Lagrange and average interpolation over 3D anisotropic elements.

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Abstract. An average interpolation is introduced for 3-rectangles and tetrahedra, and optimal order error estimates in the H^1 norm are proved. The constant in the estimate depends "weakly" (improving the results given in [8]) on the uniformity of the mesh in each direction. For tetrahedra, the constant also depends on the maximum angle of the element.
On the other hand, merging several known results [1, 8, 10, 12], we prove optimal order error for the $\mathcal{P}_1$-Lagrange interpolation in $W^{1,p}$, $p > 2$, with a constant depending on p as well as the maximum angle of the element. Again, under the *maximum angle condition*, optimal order error estimates are obtained in the H^1 norm for higher degree interpolations.

Keywords. Lagrange interpolation, average interpolation, anisotropic elements, maximum angle condition.

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1 Introduction

The clasical error analysis (see for example [6, 5]) for several kinds of interpolation operators assumes the so called regularity of the elements (i.e. bounded ratio between outer and inner diameter of the elements) in order to ensure optimal order error estimates. This condition allows mesh refinements for which the quotient between outer and inner diameter of the elements remains bounded. However, anisotropic or narrow elements, for which the regularity does not hold, arises naturally in order to approximate solutions of problems with a strong directional dependent behavior. Several results allows to drop the regularity condition for rectangular elements as well as for isoparametric quadrilaterals [2, 3, 11, 14, 15]. On the other hand, for triangles, a well known result [4, 9] shows that the regularity can be replaced by the weaker *maximum angle condition* (i.e. maximum angle bounded away from π). In [10], the author extend this condition to tetrahedra requiring that both angles inside and between faces, remains away from π, and proves optimal order error in the $W^{1,\infty}$ norm with a constant depending only on the maximum angle for the linear Lagrange interpolation. However, interesting counterexamples are given in [3, 12], showing that this result does not hold in the useful H^1 norm, for functions belonging only to H^2. A similar fact is showed in [12] for trilinear interpolation over $3-rectangles$. Indeed, the constant in the error estimate deteriorates as one compress the reference element in a direction given by one of its edges. Nonetheless, again in [12], it is proved that more regular functions and higher degree interpolations are compatible with some class of anisotropic elements. In particular with general $3-rectangles$ as well as with tetrahedra obtained by arbitrary scalings of the reference element followed by linear transformations defined by matrices of a uniform bounded condition number. For this kinds of tetrahedra uniform error estimates in the $W^{1,p}$ norm, $p > 2$, for linear elements, are proved in a recent work [8]. The constant blows up as $p \to 2$ in accordance with the counterexamples mentioned above.
The connection between the class of tetrahedra defined in [12] and those defined by the *maximum angle condition* was clarified in [1], in particular, the latter results greater than the former. The first section of

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this paper is devoted to show (generalizing [8, 10, 12]) that optimal order error hold for the $\mathcal{P}_1$-Lagrange interpolation, in the $W^{1,p}$ norm, $p > 2$, as well as in the H^1 norm for higher degree interpolations, in both cases under the *maximum angle condition*. This result was recently obtained (with a different approach) in [3]. However our version shows (following [8]), for linear elements, the behaviour of the constant given in the estimate, when $p \to 2$.
On the other hand, for singular solutions, Lagrange interpolation can not be used since pointwise values becomes meaningless. To overcome this difficult average interpolation was introduced (see [7, 13]), and again, optimal order error can be proven, under *regularity assumptions* on the elements. However, in the above mentioned work [8], Durán constructs an average interpolation over non regular $3 - rectangles$ and shows that the error results independent of the relation between the length of the edges. Nonetheless his technique made use of the quasi-uniformity of the mesh in each direction. Another interesting technique is developed in [3], where the author modifies the Scott-Zhang [13] average interpolation obtaining uniform error estimates for some family of anisotropic elements. However, the meshes are of "tensor product type", and in the three dimensional case, further restrictions on the elements are required. Indeed, the size of the element is arbitrary only in one direction, since the error estimate depends on the relation between the lengths of the edges in the remaining directions.
Results of this kind show that numerical approximations, by finite elements, of singular solutions, behaves better than Lagrange interpolation.
In Section 3.2 we define an average interpolant operator over $3 - rectangles$ and tetrahedra and prove optimal order error in the H^1 norm. The average interpolation is defined interpolating an adequate regularization of the involved function. Since Lagrange interpolation has a "good" behaviour over regular spaces, it seems very natural to regularize before interpolate. The most generalized way to regularize consists in using the so called "mollifiers", and we will see that, by using this technique, anisotropic estimates are easily obtained. However, this approach leads to the same kind of restrictions required in [8]. In order to overcome this difficult we will introduce (see Section 3.1) some appropriate modification of the clasical "mollifiers" procedure. With this approach only a "weak" restriction on the mesh is required.

2 Lagrange interpolation

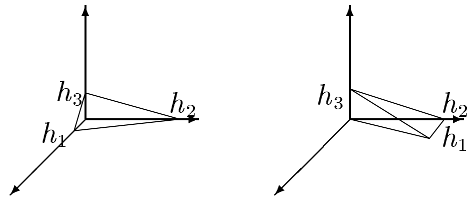
In this section we obtain results for the Lagrange interpolation over tetrahedra just merging several known results [1, 8, 12]. We begin by recalling a characterization of the *maximum angle condition* for tetrahedra given in [1]. Using this result, and following closely [8], we show, generalizing [10], that optimal order error in $W^{1,p}$, $p > 2$, holds for the $\mathcal{P}_1$-Lagrange interpolation with a constant depending on p as well as on the maximum angle. Next, for $p = 2$, but increasing the regularity of the interpolated function, and by means of the characterization mentioned above, we get, using Theorem 1 of [12], optimal order error in H^1 for the $\mathcal{P}_k$, $k \geq 2$, Lagrange interpolation, also under the *maximum angle condition*.

Let us start introducing some notation

With $\mathbf{e}_i$, $1 \leq i \leq 3$, representing the canonical vectors, and for a given positive reals h_1, h_2, h_3 we define, using *c.h.* as the convex hull, the tetrahedra (see Figure 1)

$$K_1(h_1, h_2, h_3) := c.h.\{\mathbf{0}, h_1\mathbf{e}_1, h_2\mathbf{e}_2, h_3\mathbf{e}_3\}, \qquad K_2(h_1, h_2, h_3) := c.h.\{\mathbf{0}, h_1\mathbf{e}_1 + h_2\mathbf{e}_2, h_2\mathbf{e}_2, h_3\mathbf{e}_3\}$$

For a given vector $\mathbf{v} \in \mathbb{R}^3$, and matrix $B \in \mathbb{R}^{3\times 3}$, $\|\mathbf{v}\|$ and $\|B\|$ means the euclidean norm, and the norm subordinated to the euclidean norm respectively. With $\kappa(B)$ we denote the condition number, once more in the euclidean norm, i.e. $\kappa(B) = \|B\|\|B^{-1}\|$. We use the standard notation $W^{m,p}(K)$ (also $H^m(K)$ if $p = 2$) for the Sobolev space of $L^p(K)$ functions with $L^p(K)$ distributional derivatives up to the order m, and for $u \in W^{m,p}(K)$ we write $\|u\|_{m,p,K}$ and $|u|_{m,p,K}$ to denote its usual norm and seminorm respectively.



a) $K_1(h_1, h_2, h_3)$ b) $K_2(h_1, h_2, h_3)$

Figure 1

2.1 The maximum angle condition

In [10], the author defines the *maximum angle condition*

Definition 2.1 *A tetrahedron K satisfies the "maximum angle condition" with a constant $\overline{\psi} < \pi$, or shortly $MAC(\overline{\psi})$, if the angles between faces and the angles inside faces of K are bounded above by $\overline{\psi}$.*

Under this definition the author proves optimal order error estimates in $W^{1,\infty}$, with a constant depending only on the maximum angle $\overline{\psi}$, for the linear Lagrange interpolation. His argument depends strongly on the fact that he is working in the infinite norm. Indeed, for $u \in W^{2,\infty}(K)$ and calling $\omega = u - \Pi(u)$, with Π the $\mathcal{P}_1$ Lagrange interpolation, one has $\frac{\partial \omega}{\partial \mathbf{v}_i}(q) = 0$ for certain q belonging to the edge parallel to the direction given by $\mathbf{v}_i$. Then for any $r \in K$ one can write

$$\frac{\partial \omega}{\partial \mathbf{v}_i}(r) = \frac{\partial \omega}{\partial \mathbf{v}_i}(r) - \frac{\partial \omega}{\partial \mathbf{v}_i}(q) = \int_q^r \frac{\partial^2 \omega}{\partial \eta \partial \mathbf{v}_i}(s) ds \tag{2.1}$$

where η defines the direction of the segment joining r and q. So

$$\|\frac{u - \Pi(u)}{\partial \mathbf{v}_i}\|_{0,\infty,K} = \|\frac{\partial \omega}{\partial \mathbf{v}_i}\|_{0,\infty,K} \leq h \|\frac{\partial \omega}{\partial \mathbf{v}_i}\|_{1,\infty,K} \leq h|u|_{2,\infty,K} \tag{2.2}$$

and the result given in [10], follows showing that the *maximum angle condition* ensures the existence of three "uniformly linearly independent" edges. Indeed, the author proves that it is possible to choose three edges such that the *unitary* vectors parallels to them, say t_1, t_2, t_3, verifies

$$|det(M)| \geq m(\overline{\psi})^3 \tag{2.3}$$

where M is the matrix made up with t_i as columns and $m(\overline{\psi}) = min\{sin(\frac{\pi - \overline{\psi}}{2}), sin(\overline{\psi})\}$. Finally (2.3) together with (2.2) allows to get bounds over the full seminorm $|w|_{1,\infty}(K)$.

The last argument does not longer applies to estimate the error in $W^{1,p}(K)$ with $p \neq \infty$.

In [1] we study the *maximum angle condition* finding an analytic, rather than geometric, characterization of the class of elements defined by this property. The next lemma states, in a suitable form a result given in [1].

Lemma 2.1 *If a tetrahedrum K satisfies $MAC(\overline{\psi})$ then there exist positive numbers h_1, h_2, h_3, a constant $C = C(\overline{\psi})$, and a linear transformation $F(x) = Bx + b$, such that $F(K_1) = K$ or $F(K_2) = K$ and $\|B\|, \|B^{-1}\| \leq C$. Where K_1 and K_2 are as in Figure 1).*

Proof. See the proof of Lemma 5.9 of [1]. □

Remark 2.1 *As $\|B\|$ and $\|B^{-1}\|$ are bounded by $C(\overline{\psi})$ then, one can easily get,*

$$\frac{1}{C(\overline{\psi})} diam(K) \leq diam(F^{-1}(K)) \leq C(\overline{\psi}) diam(K) \tag{2.4}$$

and so, Lemma 2.1, allows us to reduce the study of the Lagrange interpolation under the maximum angle condition to the cases given in the Figure 1, just changing variables.

Now we give a definition and a simple result which will be useful in Section 3.2

Definition 2.2 *For a given tetrahedron K, the directions t_i, $1 \leq i \leq 3$ for which (2.3) hold, will be called principal directions. We will also use principal edges (resp.: principal lenghts) to denote the edges (resp.: lenghts of the edges) parallels to these directions.*

Lemma 2.2 *Let K be a tetrahedron under $MAC(\overline{\psi})$, then calling h_1, h_2, h_3 its principal lengths we have*

$$vol(K) \geq \frac{1}{6} h_1 h_2 h_3 m(\overline{\psi})^3 \tag{2.5}$$

Proof. Follows immediately from (2.3). □

2.2 Error estimates for $\mathcal{P}_1$-Lagrange Interpolation

In [8] Theorem 2.1, the author proves optimal order error in $W^{1,p}(K)$, $p > 2$, with a constant which blows-up as $p \to 2$, for the $\mathcal{P}_1$-Lagrange interpolation and for the family of tetrahedra given in Figure 1 a). His proof applies, step by step, for the family showed in Figure 1 b, and we do not repeat his argument.

Theorem 2.1 *Let $K = K_1(h_1, h_2, h_3)$, or $K = K_2(h_1, h_2, h_3)$ for arbitrary $h_1, h_2, h_3 > 0$, and $p > 2$, then there exists $C = C(p)$ such that*

$$\|\frac{\partial(u - \Pi(u))}{\partial x_i}\|_{0,p,K} \leq C(p) \sum_{j=1}^{3} h_j \|\frac{\partial^2 u}{\partial x_j \partial x_i}\|_{0,p,K} \tag{2.6}$$

Remark 2.2 *The constant $C(p)$ depends strongly on the trace theorem (see [8]). In particular, for $p \sim 2$, it holds $C(p) \sim \frac{C}{(p-2)^{\frac{p}{2}}}$.*

From this result, one obtains, in view of Remark 2.1

Theorem 2.2 *Let K be a tetrahedron under $MAC(\overline{\psi})$, $h = diam(K)$, then there exists a constant $C = C(\overline{\psi}, p)$, such that*

$$\|u - \Pi(u)\|_{1,p,K} \leq Ch|u|_{2,p,K} \tag{2.7}$$

2.3 Error estimates for $\mathcal{P}_k$ Lagrange interpolation with $k \geq 2$.

A very general result for higher degree anisotropic elements can be found in [12]. It is straightforward to check hypothesis $II, ..., VIII$, given there ([12], p.107), when one takes as the reference element $T_0 = K_1 := K_1(1,1,1)$ or $T_0 = K_2 := K_2(1,1,1)$, as well as approximating spaces and degrees of freedom given by the elements of type (k), $k \geq 2$ (we are using the notation of [6]). So, we can state, as a direct consequence of Theorem 1 [12], and Lemma 2.1 the following theorem

Theorem 2.3 *Let us consider the finite element space of type (k), $k \geq 2$, over tetrahedra (see [6]). Let K under $MAC(\overline{\psi})$, and Π be the corresponding Lagrange interpolation, then there exists $C = C(\overline{\psi}, K_1, K_2)$ such that*

$$|u - \Pi(u)|_{1,2,K} \leq Ch_K^{m-1}|u|_{m,2,K} \tag{2.8}$$

with $m = k + 1$.

Proof. From Lemma 2.1 there exist $h_1, h_2, h_3 > 0$, a constant $C = C(\overline{\psi})$, and a linear transformation $F(x) = Bx + b$, such that $F(K_1(h_1, h_2, h_3)) = K$ or $F(K_2(h_1, h_2, h_3)) = K$, and $\|B\|, \|B^{-1}\| \leq C$. Without loss of generality we can assume $F(K_2(h_1, h_2, h_3)) = K$, then, by means of the scaling given by the diagonal matrix D, $D_{ii} = h_i$, we may write $FD(K_2) = K$.
Now, in order to match our notation with that given in [12], we write $B = h_K S_K^t$, with $S_K^t := \frac{1}{h_k} B$, $D = D_K$, and $b = b_K$, then $\overline{F}x := h_K S_K^t D_K x + b_K$, verifies $\overline{F}(K_2) = FD(K_2) = K$.
From equation (8) of [12] one easily gets

$$|u - \Pi(u)|_{1,2,K} \leq C \frac{\Lambda_K}{\lambda_K} h_K^{m-1} |u|_{m,2,K} \tag{2.9}$$

where C depends on the reference element, K_2 in this case, and Λ_K, λ_K represent the greatest and the smallest singular values of S_K^t. Observing that

$$\frac{\Lambda_K}{\lambda_K} = \kappa(S_K^t) = \kappa(B) = \|B\| \|B^{-1}\|$$

the proof finishes by means of Lemma 2.1 together with (2.9). □

3 An average interpolation

In [8], Durán, constructs an average interpolation operator over anisotropic $3-$ *rectangles*. However, his technique can not handle meshes which are not quasi-uniform in each direction. In this section we develop a straigthforward generalization of the clasical "mollifiers" which allows us to construct an average interpolation with optimal order error in H^1, over anisotropic $3-$ *rectangle* or tetrahedra, without the restriction assumed in [8].

3.1 Regularization properties

We begin introducing some notation.

With $B_1 \subset \mathbb{R}^3$ we will denote the unitary ball. For a given scalar functions $0 < \epsilon_i(x) \in C^2(\mathbb{R})$, $1 \leq i \leq 3$, we define $\epsilon(x) = \epsilon(x_1, x_2, x_3) := (\epsilon_1(x_1), \epsilon_2(x_2), \epsilon_3(x_3))$ dropping sometimes the x, in order to simplify the notation. We use also $B_{\epsilon(x)} = B_\epsilon$ to denote the ellipsoid $B_\epsilon := \{(y_1, y_2, y_3) \in \mathbb{R}^3$ *such that* $\sum_{i=1}^3 (\frac{y_i}{\epsilon_i})^2 \leq 1\}$ and for a given $y \in \mathbb{R}^3$ we will write $\frac{y}{\epsilon} := (\frac{y_1}{\epsilon_1}, \frac{y_2}{\epsilon_2}, \frac{y_3}{\epsilon_3})$, and $\epsilon y := (\epsilon_1 y_1, \epsilon_2 y_2, \epsilon_3 y_3)$. If $\rho(x_1, x_2, x_3) \in C^2$, $\rho(x) \geq 0$ supported on B_1, verifies $\frac{1}{|B_1|} \int_{\mathbb{R}^3} \rho(x) dx = 1$, we define $\rho_{\epsilon(x)}(y) = \frac{1}{|B_{\epsilon(x)}|} \rho(\frac{y}{\epsilon(x)})$ which for a fixed x will be supported on $B_{\epsilon(x)}$. Given $A, B \subset \mathbb{R}^3$ with $A + B$ we denote the set $A + B = \{x + y, \ \ x \in A, y \in B\}$, and then for a given f defined over $\{x\} + B_{\epsilon(x)}$ we write

$$\rho_{\epsilon(x)} \hat{*} f(x) := \int_{\mathbb{R}^3} \rho_{\epsilon(x)}(y) f(x - y) dy \tag{3.1}$$

Remark 3.1 *If $\epsilon_1, \epsilon_2, \epsilon_3$ are constants, we have that $\rho_\epsilon \hat{*} f = \rho_\epsilon * f$ works like the usual convolution, moreover, taking in particular $\epsilon_1 = \epsilon_2 = \epsilon_3$ we recover the clasical mollifiers.*

Remark 3.2 *For $(y_1, y_2, y_3) \in B_1$ fixed, and ϵ_i constants, the mapping*

$$\Phi(x) = (x_1 - \epsilon_1 y_1, x_2 - \epsilon_2 y_2, x_3 - \epsilon_3 y_3)$$

can be seen as a rigid movement and, in particular, it results a "good" change of variables. This property is not longer true if ϵ_i depends on x_i, indeed, in this case Φ may be no longer one to one. In order to remedy this fact, we require along this section the following hypothesis

$$\textbf{H0} \qquad\qquad |\epsilon_i'(x)| < 1/2$$

which, as we will see, represent only a weak restriction.
Under **H0***, as one can easily verify, not only the mapping* Φ*, but its components, becomes injective, and a lower bound for its Jacobian is readily find, namely,*

$$Jac(\Phi) \geq \frac{1}{2^3}$$

Definition 3.1 *For a given set* $K \subset \mathbb{R}^3$ *we define*

$$\epsilon_K^M := sup_{x\in K}\epsilon(x) := (sup_{x\in K}\epsilon_1(x_1), sup_{x\in K}\epsilon_2(x_2), sup_{x\in K}\epsilon_3(x_3))$$

In the same way, we write

$$\epsilon_K^m := inf_{x\in K}\epsilon(x) := (inf_{x\in K}\epsilon_1(x_1), inf_{x\in K}\epsilon_2(x_2), inf_{x\in K}\epsilon_3(x_3))$$

Now we can prove the following

Lemma 3.1 *Let* $K \subset \mathbb{R}^3$*,* $f \in L^p(K + B_{\epsilon_K^M})$*, and let us assume* **H0***, then*

$$\|\rho_{\epsilon(x)}\hat{*}f\|_{0,p,K} \leq 2^{3/p}\|f\|_{0,p,K+B_{\epsilon_K^M}} \tag{3.2}$$

Proof. We show first the case $p = 1$.

$$|\rho_{\epsilon(x)}\hat{*}f(x)| \leq \int_{\mathbb{R}^3} \frac{1}{|B_{\epsilon(x)}|}\rho(\frac{y}{\epsilon(x)})|f(x-y)|dy \tag{3.3}$$

changing variables $y \leftrightarrow \frac{y}{\epsilon(x)}$ and using that $|B_{\epsilon(x)}| = \epsilon_1(x_1)\epsilon_2(x_2)\epsilon_3(x_3)|B_1|$, together with the fact that $\rho(y)$ is supported on B_1 we have, writing $\epsilon(x)y := (\epsilon(x_1)y_1, \epsilon(x_2)y_2, \epsilon(x_3)y_3)$

$$|\rho_{\epsilon(x)}\hat{*}f(x)| \leq \frac{1}{|B_1|}\int_{B_1} \rho(y)|f(x-\epsilon(x)y)|dy \tag{3.4}$$

then

$$\int_K |\rho_{\epsilon(x)}\hat{*}f(x)|dx \leq \frac{1}{|B_1|}\int_{B_1} \rho(y)\int_K |f(x-\epsilon(x)y)|dxdy \tag{3.5}$$

using now the change of variables $x \leftrightarrow x - \epsilon(x)y$, and recalling that $y \in B_1$, we get for $x \in K$, $x - \epsilon(x)y \in K + B_{\epsilon_K^M}$, and in view of **H0** (see Remark 3.2) we obtain

$$\int_K |\rho_{\epsilon(x)}\hat{*}f(x)|dx \leq 2^3\frac{1}{|B_1|}\int_{B_1} \rho(y)dy\int_{K+B_{\epsilon_K^M}} |f(x)|dx \tag{3.6}$$

but $\frac{1}{|B_1|}\int_{B_1}\rho(y)dy = 1$ and we finally find (3.2) with $p = 1$.
For any p it also follows in an standard way. In fact, for $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$|\rho_{\epsilon(x)}\hat{*}f(x)| \leq \int_{\mathbb{R}^3} \{\rho_{\epsilon(x)}^{1/p}|f(x-y)|\}\{\rho_{\epsilon(x)}^{1/q}\}dy \tag{3.7}$$

and Hölder's inequality yields

$$|\rho_{\epsilon(x)}\hat{*}f(x)| \leq \{\int_{\mathbb{R}^3} \{\rho_{\epsilon(x)}|f(x-y)|^p dy\}^{1/p}\{\int_{\mathbb{R}^3} \rho_{\epsilon(x)}dy\}^{1/q} = \{\int_{\mathbb{R}^3} \{\rho_{\epsilon(x)}|f(x-y)|^p dy\}^{1/p} \tag{3.8}$$

where we have used in the last identity $\int \rho_{\epsilon(x)}(y)dy = \frac{1}{|B_1|}\int_{\mathbb{R}^3}\rho(y)dy = 1$. Observing now that $|f|^p \in L^1(K + B_{\epsilon_K^M})$, and using the case $p = 1$, we get

$$\int_K |\rho_{\epsilon(x)}\hat{*}f(x)|^p dx \le 2^3 \int_{K+B_{\epsilon_K^M}} |f(y)|^p dy \tag{3.9}$$

and (3.2) follows. □

The convolution between two functions can be bounded, in the infinite norm, by the L^2 norms of the functions involved. In the following lemma we exploit a similar property of $\rho_\epsilon\hat{*}f$ in order to obtain an useful inequality.

Lemma 3.2 *Let $K \subset \mathbb{R}^3$, $u \in L^2(K + B_{\epsilon_K^M})$ then*

$$\|\rho_{\epsilon(x)}\hat{*}u\|_{0,\infty,K} \le C_{0,\rho}\frac{1}{|B_{\epsilon_K^m}|^{1/2}}\|u\|_{0,2,K+B_{\epsilon_K^M}} \tag{3.10}$$

where $C_{0,\rho} = \frac{\|\rho\|_{0,2,B_1}}{|B_1|^{1/2}}$.

Proof. Using Schwartz's inequality we get

$$|\rho_{\epsilon(x)}\hat{*}u(x)| \le \{\int_{B_{\epsilon(x)}} \rho^2_{\epsilon(x)}(y)dy\}^{1/2}\{\int_{B_{\epsilon(x)}} |u(x-y)|^2 dy\}^{1/2} \tag{3.11}$$

and to conclude, it will be enough to bound each one of the integrals on the right hand side.

For $x \in K$, $y \in B_{\epsilon(x)}$ we have $x - y \in K + B_{\epsilon(x)} \subset K + B_{\epsilon_K^M}$ and so

$$\{\int_{B_{\epsilon(x)}} |u(x-y)|^2 dy\}^{1/2} \le \|u\|_{0,2,K+B_{\epsilon_K^M}} \tag{3.12}$$

On the other hand the change of variables $y \leftrightarrow \frac{y}{\epsilon(x)}$ gives

$$\{\int_{B_{\epsilon(x)}} \rho^2_{\epsilon(x)}(y)dy\}^{1/2} = \frac{\epsilon_1(x_1)^{1/2}\epsilon_2(x_2)^{1/2}\epsilon_3(x_3)^{1/2}}{|B_{\epsilon(x)}|}\|\rho\|_{0,2,B_1} = \frac{1}{|B_1|^{1/2}|B_{\epsilon(x)}|^{1/2}}\|\rho\|_{0,2,B_1} \tag{3.13}$$

but $x \in K$ implies $|B_{\epsilon_K^m}| \le |B_{\epsilon(x)}|$, and this fact together with equations (3.11), (3.12), gives (3.10). □

In the following lemma the first approximation property for $\rho_\epsilon\hat{*}u$ is obtained. It is worthwhile to remark that the obtained estimate looks like the usual error estimate in average interpolant operators.

Lemma 3.3 *Let $K \subset \mathbb{R}^3$, $u \in H^1(K + B_{\epsilon_K^M})$, then*

$$\|u - \rho_{\epsilon(x)}\hat{*}u\|_{0,2,K} \le 2^{3/2}3\sum_{i=1}^{3}\epsilon_i\|\frac{\partial u}{\partial x_i}\|_{0,2,K+B_{\epsilon_K^M}} \tag{3.14}$$

where $(\epsilon_1, \epsilon_2, \epsilon_3) := \epsilon_K^M$.

Proof. For a fixed x we may write

$$|\rho_{\epsilon(x)} * u(x) - u(x)| \le \int_{R^3} \rho_{\epsilon(x)}(y)|u(x-y) - u(x)|dy \le \int_{R^3} \rho_{\epsilon(x)}(y)(\int_0^1 |\nabla u(x-ty).y|dt)dy \tag{3.15}$$

where the dot means the scalar product. Now, as $y=(y_1,y_2,y_3)\in sop(\rho_\epsilon)$, we have $|y_i|\le \epsilon_i(x_i)\le\epsilon_i$, and then from (3.15)

$$|\rho_{\epsilon(x)}\hat{*}u(x)-u(x)|\le\sum_{i=1}^{3}\epsilon_i\int_{\mathbb{R}^3}\int_0^1|\frac{\partial u(x-ty)}{\partial x_i}|\rho_{\epsilon(x)}(y)dtdy \tag{3.16}$$

changing variables $y\leftrightarrow ty$, and using that $\rho_{t\epsilon}(y)=\frac{1}{t^3}\rho_\epsilon(\frac{y}{t})$, it follows that

$$\int_{\mathbb{R}^3}\int_0^1|\frac{\partial u(x-ty)}{\partial x_i}|\rho_{\epsilon(x)}(y)dtdy=\int_0^1\int_{\mathbb{R}^3}|\frac{\partial u(x-y)}{\partial x_i}|\rho_{t\epsilon(x)}(y)dydt \tag{3.17}$$

and from (3.16), (3.17) we get

$$|\rho_{\epsilon(x)}\hat{*}u(x)-u(x)|^2\le 3\sum_{i=1}^{3}\epsilon_i^2(\int_0^1\int_{\mathbb{R}^3}|\frac{\partial u(x-y)}{\partial x_i}|\rho_{t\epsilon(x)}(y)dtdy)^2 \tag{3.18}$$

Schwartz's inequality on the variable t gives

$$(\int_0^1\int_{\mathbb{R}^3}|\frac{\partial u(x-y)}{\partial x_i}|\rho_{t\epsilon}(y)dydt)^2\le\int_0^1(\int_{\mathbb{R}^3}|\frac{\partial u(x-y)}{\partial x_i}|\rho_{t\epsilon(x)}(y)dy)^2dt=\int_0^1(\rho_{t\epsilon(x)}\hat{*}|\frac{\partial u}{\partial x_i}|)^2dt \tag{3.19}$$

and from (3.18), (3.19)

$$\int_K|\rho_{\epsilon(x)}\hat{*}u(x)-u(x)|^2dx\le 3\sum_{i=1}^{3}\epsilon_i^2\int_0^1\int_K(\rho_{t\epsilon(x)}\hat{*}|\frac{\partial u}{\partial x_i}|)^2dxdt \tag{3.20}$$

taking now $t\epsilon(x)$, instead of $\epsilon(x)$, in Lemma 3.1, we have, using that $0<t<1$

$$\int_K(\rho_{t\epsilon(x)}\hat{*}|\frac{\partial u}{\partial x_i}|)^2dx\le 2^3\int_{K+B_{\epsilon_K^M}}|\frac{\partial u}{\partial x_i}|^2dx \tag{3.21}$$

noting that the last integral does not depends on t, we get from (3.20)

$$\int_K|\rho_{\epsilon(x)}\hat{*}u(x)-u(x)|^2dx\le 2^3 3\sum_{i=1}^{3}\epsilon_i^2\int_{K+B_{\epsilon_K^M}}|\frac{\partial u}{\partial x_i}|^2dx \tag{3.22}$$

and (3.14) follows. □

Remark 3.3 *If $\epsilon_1,\epsilon_2,\epsilon_3$ are constant we have, as we said before, $\rho_\epsilon\hat{*}f=\rho_\epsilon * f$, and so, from a well known property of the convolution*

$$\frac{\partial(u-\rho_\epsilon*u)}{\partial x_i}=\frac{\partial u}{\partial x_i}-\rho_\epsilon*\frac{\partial u}{\partial x_i} \tag{3.23}$$

and the result of Lemma 3.3, can be extended straightforward in the following sense:
If $u\in H^2(K+B_{\epsilon_K^M})$ then

$$||\frac{\partial(u-\rho_\epsilon*u)}{\partial x_i}||_{0,2,K}\le 2^{3/2}3\sum_{i=1}^{3}\epsilon_j||\frac{\partial^2u}{\partial x_j\partial x_i}||_{0,2,K+B_{\epsilon_K^M}}$$

to obtain a similar result for $\hat{*}$ we need, however, an analogous of (3.23). That is in fact which we are looking for in the next lemma.

Lemma 3.4 *Let $K \subset \mathbb{R}^3$, $u \in H^1(K + B_{\epsilon_K^M})$, let us assume, once more,* **H0**. *If we define*

$$c(x_i, y_i) := 1 - \frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)} y_i \qquad d(x_i, y_i) := \frac{\epsilon_i''(x_i)}{\epsilon_i(x_i)} y_i$$

then

$$\frac{\partial \rho_{\epsilon(x)} \hat{*} u(x)}{\partial x_i} = \int_{\mathbb{R}^3} c(x_i, y_i) \rho_{\epsilon(x)}(y) \frac{\partial u}{\partial x_i}(x - y) dy \tag{3.24}$$

if moreover $u \in H^2(K + B_{\epsilon_K^M})$ then for $j \neq i$

$$\frac{\partial^2 \rho_{\epsilon(x)} \hat{*} u(x)}{\partial x_j \partial x_i} = \int_{\mathbb{R}^3} c(x_j, y_j) c(x_i, y_i) \rho_{\epsilon(x)}(y) \frac{\partial^2 u}{\partial x_j \partial x_i}(x - y) dy \tag{3.25}$$

and if $j = i$

$$\frac{\partial^2 \rho_{\epsilon(x)} \hat{*} u(x)}{\partial^2 x_i} = \int_{\mathbb{R}^3} c(x_i, y_i)^2 \rho_{\epsilon(x)}(y) \frac{\partial^2 u}{\partial^2 x_i}(x - y) dy - \int_{\mathbb{R}^3} d(x_i, y_i) \rho_{\epsilon(x)}(y) \frac{\partial u}{\partial x_i}(x - y) dy \tag{3.26}$$

Proof. A direct computation gives

$$\frac{\partial \rho_{\epsilon(x)} \hat{*} u(x)}{\partial x_i} = I_1 + I_2 + I_3 \tag{3.27}$$

where

$$I_1 = -\int_{\mathbb{R}^3} \frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)} \frac{1}{|B_{\epsilon(x)}|} \rho\left(\frac{y}{\epsilon(x)}\right) u(x - y) dy$$

$$I_2 = -\int_{\mathbb{R}^3} \frac{\epsilon_i'(x_i)}{\epsilon_i^2(x_i)} y_i \frac{1}{|B_{\epsilon(x)}|} D_i \rho\left(\frac{y}{\epsilon(x)}\right) u(x - y) dy$$

$$I_3 = \int_{\mathbb{R}^3} \frac{1}{|B_{\epsilon(x)}|} \rho\left(\frac{y}{\epsilon(x)}\right) \frac{\partial u}{\partial x_i}(x - y) dy$$

rewriting I_2

$$I_2 = -\int_{\mathbb{R}^3} \frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)} y_i \frac{1}{|B_{\epsilon(x)}|} \frac{\partial \rho(\frac{y}{\epsilon(x)})}{\partial y_i} u(x - y) dy$$

and integrating by parts

$$I_2 = \int_{\mathbb{R}^3} \frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)} \frac{1}{|B_{\epsilon(x)}|} \rho\left(\frac{y}{\epsilon(x)}\right) u(x - y) dy - \int_{\mathbb{R}^3} \frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)} y_i \frac{1}{|B_{\epsilon(x)}|} \rho\left(\frac{y}{\epsilon(x)}\right) \frac{\partial u}{\partial x_i}(x - y) dy$$

adding up this expression to I_1 and I_3, we get (3.24) from (3.27).
The equation (3.25) follows in the same way just observing that $c(x_i, y_i)$ behaves as a constant when we derive (3.24) respect to x_j ($j \neq i$).
We now check (3.26) taking the derivative in (3.24). We have

$$\frac{\partial^2 \rho_{\epsilon(x)} \hat{*} u}{\partial x_i^2} = I_1 + I_2 + I_3 + I_4 \tag{3.28}$$

where now

$$I_1 = -\int_{\mathbb{R}^3} \left(\frac{\epsilon_i'(x_i)}{\epsilon(x_i)}\right)' y_i \frac{1}{|B_{\epsilon(x)}|} \rho\left(\frac{y}{\epsilon(x)}\right) \frac{\partial u}{\partial x_i}(x - y) dy$$

$$I_2 = -\int_{\mathbb{R}^3} c(x_i, y_i) \frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)} \frac{1}{|B_{\epsilon(x)}|} \rho\left(\frac{y}{\epsilon(x)}\right) \frac{\partial u}{\partial x_i}(x - y) dy$$

$$I_3 = -\int_{\mathbb{R}^3} c(x_i, y_i) \frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)} y_i \frac{1}{|B_{\epsilon(x)}|} \frac{\partial \rho(\frac{y}{\epsilon(x)})}{\partial y_i} \frac{\partial u}{\partial x_i}(x - y) dy$$

$$I_4 = \int_{\mathbb{R}^3} c(x_i, y_i)\frac{1}{|B_{\epsilon(x)}|}\rho(\frac{y}{\epsilon(x)})\frac{\partial^2 u}{\partial x_i^2}(x-y)dy$$

integrating by parts I_3 yields

$$I_3 = I_{31} + I_{32}$$

with

$$I_{31} = \int_{\mathbb{R}^3} \frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)}(1 - 2\frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)}y_1)\frac{1}{|B_{\epsilon(x)}|}\rho(\frac{y}{\epsilon(x)})\frac{\partial u}{\partial x_i}(x-y)dy$$

$$I_{32} = -\int_{\mathbb{R}^3} c_i(x_i, y_i)\frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)}y_i\frac{1}{|B_{\epsilon(x)}|}\rho(\frac{y}{\epsilon(x)})\frac{\partial^2 u}{\partial x_i^2}(x-y)dy$$

from this expressions, together with I_1, I_2, I_4, (3.28), we get (3.26). □

Remark 3.4 *Note that for $\epsilon_i = constant$ we have $c_i(x_i, y_i) \equiv 1$, $d(x_i, y_i) \equiv 0$, and the expressions obtained in the previous lemma coincide with the usual ones for the convolution.*

For further use, we define for $1 \leq i, j \leq 3$

$$C_{1,\epsilon_j} := \|\frac{\epsilon_j'(x_j)}{\epsilon_j(x_j)}\|_{0,\infty,K}$$

$$C_j := (1 + C_{1,\epsilon_j}\epsilon_j) \tag{3.29}$$

$$C_{i,j} := C_i C_j \tag{3.30}$$

We can now face the extension of Lema 3.3 to derivatives, as we did for the convolution in Remark 3.3.

Lemma 3.5 *Assume* **H0**, *and let $K \subset \mathbb{R}^3$, $u \in H^2(K + B_{\epsilon_K^M})$. If we define*

$$\epsilon_i := (\epsilon_K^M)_i$$

then

$$\|\frac{\partial(u - \rho_{\epsilon(x)}\hat{*}u)}{\partial x_i}\|_{0,2,K} \leq 2^{3/2}\{3\sum_{j=1}^{3}\epsilon_j\|\frac{\partial^2 u}{\partial x_j \partial x_i}\|_{0,2,K+B_{\epsilon_K^M}} + C_{1,\epsilon_i}\epsilon_i\|\frac{\partial u}{\partial x_i}\|_{0,2,K}\} \tag{3.31}$$

and

$$\|\frac{\partial(u - \rho_{\epsilon(x)}\hat{*}u)}{\partial x_i}\|_{0,2,K} \leq (1 + C_i 2^{3/2})\|\frac{\partial u}{\partial x_i}\|_{0,2,K} \tag{3.32}$$

Proof. Rewriting (3.24) we have

$$\frac{\partial \rho_{\epsilon(x)}\hat{*}u}{\partial x_i} = \rho_{\epsilon(x)}\hat{*}\frac{\partial u}{\partial x_i} - \int_{\mathbb{R}^3}\frac{\epsilon_i'(x_i)}{\epsilon_i(x_i)}y_i\rho_{\epsilon(x)}\frac{\partial u}{\partial x_i}(x-y)dy \tag{3.33}$$

and since $|y_i| \leq \epsilon_i$,

$$|\frac{\partial(u - \rho_{\epsilon(x)}\hat{*}u)}{\partial x_i}| \leq |\frac{\partial u}{\partial x_i} - \rho_{\epsilon(x)}\hat{*}\frac{\partial u}{\partial x_i}| + C_{1,\epsilon_i}\epsilon_i|\rho_{\epsilon(x)}\hat{*}|\frac{\partial u}{\partial x_i}|| \tag{3.34}$$

taking L^2 norm and applying the triangle inequality we get, by means of Lemmas 3.1 and 3.3, the estimate stated in (3.31).
To prove (3.32) we observe that from (3.34)

$$\|\frac{\partial(u - \rho_{\epsilon(x)}\hat{*}u)}{\partial x_i}\|_{0,2,K} \leq \|\frac{\partial u}{\partial x_i}\|_{0,2,K} + C_i\|\rho_{\epsilon(x)}\hat{*}|\frac{\partial u}{\partial x_i}|\|_{0,2,K} \tag{3.35}$$

and we conclude by using Lemma 3.1. □

Remark 3.5 *Let us observe that (3.31) and (3.32) looks like an usual interpolation error estimate.*

In the next lemma we bound the derivatives of $\rho_{\epsilon(x)}\hat{*}u$ in terms of appropiatre seminorms of u.

Lemma 3.6 *Assume that* **H0** *holds. Let $K \subset \mathbb{R}^3$, $u \in H^2(K + B_{\epsilon_K^M})$, and ϵ_i, $C_{i,j}$ as before. If we define*

$$C_{2,\epsilon_i} := \|\frac{\epsilon_i''(x_i)}{\epsilon_i(x_i)}\|_{0,\infty,K}$$

then

$$\|\frac{\partial^2 \rho_{\epsilon(x)}\hat{*}u}{\partial x_j \partial x_i}\|_{0,2,K} \leq C_{i,j}\|\frac{\partial^2 u}{\partial x_j \partial x_i}\|_{0,2,K+B_{\epsilon_K^M}} \tag{3.36}$$

if $i \neq j$, and

$$\|\frac{\partial^2 \rho_{\epsilon(x)}\hat{*}u}{\partial x_i^2}\|_{0,2,K} \leq 2^{3/2}\{C_{i,i}\|\frac{\partial^2 u}{\partial x_i^2}\|_{0,2,K+B_{\epsilon_K^M}} + \epsilon_i C_{2,\epsilon_i}\|\frac{\partial u}{\partial x_i}\|_{0,2,K+B_{\epsilon_K^M}}\} \tag{3.37}$$

Proof. Follows easily from Lemmas 3.1, 3.4. We show by example (3.37).
From (3.26) and using that $|y_i| \leq \epsilon_i$, one gets

$$|\frac{\partial^2 \rho_{\epsilon(x)}\hat{*}u(x)}{\partial x_i^2}| \leq C_{i,i}\int_{\mathbb{R}^3} \rho_{\epsilon(x)}(y)|\frac{\partial^2 u}{\partial x_i^2}(x-y)|dy + C_{2,\epsilon_i}\epsilon_i \int_{\mathbb{R}^3} \rho_{\epsilon(x)}(y)|\frac{\partial u}{\partial x_i}(x-y)|dy \tag{3.38}$$

Taking the L^2 norm in both sides we finish the proof by means of Lemma 3.1. $\square$
In the following lemma we look for similar bounds as that of the previous one but in the infinite norm.

Lemma 3.7 *Assume* **H0***, and let $K \subset \mathbb{R}^3$, $u \in H^2(K + B_{\epsilon_K^M})$. Then, with the notation defined above it holds*

$$\|\frac{\partial^2 \rho_{\epsilon(x)}\hat{*}u}{\partial x_j \partial x_i}\|_{0,\infty,K} \leq C_{0,\rho}\frac{1}{|B_{\epsilon_K^m}|^{1/2}}C_{i,j}\|\frac{\partial^2 u}{\partial x_j \partial x_i}\|_{0,2,K+B_{\epsilon_K^M}} \tag{3.39}$$

for $i \neq j$,

$$\|\frac{\partial^2 \rho_{\epsilon(x)}\hat{*}u}{\partial x_i^2}\|_{0,\infty,K} \leq C_{0,\rho}\frac{1}{|B_{\epsilon_K^m}|^{1/2}}\{C_{i,i}\|\frac{\partial^2 u}{\partial x_i^2}\|_{0,2,K+B_{\epsilon_K^M}} + \epsilon_i C_{2,\epsilon_i}\|\frac{\partial u}{\partial x_i}\|_{0,2,K+B_{\epsilon_K^M}}\} \tag{3.40}$$

and

$$\|\frac{\partial \rho_{\epsilon(x)}\hat{*}u}{\partial x_j}\|_{0,\infty,K} \leq C_{0,\rho}\frac{1}{|B_{\epsilon_K^m}|^{1/2}}C_j\|\frac{\partial u}{\partial x_j}\|_{0,2,K+B_{\epsilon_K^M}} \tag{3.41}$$

Proof. Follows arguing like in Lemma 3.6. In fact, to obtain, for example, (3.40), we proceed as before until we get (3.38) using then Lema 3.2, instead of Lema 3.1. Inequality (3.39) follows analogously. Finally, (3.41) follows similarly from (3.24) and Lemma 3.2. $\square$

The next section is devoted to construct an average interpolation which has optimal order error in H^1 whenever the *Lagrange interpolation* verifies this property over *more regular spaces*.

3.2 Construction of the average interpolation.

During this subsection we will use K to denote, either, a general tetrahedron or a $3-rectangle$. In the latter case we suppose, for simplicity, that its edges are parallels to the coordinate axis (see Figure 2 a) and we call h_i as well as h_i^K its diameters in the x_i direction. Also we use $\mathcal{T}_1$ to denote a triangulation made up using $3-rectangles$ of the kind mentioned above, and $\mathcal{T}_2$ for a triangulation made up using tetrahedra whit its principal directions (see Definition 2.2) given by the canonical vectors. We call again h_i, as well as h_i^K, the respective principal lengths (see Figure 1). Let us mention that, for a given $\mathcal{T}_1$, it is possible to obtain a $\mathcal{T}_2$ just splitting adequately each $K \in \mathcal{T}_1$ into tetrahedra. In Figure 2 b we show one way to do

that, dividing a half of a $3-rectangle$ by using 3 tetrahedra, in this case any of the involved tetrahedra verifies $MAC(\pi/2)$. More general meshes of tetrahedra could be handled with the same technique (see Theorem 3.2).
Our goal is to define an average interpolation with uniform error independently of the quotients $\frac{h_i^K}{h_j^K}$, and with a weak local restriction over $\frac{h_i^K}{h_i^{K'}}$ when K and K' are neighbour elements.

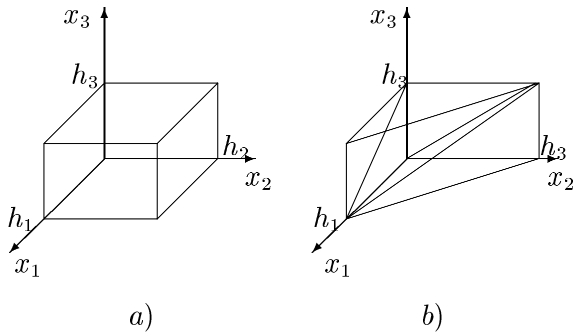


Figure 2

Now, in order to define the average interpolation, let us consider a given $\epsilon(x)$ and an arbitrary $u \in H^2(K + B_{\epsilon_K^M})$.
We write

$$\overline{u} = \rho_{\epsilon(x)} \hat{*} u$$

with $\hat{*}$ as in the preceding subsection, and define

$$P(u) = \Pi(\overline{u})$$

with Π, either, the $\mathcal{P}_1$, or the trilinear, Lagrange interpolation, depending on the nature of K.
The idea behind the definition of the operator P is quite simple. In fact, as the Lagrange interpolation error, for regular functions, has a "good" behavior, even over narrow elements, it seems reasonable to regularize before interpolate.
Indeed, we may write

$$\|\frac{\partial(u - P(u))}{\partial x_j}\|_{0,2,K} \leq \|\frac{\partial(u-\overline{u})}{\partial x_j}\|_{0,2,K} + \|\frac{\partial(\overline{u} - P(u))}{\partial x_j}\|_{0,2,K} \tag{3.42}$$

and from Lema 3.5, we know that

$$\|\frac{\partial(u-\overline{u})}{\partial x_j}\|_{0,2,K} \leq 2^{3/2}\{3\sum_{j=1}^{3} \epsilon_i \|\frac{\partial^2 u}{\partial x_j \partial x_i}\|_{0,2,K+B_{\epsilon_K^M}} + C_{1,\epsilon_j}\epsilon_j \|\frac{\partial u}{\partial x_j}\|_{0,2,K}\} \tag{3.43}$$

on the other hand, for $3-rectangles$, Lagrange interpolation has bounds of the type,

$$\|\frac{\partial(\overline{u} - \Pi(\overline{u}))}{\partial x_j}\|_{0,\infty,K} \leq C_L \sum_{i=1}^{3} h_i \|\frac{\partial^2 \overline{u}}{\partial x_i \partial x_j}\|_{0,\infty,K} \tag{3.44}$$

with h_i (see Figure 2 a) the diameter of K in the coordinates directions $\mathbf{e}_i$, as well as "no directional" bounds for general tetrahedra

$$\|\frac{\partial(\overline{u} - \Pi(\overline{u}))}{\partial x_j}\|_{0,\infty,K} \leq C_L h |\overline{u}|_{2,\infty,K} \tag{3.45}$$

with h the diameter of K. Also, the constant C_L, depends on the maximum angle for tetrahedra [10], and is independent of the shape of K for $3-rectangles$.

Remark 3.6 *Bounds similar to (3.44) hold for tetrahedra with its principal directions parallel to the coordinate axis and so for any $K \in \mathcal{T}_2$. Also it is easy to see that this kind of elements verify $MAC(\pi/2)$ uniformly. However, for general tetrahedra we may use (3.45).*

From (3.44) one gets,

$$\|\frac{\partial(\overline{u}-\Pi(\overline{u}))}{\partial x_j}\|_{0,2,K} \leq |K|^{1/2}\|\frac{\partial(\overline{u}-\Pi(\overline{u}))}{\partial x_j}\|_{0,\infty,K} \leq C_L|K|^{1/2}(\sum_{i=1}^{3} h_i\|\frac{\partial^2\overline{u}}{\partial x_i \partial x_j}\|_{0,\infty,K}) \tag{3.46}$$

and recalling the definition of $\overline{u}$ we obtain, by means of Lemma 3.7 and the last equation

$$\begin{aligned} \|\frac{\partial(\overline{u}-\Pi(\overline{u}))}{\partial x_j}\|_{0,2,K} \quad &\leq \quad C_L C_{0,\rho}(\frac{|K|}{|B_{\epsilon_K^m}|})^{1/2}\{\sum_{i=1}^{3} C_{i,j}h_i\|\frac{\partial^2 u}{\partial x_i \partial x_j}\|_{0,2,K+B_{\epsilon_K^M}} \\ &+ \quad C_{2,\epsilon_i}\epsilon_j h_j\|\frac{\partial u}{\partial x_j}\|_{0,2,K+B_{\epsilon_K^M}}\} \end{aligned} \tag{3.47}$$

Now, from equations (3.43), and (3.47), it is possible to get bounds for $\|\frac{\partial(u-P(u))}{\partial x_j}\|_{0,2,K}$, using (3.42). However, we have to relate the magnitudes ϵ_i and h_i. In order to do that, we need the following hypothesis

Definition 3.2 *Let us consider a triangulation $\mathcal{T}_i$, $1 \leq i \leq 2$, of a polihedral domain Ω, a function $\epsilon(x)$ defined as in the previous subsection, and a positive real number N. We say that $\mathcal{T}$ and $\epsilon(x)$ verifies* **H1** *with a constant N, or shortly* $\mathbf{H1}(N)$*, if and only if, for any $K \in \mathcal{T}$, and any $x \in K$, it hold*

$$\frac{1}{N}\epsilon_i(x_i) \leq h_i \leq N\epsilon_i(x_i) \qquad 1 \leq i \leq 3 \tag{3.48}$$

Remark 3.7 *From (3.48) one easy gets*

$$|K| \leq \frac{N^3}{|B_1|}|B_{\epsilon(x)}|$$

for all $K \in \mathcal{T}_i$, and any $x \in K$. In particular

$$\frac{|K|}{|B_{\epsilon_K^m}|} \leq \frac{N^3}{|B_1|} \tag{3.49}$$

In order to simplify the notation, let us define for $1 \leq i, j \leq 3$

$$\hat{C}_i = (1 + C_{1,\epsilon_i}Nh_i)$$

$$\hat{C}_{i,j} = \hat{C}_i\hat{C}_j$$

note that if (3.48) holds, we have (see (3.29), (3.30)) $C_i \leq \hat{C}_i$ and $C_{i,j} \leq \hat{C}_{i,j}$.
We can now state the following Theorem. We emphasize the dependence of the constants in order to examine further examples.

Theorem 3.1 *Let us consider a triangulation $\mathcal{T}_s$, $s = 1, 2$, and ϵ under* $\mathbf{H1}(N)$*. Let us assume* **H0** *for ϵ, then, for any $K \in \mathcal{T}_s$, we have*

$$\|\frac{\partial u - P(u)}{\partial x_j}\|_{0,2,K} \leq \sum_{i=1}^{3} A_i h_i\|\frac{\partial^2 u}{\partial x_i \partial x_j}\|_{0,2,K+B_{\epsilon_K^M}} + B_j h_j\|\frac{\partial u}{\partial x_j}\|_{0,2,K+B_{\epsilon_K^M}} \tag{3.50}$$

with $A_i = (3N2^{3/2} + C_L C_{0,\rho}\frac{N^{3/2}}{|B_1|^{1/2}}\hat{C}_{i,j})$, $B_j = (N2^{3/2}C_{1,\epsilon_j} + C_L C_{0,\rho}\frac{N^{3/2}}{|B_1|^{1/2}}C_{2,\epsilon_j}Nh_j)$ *and also*

$$\|\frac{\partial u - P(u)}{\partial x_j}\|_{0,2,K} \le D_j\|\frac{\partial u}{\partial x_j}\|_{0,2,K+B_{\epsilon_K^M}} \tag{3.51}$$

with $D_j = 1 + \hat{C}_j(2^{3/2} + C_L C_{0,\rho}\frac{N^{3/2}}{|B_1|^{1/2}})$.

Proof. From (3.43) and (3.47) we get, using the bounds (3.48), (3.49),

$$\|\frac{\partial(u-\overline{u})}{\partial x_j}\|_{0,2,K} \le N2^{3/2}\{3\sum_{i=1}^{3} h_i\|\frac{\partial^2 u}{\partial x_j \partial x_i}\|_{0,2,K+B_{\epsilon_K^M}} + C_{1,\epsilon_j}h_j\|\frac{\partial u}{\partial x_j}\|_{0,2,K}\} \tag{3.52}$$

and

$$\begin{aligned}\|\frac{\partial(\overline{u} - \Pi(\overline{u}))}{\partial x_j}\|_{0,2,K} \quad &\le \quad C_L C_{0,\rho}\frac{N^{3/2}}{|B_1|^{1/2}}\{\sum_{i=1}^{3}\hat{C}_{i,j}h_i\|\frac{\partial^2 u}{\partial x_i \partial x_j}\|_{0,2,K+B_{\epsilon_K^M}} \\ &+ \quad C_{2,\epsilon_j}Nh_j^2\|\frac{\partial u}{\partial x_j}\|_{0,2,K+B_{\epsilon_K^M}}\}\end{aligned} \tag{3.53}$$

adding up (3.52), and (3.53), (3.50) follow, by means of (3.42).
In order to obtain (3.51) we use the same idea bounding the rigth hand side terms of (3.42). This can be done for the first term by means of equation (3.32), and (3.48), getting

$$\|\frac{\partial(u-\overline{u})}{\partial x_j}\|_{0,2,K} \le (1 + \hat{C}_j 2^{3/2})\|\frac{\partial u}{\partial x_j}\|_{0,2,K} \tag{3.54}$$

For the second term we write again, using the Lagrange interpolation estimate,

$$\|\frac{\partial\overline{u} - \Pi(\overline{u})}{\partial x_j}\|_{0,2,K} \le |K|^{1/2}\|\frac{\partial\overline{u} - \Pi(\overline{u})}{\partial x_j}\|_{0,\infty,K} \le C_L|K|^{1/2}\|\frac{\partial\overline{u}}{\partial x_j}\|_{0,\infty,K} \tag{3.55}$$

And now by means of (3.41), (3.48) and (3.49), we obtain

$$\|\frac{\partial\overline{u} - \Pi(\overline{u})}{\partial x_j}\|_{0,2,K} \le C_L C_{0,\rho}\frac{N^{3/2}}{|B_1|^{1/2}}\hat{C}_j\|\frac{\partial u}{\partial x_j}\|_{0,2,K+B_{\epsilon_K^M}} \tag{3.56}$$

and (3.51) follows from (3.54), (3.56) and (3.42). $\square$

In the following Remarks we examine the scope of the preceding result.

Remark 3.8 *When one looks for "global" estimates, the following terms have to be bounded*

$$\sum_{K\in\mathcal{T}}|u|_{2,2,K+B_{\epsilon_K^M}} \qquad \text{and} \qquad \sum_{K\in\mathcal{T}}|u|_{1,2,K+B_{\epsilon_K^M}}. \tag{3.57}$$

Then $K + B_{\epsilon_K^M}$ *should not intersect a "big" number of elements. From* **H0** *and (3.48), we can easily see that this number can be bounded in terms of* N *(independently of* K*).*

Remark 3.9 *From Theorem 3.1 we easily get uniform error estimates for meshes which are quasi-uniform in each direction. In fact, for a given triangulation* $\mathcal{T}_l$, $l = 1, 2$, *let us call* $s_j := sup_{K,K'\in\mathcal{T}_l}\frac{h_j^K}{h_j^{K'}}$, *for* $1 \le j \le 3$, *then, for any fixed* K, *the choice* $\epsilon_j(x) = h_j^K = constant$, *gives* $C_{1,\epsilon_j} = C_{2,\epsilon_j} \equiv 0$, *and taking*

$N = max\{s_j\}_{1\leq j\leq 3} \geq 1$, we get $B_j \equiv 0$ and $A_i \leq (max\{s_j\}_{1\leq j\leq 3})^{3/2}(9 + \frac{C_L C_{0,\rho}}{|B_1|})$ and by means of (3.50) one gets

$$\|\frac{\partial u - P(u)}{\partial x_j}\|_{0,2,K} \leq (max\{s_j\}_{1\leq j\leq 3})^{3/2}(9 + \frac{C_L C_{0,\rho}}{|B_1|})\{\sum_{i=1}^{3} h_i \|\frac{\partial^2 u}{\partial x_i \partial x_j}\|_{0,2,K+B_{\epsilon_K^M}}\} \tag{3.58}$$

which results uniform whenever s_i remains bounded, and without any restriction over $\frac{h_i^K}{h_j^K}$.

Remark 3.10 *The result shown in the last remark is similar to that obtained in [8]. However, our technique essentially replace the restrictions required by the boundedness of the numbers s_j, $1 \leq j \leq 3$, by the local ones*

$$\mathbf{H1} \qquad \textit{and} \qquad C_{1,\epsilon_i}, C_{2,\epsilon_i} \leq C \tag{3.59}$$

allowing the use of several non uniform meshes. Indeed, let us consider, for example, a domain $\Omega \subset \mathbb{R}^3$ such that $0 \leq x_1 \leq 10$ whenever $x = (x_1, x_2, x_3) \in \Omega$ and the "non uniform" mesh $\mathcal{T}_l$ made up in such a way that $K \in \mathcal{T}_l$ and $x \in K$ implies $h_K^1 \sim (\frac{1}{2})^{x_1}$. Defining $\epsilon_1(x_1) := (\frac{1}{2})^{x_1}$ we find $C_{1,\epsilon_1} = |ln(\frac{1}{2})|$ and $C_{2,\epsilon_1} = ln(\frac{1}{2})^2$, showing that the estimate (3.50) does not deteriorates, however $max_{K,K'\in\mathcal{T}} \frac{h_K^1}{h_{K'}^1} \sim 2^{10}$. Another interesting remark, is that the constant D_j in the estimate (3.51) remains bounded under the weaker assumption

$$\mathbf{H1} \qquad \textit{and} \qquad |\epsilon_j'| \leq C$$

allowing uniform bounds for more general meshes.

Remark 3.11 *Let us note that (3.59) implies* **H0** *for practical purposes. In fact, as $h_i \to 0$ one gets*

$$|\epsilon_i'(x_i)| \leq C\epsilon_i(x_i) \leq CNh_i << \frac{1}{2}$$

The argument shown for $\mathcal{T}_2$ applies also for more general meshes of tetrahedra, just changing the estimates of the Lagrange interpolation, and taking care of certain aspects which relates the geometry of the ellipsoids defined in the preceding subsection with the geometry of the elements.
For example, for a given triangulation $\mathcal{T}$ we could not require the same principal directions for every $K \in \mathcal{T}$ nor the orthogonality between t_i and t_j. In the latter case we have to use (3.45) instead of (3.44) for the Lagrange interpolation error. On the other hand, for general meshes, hypothesis $H1(N)$ does not relate any more the shape of K and B_ϵ, therefore we restrict ourselves to the meshes defined in the following

Definition 3.3 *We say that a triangulation $\mathcal{T}$ made of tetrahedra is a perturbation of $\mathcal{T}_2$, and we note it by $\mathcal{T}_p$ if and only if for any $K \in \mathcal{T}$ and any coordinate axis x_j, $1 \leq j \leq 3$, there exist a unique principal direction, say $t_j(K)$ (renumbering if is needed), such that the angle between them is less than or equal to $\pi/4$. For any $K \in \mathcal{T}_p$ we call again h_i as well as h_i^K the respective lenghts of the edge associated with $t_i(K)$, moreover we say that $\mathcal{T}_p$ verifies* **H1**(N) *whenever (3.48) holds.*

And now a similar result to that given in Theorem 3.1 can be proven. We just state it without proof.

Theorem 3.2 *Let us consider a triangulation $\mathcal{T}_p$, and ϵ under* **H1**(N). *Let us assume* **H0** *for ϵ, then, for any $K \in \mathcal{T}_p$, we have*

$$|u - P(u)|_{1,2,K} \leq Ah(|u|_{2,2,K+B_{\epsilon_K^M}} + |u|_{1,2,K+B_{\epsilon_K^M}}) \tag{3.60}$$

with $A = A(N, C_L(\overline{\psi}), C_{0,\rho}, C_{1,\epsilon_j}, C_{2,\epsilon_j})$ and also

$$|u - P(u)|_{1,2,K} \leq C|u|_{1,2,K+B_{\epsilon_K^M}} \tag{3.61}$$

with $C = C(N, C_L(\overline{\psi}), C_{0,\rho}, C_{1,\epsilon_j})$.

Remark 3.12 *One is tempted to replace* **H1** *by the weaker couple*

$$\frac{1}{N}\epsilon_K^M \leq h_K \leq N\epsilon_K^M \qquad \textit{and} \qquad \frac{|K|}{|B_{\epsilon_K^m}|} \leq \frac{N^3}{|B_1|}. \tag{3.62}$$

Indeed, these are the unique bounds we need in order to obtain the result given in the last Theorem. However, under this assumption, the result may have not a finite element value, since terms like (3.57) could not be properly bounded due to the fact that $K + B_{\epsilon_K^M}$ *may intersect an increasing number of neighboring elements when anisotropic elements are allowed.*

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