Singularities of logarithmic foliations

Fernando Cukierman [1], Marcio G. Soares [2] and Israel Vainsencher [3]

ABSTRACT

A logarithmic 1-form on $\mathbb{CP}^n$ can be written as

$$\omega = \left(\prod_0^m F_j\right) \sum_0^m \lambda_i \frac{dF_i}{F_i} = \lambda_0 \widehat{F}_0\, dF_0 + \cdots + \lambda_m \widehat{F}_m\, dF_m$$

with $\widehat{F}_i = \left(\prod_0^m F_j\right)/F_i$ for some homogeneous polynomials F_i of degree d_i and constants $\lambda_i \in \mathbb{C}^\star$ such that $\sum \lambda_i d_i = 0$. For general F_i, λ_i, the singularities of ω consist of a schematic union of the codimension 2 subvarieties $F_i = F_j = 0$ together with, possibly, finitely many isolated points. This is the case when all F_i's are smooth and in general position. In this situation, we give a formula which prescribes the number of isolated singularities.

1. INTRODUCTION

The search for numerical invariants attached to algebraic foliations goes back to Poincaré [13]. He was interested in determining bounds for the degree of curves left invariant by a polynomial vector field on $\mathbb{C}^2$.

Recent work treat the question by establishing relations for the number of singularities of the foliation and certain Chern numbers and then using positivity of certain bundles. For a survey of recent results, see [4], [7], [10], [14].

A foliation of dimension r on a smooth variety X of dimension n is a coherent subsheaf $\mathcal{F}$ of the tangent sheaf TX of generic rank r, locally split in codimension ≥ 2.

If $r = n-1$ (codimension one foliations), the foliation corresponds to a global section of $\Omega^1_X \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$.

Suppose $X = \mathbb{CP}^n$, with homogeneous coordinates $x_0, \ldots, x_n$. Recall Euler's sequence,

$$\Omega^1_{\mathbb{CP}^n}(1) \to \mathcal{O}^{\oplus n+1} \to \mathcal{O}(1).$$

A global section ω of

$$\Omega^1_{\mathbb{CP}^n}(d) \subset \mathcal{O}^{\oplus n+1}(d-1)$$

can be written as

$$\omega = \textstyle\sum_0^n F_i dx_i$$

where F_i is a homogeneous polynomial of degree $d-1$, subject to the condition

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$$\sum F_i x_i = 0$$

(contraction by the radial vector field on $\mathbb{C}^{n+1}$).

The degree of a codimension one foliation $\mathcal{F}$, $\deg \mathcal{F}$, is the number of tangencies of the leaves of $\mathcal{F}$ with a generic one-dimensional linear subspace of $\mathbb{CP}^n$. A simple calculation shows that $\deg \mathcal{F} = d-2$ if the 1-form defining $\mathcal{F}$ has components F_i of degree $d-1$. The form ω is integrable if $\omega \wedge d\omega = 0$.

Integrable 1–forms make up a Zariski closed subset of $\mathbb{P}\,(H^0(\Omega^1(d))$. We denote by $Fol(\mathbb{CP}^n; d)$ the space of codimension one integrable holomorphic foliations of degree $d-2$ of $\mathbb{CP}^n$.

Not much is known about the dimensions nor the number of irreducible components of $Fol(\mathbb{CP}^n; d)$ (but see [8] and [9]).

When ω can be written as

$$\omega = \prod_0^m F_j \sum_0^m \lambda_i \frac{dF_i}{F_i} = \lambda_0 \widehat{F}_0 dF_0 + \cdots + \lambda_m \widehat{F}_m dF_m$$

for some homogeneous polynomials F_i of degree d_i and $\lambda_i \in \mathbb{C}^\star$ such that $\sum \lambda_i d_i = 0$, we say ω is logarithmic of type $\underline{d} = d_0, \ldots, d_m$. Given positive integers $d_0, \ldots, d_m$, set $d = \sum_{i=0}^m d_i$ and consider the hyperplane

$$\mathbb{CP}(m-1, \underline{d}) = \{(\lambda_0, \ldots, \lambda_m) \in \mathbb{CP}^m \mid \Sigma\ d_i \lambda_i = 0\}.$$

Define a rational map Ψ by

$$\begin{array}{ccc} \mathbb{CP}(m-1,\underline{d}) \times \prod\limits_{i=0}^{m} \mathbb{P}\,(H^0(\mathbb{CP}^n, \mathcal{O}(d_i))) & \stackrel{\Psi}{\longrightarrow} & Fol(\mathbb{CP}^n; d) \\ ((\lambda_0, \ldots, \lambda_m), (F_0, \ldots, F_m)) & \longmapsto & (\prod\limits_{j=0}^{m} F_j) \sum\limits_{i=0}^{m} \lambda_i \dfrac{dF_i}{F_i} \end{array}$$

The closure of the image of Ψ is the set $Log_n(\underline{d})$ of logarithmic foliations of type $\underline{d}$ (of degree $d-2$) of $\mathbb{CP}^n$. Recall the following result.

Theorem. (Calvo-Andrade [5]) *For fixed d_i and $n \geq 3$, logarithmic foliations form an irreducible component of the space of codimension one integrable holomorphic foliations of $\mathbb{CP}^n$ of degree $d-2$* (with $d = \sum d_i$).

The singular scheme of the foliation defined by $\omega \in H^0(\Omega^1(d))$ is the scheme of zeros of ω. This is the closed subscheme with ideal sheaf given by the image of the co-section $\omega^\vee : (\Omega^1(d))^\vee \to \mathcal{O}$.

For ω general in $H^0(\Omega^1(d))$, there are just finitely many singularities, to wit (cf. Jouanolou, [12, p. 87, Th. 2.3], setting in his notation, $m = d-1$, $r = n$) ,

$$\int_{\mathbb{CP}^n} c_n\left(\Omega^1(d)\right) = \sum_0^n (-1)^i \binom{n+1}{i} d^{n-i}.$$

On the other hand of course, a *general* ω is *not* integrable.

Theorem. (Jouanolou [12]) *For integrable ω, the singular set must contain a codimension 2 component.*

It is easy to see that, for logarithmic (hence integrable) forms

$$\omega = \lambda_0 \widehat{F}_0 \, dF_0 + \cdots + \lambda_m \widehat{F}_m \, dF_m$$

the singular set contains the union of all codimension 2 subsets

$$F_i = F_j = 0, \; i \neq j.$$

It is worth mentioning that Jouanolou describes examples of integrable 1-forms with singular schemes containing positive dimensional components of "wrong" positive dimension. We found no hint as to the existence of isolated singularities for general enough foliations.

Let D_i be the divisor associated to F_i. We assume the following genericity conditions to hold:

$$(1) \qquad \begin{cases} \text{the } \; D_i\text{'s}, i = 0, \ldots, m, \; \text{are smooth and in general position.} \\ \lambda_i \neq 0, \;\; i = 0, \ldots, m. \end{cases}$$

Remark that (1) defines a Zariski open subset of

$$\mathbb{CP}(m-1, \underline{d}) \times \prod_{i=0}^{m} \mathbb{P}\left(H^0(\mathbb{CP}^n, \mathcal{O}(d_i))\right)$$

Before stating our main result recall that the complete symmetric function σ_ℓ, of degree ℓ in the variables $X_1, \ldots, X_k$ is defined by: $\sigma_0 = 1$ and, for $\ell \geq 1$,

$$\sigma_\ell(X_1, \ldots, X_k) = \sum_{i_1 + \cdots + i_k = \ell} X_1^{i_1} \ldots X_k^{i_k}.$$

We then have

Theorem. *Let $\mathcal{F}$ be a logarithmic foliation on $\mathbb{CP}^n$ of type $\underline{d} = d_0, \ldots, d_m$, given by*

$$\omega = \lambda_0 \widehat{F}_0 \, dF_0 \; + \cdots + \; \lambda_m \widehat{F}_m \, dF_m$$

and satisfying (1)*. Then the singular scheme $S(\mathcal{F})$ of $\mathcal{F}$ can be written as a disjoint union*

$$S(\mathcal{F}) = Z \cup R$$

where

$$Z = \bigcup_{i<j} D_i \cap D_j$$

and R is finite, consisting of

$$N(n, \underline{d}) = \sum_{i=0}^{n} (-1)^i \binom{n+1}{i} \sigma_{n-i}(\underline{d})$$

points counted with natural multiplicities. Moreover,

(1) $N(n,\underline{d}) = 0$ *if* $n \geq m$ *and* $d_i = 1$ *for all* i.
(2) $N(n,\underline{d}) = \binom{m}{n+1}$ *if* $n < m$ *and* $d_i = 1$ *for all* i.
(3) $N(n,\underline{d}) > 0$ *whenever* $d_i \geq 2$ *for some* i.

It will be shown below, see formula (8) in 4.3, that

$$N(n,\underline{d}) = \text{ the coefficient of } h^n \text{ in } \frac{(1-h)^{n+1}}{\Pi_0^m(1-d_ih)}$$

from which we deduce:

1.0.1. *Example.* If $d_i = 1$ for all i then $\dfrac{(1-h)^{n+1}}{\Pi_0^m(1-d_ih)}$ reduces to $\dfrac{(1-h)^n}{(1-h)^m}$ and we have items (1) and (2) of theorem:

(1) $n \geq m$. In this case $\dfrac{(1-h)^n}{(1-h)^m}$ is a polynomial of degree $n-m < n$ and hence the coefficient of h^n vanishes, so that there are no isolated zeros.
(2) $n < m$. In this case $\dfrac{(1-h)^n}{(1-h)^m}$ reads $\dfrac{1}{(1-h)^{m-n}}$ and it's easily seen that the coefficient of h^n is $\binom{m}{n+1}$.

2. Proof of the theorem

We will show that, if a point is non isolated in $S(\mathcal{F})$, then it lies in $D_i \cap D_j$ for some $i < j$. Indeed, let C be an irreducible component of $S(\mathcal{F})$ of dimension $1 \leq \dim C \leq n-2$. By ampleness and general position, we may pick a point $p \in C$ lying in the intersection of precisely k of the divisors D_i, $1 \leq k \leq \min\{n, m+1\}$. Let f_i be a local equation for D_i at p. Near p, the foliation $\mathcal{F}$ is given by the 1–form

$$\varpi = \widehat{f} \sum_{i=0}^{m} \lambda_i \frac{df_i}{f_i}.$$

Renumbering the indices we may assume $p \in D_0 \cap \cdots \cap D_{k-1}$. The local defining equations $f_i = 0$ of the D_i's, for $i = 0, \ldots, k-1$, are part of a regular system of parameters, *i.e.*, $df_0, \ldots, df_{k-1}$ are linearly independent at p. Write $\widetilde{g} = f_k \cdots f_m$. Since $p \notin D_j$, $k \leq j \leq m$, we may assume $\tilde{g}$ vanishes nowhere around p and write ϖ as

$$\varpi = f_0 \cdots f_{k-1}\, \tilde{g} \left[\sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + \sum_{i=k}^{m} \lambda_i \frac{df_i}{f_i} \right] = f_0 \cdots f_{k-1}\, \tilde{g} \left[\sum_{j=0}^{k-1} \lambda_j \frac{df_j}{f_j} + \eta \right],$$

where $\eta = \sum_{i=k}^{m} \lambda_i \dfrac{df_i}{f_i}$ is a holomorphic closed form near p. Since η is closed, by the formal Poincaré lemma it is exact near p, say $\eta = d\xi$. Set $\vartheta = \varpi/\widetilde{g}$. Then $\mathcal{F}$ is defined around p by

$$\vartheta = f_0 \cdots f_{k-1}\left[\sum_{j=0}^{k-1}\lambda_j \frac{df_j}{f_j} + d\xi\right] = f_0 \cdots f_{k-1}\left[\lambda_0 \frac{d(\exp[\xi/\lambda_0]f_0)}{\exp[\xi/\lambda_0]f_0} + \sum_{j=1}^{k-1}\lambda_j \frac{df_j}{f_j}\right].$$

Set $z_0 = \exp[\xi/\lambda_0]f_0$ and $z_1 = f_1, \ldots, z_{k-1} = f_{k-1}$. Since $u = \exp[\xi/\lambda_0]$ is a unit, we have that also $z_0, \ldots, z_{k-1}$ are part of a regular system of parameters at p. Now ϑ can be written as

$$\vartheta = \frac{z_0}{u}\, z_1 \cdots z_{k-1}\left[\lambda_0 \frac{dz_0}{z_0} + \sum_{j=1}^{k-1}\lambda_j \frac{dz_j}{z_j}\right].$$

Thus $\mathcal{F}$ is defined around p by the 1–form

$$(2) \qquad \widetilde{\vartheta} = z_0\, z_1 \cdots z_{k-1}\left[\lambda_0 \frac{dz_0}{z_0} + \sum_{j=1}^{k-1}\lambda_j \frac{dz_j}{z_j}\right] = \sum_{j=0}^{k-1}\lambda_j\, z_0 \cdots \widehat{z_j} \cdots z_{k-1}\, dz_j.$$

If $k = 1$, (2) shows that the foliation is defined near p by dz_0 and then is non-singular at p. Hence we necessarily have $k \geq 2$. Note that the ideal of the scheme of zeros of $\widetilde{\vartheta}$ (as well as of ω) near p is generated by the k monomials $z_0 \cdots \widehat{z_j} \cdots z_{k-1}$ with $0 \leq j \leq k-1$. That is just the scheme union $\cup_{i,j} D_i \cap D_j$, for $0 \leq i < j \leq k-1$. Thus C must be contained in $D_i \cap D_j$, for some $i < j$, and therefore C is an irreducible component of $D_i \cap D_j$ and $\dim C = n - 2$. □

The formula for the finite part is proved in the next section in a slightly more general context.

2.1. **Remark.** The argument above shows that the codimension two part, $Z = \bigcup D_{ij}$, of the singular scheme of a general logarithmic foliation is equal to the singular scheme of the normal crossing divisor $\bigcup D_i$. This will enable us to use Aluffi's formula for the Segre class. We also note that, since D_{ij} is smooth and connected, the component C is actually equal to some D_{ij}.

3. FORMULAS

Let $\mathcal{E} \to X$ be a holomorphic vector bundle of rank n over a complex projective smooth variety of dimension n. Let $s : X \to \mathcal{E}$ be a section. Assume
(1) the scheme of zeros W of s is a disjoint union

$$W = Z \cup R$$

with R finite;
(2) there are Cartier divisors $D_0, \ldots, D_m$, $m \geq 1$, such that

$$Z = \bigcup_{i<j} D_{ij}$$

as schemes, where

$$D_{ij} = D_i \cap D_j;$$

(3) for all choices of indices

$$I_r = (0 \le i_1 < \cdots < i_r \le m)\,,$$

the intersection $D_{I_r} = \bigcap_{i \in I_r} D_i$ is transversal.

We are mainly interested in the case where $X = \mathbb{CP}^n$ and the section s is a logarithmic form as in the Theorem in p. 3.

We give an expression for the number of points in R, counted with natural multiplicities, in terms of the intersection numbers

$$D^J \cdot c_j(\mathcal{E})$$

with

$$J = (j_0, \ldots, j_m),\ D^J = D_0^{j_0} \cdots D_m^{j_m},\ |J| + j = n.$$

When $Z = \bigcup_{i<j} D_{ij}$ is a disjoint union, the formula is but a simple direct application of usual excess intersection techniques as reviewed below.

Disjointness implies that Z is a local complete intersection with explicitly known normal bundle.

The ideal of W is the image $\mathcal{I}(W)$ of the co-section

$$s^\vee : \mathcal{E}^\vee \to \mathcal{O}.$$

It can be written as

$$\mathcal{I}(W) = \mathcal{I}(Z) \cdot \mathcal{I}(R).$$

Locally, it is of the form $\mathcal{I} = \langle z_0, z_1 \rangle \cdot \mathfrak{m}$, where z_0, z_1 are equations for the pair of transversal divisors cutting Z, and $\mathfrak{m}$ denotes an ideal of finite co-length corresponding to the finite part $R \subset W$. (Note that $\mathfrak{m} = \langle 1 \rangle$ if R is disjoint from the present coordinate chart.)

Let $\pi : X' \to X$ be the blowup along Z. Put $E' = \pi^{-1}(Z)$, the exceptional divisor. The pullback $\pi^\star s^\vee$ of the co-section maps $\pi^\star \mathcal{E}^\vee$ onto

$$\mathcal{O}(-E') \cdot \mathcal{I}(R').$$

($R' = \pi^{-1} R$). We get an induced map of sheaves

$$(s')^\vee : \pi^\star \mathcal{E}^\vee \otimes \mathcal{O}(E') \twoheadrightarrow \mathcal{I}(R') \subseteq \mathcal{O}.$$

Dualizing, we find a section s' of

$$\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}(-E') \tag{3}$$

whose scheme of zeros is precisely $R' \simeq R$, the finite part.

Indeed, since R is disjoint from the blowup center, $\pi : X' \to X$ is an isomorphism in a neighborhood of R'. Hence, the length of $\mathcal{O}_{X'}/\mathcal{I}(R')$ is the same as for R. This implies the formula for the degree of the zero cycle,

$$\deg\,[R] = \deg\,[R'] = \int_{X'} c_n(\mathcal{E}'). \tag{4}$$

To compute it explicitly, recall that the exceptional divisor E' is the projective bundle $\mathbb{P}\left(\mathcal{N}_{Z/X}\right)$ of the normal bundle of Z in X. The restriction of $\mathcal{N}_{Z/X}$ to each D_{ij} is the restriction of $\mathcal{O}(D_i) \oplus \mathcal{O}(D_j)$. Let $\iota : E' \hookrightarrow X'$ be the inclusion. We recall from [11, B.6, p. 435] a couple of facts that follow from the construction of the blowup as $\mathrm{Proj}(\oplus \mathcal{I}^k)$ of the Rees algebra of the ideal sheaf $\mathcal{I} = \mathcal{I}(Z)$. The natural relatively ample line bundle $\mathcal{O}_{X'}(1)$ is presently the image of $\pi^\star \mathcal{I} \to \pi^\star \mathcal{O}_X = \mathcal{O}_{X'}$, thus it is equal to the exceptional ideal sheaf $\mathcal{O}_{X'}(-E')$. The exceptional divisor $E' \subset X'$ is identified to the projectivization of the normal cone, $\mathrm{Proj}(\oplus \mathcal{I}^k/\mathcal{I}^{k+1})$. Accordingly, we have the identification $\iota^\star \pi^\star \mathcal{I} = \mathcal{I}/\mathcal{I}^2 \to \iota^\star \mathcal{O}_{X'}(1)$. The latter is but the hyperplane bundle $\mathcal{O}_{E'}(1)$ of the $\mathbb{CP}^1$–bundle $E' = \mathbb{P}\left(\mathcal{N}_{Z/X}\right) \to Z$. We may compute the self-intersection (cf. [11, 2.6, p. 44]),

$$
\begin{aligned}
(E')^2 &= \iota_\star(\iota^\star E') = \iota_\star(\iota^\star c_1(\mathcal{O}_{X'}(E')) \cap [X']) \\
&= \iota_\star(\iota^\star c_1(\mathcal{O}_{X'}(-1)) \cap [X']) \\
&= -\iota_\star(\xi \cap [E'])
\end{aligned}
$$

with

$$\xi = c_1(\mathcal{O}_{E'}(1)).$$

Recall that the push-forward of powers of the hyperplane class ξ of the $\mathbb{CP}^1$–bundle $E' = \mathbb{P}\left(\mathcal{N}_{Z/X}\right) \to Z$ are expressed (cf. [11, p. 47]) by Segre classes:

$$\pi_\star(\xi^{j+1}) = s_j(\mathcal{N}_{Z/X}) \ \forall\, j \in \mathbb{Z}.$$

Writing $[D_{ij}]$ for the cycle class of $D_i \cap D_j$ in the Chow (or homology) group $A_\star X$, we have, for $r \geq 0$,

$$(E')^{r+1} = \iota_\star\left(\iota^\star (E')^r\right) = \iota_\star((-\xi)^r \cap [E']).$$

We may write

$$
\begin{aligned}
\pi_\star\left((E')^{r+1}\right) &= \pi_\star \iota_\star((-\xi)^r \cap [E']) \\
&= (-1)^r \sum_{i<j} s_{r-1}\left(\mathcal{O}(D_i) \oplus \mathcal{O}(D_j)\right) \cap [D_{ij}]
\end{aligned}
$$

in the group $A_m X$ of cycles of dimension $m = n - 2 - k$.

Put

$$
\begin{aligned}
s_{kij} &= s_k\left(\mathcal{O}(D_i) \oplus \mathcal{O}(D_j)\right) \cap [D_{ij}] \\
&= (-1)^k D_i \cdot D_j \cdot \sum_{u=0}^{k} D_i^u D_j^{k-u}.
\end{aligned}
$$

Since $s_j = 0$ for $j < 0$, we also have

$$\pi_\star((E')) = 0.$$

It follows from (4) and (3) that

$$\begin{aligned}
\deg\,[R] &= \int_X \pi_\star c_n(\mathcal{E}') \\
&= \int_X \sum_{r=0}^{n} c_{n-r}(\mathcal{E}) \cdot \pi_\star((-E')^r) \\
&= \int_X c_n(\mathcal{E}) + \sum_{r=1}^{n-1} (-1)^{r+1} c_{n-1-r}(\mathcal{E}) \cdot \pi_\star((E')^{r+1}) \\
&= \int_X c_n(\mathcal{E}) \; - \; \sum_{r=1}^{n-1}\sum_{i<j} c_{n-1-r}(\mathcal{E}) \cdot s_{(r-1)ij} \\
&= \int_X c_n(\mathcal{E}) - \sum_{r=1}^{n-1} (-1)^{r-1} c_{n-1-r}(\mathcal{E}) \sum_{i<j}\sum_{u=0}^{r-1} D_i^{u+1} D_j^{r-u}\,.
\end{aligned}$$

The idea now is to reduce the general case to the above situation. This will be done by a sequence of blowups along smooth centers with known normal bundles.

We explain how the reduction works, say in the case when all 4-fold intersections are empty, for the sake of simplicity. The general case is entirely similar. Thus assume

$$\forall\; I_4 = (0 \le i_0 < i_2 < i_3 < i_4 \le m)\,,$$

we have

$$D_{I_4} := \textstyle\bigcap_{i \in I_4} D_i = \emptyset.$$

(This is the case if, for instance, $\dim X = 3$.) It follows that for all choices of triple indices,

$$I_3 = (i < j < k) \ne I_3' = (i' < j' < k'),$$

we must have

$$D_{I_3} \cap D_{I_3'} = \emptyset.$$

Now the union T of all triple intersections D_{I_3} is smooth.

Let $\pi : X' \to X$ be the blowup along T. The strict transform D'_{ij} is equal to the blowup of D_{ij} along the disjoint union of Cartier divisors D_{ijk}, hence $D'_{ij} \simeq D_{ij}$ holds. Moreover, since $D_{ij} \cap D_{jk}$ is a union of connected components of the blowup center, it follows that $D'_{ij} \cap D'_{jk} = \emptyset$. We also have that the D'_i meet transversally.

Look at the pullback $\pi^{-1}W$ of the zero scheme of the section s. We will take coordinates on X in a neighborhood of a point $0 \in D_{123}$, say. Near 0, W is equal to the union $D_{12} \cup D_{13} \cup D_{23}$. Let $z_i = 0$ be a local equation of D_i. Then the ideal of W near 0 is equal to the intersection

$$\langle z_1, z_2 \rangle \cap \langle z_1, z_3 \rangle \cap \langle z_2, z_3 \rangle = \langle z_1 z_2, z_1 z_3, z_2 z_3 \rangle\,.$$

The blowup center, T, is locally given by $\langle z_1, z_2, z_3 \rangle$. The restriction of X' over the present affine neighborhood of the point 0 is covered by three affine open subsets, one for each choice of z_i as a generator of the exceptional ideal $\mathcal{O}(-E')$.

Say we take z_1 as a local generator. We may write $z_i = z_1 z_i'$, $i = 1, 2$. Here z_i' is a local equation of the strict transform of D_i.

The pullback of W is given by the ideal

$$\mathcal{I}(\pi^{-1}W) = z_1^2 \langle z_2', z_3', z_2' z_3' \rangle = z_1^2 \langle z_2', z_3' \rangle .$$

This is twice the exceptional ideal, times the ideal of the strict transform of D_{23}.

Note that the strict transforms of D_{13} and of D_{12} are empty in the present neighborhood of X'. Thus the D_{ij}' are presently disjoint.

The local expression shows that the image $\mathcal{I}(W)\mathcal{O}_{X'}$ of the co-section

$$\pi^\star s^\vee : \mathcal{E}^\vee \to \mathcal{O}_{X'}$$

is of the form

$$\mathcal{I}(W)\mathcal{O}_{X'} = \mathcal{O}(-2E') \cdot \mathcal{I}(Z') \cdot \mathcal{I}(R'),$$

where the finite piece $R' = \pi^{-1}(R) \simeq R$ and $Z' = \cup D_{ij}'$ is the disjoint union of pairwise transversal intersections of Cartier divisors D_i'.

Hence, we may apply the previous case to the section $s' = s \otimes \mathcal{O}(-2E')$ of $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}(-2E')$. We find

$$\begin{aligned} \deg[R] &= \deg[R'] \\ &= \int_{X'} c_n(\mathcal{E}') - \sum_{r=0}^{n-1} (-1)^{r-1} c_{n-1-r}(\mathcal{E}') \sum_{i<j} \sum_{u=0}^{r-1} (D_i')^{u+1} \cdot (D_j')^{r-u}. \end{aligned} \tag{5}$$

Let E_i' denote the sum of the (disjoint) exceptional divisors over all D_{I_3} with $i \in I_3$. Using the formulas $D_i' = \pi^\star D_i - E_i'$ and universal formulas for $c(\mathcal{E} \otimes \mathcal{O}(-2E'))$ and applying $\pi_\star$, the above expression can be written in terms of the intersection numbers $D^J \cdot c_j(\mathcal{E})$.

In general, let r be the smallest integer such that for all possible choices of indices

$$I_{r+2} = (0 \le i_0 < i_1 < \cdots < i_{r+1} \le m),$$

we have

$$D_{I_{r+2}} := \bigcap_{i \in I_{r+2}} D_i = \emptyset.$$

If $m \ge 2$, we have $r \le \min(n-1, m-1)$ because $\dim X = n$ and the divisors are in general position. Of course if $r \ge m$ no I_{r+2} exists! If $m = 1$, set $r = 1$.

We then have that the union

$$Z_{r+1} = \bigcup_{I_{r+1}} D_{I_{r+1}}$$

of all $(r+1)$-fold intersections among D_i's is smooth. Let $\pi^1 : X^1 \to X$ be the blowup along Z_{r+1}. A local analysis as performed above shows that the strict transforms D_i^1 are in general position and the intersections $D_{I_{r+1}}^1$ are empty. Moreover, there is a section s^1 of $\mathcal{E}^1 = \mathcal{E} \otimes \mathcal{O}(-rE^1)$ with zeroes scheme W^1 equal to the disjoint union $Z^1 \cup R^1$, with $R^1 = (\pi^1)^{-1}(R) \simeq R$. Here Z^1 is the scheme union of the pairwise intersections D^1_{ij}. Continuing this way, we construct a sequence of blowups,

$$X^r \xrightarrow{\pi^r} \cdots \xrightarrow{\pi^2} X^1 \xrightarrow{\pi^1} X$$

such that ultimately the bundle

$$\mathcal{E}^r = \mathcal{E} \otimes \mathcal{O}(-rE^1 - (r-1)E^2 - \cdots - E^r)$$

is endowed with a section s^r whose scheme of zeros is exactly

$$R^r = (\pi^r)^{-1} \cdots (\pi^2)^{-1}(\pi^1)^{-1}(R) \simeq R.$$

Thus, we get the formula

$$\deg(R) = \int_X \pi^1_\star \cdots \pi^r_\star\left(c_n(\mathcal{E}^r)\right).$$

The right hand side may clearly be written in terms of the intersection numbers $D^J \cdot c_j(\mathcal{E})$.

4. Examples

Set for short $c_i = c_i\mathcal{E}$. Let

$$\sigma_i = \sigma_i(\underline{D}) = \sum_{i_0+\cdots+i_m=i} D_0^{i_0} \cdots D_m^{i_m}$$

denote the sum of all monomials of degree i in the classes of the D_i.

4.1. $m = 1$. We find

$n = 3:$ $\quad \deg(R) = c_3 - D_0D_1c_1 + D_0^2D_1 + D_0D_1^2.$

$n = 4:$ $\quad \deg(R) = c_4 - D_0D_1c_2 + (D_0^2D_1 + D_0D_1^2)\,c_1 - (D_0^3D_1 + D_0^2D_1^2 + D_0D_1^3).$

These first few cases suggest the formula for general n, still with $m = 1$,

$$\deg(R) = c_n - \sum_1^{n-2}(-1)^{n-i}\sigma_{n-i}(\underline{D})c_i - (-1)^n\sigma_n(\underline{D}).$$

which will be generalized in the sequel.

4.2. **Aluffi's formula.** This was explained to us by P. Aluffi. In fact, nearly closed formula can be achieved using Fulton's residual intersection formula (RIF) [11, 9.2.3, p. 163], instead of the above blowup sequence. It requires the knowledge of the Segre class of the excess locus $Z = \bigcup D_{ij}$. This is rendered feasible thanks to Aluffi's formula for the Segre class of the singular scheme of a normal crossing divisor $D = \sum D_i$, (cf. [2], proof of Lemma II.2). The formula reads

$$s(Z, X) = \left(\left(1 - \frac{1 - D}{\prod_0^m (1 - D_i)}\right) \cap [X]\right) \otimes_X \mathcal{O}(D).$$

The right hand side uses Aluffi's $\cdot \otimes L$ operation on the Chow group introduced in [1]: if a_i is a class of codimension i in the Chow group, and L is a line bundle, then

$$a_i \otimes L = \frac{a_i}{c(L)^i}.$$

We have

$$s(Z, X) \ = [X] - \left(\left(\frac{1 - D}{\prod_0^m (1 - D_i)}\right) \cap [X]\right) \otimes_X \mathcal{O}(D) \tag{6}$$

The operation $\cdot \otimes L$ behaves well with respect to Chern classes of 'rank 0 bundles'(!). That is: if E, F are bundles of the same rank, then

$$((c(E)/c(F)) \cap a) \otimes L = (c(E \otimes L)/c(F \otimes L)) \cap (a \otimes L).$$

We have to pretend that the fraction in (6) is the quotient of the Chern classes of two bundles of the same rank, so regard the second piece as

$$\left(\frac{(1 - D) \cdot 1^m}{\prod_0^m (1 - D_i)} \cap [X]\right) \otimes_X O(D)$$

that is, view the numerator as the Chern class of the bundle $\mathcal{O}(-D) \oplus \mathcal{O}^{\oplus m}$. Tensoring by $\mathcal{O}(D)$, the numerator turns from

$$(1 - D) \cdot 1^m, \ \text{ into } \ (1 - D + D)(1 + D)^m = (1 + D)^m;$$

the denominator goes from $\prod(1 - D_i)$ to $\prod(1 + D - D_i)$; and again nothing happens to the term $[X]$, because it is of codimension 0. Bottom line,

$$s(Z, X) = [X] - \frac{(1 + D)^m}{\prod_0^m (1 + D - D_i)} \cap [X].$$

We apply Fulton's RIF, in his notation, to the regular embedding $i : X \to Y$ with X as above, and i equal to the zero section of $Y := \mathcal{E}$; we take for $f : V = X \to Y = \mathcal{E}$ the given section s as in the beginning of §3. Now we have, in one hand, $X \cdot V = c_n(\mathcal{E})$ by [11, Ex. 3.3.2, p. 67 or 6.3.4, p. 105]. Presently, the residual intersection class $\mathbb{R}$ is equal to the class of the finite part R since the latter is disjoint from Z. Hence we may write

$$[\mathbb{R}] = c_n(\mathcal{E}) \cap [X] - [c(\mathcal{E}) \cap s(Z, V)]_n,$$

where $[\cdot]_n$ denotes the $n-$codimensional part of a cycle. We get,

$$\begin{aligned}[\mathbb{R}] &= [c(\mathcal{E}) \cap ([X] - s(Z,V))]_n \\ &= c(\mathcal{E}) \cap \left[\frac{(1+D)^m}{\prod_0^m (1+D-D_i)}\right]_n .\end{aligned}$$

Hence

$$\deg R = \int_X \left[\frac{c(\mathcal{E})(1+D)^m}{\prod_0^m (1+D-D_i)}\right]. \tag{7}$$

4.2.1. *Remark.* Let us recall a nice observation in [3] to the effect that, if F is a virtual sheaf of rank $n-1$ then $c_n(F \otimes L) = c_n(F)$ for any line bundle L. We may write

$$\begin{aligned}\frac{c(\mathcal{E})(1+D)^m}{\prod_0^m (1+D-D_i)} &= c\left(\mathcal{E} + \mathcal{O}(D)^{\oplus m} - \bigoplus_0^m \mathcal{O}(D-D_i)\right) \\ &= c\Big(\underbrace{\Big(\mathcal{E}\otimes\mathcal{O}(-D) + \mathcal{O}^{\oplus m} - \bigoplus_0^m \mathcal{O}(-D_i)\Big)}_{\text{rank}\, =\, n-1} \otimes \mathcal{O}(D)\Big).\end{aligned}$$

Thus, in degree n we find

$$\left[\frac{c(\mathcal{E})(1+D)^m}{\prod_0^m (1+D-D_i)}\right]_n = \left[\frac{c(\mathcal{E}\otimes\mathcal{O}(-D))}{\prod_0^m (1-D_i)}\right]_n .$$

This can be expanded as

$$\sum_0^n c_i(\mathcal{E}\otimes\mathcal{O}(-D))\sigma_{n-i}(\underline{D}) = \sum_0^n \sum_0^i \binom{n-j}{i-j} c_j(\mathcal{E})(-D)^{i-j}\sigma_{n-i}(\underline{D}).$$

4.2.2. *Remark.* The preprint by F. Catanese, S. Hoşten, A. Khetan and B. Sturmfels [6] also contains a similar formula, deduced by different methods and in the context of another subject, namely, algebraic statistics.

4.3. **Foliations on $\mathbb{CP}^n$.** For $\mathcal{E} = \Omega^1_{\mathbb{CP}^n}(d)$, the above reduces to

$$\begin{aligned}\deg R = \text{coefficient of } h^n \text{ in } & \left[\frac{(1-h)^{n+1}}{\prod_0^m (1-d_i h)}\right] \\ &= \sum_{i=0}^n (-1)^i \binom{n+1}{i} \sigma_{n-i}(\underline{d})\end{aligned} \tag{8}$$

with σ_{n-i} the complete symmetric function of degree $n-i$ in $d_0, \ldots, d_m$.

One further application of Remark 4.2.1 yields the following positivity result.

4.3.1. **Proposition.** *Assume at least one* $d_i \geq 2$ *(and of course all* $d_i \geq 1$*). Then we have* $deg\, Z > 0$*.*

Proof. We show that, under the change of variables $d_i = e_i + 1$, the formula (8) becomes

$$\deg R = \sum_0^n \binom{m-1}{i} \sigma_{n-i}(\underline{e}).$$

The latter is obviously > 0 if some $e_i > 0$. To show the last equality, we use 4.2.1 to write

$$\begin{aligned} & c_n \left(\mathcal{O}(-h)^{\oplus n+1} - \bigoplus_0^m \mathcal{O}(-d_i h) + \mathcal{O}^{\oplus m-1} \right) \\ &= c_n \left(\mathcal{O}^{\oplus n+1} - \bigoplus_0^m \mathcal{O}(h - d_i h) + \mathcal{O}(h)^{\oplus m-1} \right) \\ &= \left[c \left(\mathcal{O}(h)^{\oplus m-1} - \bigoplus_0^m \mathcal{O}(h - d_i h) \right) \right]_n \\ &= \left[\frac{(1+h)^{m-1}}{\prod_0^m (1 - e_i h)} \right]_n \\ &= \sigma_n(\underline{e}) + (m-1)\sigma_{n-1}(\underline{e}) + \tbinom{m-1}{2} \sigma_{n-2}(\underline{e}) + \cdots \end{aligned}$$

□

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Fernando Cukierman
Dep. de Matematica - FCEyN
UBA - Ciudad Universitaria
1428 Ciudad de Buenos Aires
Argentina
fcukier@dm.uba.ar

Marcio G. Soares
Dep. Matemática - UFMG
Av. Antonio Carlos 6627
31 270-901 Belo Horizonte
Brasil
msoares@mat.ufmg.br

Israel Vainsencher
Dep. Matemática - UFMG
Av. Antonio Carlos 6627
31 270-901 Belo Horizonte
Brasil
israel@mat.ufmg.br