A NOTE ON RESULTANTS

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ABSTRACT. It is known that the resultant hypersurface associated to a vector bundle may be defined by the determinant of a Cayley-Koszul complex. It is also known that this determinant vanishes to order one along the resultant. In this note we give a new proof of this fact.

1. Let X be a smooth connected projective algebraic variety of dimension $n \geq 1$ over an algebraically closed field k. We fix a vector bundle E of rank r on X and denote $P = \mathbf{P}H^0(X, E)$ the projective space of global sections of E. We define the resultant of E

$$R_E \subset P$$

as the subset of global sections of E that have a zero $x \in X$.

2. As examples of this construction we mention

a) The classical resultant of a system of homogeneous polynomial equations. Here $X = \mathbf{P}^r$ is a projective space and E is a direct sum of line bundles.

b) The dual variety of a projective variety $X \subset \mathbf{P}^r$. Here $E = P^1\mathcal{O}_X(1)$ is the bundle of principal parts of order one of sections of $\mathcal{O}_X(1)$. This example includes Discriminant varieties (dual of Veronese varieties) and Hyperdeterminants (dual of Segre varieties). See [1], [2] for details on these and other examples.

3. Since R_E is the image of the projection

$$\pi_1 : R'_E = \{(s, x) \in P \times X \;/\; s(x) = 0\} \to P$$

it follows that $R_E \subset P$ is closed. We now assume that $r = n+1$ and the projection $\pi_1 : R'_E \to R_E$ is birational (the general section of E with a zero has only one zero); it then follows by a simple dimension count that $R_E \subset P$ is a hypersurface.

4. We claim that the degree of R_E is equal to $\int_X c_n(E)$. To see this, let $L = \{as_0 + bs_1\}$ be a general pencil in P, so that $\deg(R_E) = \mathrm{card}(R_E \cap L)$. Our assumption on E implies that the number of $(a : b) \in \mathbf{P}^1$ such that $as_0 + bs_1$ has a zero $x \in X$ coincides with the number of $x \in X$ such that $s_0(x), s_1(x) \in E(x)$ are linearly dependent, which is well known to be $\int_X c_n(E)$.

Date: October 1998.

1991 *Mathematics Subject Classification.* 14M12, 14N10.

Key words and phrases. Resultant, Determinant, Complex.

We thank Alicia Dickenstein for useful conversations.

5. For each $s \in H^0(X, E)$ denote by $K(E, s)$ the Koszul complex, obtained from multiplication by s in the exterior algebra of E

$$0 \to \mathcal{O}_X \stackrel{\wedge s}{\to} E \stackrel{\wedge s}{\to} \wedge^2 E \stackrel{\wedge s}{\to} \dots \stackrel{\wedge s}{\to} \wedge^r E \to 0$$

Since a Koszul complex is exact iff it is exact in degree zero, the resultant R_E is equal to the set of sections s such that $K(E, s)$ is not exact. For variable s, the complex above may be viewed as a complex $K(E)$ on $P \times X$; in fact, as the Koszul complex $K(E(1), \sigma)$ where $E(1) = \pi_1^* \mathcal{O}_P(1) \otimes \pi_2^* E$ and $\sigma \in H^0(P \times X, E(1))$ is defined by $\sigma(s, x) = s(x)$. It follows that

$$R_E = \text{support } \text{Div} R\pi_{1*} K(E)$$

According to [3], Prop. 9 (b), for a line bundle L,

$$\text{Div} R\pi_{1*} K(E) = \text{Div} R\pi_{1*} (K(E) \otimes L)$$

We take $L = \pi_2{}^* O_X(m)$ where $\mathcal{O}_X(1)$ is some very ample line bundle on X. Choosing m large so that $H^i(X, (\wedge^j E)(m)) = 0$ $(i > 0, j \geq 0)$ we obtain that $R\pi_{1*} K(E) \otimes L$ is represented by the complex

$$0 \to H^0(\mathcal{O}_X(m)) \otimes \mathcal{O}_P \stackrel{\wedge s}{\to} H^0(E(m)) \otimes \mathcal{O}_P(1) \stackrel{\wedge s}{\to} \dots \stackrel{\wedge s}{\to} H^0((\wedge^r E)(m)) \otimes \mathcal{O}_P(r) \to 0$$

that will be denoted $K_m = K_m(E)$ (see Cayley-Koszul complex, [2]). It follows that the divisors $\text{Div}(K_m)$ and R_E have the same support, and then there exists an integer $d = d(X, E, m)$ such that

$$\text{Div}(K_m) = d \ R_E$$

Our purpose in this note is to prove

Proposition: With the notation introduced above, $d = 1$ for $m >> 0$.

Remark: This fact is proved in [2] via derived categories. Here we provide an alternative proof through Chern class calculations.

Proof: Denote $D_m = \text{Div}(K_m)$. By [2], Appendix A, Corollary 15,

$$\deg(D_m) = \sum_{i=0}^{r} (-1)^i \ i \ \dim H^0((\wedge^i E)(m))$$

which equals, for $m >> 0$, $\sum_{i=0}^{r} (-1)^i \ i \ \chi((\wedge^i E)(m))$. By 4. above, we need to prove

$$\sum_{i=0}^{r} (-1)^i \ i \ \chi((\wedge^i E)(m)) = \int_X c_n(E)$$

By the Hirzebruch-Riemann-Roch theorem, we are reduced to checking the equality involving Chern classes

$$\int_X (\sum_{i=0}^{r} (-1)^i \; i \; \mathrm{ch}(\wedge^i E)(m)).\mathrm{Td}(T_X) = \int_X c_n(E) \tag{1}$$

Denote by $x_1, \ldots, x_r$ the Chern roots of E and $y = c_1 \mathcal{O}_X(1)$. Then we have

$$\begin{aligned} \sum_{i=0}^{r} (-1)^i \mathrm{ch}((\wedge^i E)(m)) \; t^i &= \sum_{i=0}^{r} (-1)^i \sum_{j_1 < \ldots < j_i} \exp(x_{j_1} + \ldots + x_{j_i} + m \; y) \; t^i \\ &= \exp(my) \prod_{i=1}^{r} (1 - t \exp(x_i)) \end{aligned}$$

Taking derivative with respect to t and setting $t = 1$ we obtain

$$\sum_{i=0}^{r} (-1)^i \; i \; \mathrm{ch}((\wedge^i E)(m)) = -\exp(my) \sum_{i=1}^{r} \exp(x_i) \prod_{j \neq i} (1 - \exp(x_j))$$

The last expression equals $\sum_{i=1}^{r} \prod_{j \neq i} x_j = c_n(E)$, plus terms of degree $> n$. Multiplying by $\mathrm{Td}(T_X) \in 1 + t \; k[[t]]$ and taking component of degree n we obtain (1), as wanted.

References

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