Therefore, we only have to estimate the third term in (4.39):

$$\int_{\Omega_h\setminus\Omega} f(\Pi\Phi^E) = \int_{\Omega_h\setminus\Omega} f(\Pi\Phi^E - \Phi^E) + \int_{\Omega_h\setminus\Omega} f\Phi^E. \tag{4.40}$$

Now, the L^2 interpolation error estimate given in Theorem 2.2 says that

$$\|\Pi\Phi^E - \Phi^E\|_{L^2(\Omega_h)} \leq Ch^2\|\Phi^E\|_{H^2_\alpha(\Omega_h)}$$

and then, using (3.21) and (3.22) we get

$$\int_{\Omega_h\setminus\Omega} f(\Pi\Phi^E - \Phi^E) \leq Ch^2\|e\|_{L^2(\Omega_h)}\|f\|_{L^2(\Omega_h\setminus\Omega)}.$$

Now, for the second term in (4.40) we use (3.23) and the fact that for $p > 1$ functions in $W^{2,p}(T_U)$ are bounded, together with (3.21), (3.22) and Lemma 2.3, to obtain

$$\begin{aligned}\int_{\Omega_h\setminus\Omega} f\Phi^E &\leq C\|\Phi^E\|_{L^\infty(T_U)}|\Omega_h \setminus \Omega|^{\frac{1}{2}}\|f\|_{L^2(\Omega_h\setminus\Omega)} \\ &\leq Ch\|\Phi^E\|_{H^2_\alpha(T_U)}\|f\|_{L^2(\Omega_h\setminus\Omega)} \leq Ch\|f\|_{L^2(\Omega_h\setminus\Omega)}\|e\|_{L^2(\Omega)}\end{aligned}$$

and the theorem follows. □

5. L^2 error estimates between u^E and u_h in Ω_h

In this section we obtain L^2 error estimates between the extended function u^E and the numerical solution u_h in the polygonal domain Ω_h. The results given below allow us, in particular, to obtain a precise computation of terms like $\|u_h\|_{L^2(\Omega_h\setminus\Omega)}$ which, for example, provides an optimal bound for the error between $\|u\|_{L^2(\Omega)}$ and $\|u_h\|_{L^2(\Omega_h)}$. On the other hand, estimates for the error between u^E and u_h are useful in the analysis of the error of eigenvalue problems [17].

The approach follows the lines of the previous sections, however, several extra complications arise since the dual problem is posed over the polygonal domain Ω_h. The main result of this section is Theorem 5.1, which is more general than Theorem 4.2. However, we want to remark that Theorem 5.1 relies on the Assumption 1 below, which is not necessary for the estimates in $L^2(\Omega)$ obtained in the previous sections.

We recall that error estimates between the extended function u^E and the numerical solution u_h in the $H^1(\Omega_h)$ norm have been obtained in Theorem 4.1.

We want to use duality arguments similar to those in the previous section. For this reason we introduce the following auxiliary problem closely related to (3.20):

For any h let $\Phi_h \in H^1(\Omega_h)$ be the solution of

$$\begin{cases} -\Delta\Phi_h = u^E - u_h\,, & \text{in } \Omega_h \\ \dfrac{\partial\Phi_h}{\partial\nu} = 0\,, & \text{on } \Gamma_1 \cup \Gamma_{3,h} \\ \Phi_h = 0\,, & \text{on } \Gamma_2 \end{cases} \tag{5.41}$$

where ν denotes the outside normal.

A priori estimates for (5.41) in fractional and weighted Sobolev spaces are well-known. Calling ω_h the maximum inner angle of Ω_h, and taking

$$\begin{cases} r_h = 1 - C\alpha(\alpha-1)h^{\frac{\alpha-1}{3-\alpha}}, & \text{for} \quad \alpha < 2 \\ r_h = 1 - C\alpha(\alpha-1)h, & \text{for} \quad \alpha \geq 2 \end{cases}, \tag{5.42}$$

for a suitable C we can assume, from Lemma 2.4, that $r_h < \pi/\omega_h$ and, hence, we have that $\Phi_h \in H^{1+r_h}(\Omega_h)$ [15] (with $r_h = 1$ if Ω_h is convex), and that [15, page 388] Φ_h belongs to the weighted Sobolev space $H^{2,\gamma_h}(\Omega_h)$ defined by:

$$H^{2,\gamma_h}(\Omega_h) = \left\{ v : \hat{r}^{\gamma_h} D^\beta v \in L^2(\Omega_h) \quad \forall \beta, |\beta| \leq 2 \right\},$$

where

$$\hat{r} = \min_{1 \leq j \leq n} r_j \tag{5.43}$$

with $r_j = \sqrt{(x-x_j)^2 + (y-x_j^\alpha)^2}$ and

$$\begin{cases} \gamma_h = C_\alpha h^{\frac{\alpha-1}{3-\alpha}}, & \text{for} \quad \alpha < 2 \\ \gamma_h = C_\alpha h & \text{for} \quad \alpha \geq 2 \end{cases}. \tag{5.44}$$

The following a-priori estimates also holds:

$$\|\Phi_h\|_{H^{1+r_h}(\Omega_h)} \leq C_h \|u^E - u_h\|_{L^2(\Omega_h)} \tag{5.45}$$

$$\|\Phi_h\|_{H^{2,\gamma_h}(\Omega_h)} \leq C_h \|u^E - u_h\|_{L^2(\Omega_h)}. \tag{5.46}$$

For (5.45) we refer the reader to [15], while (5.46) can be found in [7, 10].

The constants C_h in (5.45) and (5.46) may change with the number of vertices of the polygonal domain Ω_h (and hence with h). On the other hand, as mentioned in [16], the classical proof for (5.45) provides a very poor control of the constant C_h (see Remark 4.3.2.6 in [15]). However, in [16] it is also mentioned that for Lipschitz domains Ω with piecewise C^2 boundary the uniform boundedness of C_h with respect to h is a plausible hypothesis for reasonable triangulations, since the constants C_h could be bounded [16, page 141] via a boundary integral formulation. As far as we know, there is not an explicit proof of this fact in the literature, even for regular domains Ω.

Although our domain Ω is not Lipschitz, it has a C^2 piecewise boundary and, in view of (1.3), the a-priori estimate for Ω, we consider also plausible the following assumption:

Assumption 1. *Our family of triangulations $\{\mathcal{T}_h\}$ is such that the constants C_h in (5.45) and in (5.46) are uniformly bounded with respect to h. For this reason we will drop the subindex h in C_h in further references to (5.45) and (5.46).*

In order to obtain L^2 error estimates using the auxiliary problem (5.41), we will need some embedding results in Ω_h for the solution Φ_h. Since Ω_h is not uniformly Lipschitz in h (in fact, $\Omega_h \to \Omega$, and Ω is not a Lipschitz domain), the classical embedding theorems for Lipschitz domains do not apply, neither do the general results for cusps given in [20] since Φ_h belongs to a weighted Sobolev space. As a consequence, we will extend Φ_h to some fixed Lipschitz domain in a certain weighted Sobolev space (we recall that the extension results given in [20] do not apply in our case), and then we will get proper embedding results. Therefore, we will follow the approach given in Lemma 3.1 of [2].

We first extend Φ_h from Ω_h to the following domain $\mathcal{D}_h$ (see Figure 4),

$$\mathcal{D}_h = \{(x,y) \in \mathbb{R}^2 : -x < y < g_h(x), 0 < x < 1\}$$

where $g_h : [0,1] \to \Gamma_{3,h}$ is a parametrization of $\Gamma_{3,h} := \cup_j \Gamma^j_{3,h}$, and we show that the extension

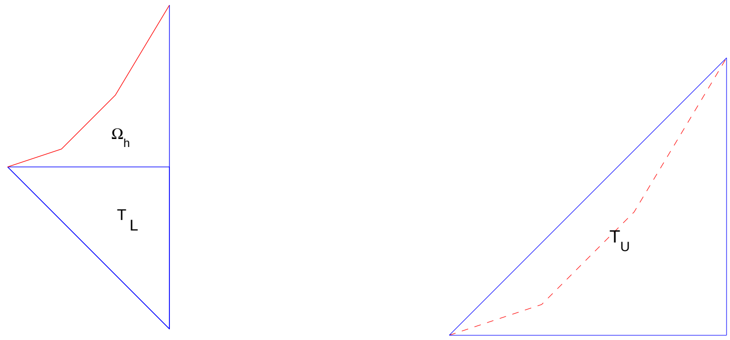

FIGURE 4. Left: Domain $\mathcal{D}_h$. Right: Triangle T_U.

belongs to the space

$$H^{2,\gamma_h}_{\alpha+A(h)}(\mathcal{D}_h) = \left\{ v : r^{\frac{\alpha-1}{2}+\frac{A(h)}{2}} \rho^{\gamma_h} D^\beta v \in L^2(\mathcal{D}_h) \ \ \forall \beta, |\beta| \le 2 \right\}$$

where $A(h) = 6\gamma_h$, $r = \sqrt{x^2+y^2}$ and

$$\rho = \min_{1\le j\le n} \{r_j, d_j\}, \tag{5.47}$$

with $r_j = \sqrt{(x-x_j)^2 + (y-x_j^\alpha)^2}$ and $d_j = \sqrt{(x-x_j)^2+(y+x_j)^2}$.

In the next lemma we find equivalent expressions for the distances involved in the weights.

Lemma 5.1. *Let us denote by $d_{\Gamma_3}(x,\eta)$ the distance from $(x,\eta) \in \Omega$ to Γ_3. Then,*

$$d_{\Gamma_3}(x,\eta) \le x^\alpha - \eta \le C d_{\Gamma_3}(x,\eta), \tag{5.48}$$

with C depending only on α.

What is more, a similar discrete version of this property holds. Indeed, for any sequence $0 = x_0 < x_1 < \cdots < x_N = 1$, if we define $\Omega_j = \{(x,\eta) : x_{j-1} \le x \le x_j, 0 \le \eta \le x^\alpha\}$, $1 \le j \le N$, then for any $(x,\eta) \in \Omega_j$ there exists a constant C depending only on α such that

$$\rho(x,\eta) \le \min_{i=j-1,j} \{r_i(x,\eta)\} \le C\rho(x,\eta) \tag{5.49}$$

where $r_i(x,\eta)$ stands for the distance from (x,η) to (x_i, x_i^α), and $\rho(x,\eta) = \min_{1\le i\le N}\{r_i(x,\eta)\}$.

Proof. It is clear that $d_{\Gamma_3}(x,\eta) \le x^\alpha - \eta$. On the other hand, denoting by $P_* \in \Gamma_3$, $P_* = (x_*, x_*^\alpha)$ the point for which $d_{\Gamma_3}(x,\eta) = \|P_* - (x,\eta)\|$, and by L the line joining the point P_* with $(x, x^\alpha) \in \Gamma_3$, we get that d_L, the distance from (x,η) to L, verifies $d_L(x,\eta) \le d_{\Gamma_3}(x,\eta)$ (since $P_* \in L \cap \Gamma_3$).

Let us consider the point $Q_* \in L$ such that $d_L(x,\eta) = \|Q_* - (x,\eta)\|$, and the triangle given by the points $Q_*, (x,\eta)$ and (x,x^α). This triangle has a right angle at Q_* and the angle θ placed at (x,x^α) is clearly bounded by below by some fixed $\theta_0 > 0$ depending only on α. Now (5.48) follows because of the following inequalities

$$\begin{aligned} d_{\Gamma_3}(x,\eta) \ &\ge \ d_L(x,\eta) = \|(Q_* - (x,\eta))\| = \|(x,x^\alpha) - (x,\eta)\| \sin(\theta) \\ &\ge \ \|(x,x^\alpha) - (x,\eta)\| \sin(\theta_0) = \sin(\theta_0)(x^\alpha - \eta). \end{aligned}$$

Let us now consider (5.49). A direct calculation shows that the function $h : (0,1) \to \mathbb{R}$, $h(t) = (t-x)^2 + (t^\alpha - \eta)^2$, decreases before its global minimum and increases after that. Indeed, if x_* verifies $h'(x_*)=0$, with $h'(t)=2(t-x)+2\alpha(t^\alpha-\eta)t^{\alpha-1}$ hence $x_*\in[\eta^{1/\alpha},x]$ since obviously $h'(t)<0$ for $t<\eta^{1/\alpha}$ and $h'>0$ for $t>x$. On the other hand, $h''(t)>0$ for $\eta^{1/\alpha}\le t\le x$, which shows the existence of an unique $x_*\in[\eta^{1/\alpha},x]$ global minimum of h.

For $(x,\eta)\in\Omega_j$ and $P_*=(x_*,x_*^\alpha)$, the point for which $d_{\Gamma_3}(x,\eta)=\|P_*-(x,\eta)\|=\sqrt{h(x_*)}$, we consider the index l such that $P_*\in\Omega_l$. If $l=j$, then $\min_{j-1\le i\le j}\{r_i(x,\eta)\}=\min_{1\le i\le N}\{r_i(x,\eta)\}$ (since h is increasing for $t>x_*$, and decreasing if $t<x_*$). If $l\neq j$ then, without loss of generality we may assume $l<j$, and we write

$$\rho(x,\eta)=\min_{1\le i\le N}\{r_i(x,\eta)\}=\min_{l-1\le i\le l}\{r_i(x,\eta)\}\le\sqrt{h(x)}=x^\alpha-\eta\le Cd_{\Gamma_3}(x,\eta),$$

where we have used that $x\ge x_l$, the point $(x,x^\alpha)\in\Gamma_3$, h is increasing in the range $[x_l,x]$, and (5.48). Now, (5.49) follows from the fact that $d_{\Gamma_3}(x,\eta)\le\min_{j-1\le i\le j}\{r_i(x,\eta)\}$. □

Remark 5.1. *It is easy to see that for $(x,\eta)\in\Omega_h\subset\mathcal{D}_h$, $\rho=\hat r$ where $\hat r$ and ρ are defined in (5.43) and (5.47) respectively.*

We are now ready to extend Φ_h to $\mathcal{D}_h$.

Lemma 5.2. *Given $v\in H^{2,\gamma_h}(\Omega_h)$ such that $\frac{\partial v}{\partial\nu}=0$ on Γ_1, there exists a function $\tilde v\in H^{2,\gamma_h}_{\alpha+A(h)}(\mathcal{D}_h)$ such that $\tilde v|_{\Omega_h}=v$ and*

$$\|\tilde v\|_{H^{2,\gamma_h}_{\alpha+A(h)}(\mathcal{D}_h)}\le C\|v\|_{H^{2,\gamma_h}(\Omega)},$$

where $A(h)=6\gamma_h$ and, in particular, $A(h)\to0$ when $h\to0$.

Proof. The proof follows the ideas given in Lemma 3.1 of [2]. We extend v by reflection in the following way.

For any $(x,y)\in\mathcal{D}_h$ with $y\le0$, let us define $\eta=-x^{\alpha-1}y$. Observe that the function $(x,y)\to(x,\eta)$ maps T_L onto $\Omega\subset\Omega_h$ (see Figure 4), and therefore, calling $T_L:=\mathcal{D}_h\setminus\overline{\Omega_h}=\{0\le x\le1,\,-x\le y<0\}$,

we can define

$$\begin{cases}\tilde v(x,y)=v(x,y), & \text{for}\quad (x,y)\in\Omega_h\\ \tilde v(x,y)=v(x,\eta), & \text{for}\quad (x,y)\in T_L\end{cases}$$

We notice that for $(x,y)\in T_L$ we have $r=\sqrt{x^2+y^2}\sim x$ and, therefore, we can replace the weight $r^{\alpha-1+A(h)}$ by $x^{\alpha-1+A(h)}$ in our estimates.

Now, it is clear that

$$\int_{T_L}\tilde v^2(x,y)x^{\alpha-1+A(h)}\rho^{2\gamma_h}(x,y)dxdy\le A+B \tag{5.50}$$

with

$$A=\int_{T_{L1}}\tilde v^2(x,y)x^{\alpha-1+A(h)}\left[(x-x_1)^2+(y+x_1)^2\right]^{\gamma_h}dxdy \tag{5.51}$$

and

$$B=\sum_{j=2}^N B_j \tag{5.52}$$

$$B_j=\int_{T_{Lj}}\tilde v^2(x,y)x^{\alpha-1+A(h)}\min_{i=j-1,j}\left[(x-x_i)^2+(y+x_i)^2\right]^{\gamma_h}dxdy$$

where $T_{Lj}=\{x_{j-1}\le x\le x_j,\,-x\le y\le0\}$ (notice that we have used the fact that $\rho\le d_j=\sqrt{(x-x_j)^2+(y+x_j)^2}$, for any j).

Changing variables, and taking into account that $\alpha < 3$, we get for $j > 1$

$$B_j = \int_{\Omega_j} v^2(x,\eta) x^{A(h)} \min_{i=j-1,j} \left\{(x-x_i)^2 + (-\frac{\eta}{x^{\alpha-1}} + x_i)^2\right\}^{\gamma_h} dxd\eta$$
$$\leq C \int_{\Omega_j} v^2(x,\eta) x^{A(h)-4\gamma_h} \min_{i=j-1,j} \left\{(x-x_j)^2 + (-\eta + x_j x^{\alpha-1})^2\right\}^{\gamma_h} dxd\eta$$

where $\Omega_j = \{x_{j-1} \leq x \leq x_j,\ \ 0 \leq \eta \leq x^\alpha\}$. Similarly,

$$A \leq C \int_{\Omega_1} v^2(x,\eta) x^{A(h)-4\gamma_h} \left\{(x-x_1)^2 + (-\eta + x_1 x^{\alpha-1})^2\right\}^{\gamma_h} dxd\eta.$$

Since

$$(-\eta + x_j x^{\alpha-1})^2 \leq C[(\eta - x_j^\alpha)^2 + (x_j^\alpha - x_j x^{\alpha-1})^2],$$

using the mean value theorem, Lemma 2.1, the fact that $x, x_j \leq 1$, and $1 < \alpha$, we obtain for $j > 1$

$$(x_j^\alpha - x_j x^{\alpha-1})^2 \leq C x_j^2 x^{2(\alpha-2)} (x-x_j)^2 \leq C \max\{x^{2(\alpha-2)}, 1\}(x-x_j)^2 \leq x^{-2}(x-x_j)^2$$

and, hence,

$$B_j \leq C \int_{\Omega_j} v^2(x,\eta) x^{A(h)-6\gamma_h} \min_{i=j-1,j} \left\{r_i(x,\eta)^2\right\}^{\gamma_h} dxd\eta$$

where $r_i(x,\eta) = \sqrt{(x-x_i)^2 + (\eta - x_i^\alpha)^2}$. Using that $A(h) = 6\gamma_h$ and Lemma 5.1 we get

$$B_j \leq C \int_{\Omega_j} v^2(x,\eta) \rho^{2\gamma_h} dxd\eta. \tag{5.53}$$

Similarly, for $j = 1$ we have that

$$(x_1^\alpha - x_1 x^{\alpha-1})^2 \leq C \max\{x^{2(\alpha-2)}, 1\}(x-x_1)^2 \leq C x^{-2}(x-x_1)^2$$

As a consequence,

$$A \leq C \int_{\Omega_1} v^2(x,\eta) x^{A(h)-6\gamma_h} \left\{r_1(x,\eta)^2\right\}^{\gamma_h} dxd\eta$$

let us notice that, for $(x,\eta) \in \Omega_1$, it is clear that $r_1(x,\eta) = \rho(x,\eta)$, then

$$A \leq C \int_{\Omega_1} v^2(x,\eta) \rho(x,\eta)^{2\gamma_h} dxd\eta.$$

From the previous inequality, (5.50) and (5.53), we have

$$\int_{T_L} \tilde{v}^2(x,y) x^{\alpha-1+A(h)} \rho^{2\gamma_h}(x,y) dxdy \leq \int_\Omega v^2(x,\eta) \rho(x,\eta)^{2\gamma_h} dxd\eta.$$

Bounds for the first and second derivatives of $\tilde{v}$ follow similarly using the same ideas given in Lemma 3.1 of [2] and the estimates given above. Therefore, we have proved that $\tilde{v} \in H^{2,\gamma_h}_{\alpha+A(h)}(T_L)$, and that

$$\|\tilde{v}\|_{H^{2,\gamma_h}_{\alpha+A(h)}(T_L)} \leq C \|v\|_{H^{2,\gamma_h}(\Omega)}.$$

On the other hand, using that $\frac{\partial v}{\partial \nu} = 0$ on Γ_1, it is easy to see that $\tilde{v} \in H^{2,\gamma_h}_{\alpha+A(h)}(\mathcal{D}_h)$ concluding the proof. □

From the previous Lemma we conclude that any $\tilde{\Phi}_h \in H^{2,\gamma_h}(\Omega_h)$ has an extension $\tilde{\Phi}_h^E$ belonging to $H^{2,\gamma_h}_{\alpha+A(h)}(\mathcal{D}_h)$. Since $\mathcal{D}_h$ is uniformly Lipschitz, and the weights involved belong to the Muckenhoupt class A_2, we can use Chua's results [13] with the same arguments given in [2], and then $\tilde{\Phi}_h^E$ (and hence Φ_h) can be extended to $\mathbb{R}^2$. More precisely, there exists a function Φ_h^E belonging to

$$H^{2,\gamma_h}_{\alpha+A(h)}(\mathbb{R}^2) = \left\{ v : r^{\frac{\alpha-1}{2}+\frac{A(h)}{2}} \rho^{\gamma_h} D^\beta v \in L^2(\mathbb{R}^2) \;\; \forall \beta, |\beta| \le 2 \right\} \tag{5.54}$$

such that

$$\|\Phi_h^E\|_{H^{2,\gamma_h}_{\alpha+A(h)}(\mathbb{R}^2)} \le C \|\Phi_h\|_{\hat{H}^{2,\gamma_h}(\Omega_h)}. \tag{5.55}$$

Remark 5.2. *The extension result given in (5.55) agrees with (1.5), in the sense that when h goes to zero $\gamma_h \to 0$, $A(h) \to 0$, and $\Omega_h \to \Omega$. We emphasize the fact that this sort of extension cannot be obtained in a direct way from the results given in* [20] *due to the weights involved in the space of functions.*

In what follows we will make use of Φ_h^E restricted to the domain T_U (see (2.17) and Figure 4). Let us notice that $\Omega_h \subset T_U$ for any $0 < h$ and, for $(x, y) \in T_U$, we have that $min_{1\le j\le n} r_j \le min_{1\le j\le n} d_j$, and $r \sim x$. Therefore, we can state the following result (see (5.46), and Assumption 1).

Lemma 5.3. *There exists an extension Φ_h^E of Φ_h (the solution of (5.41)) belonging to the space*

$$H^{2,\gamma_h}_{\alpha+A(h)}(T_U) = \left\{ v : x^{\frac{(\alpha-1)}{2}+\frac{A(h)}{2}} \rho^{\gamma_h} D^\beta v \in L^2(T_U) \;\; \forall \beta, |\beta| \le 2 \right\}.$$

where

$$T_U = \{(x, y) \in \mathbb{R}^2 : 0 < y < x \quad 0 < x < 1\}$$

$\rho = \min_{1\le j\le n} \{r_j\}$, *with* $r_j = \sqrt{(x - x_j)^2 + (y - x_j^\alpha)^2}$. *Moreover,*

$$\|\Phi_h^E\|_{H^{2,\gamma_h}_{\alpha+A(h)}(T_U)} \le \|\Phi_h\|_{H^{2,\gamma_h}(\Omega_h)} \le C\|u^E - u_h\|_{L^2(\Omega_h)}.$$

Lemma 5.4. *Let Φ_h be the solution of (5.41), and Φ_h^E be the extension defined in Lemma 5.3. For h small enough we have*

(1) $\Phi_h^E \in W^{2,p}(T_U)$ *for* $1 \le p < \frac{4}{1+\alpha}$. *Moreover,*

$$\|\Phi_h^E\|_{W^{2,p}(T_U)} \le C\|\Phi_h^E\|_{H^{2,\gamma_h}_{\alpha+A(h)}(T_U)} \tag{5.56}$$

with a constant C independent of h.

(2) $\nabla\Phi_h x^\beta \in W^{1,s}(T_U)$, *for* $\beta > \frac{\alpha-1}{2}$, *and* $s = 2 - \epsilon$, *with* $\epsilon > 4\gamma_h$. *Moreover,*

$$\|\nabla\Phi_h^E x^\beta\|_{W^{1,s}(T_U)} \le C\|\Phi_h^E\|_{H^{2,\gamma_h}_{\alpha+A(h)}(T_U)} \tag{5.57}$$

with a constant C independent of h.

(3) *With β and s as in (2), we have that* $\nabla\Phi_h^E x^\beta \in L^{s^*}(T_U)$, *with* $s^* = \frac{2s}{2-s} = \frac{2(2-\varepsilon)}{\varepsilon}$. *Moreover,*

$$\|\nabla\Phi_h^E x^\beta\|_{L^{s^*}(T_U)} \le \frac{C}{\varepsilon}\|\Phi_h^E\|_{H^{2,\gamma_h}_{\alpha+A(h)}(T_U)} \tag{5.58}$$

with a constant C independent of h.

Proof. Let us write

$$\int_{T_U} |v|^p = \int_{T_U} |v|^p x^{\frac{p(\alpha-1+A(h))}{2}} \rho^{p\gamma_h} x^{-\frac{p(\alpha-1+A(h))}{2}} \rho^{-p\gamma_h}$$

for some $p < \frac{4}{\alpha+1} < 2$ fixed. Applying Hölder's inequality with exponent $2/p$ and its dual exponent we obtain

$$\int_{T_U} |v|^p \leq \Big(\int_{T_U} |v|^2 x^{(\alpha-1+A(h))} \rho^{2\gamma_h} \Big)^{\frac{p}{2}} \Big(\int_{T_U} x^{-\frac{p}{2-p}(\alpha-1+A(h))} \rho^{-\frac{2p}{2-p}\gamma_h} \Big)^{\frac{2-p}{2}} \tag{5.59}$$

Calling $s_h = \frac{2p}{2-p}\gamma_h$, and applying Hölder's inequality again with $\frac{1}{1-2s_h}$ (its dual exponent is $\frac{1}{2s_h}$) we have

$$\Big(\int_{T_U} x^{-\frac{p(\alpha-1+A(h))}{2-p}} \rho^{-\frac{2p}{2-p}\gamma_h} \Big)^{\frac{2-p}{2}} \leq I_1 I_2 \tag{5.60}$$

where

$$I_1 = \Big(\int_{T_U} x^{-(\alpha-1+A(h))\frac{p}{(2-p)(1-2s_h)}} \Big)^{\frac{2-p}{2}(1-2s_h)}$$

and

$$I_2 = \Big(\int_{T_U} \rho^{-\frac{1}{2}} \Big)^{(2-p)s_h}.$$

Now, since $1 \leq p < \frac{4}{1+\alpha}$, for h small enough we can assume that $1 \leq p < \frac{4}{1+\alpha+14\gamma_h}$, and using that $A(h) = 6\gamma_h$, one can easily check that $(\alpha - 1 + A(h))\frac{p}{(2-p)(1-2s_h)} < 2$, which is precisely the condition that implies

$$I_1 \leq C$$

with $C = C(\alpha)$.

On the other hand, since

$$\rho \geq d_{\Gamma_3}, \tag{5.61}$$

where d_{Γ_3} is the distance function to Γ_3, and taking into account that $\int_{T_U} \frac{1}{d^s} < C$ for any $s < 1$ (see for instance [15]), we get that $\int_{T_U} \rho^{-\frac{1}{2}} \leq C$. As a consequence, we have proved that for any function v and $1 \leq p < \frac{4}{\alpha+1+14\gamma_h}$

$$\|v\|_{L^p(T_U)} \leq C \|v x^{\frac{\alpha-1+A(h)}{2}} \rho^{\gamma_h}\|_{L^2(T_U)}.$$

Thanks to Lemma 5.3, we conclude that $\Phi_h^E \in W^{2,p}(T_U)$, and (1) follows.

Our next goal is to prove (2). Take $\beta > \frac{\alpha-1}{2}$, then, for h small enough, we also have $\beta > \frac{(\alpha-1+A(h))}{2}$. Let $s = 2 - \epsilon$, with ϵ to be chosen below. Following similar arguments as those of Lemma 4.1 of [2], we have that

$$D^2\Phi_h^E x^\beta \in L^{2-\epsilon}(T_U).$$

Indeed, since $D^2\Phi_h^E x^\beta \rho^{\gamma_h} \in L^2(T_U)$, we can write for fixed ϵ,

$$\int_{T_U} |D^2\Phi_h^E|^s x^{s\beta} \leq \Big(\int_{T_U} \Big(D^2\Phi_h^E x^\beta \rho^{\gamma_h} \Big)^2 \Big)^{\frac{2-\epsilon}{2}} \Big(\int_{T_U} \rho^{-\frac{2\gamma_h}{\epsilon}} \Big)^{\frac{\epsilon}{2}}$$

and the last integral in the previous inequality is finite, taking for instance

$$4\gamma_h < \epsilon \tag{5.62}$$

and using (5.61). On the other hand,

$$\nabla\Phi_h^E x^{\beta-1} \in L^2(T_U).$$

In fact, from (5.56), and embedding results for the planar Lipschitz domain T_U

$$\nabla\Phi_h^E \in L^{p*}(T_U)$$

with $p^* = \frac{2p}{2-p}$ and $1 \le p < \frac{4}{\alpha+1}$. Now, by Hölder's inequality with exponent $p^*/2$ and its conjugate exponent $\frac{p}{2(p-1)}$ we get

$$\int_{T_U} |\nabla\Phi_h^E|^2 x^{2(\beta-1)} \le \left(\int_{T_U} |\nabla\Phi_h^E|^{p^*}\right)^{\frac{2}{p^*}} \left(\int_{T_U} x^{p(\beta-1)/(p-1)}\right)^{\frac{2(p-1)}{p}}.$$

A straightforward computation shows that the condition for the last integral to be finite is

$$p(\beta-1)/(p-1)+2>0$$

or, equivalently,

$$p > \frac{2}{\beta+1}.$$

Choosing p such that

$$\frac{2}{\beta+1} < p < \frac{4}{1+\alpha}$$

which is possible since $\beta > \frac{\alpha-1}{2}$, (2) follows.

The proof of (3) is now direct using the imbedding $L^{s^*}(T_U) \subset W^{1,s}(T_U)$, $s^* = \frac{2s}{2-s} = \frac{2(2-\epsilon)}{\epsilon}$, the explicit dependence on s of the constant (see the proof of Theorem 1 in [14, page 277]), and the result obtained in (2).

In fact,

$$\|\nabla\Phi_h^E x^\beta\|_{L^{s^*}(T_U)} \le \frac{C}{2-s}\|\nabla\Phi_h^E x^\beta\|_{W^{1,s}(T_U)} \le \frac{C}{\epsilon}\|\Phi_h^E\|_{H_\alpha^{2,\gamma_h}(T_U)}$$

for $s = 2-\epsilon$, with ϵ verifying (5.62). ☐

Lemma 5.5. *Let Φ_h be the solution of (5.41), then there exists a constant C such that*

$$\|\nabla\Phi_h\|_{L^2(\Omega_h\backslash\Omega)} \le Ch\log(1/h)\|u^E - u_h\|_{L^2(\Omega_h)}$$

and

$$\|\Phi_h\|_{L^2(\Omega_h\backslash\Omega)} \le Ch\|u^E - u_h\|_{L^2(\Omega_h)}.$$

Proof. Let β, s, and $\epsilon > 4\gamma_h$, as in Lemma 5.4. Applying Hölder's inequality with $s^*/2 = \frac{2-\epsilon}{\epsilon}$ and its dual exponent $q = \frac{s^*}{s^*-2} = \frac{2-\epsilon}{2-2\epsilon}$, we have

$$\int_{\Omega_h\backslash\Omega} |\nabla\Phi_h|^2 \le \Big(\int_{\Omega_h\backslash\Omega} |\nabla\Phi_h^E|^{s^*} x^{\beta s^*}\Big)^{\frac{2}{s^*}} \Big(\int_{\Omega_h\backslash\Omega} x^{-2\beta q}\Big)^{\frac{1}{q}}, \tag{5.63}$$

and, therefore, from (3) in Lemma 5.4 we obtain

$$\int_{\Omega_h\backslash\Omega} |\nabla\Phi_h|^2 \le \frac{C}{\epsilon^2}\,\|\Phi_h^E\|^2_{H^{2,\gamma_h}_{\alpha+A(h)}(T_U)} \Big(\int_{\Omega_h\backslash\Omega} x^{-2\beta q}\Big)^{\frac{1}{q}}. \tag{5.64}$$

From Lemma 2.2 we get (observe that the constant given in that Lemma remains bounded when $q \to 1$, and in the present context $q = \frac{2-\epsilon}{2-2\epsilon}$ while ϵ will be chosen such that $\epsilon \to 0$ when $h \to 0$)

$$\int_{\Omega_h\backslash\Omega} |\nabla\Phi_h|^2 \le \frac{C}{\epsilon^2}\,\|\Phi_h^E\|^2_{H^{2,\gamma_h}_{\alpha+A(h)}(T_U)} h^{\frac{2}{q}} \tag{5.65}$$

and, since $\frac{1}{q} = \frac{2-2\epsilon}{2-\epsilon} = 1 - \frac{\epsilon}{2-\epsilon}$,

$$\int_{\Omega_h\backslash\Omega} |\nabla\Phi_h|^2 \le C\left(\frac{h^{1-\frac{\epsilon}{2-\epsilon}}}{\epsilon}\right)^2 \|\Phi_h^E\|^2_{H^{2,\gamma_h}_{\alpha+A(h)}(T_U)}.$$

Let us now take $\epsilon = -\frac{1}{\log(h)}$. It is clear that for h small enough, ϵ verifies $\epsilon > 4\gamma_h$ for any choice of γ_h in (5.44). Taking into account that $1 - \frac{\epsilon}{2-2\epsilon} \sim 1 - \frac{1}{2}\epsilon$ for $\epsilon \to 0$, we get by standard arguments

$$\int_{\Omega_h \setminus \Omega} |\nabla \Phi_h|^2 \leq C h^2 \log^2(1/h) \|\Phi_h^E\|^2_{H^{2,\gamma_h}_{\alpha+A(h)}(T_U)} \tag{5.66}$$

and the first estimate of the Lemma follows from Lemma 5.3.

The estimate for $\int_{\Omega_h\setminus\Omega} |\Phi_h|^2$ follows immediately. Since for $p > 1$, functions in $W^{2,p}(T_U)$ are bounded, using (5.56) we can write

$$\int_{\Omega_h \setminus \Omega} |\Phi_h|^2 \leq \|\Phi_h^E\|^2_{L^\infty(T_U)} |\Omega_h \setminus \Omega| \leq C \|\Phi_h^E\|^2_{W^{2,p}(T_U)} |\Omega_h \setminus \Omega| \leq C \|\Phi_h^E\|^2_{H^{2,\gamma_h}_{\alpha+A(h)}(T_U)} |\Omega_h \setminus \Omega|$$

and the proof concludes using Lemma 5.3 and Lemma 2.3. □

Now we are ready to obtain error bounds in the L^2 norm. Mainly due to Lemma 5.5, it will not be possible (at least with the present approach, see Remark 3.1) to improve the logarithmic factor $\log h$ in the estimates. For this reason, in the intermediate computations we will replace terms like h^{r_h}, with r_h given by (5.42), by $C\sqrt{\log(1/h)}h$. This can be done thanks to the bound

$$h^{r_h} \leq C\sqrt{\log(1/h)}h, \tag{5.67}$$

that holds for $r_h \sim 1 - Ch^s$ with any $0 < s \leq 1$, as one can easily check from the fact that

$$\lim_{h\to 0} h^{h^s} = 1.$$

Our next goal is to obtain error estimates in $L^2(\Omega_h)$.

Theorem 5.1. *Let u be the solution of (1.2) and u_h be the solution of (1.4). Assume $\alpha < 3$, and $f \in L^2(\mathbb{R}^2)$. Then,*

$$\|u^E - u_h\|_{L^2(\Omega_h)} \leq Ch^2 \log(1/h) \|f\|_{L^2(\Omega)} + Ch\|f\|_{L^2(\Omega_h\setminus\Omega)}.$$

Proof. Let $e = u^E - u_h$ and Φ_h be the solution of (5.41). We have that

$$\int_{\Omega_h} e^2 = \int_{\Omega_h} (-\Delta \Phi_h) e = \int_{\Omega_h} \nabla \Phi_h \nabla e = \int_{\Omega_h} \nabla(\Phi_h - \Pi\Phi_h)\nabla e + \int_{\Omega_h} \nabla(\Pi\Phi_h)\nabla e. \tag{5.68}$$

From (1.2) and (1.4) we get

$$\int_{\Omega_h} \nabla e \nabla v = \int_{\Omega_h\setminus\Omega} \nabla u^E \nabla v - \int_{\Omega_h\setminus\Omega} f v \quad \forall v \in V_h.$$

Hence,

$$\int_{\Omega_h} e^2 = \int_{\Omega_h} \nabla(\Phi_h - \Pi\Phi_h)\nabla e + \int_{\Omega_h\setminus\Omega} \nabla u^E \nabla(\Pi\Phi_h) - \int_{\Omega_h\setminus\Omega} f(\Pi\Phi_h). \tag{5.69}$$

From standard estimates for the Lagrange interpolation using finite triangular elements verifying the maximal angle condition, and (5.67), we get

$$\|\nabla(\Phi_h - \Pi\Phi_h)\|_{L^2(\Omega_h)} \leq Ch^{r_h}\|\Phi_h\|_{H^{1+r_h}(\Omega_h)} \leq Ch\sqrt{\log(1/h)}\|\Phi_h\|_{H^{1+r_h}(\Omega_h)}$$

which under Assumption 1 for (5.45) yields

$$\|\nabla(\Phi_h - \Pi\Phi_h)\|_{L^2(\Omega_h)} \leq Ch\sqrt{\log(1/h)}\|e\|_{L^2(\Omega_h)}. \tag{5.70}$$

Therefore, the first term of (5.69) can be bounded using Theorem 4.1. Indeed,

$$\int_{\Omega_h} \nabla(\Phi - \Pi\Phi_h)\nabla e \le Ch\sqrt{\log(1/h)} \left\{ h\sqrt{\log(1/h)}\|f\|_{L^2(\Omega)} + h^{\frac{2}{\alpha+1}}\|f\|_{L^2(\Omega_h\setminus\Omega)} \right\} \|e\|_{L^2(\Omega_h)} \tag{5.71}$$

For the second term in (5.69), using the estimates given in Lemma 2.7 and (1.3) we know that

$$\int_{\Omega_h\setminus\Omega} \nabla u^E \nabla(\Pi\Phi_h) \le Ch\sqrt{\log(1/h)}\|f\|_{L^2(\Omega)}\{\|\nabla(\Pi\Phi_h - \Phi_h)\|_{L^2(\Omega_h\setminus\Omega)} + \|\nabla\Phi_h\|_{L^2(\Omega_h\setminus\Omega)}\}.$$

Using (5.70), Lemma 5.5, and Assumption 1 for (5.45), we get

$$\int_{\Omega_h\setminus\Omega} \nabla u^E \nabla(\Pi\Phi_h) \le Ch^2\log(1/h)\|f\|_{L^2(\Omega)}\|e\|_{L^2(\Omega_h)}. \tag{5.72}$$

Therefore, we only have to estimate the third term in (5.69)

$$\int_{\Omega_h\setminus\Omega} f(\Pi\Phi_h) = \int_{\Omega_h\setminus\Omega} f(\Pi\Phi_h - \Phi_h) + \int_{\Omega_h\setminus\Omega} f\Phi_h \tag{5.73}$$

Now, L^2 interpolation error estimates give

$$\|\Pi\Phi_h - \Phi_h\|_{L^2(\Omega_h)} \le Ch^{1+r_h}\|\Phi_h\|_{H^{1+r_h}(\Omega_h)}$$

and then, (5.45) with Assumption 1 and (5.67) give

$$\int_{\Omega_h\setminus\Omega} f(\Pi\Phi_h - \Phi_h) \le Ch^2\sqrt{\log(1/h)}\|e\|_{L^2(\Omega_h)}\|f\|_{L^2(\Omega_h\setminus\Omega)}.$$

Now, for the second term in (5.73), by using Lemma 5.5 we have

$$\int_{\Omega_h\setminus\Omega} f\Phi_h \le Ch\|f\|_{L^2(\Omega_h\setminus\Omega)}\|e\|_{L^2(\Omega_h)}. \tag{5.74}$$

So, from (5.69), (5.71), (5.72) and (5.74) we get the estimate of the theorem, taking into account that the term arising from (5.71)

$$h^{1+\frac{2}{\alpha+1}}\sqrt{\log(1/h)}\|f\|_{L^2(\Omega_h\setminus\Omega)}$$

is bounded, up to a multiplicative constant, by the term

$$h\|f\|_{L^2(\Omega_h\setminus\Omega)}$$

given in (5.74). □

6. Numerical examples

Now we show that meshes verifying hypotheses (1)-(3) and (Ha)-(Hb) can be constructed by the same method given in [2].

(1) Introduce the partition of the interval $(0,1)$ given by

$$x_j = \left(\frac{j}{n}\right)^{\frac{2}{3-\alpha}} \qquad 0 \le j \le n.$$

(2) Take the points $(x_j, 0)$ in Γ_1, (x_j, x_j^α) in Γ_3, and for $j > 1$, divide each of the vertical lines $\{(x_j, y) : 0 \le y \le x_j^\alpha\}$ uniformly into subintervals such that each of them has length $\sim x_j - x_{j-1}$.

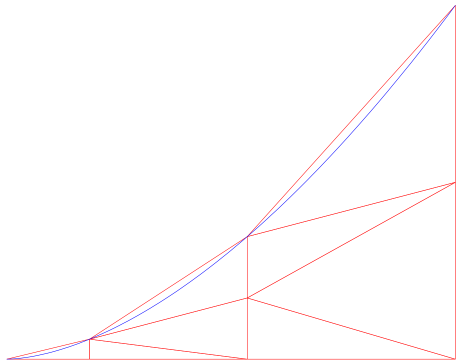

FIGURE 5. Graded mesh with $\alpha = 2$ and $n = 3$

Figure 5 shows an example of one of these meshes.

We observe that is clear that the meshes constructed in this way satisfy hypotheses (1), (2), (3) and (Ha). Moreover, these meshes satisfy the additional condition (Hb). Indeed, the first triangle T_1 has vertices $(0,0), (x_1, 0)$ and (x_1, x_1^α) and so the triangle $\tilde{T}_1$ has vertices $(\frac{x_1}{2}, 0)$, $(x_1, 0)$ and $(x_1, \frac{x_1^\alpha}{2})$ and then, in order to check that this triangle does not intersect Ω_h^1, we analyze the function

$$g(x) = x^\alpha - x_1^{\alpha-1}(x - \frac{x_1}{2}).$$

Hence, the hypothesis holds if we prove that $g(x) > 0$ for $0 \le x \le x_1$. An easy calculation shows that g is convex and has a minimum in $x^* = \frac{x_1}{\alpha^{\frac{1}{\alpha-1}}}$ and

$$g(x^*) = x_1^\alpha \left(\frac{1}{2} + \frac{1}{\alpha^{\frac{\alpha}{\alpha-1}}} - \frac{1}{\alpha^{\frac{1}{\alpha-1}}} \right)$$

which is positive for $1 < \alpha < 3$.

Similar arguments can be used for the rest of the triangles T_j, $2 \le j \le n$.

If N is the number of nodes in the partition $\mathcal{T}_h$, it can be proved that $h^2 \sim 1/N$ [15, page 393],[21]. Therefore, if f is assumed to be zero outside Ω, we have the following error estimate in terms of the number of nodes,

$$\|u - u_h\|_{L^2(\Omega)} \le C \frac{\log N}{N} \|f\|_{L^2(\Omega)}.$$

Observe that this estimate is quasi-optimal. Indeed, up to the logarithmic factor, the order with respect to the number of nodes is the same as that obtained for a smooth problem using quasi-uniform meshes.

We end this section by considering the same example presented in [2]. Here, we compare the L^2 order obtained by using uniform and graded meshes. Let us notice that we take a non-homogeneous Neumann condition, for which we know the analytical solution, and hence, the exact error. However, similar results are obtained for the same source term f taking $g = 0$, and computing an estimated order of convergence from successive refinements.

Example 6.1. *Consider the problem (1.1), with*

$$f(x, y) = s(s-1)(1 + y^2/2)x^{s-2} + x^s - 1$$

and

$$g(t, t^\alpha) = \frac{-s\alpha t^{\alpha+s-2}(1 + t^{2\alpha}/2) + (1 - t^s)t^\alpha}{\sqrt{1 + \alpha^2 t^{2(\alpha-1)}}}.$$

value of s	order in number of nodes	order in h
0.55	0.769	1.497
0.6	0.785	1.528
0.65	0.801	1.561
0.7	0.820	1.597
0.75	0.842	1.640
0.8	0.869	1.693
0.85	0.904	1.761
0.9	0.949	1.847
0.95	1.001	1.951

TABLE 1. L^2 order using quasi-uniform meshes for $\alpha = 2$

value of s	order in number of nodes	order in h
0.55	1.090	2.024
0.6	1.086	2.018
0.65	1.084	2.013
0.7	1.081	2.009
0.75	1.080	2.006
0.8	1.078	2.003
0.85	1.077	2.001
0.9	1.076	1.999
0.95	1.076	1.999

TABLE 2. L^2 order using graded meshes for $\alpha = 2$

Then, the solution is

$$u(x, y) = (1 - x^s)(1 + y^2/2)$$

and an easy calculation shows that $u \in H^2(\Omega)$ *whenever* $s > \frac{3-\alpha}{2}$.

We solve using quasi-uniform meshes and graded meshes. Table 1 and Table 2 show the order of the error in the L^2 *norm, in terms of the number of nodes and in terms of the mesh size for both kind of meshes. Although the solution is in* $H^2(\Omega)$*, for all the values of* s *considered, the order of convergence is not optimal when quasi-uniform meshes are used. On the other hand, the optimal order of convergence is recovered by using appropriate graded meshes according to our theoretical results.*

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