ON HIGHER WEIERSTRASS POINTS

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This paper deals with flexes of the n-canonical linear series, or n-Weierstrass points, on a compact Riemann surface.

Based on the theory of Limit Linear Series of Eisenbud and Harris we identify the limit of the n-canonical linear series, and hence the limits of its flexes, as a smooth curve degenerates to a stable curve of compact type. This allows us to prove some propositions that have known analogues in the case of 1-Weierstrass points.

After some preliminaries in the first four sections, we consider in §5 the "Hurwitz problem": which sequences occur as vanishing sequences of n-Weierstrass points? By a regeneration argument we prove (Theorem (5.3)) that every sequence of low enough weight occurs in the right dimension. Also, we reprove Lax's result [**L**] Theorem 3, to the effect that a general curve of genus $g \geq 3$ has only ordinary n-Weierstrass points.

In §6 we prove that the hypersurfaces $\mathcal{W}_g^n$ of n-Weierstrass points in the moduli spaces of pointed curves are irreducible; see Theorem (6.1) for a more precise statement. We do this by showing that the monodromy of $\mathcal{W}_g^n \to \mathcal{M}_g$ is transitive; the basic idea to generate monodromy being as in [**E-H,3**], namely, to vary the limit n-canonical series of a fixed curve.

In the last section we compute the linear equivalence class of the closure of $\mathcal{W}_g^n$ in the moduli space of stable pointed curves. This adds some more members to our collection of effective divisor classes on moduli space (see Remark (7.8)).

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§1 Flexes

Let X be a smooth irreducible complete algebraic curve over the complex numbers, and L a g^r_d on X (that is, $L = (V, \mathcal{L})$ where $\mathcal{L}$ is a line bundle on X of degree d and V is a linear space of sections of $\mathcal{L}$, of dimension $r+1$).

For each point $p \in X$ we consider the *vanishing sequence* of L at p

$$a^L(p) = (a^L_0(p) < \cdots < a^L_r(p))$$

defined as the sequence of orders of vanishing of elements of V at p, the *ramification sequence*

$$\alpha^L(p) = (\alpha^L_0(p) \leq \cdots \leq \alpha^L_r(p))$$

where $\alpha^L_i(p) = a^L_i(p) - i$, and the *weight* of L at p defined by

$$w^L(p) = \sum_{i=0}^{r} \alpha^L_i(p).$$

The point p is said to be a *flex* for L if $w^L(p) \neq 0$ (equivalently, p is a flex for L iff there is an element in V vanishing to order at least $r+1$ at p).
The *Plücker formula* (see [**ACGH**] Exercise C-13, page 39) says that the number of L-flexes is finite and

$$\sum_{p \in X} w^L(p) = (r+1)(d + rg - r)$$

where g is the genus of X.
In particular, if L is a complete non-special linear series then the weighted number of L-flexes is $g(r+1)^2$.
We remark that if L is a complete linear series of degree $d \geq 2g-1$ then, for any $p \in X$,

$$a^L_i(p) = i, \quad \text{for } 0 \leq i \leq d - 2g$$

and the vanishing sequence of L at p is given by the g integers $a^L_{d-2g+1}(p), \ldots, a^L_{d-g}(p)$.

§2 n-Weierstrass points

Let ω_X denote the canonical line bundle on the curve X.
In this paper we will study *n-Weierstrass points* which are, by definition, flexes of the complete linear series associated to the line bundle $\omega_X^{\otimes n}$. We will denote $W^n(X)$ the set of n-Weierstrass points of X.
For $n \geq 2$, $h^0(X, \omega_X^{\otimes n}) = (2n-1)(g-1)$ and then the weighted number of n-Weierstrass points is, by the Plücker formula, $(2n-1)^2 g(g-1)^2$.

(2.1) **Examples:**

(i) If X has genus 3 then $W^1(X) \subset W^2(X)$:
in case X is not hyperelliptic, let ω_0, ω_1, ω_2 be regular differentials vanishing at $p \in X$ to orders 0, 1 and ≥ 3, respectively; then the monomials of degree 2 in these differentials give a vanishing sequence $\geq (0, 1, 2, 3, 4, 6)$.

(ii) Let X be the hyperelliptic curve obtained by normalization of the plane curve $y^2 = \prod_{j=1}^{2g+2}(x - x_j)$. The n-differentials

$$(x - x_j)^i (dx/y)^n,\ 0 \leq i \leq n(g-1),$$
$$\text{and} \qquad y(x - x_j)^i (dx/y)^n,\ 0 \leq i \leq n(g-1) - (g+1)$$

are regular and form a basis of $H^0(X, \omega_X^{\otimes n})$ giving the vanishing sequence at x_j

$$a_i = i\ (0 \leq i \leq \alpha), \quad a_{\alpha+i} = \alpha + 2i\ (1 \leq i \leq g)$$

where we let $\alpha = 2n(g-1) - 2g$.
Then, every 1-Weierstrass point is a n-Weierstrass point of weight $\binom{g+1}{2}$ for $n \geq 2$.

(2.2) If L is a g^r_d on X and $p \in X$, then it is traditional to introduce the set

$$G^L(p) = \{a_0^L(p) + 1, \ldots, a_r^L(p) + 1\}$$

of *gaps* of L at p, and $N^L(p) = \mathbb{N} - G^L(p)$, the set of non-gaps.
When L is the complete canonical series, Serre duality implies that

$$N^{\omega_X}(p) = \{-ord_p(f),\ f \in H^0(X - \{p\}, \mathcal{O}_X)\} \qquad (2.3)$$

which is a subsemigroup of $\mathbb{N}$.
If L is a complete linear series then $N^{\omega_X}(p)$ acts on $N^L(p)$ in the sense that

$$N^{\omega_X}(p) + N^L(p) \subset N^L(p). \qquad (2.4)$$

(2.5) To consider n-Weierstrass points on variable curves, let $\mathcal{C}_g$ be the coarse moduli space of genus g pointed curves over the complex numbers [**K**]. The main objects of our study will be the hypersurfaces

$$\mathcal{W}_g^n = \{(X,p)/p \text{ is a } n\text{-Weierstrass point of } X\} \subset \mathcal{C}_g$$

Using (2.3) one may parametrize $\mathcal{W}_g^1$ by a Hurwitz scheme [**F**]; the irreducibility of the divisor of Weierstrass points then follows from the classical theorem about irreducibility of the Hurwitz scheme. To analize the irreducibility of $\mathcal{W}_g^n$ for $n > 1$ we seem to need different techniques. Eisenbud and Harris [**E-H,3**] gave an algebraic proof of the irreducibility of $\mathcal{W}_g^1$ (they proved the stronger statement that the monodromy of $\mathcal{W}_g^1 \to \mathcal{M}_g$ is the full symmetric group) and our treatment is based on their ideas.

§3 Limit linear series

(3.1) The main technique for our study of $\mathcal{W}_g^n$ is the theory of limit linear series of Eisenbud-Harris. In order to set up notation we recall some basic facts from this theory. We refer to [**E-H,1**] for proofs and more details.

Consider the following situation: $f : X \to B$ is a flat, proper morphism, where X is a smooth surface and B is the spectrum of a discrete valuation ring (with the complex numbers as residue field). We assume that the generic fiber X_b of f is smooth and the special fiber X_0 is reduced, connected, with nodes as only singularities and of compact type (the dual graph of X_0 is a tree). Also, let $\mathcal{L}$ be a line bundle on X. We shall refer to such a collection of data as a *limit series situation* (LSS).

Given a LSS we may, first of all, single out an irreducible component Y of X_0 and twist $\mathcal{L}$ by a divisor supported on X_0 to obtain a (unique) line bundle $\mathcal{L}_Y$ such that

(3.2) $$\deg(\mathcal{L}_Y|_Z) = 0$$

for every irreducible component Z of X_0 different from Y. Next, we consider

(3.3) $$(f_*\mathcal{L}_Y)(0) \to H^0(X_0, \mathcal{L}_Y|_{X_0}) \to H^0(Y, \mathcal{L}_Y|_Y)$$

where both maps are restriction; it follows from (3.2) that these maps are injective.
Also, since $f_*\mathcal{L}_Y$ is torsion-free and B is the spectrum of a DVR, it follows that $f_*\mathcal{L}_Y$ is free and

(3.4) $$\dim\ (f_*\mathcal{L}_Y)(0) = \dim\ H^0(X_b, \mathcal{L}|_{X_b}) := r + 1$$

If we let

(3.5) $$d := \deg(\mathcal{L}|_{X_b}) = \deg(\mathcal{L}_Y|_Y)$$

and V_Y denotes the image of the composite map (3.3), then $L_Y = (V_Y, \mathcal{L}_Y|_Y)$ is a g_d^r on Y. The collection

(3.6) $$L = \{L_Y\}_{Y \text{ irred comp of } X_0}$$

is called the *limit linear series* associated to the LSS. The series L_Y is called the Y-aspect of L.
It is shown in [**E-H,1**] Proposition (2.2) that if Y and Z are two components of X_0 intersecting at the point p then the vanishing sequences of L_Y and L_Z are related by

$$a_i^{L_Y}(p) + a_{r-i}^{L_Z}(p) \geq d \tag{3.7}$$

If equality holds in (3.7) for all i and all p, we say that L is *refined.* This turns out to be equivalent to the condition that no flex for $\mathcal{L}|_{X_b}$ on X_b specializes to a node of X_0 (see [**E-H,1**] Proposition (2.5)).

(3.8) **Definitions:**

(i) A *limit linear series* (of degree d and dimension r) on a curve X_0 of compact type is a collection $L = \{L_Y\}$ of g^r_d's, one for each irreducible component Y of X_0, satisfying the compatibility conditions (3.7).

(ii) A limit linear series is said to be *smoothable* if it arises from a LSS as described above.

(iii) If X_0 has arithmetic genus g, a *limit canonical series* on X_0 is a limit g^{g-1}_{2g-2} on X_0.

We end this section with the trivial but useful remark that in a LSS the flexes in the generic fiber specialize either to nodes or to flexes of the associated limit series.

§4 Limit n-canonical series

Let X_0 be a genus g curve of compact type. Fix $n \in \mathbb{N}$, $n > 1$, and let $d = 2n(g-1)$ and $r + 1 = (2n-1)(g-1)$.

(4.1) **Definition:** A *limit n-canonical series* on X_0 is a limit g^r_d on X_0,$\{(V_Y, \mathcal{L}_Y)\}$, such that there exists a limit canonical series $\{(U_Y, \mathcal{K}_Y)\}$ satisfying

(a) $\mathcal{L}_Y = \mathcal{K}_Y^{\otimes n}$ and

(b) $\mu(U_Y^{\otimes n}) \subset V_Y$ where $\mu : H^0(Y, \mathcal{K}_Y)^{\otimes n} \to H^0(Y, \mathcal{L}_Y)$ is the multiplication map.

Remark: it is proved in [**E-H,1**] Theorem (4.1) that all limit canonical series on a fixed curve have the same line bundles. By (a) the same is true for limit n-canonical series.

(4.2) **Proposition:** In a limit series situation (as in §3) with $\mathcal{L} = \omega_{X|B}^{\otimes n}$, the associated limit series is a limit n-canonical series.

Proof:

The Proposition follows easily from considering, for each component Y of X_0, the commutative diagram:

$$\begin{array}{ccc} (f_*\mathcal{L}_Y(0))^{\otimes n} & \longrightarrow & f_*(\mathcal{L}_Y^{\otimes n})(0) \\ \downarrow & & \downarrow \\ H^0(X_0, \mathcal{L}_Y|_{X_0})^{\otimes n} & \longrightarrow & H^0(X_0, \mathcal{L}_Y^{\otimes n}|_{X_0}) \\ \downarrow & & \downarrow \\ H^0(Y, \mathcal{L}_Y|_Y)^{\otimes n} & \longrightarrow & H^0(Y, \mathcal{L}_Y^{\otimes n}|_Y) \end{array}$$

Q.E.D.

Our next goal is to determine all limit n-canonical series on some curves of compact type. We consider now the case of curves with two components, and we treat in Proposition (4.6) general curves in each stratum of the boundary of $\bar{\mathcal{M}}_g$.

(4.3) **Proposition:** Let $X = Y \cup_p Z$ be the union of two smooth curves Y and Z meeting at a point p. Let $L = \{(V_Y, \mathcal{L}_Y), (V_Z, \mathcal{L}_Z)\}$ be a limit n-canonical series on X, with $n \geq 2$. Then

(i) $\mathcal{L}_Y = \omega_Y^{\otimes n}(2ng_Z\ p)$ (and $\mathcal{L}_Z = \omega_Z^{\otimes n}(2ng_Y\ p)$).

(ii) If $g_Z \geq 2$ and $p \notin W^n(Z)$ (notation as in §2) then

$$V_Y = H^0(Y, \omega_Y^{\otimes n}((2n-1)g_Z\ p)) \subset H^0(Y, \mathcal{L}_Y)$$

(iii) If $g_Z = 1$ then

$$V_Y = H^0(Y, \omega_Y^{\otimes n}((2n-2)p)) + \mathbb{C}.\eta^n \subset H^0(Y, \mathcal{L}_Y)$$

where $\eta \in H^0(\omega_Y(2p)) - H^0(\omega_Y)$.

(iv) If $p \notin W^n(Z)$ and $p \in W^n(Y)$ is ordinary (i.e. weight one) then V_Y is determined as in (ii) or (iii) and the possible V_Z's are parametrized by $\mathbb{P}^1$.

Proof:

(i) follows from the fact [**E-H,1**] Theorem (4.1) that the line bundle of the Y-aspect of any limit canonical series on X is $\omega_Y(2g_Z\ p)$.

(ii) Let $a^Y(p)$ denote the vanishing sequence at p of the Y-aspect of L. Since $p \notin W^n(Z)$ and $g_Z \geq 2$ it follows that p is not a flex for $\mathcal{L}_Z$, and then

$$a_r^Z(p) \leq h^0(Z, \mathcal{L}_Z) - 1 = d - g_Z$$

and combining with the compatibility condition $a_0^Y(p) + a_r^Z(p) \geq d$, we obtain

$$a_0^Y(p) \geq g_Z$$

which implies that

$$V_Y \subset H^0(Y, \mathcal{L}_Y(-g_Z\ p)) = H^0(Y, \omega_Y^{\otimes n}((2n-1)g_Z\ p))$$

and since both spaces have the same dimension, the desired equality holds.

(iii) Since $V_Z \subset H^0(\mathcal{O}_Z(d\ p))$ and $\mathcal{O}_Z(d\ p)$ has vanishing sequence $(0, 1, 2, \ldots, d-2, d)$ at p, we have $a_{r-1}^Z(p) \leq d-2$, and then, by the compatibility conditions,

$$a_1^Y(p) \geq 2$$

that is, V_Y has at least a cusp at p.

On the other hand, from [**E-H,1**] Theorem (4.1) (see also the proof of (4.3) in [**E-H,2**]), the Y-aspect of any limit canonical series on X is $H^0(\omega_Y(2p))$; then, by Definition (4.1), V_Y contains the image of $H^0(\omega_Y(2p))$ under the multiplication map. Choose $\eta \in H^0(\omega_Y(2p)) - H^0(\omega_Y)$ and we obtain that V_Y contains η^n. Then,

$$V_Y \supset H^0(Y, \omega_Y^{\otimes n}((2n-2)p)) + \mathbb{C}.\eta^n$$

and since both spaces have the same dimension, equality holds.

(iv) For $g_Z \geq 2$, combining (ii) with the fact that $p \in W^n(Y)$ is ordinary, we obtain

$$a^Y(p) = (g_Z, g_Z + 1, \ldots, g_Z + r - 1, g_Z + r + 1)$$

and the compatibility conditions then imply that

$$a_0^Z(p) \geq g_Y - 1, \quad a_i^Z(p) \geq g_Y + i, \quad (i \geq 1).$$

If $g_Z = 1$ it follows from (iii) that

$$a^Y(p) = (0, 2, 3, \ldots, r, r+2)$$

and then

$$a^Z(p) \geq (g-2, g, g+1, \ldots, d-2, d)$$

In both cases, if we let $\mathcal{K} = \omega_Z^{\otimes n}((2n-1)g_Y + 1)p$, then

$$V_Z((-g_Y + 1)p) \subset H^0(Z, \mathcal{K})$$

is an arbitrary codimension one linear space with a cusp at p. Then, the possible V_Z's are in bijective correspondence with the points of the tangent line T to Z at p in the embedding $Z \hookrightarrow \mathbb{P}H^0(Z, \mathcal{K})^*$ by the very ample line bundle $\mathcal{K}$. **Q.E.D.**

Remark: Using Noether's theorem (i.e. the map $H^0(C, \omega_C)^{\otimes n} \to H^0(C, \omega_C^{\otimes n})$ is surjective for non-hyperelliptic curves), it is not hard to see that in (iii) V_Y can be described as the image of $H^0(Y, \omega_Y(2p))^{\otimes n}$ in $H^0(Y, \omega_Y^{\otimes n}(2np))$.

(4.4) **Corollary:** Let $X = Y \cup_p Z$ be the union of two smooth curves Y and Z meeting at a point p. Assume that $p \notin W^n(Y) \cup W^n(Z)$. Then, in every limit series situation with X as central fiber the n-Weierstrass points of the generic fiber specialize to the flexes of the unique limit n-canonical series on X. In particular, $(2n-1)^2(g-1)^2 g_Y$ n-Weierstrass points specialize to points in Y, the other $(2n-1)^2(g-1)^2 g_Z$ specialize to points in Z, and none of them specializes to p.

When $p \in W^n(Y)$ has weight one and $p \notin W^n(Z)$ every point in Z is a limit of n-Weierstrass points of smooth curves. In each LSS either p is a limit of n-Weierstrass points or $(2n-1)^2(g-1)^2 g_Z + 1$ n-Weierstrass points specialize to $Z - \{p\}$ and $(2n-1)^2(g-1)^2 g_Y - 1$ n-Weierstrass points specialize to $Y - \{p\}$.

Proof:

The assertion in the case $p \notin W^n(Y) \cup W^n(Z)$ follows from the description of the unique n-canonical series on X given in Proposition (4.3), together with the Plücker formula and the fact that the n-canonical series is refined (it is refined because the vanishing sequence at p of the Y-aspect is $g_Z + (0, 1, \dots, r)$, when $g_Z > 1$, and similarly for the Z-aspect. If $g_Z = 1$ or $g_Y = 1$, use Proposition (4.3)(iii). See also (6.7).) The other case follows from (4.3) combined with (4.5) below. **Q.E.D.**

(4.5) **Proposition:** The limit n-canonical series in Propositon (4.3) (iv) occur as limits of n-canonical series of smooth curves.

Proof:

The strategy is to follow [**E-H,1**] Proposition (3.3) to try to produce a variety $\mathcal{K}^n(\mathcal{B})$ parametrizing limit n-canonical series for a family of curves $\pi : \mathcal{X} \to \mathcal{B}$, and then deduce the proposition by a dimension count.

We first consider the case when both g_Y and g_Z are greater than 1. Remember that $p \notin W^n(Z)$ and $p \in W^n(Y)$ has weight 1. Let $\pi : \mathcal{X} \to \mathcal{B}$ be the Kuranishi family of the curve $X = Y \cup_p Z$. d will be $2n(g-1)$ and $r = (2n-1)(g-1) - 1$. Let $\omega := \omega_{\mathcal{X}/\mathcal{B}}$ be the relative dualizing sheaf of the family. After twisting by suitable divisors in $\mathcal{X}$, we will get two different line bundles Ω_Y and Ω_Z on $\mathcal{X}$ such that they restrict to $\omega_Y(2g_Z p)$ and $\omega_Z(2g_Y p)$ on Y and Z respectively.

Take D to be a relative ample divisor of π with high intersection number with both Y and Z. Put $\deg D = e$. Then $\pi_*(\Omega_Y^{\otimes n}(D))$ and $\pi_*(\Omega_Z^{\otimes n}(D))$ are two vector bundles on $\mathcal{B}$ of rank $r + 1 + e$. Let G_Y and G_Z be the Grassmannian bundle of $r+1$ dimensional subspaces in $\pi_*(\Omega_Y^{\otimes n}(D))$ and $\pi_*(\Omega_Z^{\otimes n}(D))$ respectively. Denote by F_Y, F_Z the projective frame bundles associated to G_Y and G_Z. Now we take the fibre product of F_Y, F_Z and G_Y, G_Z over $\mathcal{B}$ to get F and G. Thus, a point in F consists of

$$\begin{cases} \text{a point } q \in \mathcal{B}, \quad \text{line bundles } \Omega_Y^{\otimes n}(D)\big|_{\mathcal{X}_q} \text{ and } \Omega_Z^{\otimes n}(D)\big|_{\mathcal{X}_q} \\ \text{two } r+1 \text{ dimensional subspaces } V_Y, V_Z \text{ of } H^0(\mathcal{X}_q, \Omega_Y^{\otimes n}(D)) \text{ and } H^0(\mathcal{X}_q, \Omega_Z^{\otimes n}(D)) \\ \text{two bases for } V_Y, V_Z \text{ up to projective equivalence.} \end{cases}$$

Here projective equivalence means that $(s_0, s_1, \cdots) \sim (\lambda_0 s_0, \lambda_1 s_1, \cdots)$ with $\lambda_0, \lambda_1, \cdots$ all in $\mathbb{C}^*$.

We then have $\dim G = \dim \mathcal{B} + \sum_{Y,Z} (r+1)e = \dim \mathcal{B} + 2(r+1)e$, and $\dim F = \dim G + 2r(r+1)$. To define the variety of limit n-canonical series $\bar{\mathcal{K}}^n(\mathcal{B})$ in F we need the following two sets of equations :

(1) Vanishing on the multi-section D, and

(2) compatibility conditions.

From the proof of Proposition (4.3), we see that a point satisfying these conditions corresponds to a limit n-canonical series. For each section, (1) contributes e equations, so (1) accounts for $(r+1)e$ equations in total. Each of the compatibility conditions gives an equation on each section of the projective space bundle of fibre dimension $r+e$, so (2) accounts for $(r+1)(r+e)$ equations in total. Thus each component of $\bar{\mathcal{K}}^n(\mathcal{B})$ has dimension

$$\begin{aligned}
&\geq \dim F - \text{ number of equations defining } \bar{\mathcal{K}}^n(\mathcal{B}) \\
&= \dim B + 2(r+1)e + 2r(r+1) - (r+1)e - (r+1)(r+e) \\
&= \dim B + r(r+1).
\end{aligned}$$

We now let $\mathcal{K}^n(\mathcal{B})$ be the image of $\bar{\mathcal{K}}^n(\mathcal{B})$ in G. As in **[E-H,1]**, we can see that on some open set F' of F, the fiber dimension of $F \to G$ is no greater than $r(r+1)$, therefore we see that each component of $\mathcal{K}^n(\mathcal{B})$ has dimension $\geq \dim \mathcal{B}$. To conclude smoothing, let us denote by Δ' the locus in $\mathcal{B}$ of singular curves with attachment point being a n-Weierstrass point on one of the components; notice that the codimension of Δ' in $\mathcal{B}$ is 2. If there were a limit n-canonical series as in proposition (4.3) (iv) that is not smoothable, then we would have a component of $\mathcal{K}^n(\mathcal{B})$ lying entirely over Δ', but the fibre dimension of that component over Δ' is 1 by (4.3) (iv). This implies that the dimension of that particular component is at most $\dim \mathcal{B} - 2 + 1 = \dim \mathcal{B} - 1$ which contradicts the above inequality and hence establishes the smoothing.

The remaining case is when $g_Z = 1$ and $g_Y > 1$. The construction in the previous case does not take into account condition (b) of definition (4.1) which plays an essential role in fixing the Y-aspect of the n-canonical series. While it is easy to write an equation for this condition, this procedure spoils the dimension count. We make the following detour : in addition to the relative ample divisor D, we take another section $s : \mathcal{B} \to \mathcal{X}$ which intersects Z. We then define the variety $\mathcal{K}^n(\mathcal{B}, s)$ that parametrizes r-dimensional subspaces of the limit n-canonical series vanishing on s. More precisely, fix a point q (other than p) on Z, and let $\pi : \mathcal{X} \to \mathcal{B}$ be the versal deformation family for the curve X with the point q. There is a canonical section $s : \mathcal{B} \to \mathcal{X}$ that intersects X at q. As before, we take a relative ample divisor D having high intersection number with Y and Z. We then follow the proof in the previous case to construct $F \to G$ over $\mathcal{B}$, the only difference is : instead of considering

$(r+1)$-dimensional subspaces, we consider r-dimensional ones. Thus we have, in this case,

$$\begin{aligned}\dim G &= \dim \mathcal{B} + \sum_{Y,Z} r(e+1) \\ &= \dim \mathcal{B} + 2r(e+1) \qquad \text{and} \\ \dim F &= \dim G + 2(r-1)r.\end{aligned}$$

To define $\bar{\mathcal{K}}^n(\mathcal{B}, s)$ in F, we need the following sets of equations :

(1) Vanishing on D and s , and
(2) Compatibility conditions.

The first condition contributes $e+1$ equations for each section, so we get $r(e+1)$ equations in total. The second condition accounts for $r(r+e)$ equations. Thus each component of $\bar{\mathcal{K}}^n(\mathcal{B}, s)$ has dimension

$$\begin{aligned}&\geq \dim F - \text{ number of equations defining } \mathcal{K}^n(\mathcal{B}, s) \\ &= \dim \mathcal{B} + 2r(e+1) + 2r(r-1) - r(e+1) - r(r+e) \\ &= \dim \mathcal{B} + r(r-1).\end{aligned}$$

Take $\mathcal{K}^n(\mathcal{B}, s)$ to be the image of $\bar{\mathcal{K}}^n(\mathcal{B}, s)$. As in the first case, we can then conclude that each component of $\mathcal{K}^n(\mathcal{B}, s)$ has dimension $\geq \dim \mathcal{B}$.

Now smoothing follows from a similar dimension count. Let us consider the fiber dimension of $\mathcal{K}^n(\mathcal{B}, s)$ over the point (X, q) in $\mathcal{B}$. Points in the fiber parametrize limit linear subseries of the limit n-canonical series that vanish at q. For the Y-aspect, this is equivalent to saying that the subseries has a base point at p. From the description of the aspect given in Proposition (4.3) (iii), we see that there is only one such subseries, namely $\left|\omega_Y^{\otimes n}((2n-2)p)\right| + 2p$. On the elliptic tail Z, the aspects are to be chosen from the subseries $V_{Z,q}$ of sections of $|\mathcal{O}_Z(((2n-1)g_Y+1)p)| + (g_Y-1)p$ vanishing at q. Notice that sections in $V_{Z,q}$ have vanishing order $< 2ng_Y$ at p, so each member of the 1-parameter family in (4.3) (iv), when restricted to $V_{Z,q}$, gives rise to a Z-aspect for a point in $\mathcal{K}^n(\mathcal{B}, s)$ over (X, q). Conversely the Z-aspect of each point in the fiber arises in this fashion. Thus the fibre of $\mathcal{K}^n(\mathcal{B}, s)$ over the point (X, q) in $\mathcal{B}$ has dimension 1. Reasoning as before, there is no component of $\mathcal{K}^n(\mathcal{B}, s)$ lying entirely over the locus of singular curves in $\mathcal{B}$. Given any member of the 1-parameter family in (4.4) (iv), with Z-aspect W_Z there exists a 1-parameter family of smooth curves $\pi' : \mathcal{X}' \to \mathcal{B}'$ degenerating to X, together with a section s' that intersects Z at q, such that the subseries of the n-canonical series vanishing at s degenerates to $W_Z \cap V_{Z,q}$ on Z. To see that the complete series degenerates to the aspects we have chosen, remember that from the description of the limit canonical series on the family π', there is a section t of the relative dualizing sheaf which after suitable twisting gives a section on the Y-aspect (Z-aspect, resp.) that vanishes to order 0 ($2g_Y$, resp.) at p. Adding t^n (after twisting) to the $W_Z \cap V_{Z,q}$ recovers W_Z. **Q.E.D.**

Next we analize limit n-canonical series on general curves of compact type.

(4.6) **Proposition:** Consider a stable curve of compact type X, general among curves with the same configuration. For each $n \geq 1$, X has a unique limit n-canonical series whose Y-aspect can be described as follows:

$$\mathcal{L}_Y = \omega_Y^{\otimes n}(2n \sum_1^s g_i p_i)$$

$$V_Y = \left| \omega_Y^{\otimes n} \left(\sum_{g_i > 1} (2n-1) g_i p_i + \sum_{g_i = 1} 2(n-1) p_i \right) \right| \bigoplus \sum_{g_i = 1} \mathbb{C} \cdot \eta_i^n$$

where the first summand is understood to be the complete linear series with base points $\sum_{g_i>1} g_i p_i + \sum_{g_i=1} 2p_i$ added, and η_i is any section in the Y-aspect of the limit canonical series with vanishing order 0 at p_i. In particular, for each $g_i > 1$ the Y-aspect has a base point of order g_i at p_i and no base point at p_i for those $g_i = 1$.

Proof:

Notice that the case when X has only two components is proved in (4.3). (Indeed, we see that "general" in that case means $p \notin W^n(Y) \cup W^n(Z)$). Take a genus g curve of compact type X. Look at one of the components Y. There are two possibilities : (i) all curves attached to Y are smooth, and (ii) one of the curves attached to Y is reducible. Incidentally, case(i) will suffice for later applications, we include case(ii) for completeness.

Let us take case (i) first. Denote the smooth curves attached to Y by Z_i for $i = 1, \ldots, s$. To avoid triviality we may assume that none of them are rational. Put genus of Z_i to be g_i. We can also assume that $g_i = 1$ for $i \leq t$ and $g_i > 1$ for $i > t$. Since X is general, $p_i \notin W^n(Z_i)$

and for $i > t$ this implies that p_i is not a flex for L_i. Then $a_r^{V_i}(p_i) \leq h^0(Z_i, L_i) - 1 = d - g_i$ and by the compatibility conditions we obtain

$$a_0^{V_Y}(p_i) \geq g_i, \qquad (t < i \leq s) \tag{4.7}$$

For $i \leq t$, $a_{r-1}^{V_i}(p_i) \leq d - 2$ and then

$$a_1^{V_Y}(p_i) \geq 2, \qquad (1 \leq i \leq t) \tag{4.8}$$

Again because X is general, it is in particular aresidually generic (see [**E-H,2**]) which means that it has a unique limit canonical series with Y-aspect

$$U = \sum_{i=1}^{s} H^0(\omega_Y(g_i + 1)p_i) \subset H^0(\omega_Y(2\sum_{i=1}^{s} g_i p_i)).$$

We know from Definition (4.1) that $V \supset \mu(U^{\otimes n})$ where
$\mu : H^0(\omega_Y(2\sum_{i=1}^{s} g_i p_i))^{\otimes n} \to H^0(\mathcal{L}_Y)$ is the multiplication map. In particular,

$$\eta_i^n \in V, \qquad (1 \leq i \leq t) \tag{4.9}$$

with η_i^n as above. Combining (4.7), (4.8) and (4.9) we deduce that

$$V_Y \subset \left|\omega_Y^{\otimes n}\left(\sum_{g_i > 1}(2n-1)g_i p_i + \sum_{g_i = 1} 2(n-1)p_i\right)\right| \bigoplus \sum_{g_i=1} \mathbb{C} \cdot \eta_i^n$$

and since both sides have the same dimension, the desired equality holds. (In this case, one can easily deduce from the compatibility condition that the Z_i-aspects have to be the ones stated.)

The proof of case (ii) is also combinatorial in nature. It will be sufficent to prove that (4.7) or (4.8) holds at the attachment points. Suppose that among the curves $Z_1, \ldots, Z_s$ attached to Y, Z_1 is reducible with genus g_1. The idea here is that because Z_1 is general, the compatibility conditions will force the same inequality (4.7) or (4.8) upon p_1. We can see this by working "backwards", starting from components of Z_1 furthest away from p_1. More precisely, one can look at the dual graph of X (which is a tree) and find the node n_Y representing Y. Among those nodes with distance 1 from n_Y, there is one that represents the component of Z_1 meeting Y at p_1. Call it n_1. The components of Z_1 will correspond to the branch of nodes connected to n_1 without going through n_Y. Now, locate a leaf furthest from n_1 on this branch. There is a unique node n_0 attached to this leaf. Moreover, by our choice of the leaf, every edge in the graph coming out of n_0 is attached to a leaf, with one exception (this is the edge that joins n_0 to n_1). Let us denote the component in X that corresponds to n_0 by C_0, and let its genus be g_0. Translating the incidence property of the node into attaching property of C_0, we see that C_0 is connected to Y through some point q (possibly by some reducible curves), and that there are smooth curves $C_1, \ldots, C_m$ $(m \geq 1)$ (with genus $g'_1, \cdots, g'_m$) attached to C_0 at general points $r_1, \ldots, r_m$. We will assume $g'_i = 1$ for $i \leq k$ and $g'_i > 1$ for $k < i \leq m$. The idea is that the vanishing sequence of the C_0-aspect at q imposes the same compatibility inequality on the adjacent aspect as if $C_0 \cup \cdots \cup C_m$ were a smooth curve of genus $g' = g'_0 + \cdots g'_m$.

Because $C_1, \cdots, C_m$ are smooth and the attachment points on them are general, one can argue as in case (i) that the C_0-aspect has to satisfy (4.7) or (4.8) at $r_1, \cdots, r_m$. In other words, we have

$$V_{C_0} \subset \left|\omega_{C_0}^{\otimes n}\left(2n\bar{g}q + \sum_1^k (2n-1)g'_i r_i + \sum_{k+1}^m 2(n-1)r_i\right)\right| \bigoplus \sum_{k+1}^m \mathbb{C}\cdot(\eta'_i)^n. \tag{4.10}$$

Here $\bar{g}$ is the genus of the (reducible) curve attached to C_0 at q, i.e. $\bar{g} = g - g'$. Denote the series on the right of (4.10) by V'_0, and the first summand by W_0. We claim that because q is a general point, V'_0 is not ramified at q. To see this, take W'_0 to be the series

$$\left|\omega_{C_0}^{\otimes n}\left(\sum_1^k (2n-1)g'_i r_i + \sum_{k+1}^m 2(n-1)r_i\right)\right|.$$

Since the sections (η'_i) have vanishing order $\geq 2\bar{g}$ at q, we may remove $2\bar{g}q$ from them to form $\bar{\eta}_i$. Since q is general, we can assume q is a non-flex for the series $W'_0 \oplus \sum_{k+1}^m \mathbb{C}\cdot(\bar{\eta}_i)^n$. We claim that in this case q is not a flex for V'_0. Indeed, adding $2n\bar{g}q$ to sections in $W'_0 \oplus \sum_{k+1}^m \mathbb{C}\cdot(\bar{\eta}_i)^n$ gives sections in V'_0 with vanishing order from $2n\bar{g}$ up to $(2n-1)(g'-1)-1+2n\bar{g}$. From Riemann-Roch, the dimension of W_0 is exactly $2n\bar{g}$ higher than that of W'_0. So there are sections in W_0 with vanishing order $0, 1, \ldots, 2n\bar{g}-1$ at q. Thus q is not a flex for V'_0.

The dimension of V'_0 is $(2n-1)(g'-1)-1+2n\bar{g}$. So the vanishing sequence at q is exactly the same as that of $\left|\omega_{\bar{C}}^{\otimes n}(2n\bar{g}\bar{q})\right|$ for a smooth genus g' curve $\bar{C}$ at a general $\bar{q}$. From the combinatorial point of view, in order to get the compatibility inequality (of the same type as

(4.7) or (4.8)) at q for the component adjacent to C_0, we may as well replace $C_1\cup\cdots\cup C_m$ by $\bar{C}$ at a non-flex $\bar{q}$. Now we can repeat this argument and work our way up the branch until we arrive at the node n_1. The conclusion is therefore: if Z_1 is general, then the compatibility conditions on all components of Z_1 will join force to give a compatiblity condition at p_1 as if p_1 were a general point on a smooth Z_1 of genus g_1. So we will have (4.7) or (4.8) for the Y-aspect at p_1. Do the same for the other curves $Z_2,\cdots,Z_m$. This brings us back to the situation as in case (i), and so we conclude that the Y-aspect is again the stated one. **Q.E.D.**

§5 Existence of n-Weierstrass points

We start this section analyzing n-Weierstrass points on a general curve. To this end, let us consider the stable curve C consisting of a rational curve Y with elliptic tails $E_1,\ldots,E_g$ attached at the points $p_1,\ldots,p_g$, $(g\geq 3)$. For general attachment points the E_i-aspect of the n-canonical series $(n>1)$ is given in (4.6) to be $|\mathcal{O}((2n-1)(g-1)p_i)|+(g-1)p_i$. The flexes of this series, other than p_i, are torsion points of order $(2n-1)(g-1)$ with respect to p_i. They are all ordinary flexes (i.e. have weight 1) and there are $(2n-1)^2(g-1)^2-1$ of them. Thus the elliptic tails contribute $(2n-1)^2(g-1)^2g-g$ ordinary n-Weierstrass points to C. Plücker's formula gives the total weight as $(2n-1)^2(g-1)^2g$. So there are n-Weierstrass points on Y with total weight g. (Contrast the case of 1-Weierstrass points, where there is no flex on Y. Also, when $g=2$ one can check, see (6.8), that there are no flexes on the rational curve.)

Consider the case $g=3$. Since the description of the limit n-canonical series is invariant under the automorphisms of Y fixing the three attachment points, the flexes on Y should also be in some orbit of this action. But there is no single point on Y which is fixed under this action, therefore there cannot be n-Weierstrass points of weight higher than 1 on Y. In fact taking the attachment points to be $0,1,\infty$, the three flexes are $-1,2,\frac{1}{2}$. This has been worked out by Lax and Widland explicitly in the case when n equals 2. From this we may conclude that for a general curve of genus 3 all n-Weierstrass points are ordinary.

For $g\geq 4$, while it seems likely that for general choice of attachment points on Y, the n-Weierstrass points on Y will be ordinary, it is not clear how to show this directly. So, we follow the idea of Eisenbud and Harris of further degenerating the curve, by letting the attachment points come together. In fact we will look at a curve C obtained in the following fashion : take rational curves $C_1,\ldots,C_{g-2}$ and ellipitic curves $E,\ldots,E_g$, attach C_i to C_{i-1} to form a chain of rational curves, then attach two elliptic curves to distinct points on each of C_1 and C_{g-2}, to all other C_i's we attach one elliptic tail; this is our C. The aspects on the elliptic tails remain the same, so we still have g n-Weierstrass points on the chain of rational curves. For $i \neq 1$ or $g-2$, the C_i-aspect has dimension $(2n-1)(g-1)$ and degree $2n(g-1)$ and from Plücker's formula the total weight on C_i is $(2n-1)(g-1)g$. Now let p_1 be the attachment point for the elliptic tail on C_i, p_2 and p_3 be the attachment points for reducible curves of genus g_2 and g_3 respectively; then we have $g_2 > 1$, $g_3 > 1$ and $g_2 + g_3 = g - 1$. From the description of the C_i-aspect it is not hard to check that at p_1 the ramification sequence is $(0, 1, 1, \cdots, 1)$ with total weight $(2n-1)(g-1) - 1$. As for p_j $(j = 2, 3)$ the ramification sequence there is $(g_j, g_j, \cdots, g_j)$. Thus the total weight contribution from these attachment points is $(2n-1)(g-1)g - 1$. Consequently, there is exactly one ordinary n-Weierstrass points on C_i. Finally, using the description of the C_1 and C_{g-2} aspects, we see that the weighted number of n-Weierstrass points on each of C_1 and C_{g-2} is two. Now we only need to show that on C_1 and C_{g-2}, there are only ordinary n-Weierstrass points to complete the proof of the following proposition.

(5.1) **Proposition (Lax):** If $g \geq 3$ then a general smooth curve of genus g has only ordinary n-Weierstrass points.

Proof :

The case when $n = 1$ or $g = 3$ has been discussed.
Denote C_1 by D. We can take the attachment points p_1, p_2, p_3 to be $0, \infty, 1$ (E_1 and E_2 are attached to p_1 and p_2.) Let dz be a non-zero section in $|\omega_D(2p_2)|$. Using the description for the limit canonical series, we will regard dz as a section of the D-aspect of the canonical series. In particular dz is a section that does not vanish at ∞, while $\frac{1}{z^2}dz$ does not vanish at 0. The D-aspect of the n-canonical series has a base point of order $g-2$ at p_3. Since we are not interested in ramification at p_3, we will remove $(g-2)p_3$ from the series, and thus look at the D-aspect as the subseries of

$$\left|\omega_D^{\otimes n}((2n-1)(g-2)p_3 + 2np_1 + 2np_2)\right|,$$

spanned by $(dz)^n$, $\frac{1}{z^{2n}}(dz)^n$ and the sections with vanishing order 2 or higher at p_1 and p_2. In terms of coordinates, sections in V_D can be expressed in the form

$$\frac{p(z)}{z^k(z-1)^l}(dz)^n, \tag{5.2}$$

with $p(z)$ being a polynomial such that $p(0) \neq 0$ and $p(1) \neq 0$, and degree of $p(z)$ is $\leq k + l$, $k \leq 2n$ and $l \leq (2n-1)(g-2)$. Now, if there is a weight 2 n-Weierstrass point on D, then it should be left invariant by the automorphism on D that fixes 1 and interchanges 0 and

∞. The only such point is -1. It suffices to show that the series is not ramified at -1. As usual, r will be $(2n-1)(g-1)-1$. At a weight 2 flex the vanishing sequence is either

$$(0,1,\ldots,r-1,r+2) \qquad \text{or} \qquad (0,1,\ldots,r-2,r,r+1).$$

Let us dispose of the first possibility. In terms of (5.2), a section s with $\mathrm{ord}_{-1}(s)=r+2$ will have $p(z)$ divisible by $(z+1)^{r+2}$. Note that $r+2=(2n-1)(g-2)+2n$ and $\deg(p(z))\leq k+l\leq 2n+(2n-1)(g-2)$. Therefore we must have $k=2n$ and $l=(2n-1)(g-2)$. So up to scalar

$$s=\frac{(z+1)^{r+2}}{z^k(z-1)^l}(dz)^n.$$

One checks at once that $s-(dz)^n$ has a zero of order 1 at ∞ and hence cannot be in V_D. This shows that s cannot be in V_D.

To get rid of the second possibility, we show that there is no section s in V_D with $\mathrm{ord}_{-1}(s)=r+1$. In terms of (5.2) $p(z)$ must be divisible by $(z+1)^{r+1}$. Since $r+1\leq \deg(p(z))\leq k+l\leq r+2$, $\deg(p(z))$ must be $r+1$ or $r+2$. If $\deg(p(z))=r+1$, then $k+l$ cannot be $r+2$, for that leads to s having vanishing order 1 at ∞. But if $k+l=r+1$, then $s-(dz)^n$ will again have vanishing order 1 at ∞. Thus $\deg(p(z))\neq r+1$. Hence $p(z)$ can only be $(z+1)^{r+1}(z+a)$ for some $a\in\mathbb{C}$. This forces $k=2n$ and $l=(2n-1)(g-2)$. The only choice of a that makes $s-(dz)^n$ vanish to order >1 at ∞ is $a=-r-l-1$. One checks that $s+(-1)^l(r+1+l)\frac{1}{z^{2n}}(dz)^n$ has vanishing order 1 at 0 and hence cannot be in V_D.

This shows that n-Weierstrass points on C_1 and C_{g-2} are ordinary. A smooth curve near C will have only ordinary n-Weierstrass points. **Q.E.D.**

Now we consider the problem of describing the possible vanishing sequences of n-Weierstrass points. We refer to [**H-O**] for results about the possible weights, and also see (2.4) for some possible restrictions on the vanishing sequences.

We shall show that every vanishing sequence of low weight actually occurs, and it does so in the expected dimension. A n-Weierstrass point p will be said to be *dimensionally proper* if the locus of n-Weierstrass points near p in the versal deformation family has codimension equal to the weight at p. For example, it follows from (2.1) (ii) that a 1-Weierstrasss point of a hyperelliptic curve is not dimensionally proper as n-Weierstrass point.

(5.3) **Theorem:** Let $\beta=(\beta_0,\ldots,\beta_r)$ where $r=(2n-1)(g-1)-1$ with $0\leq\beta_i\leq\beta_{i+1}$ for all i.

(i) For $n \geq 3$, $g \geq 2$, if $w(\beta) = \sum_{i=0}^{r} \beta_i \leq g-1$, then there exists a dimensionally proper n-Weierstrass point on some smooth curve of genus g having β as its ramification sequence.

(ii) For $n = 2$, $g \geq 3$, if $w(\beta) < g-1$, then the same conclusion holds.

Proof:

(i) The proof is almost exactly the same as in [**E-H,2**]. We proceed by induction on g. For $g = 2$, $w(\beta) = 0$ or 1. The first case is trivial. According to Lax ([**L**] Theorem 4), for generic genus 2 curves, all n-Weierstrass points (other than those that are 1-Weierstrass points) are ordinary, i.e. have weight 1. It is clear that these points are dimensionally proper. (This is where the proof breaks down for $n = 2$, because all 2-Weierstrass points are 1-Weierstrass points when $g = 2$.).

Now assume $g \geq 3$. We are going to look at a two-component curve with an elliptic tail, we will construct a dimensionally proper n-Weierstrass point on it with the desired ramification sequence and then smooth the series.

Since $w(\beta) \leq g-1$ the first $(2n-1)(g-1)-(g-1) = (2n-2)(g-1)$ terms of β are 0, that is,

$$\beta_0 = \beta_1 = \cdots = \beta_{(2n-2)(g-1)-1} = 0. \tag{5.4}$$

If all the β_i's are 0, then the result is clear. Otherwise let $j =$ smallest index such that $\beta_j > 0$. Thus by (5.4) $j \geq (2n-2)(g-1)$. Define another sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{(2n-1)(g-2)-1})$ as follows :

$$\begin{cases} \alpha_k = 0 & \text{for } k = 0, 1, \ldots, j-2n \\ \alpha_{j-2n+1} = \beta_j - 1 & \\ \alpha_k = \beta_{k+2n-1} & \text{for } k = j-2n+2, \ldots, (2n-1)(g-2)-1. \end{cases} \tag{5.5}$$

We have $0 \leq \alpha_i \leq \alpha_{i+1}$ for all i and $w(\alpha) \leq g-2$. By induction hypothesis, there exists a smooth curve C of genus $g-1$ and $p \in C$ with p a dimensionally proper n-Weierstrass point on C having α as its ramification sequence.

Pick an elliptic curve E and attach E to C at p. As in Proposition (5.1) of [**E-H,2**], we shall show that for a finite number of $q \in E$ there are finitely many series with the desired ramification sequences at p and q, and no such series for generic q. The underlying bundle of the E-aspect for the n-canonical series is $\mathcal{O}_E(2n(g-1)p)$. The vanishing sequence at p for the C-aspect is :

$$(0, 2, 3, \ldots, 2n-1, \alpha_0 + 2n, \alpha_1 + 2n + 1, \ldots, \alpha_{(2n-1)(g-2)-1} + (2n-1)(g-1)). \tag{5.6}$$

The compatibility condition forces the vanishing sequence at p for the E-aspect to be

(5.7) $(d-\beta'_r-1, d-\beta'_{r-1}-1, \dots, d-\beta'_{j+1}-1, d-\beta'_j, d-\beta'_{j-1}-1, \dots, d-\beta'_1-1, d-\beta'_0)$,

where $d = 2n(g-1)$, and $\beta'_i = \beta_i + i$. Thus we are in a situation to apply the criterion of Propostion (5.2) of **[E-H,2]** with $D = d \cdot p$. According to the proposition, there exists a series with (5.7) and β as vanishing sequence at p and q if

$$\text{(5.8)} \qquad \begin{aligned} &\text{(a)} \quad a_{r-i} + b_i = d \implies a_{r-i}p + b_i q \sim D \\ &\text{(b)} \quad a_{r-i}p + (b_i+1)q \sim D \implies b_{i+1} = b_i + 1. \end{aligned}$$

where a_i and b_i are the i-th term in (5.7) and β respectively. (a) can be satisfied only if $q-p$ is of order $\beta_j + j$. Let us choose q so that $q-p$ is exactly of order $\beta_j + j$, then (b) can be satisified if there does not exist $\beta_i + i + 1$ that kills $p-q$ for any $i \neq j$.

Now $\beta_i \leq g-2$, because weight of $\beta \leq g-1$ and $\beta_j > 0$. And $i+1 \leq r+1 = (2n-1)(g-1)$, therefore

$$\beta_i + i + 1 \leq (2n-1)(g-1) + (g-2) = 2n(g-1) - 1.$$

But $\beta_j \geq 1$ and from (5.4) $j \geq (2n-2)(g-1)$, So

$$\text{(5.9)} \qquad \begin{aligned} 2(j+\beta_j) &\geq 2 + 2(2n-2)(g-1) \\ &= 2n(g-1) - 1 + (2n-4)(g-1) + 3 \\ &> 2n(g-1) - 1. \end{aligned}$$

Thus no $\beta_i + i + 1$ can kill $p-q$. This proves the existence (and the dimensional properness) of the aspect. As in the second part of the proof of Proposition (4.5), we can construct the variety $\mathcal{K}^n(\mathcal{B}, s)$, with dimension greater than or equal to that of $\mathcal{B}$. Making use of the assumption that there exist dimensionally proper n-Weierstrass points of lower weight on genus $g-1$ curves, we may pick C and p so that the stratum consisting of reducible curves with weight w n-Weierstrass point as attachment point has codimension $w+1$ in $\mathcal{B}$. From the same line of argument as in (4.4), we can see that the fiber dimension over each stratum is w, thus there is no component of $\mathcal{K}^n(\mathcal{B}, s)$ lying over the locus of singular curves, and thus we conclude smoothing for any of the limit n-canonical series on $C \cup E$. The proposition now follows from Corollary (3.7) of **[E-H,1]** as in **[E-H,2]**.

(ii) Instead of starting with $g = 2$, we start with $g = 3$. We have shown that general genus 3 curves have only ordinary 2-Weierstrass points, so the proposition is true for $g = 3$.

The rest of the proof follows as in the previous case. **Q.E.D.**

(5.10) **Remark:** If we try to improve the bound from $g-1$ to g, we need to have appropriate n-Weierstrass points on low genus curves to start the induction. But for $g=3$, there is no 2-Weierstrass point with vanishing sequence equal to $(0,1,2,3,5,6)$: On a genus 3 non-hyperelliptic curve there can only be three kinds of vanishing sequence for the canonical series, namely $(0,1,2)$, $(0,1,3)$ and $(0,1,4)$. They all produce a section of the 2-canonical series with vanishing order 4. For hyperelliptic curves, the additional possibilities are $(0,2,3)$ and $(0,2,4)$, which still produce sections of vanishing order 4 for the 2-canonical series. Then, it seems that some restrictions have to come into play if one tries to extend the bound to the case when the weight equals $g-1$.

§6 Irreducibility of $\mathcal{W}_g^n$

(6.1) **Theorem:**
(i) $\mathcal{W}_g^n$ is irreducible for $g \geq 4,\ n \geq 1$ and for $g=3,\ n \neq 2$.
(ii) $\mathcal{W}_3^2 = \mathcal{W}_3^1 + \mathcal{S}$ where $\mathcal{S}$ is an irreducible divisor.

Proof:
We shall assume $n > 1$ since the result is known for $n=1$ (see [**F**]).
(i) Let $X = Y \cup_p Z$ be the union of two smooth curves Y and Z meeting at a point p, with $g_Y = g-i$, $g_Z = i$ where $1 \leq i \leq [g/2]$. Suppose that $p \notin W^n(Z)$ and $p \in W^n(Y)$ is ordinary (such a (Y,p) exists if $g-i \geq 3, n \geq 2$ or $g-i=2, n \geq 3$, see §5). We are in the situation of Proposition (4.3) (iv) and we can describe the limits of n-Weierstrass points as smooth curves degenerate to X: in each LSS with central fiber X there are $(g-i)(2n-1)^2(g-1)^2-1$ n-Weierstrass points of the generic fiber that specialize to the flexes of $\omega_Y^{\otimes n}((2n-1)i\ p)$ away from p, and there are $i(2n-1)^2(g-1)^2+1$ n-Weierstrass points that specialize to the flexes away from p of the linear series V_q on Z. The series V_q is defined as follows: we consider the embedding

$$Z \hookrightarrow \mathbb{P}H^0(Z,\mathcal{K})^*$$

where $\mathcal{K} = \omega_Z^{\otimes n}((2n-1)g_Y+1)p$, and let T denote the tangent line to Z at p. For each point $q \in T$, V_q denotes the linear series on Z cut by hyperplanes containing q. We remark that for $q \neq p$ the corresponding limit n-canonical series is refined, whereas V_p is not refined and p should be counted as one of the limits of n-Weierstrass points, with multiplicity one.

For each $r \in Z$ let H_r denote the osculating hyperplane to Z at r. It is easy to check that, away from the finitely many flexes for $\mathcal{K}$, a point r is a flex for V_q if and only if H_r contains q (and r turns out to be an ordinary flex for V_q). Also, since V_p has only finitely many flexes, there are only finitely many r's such that H_r contains T. Then we have a well defined rational map

$$f : Z \to T$$

sending r to $H_r \cap T$. For general $q \in T$,

$$f^{-1}(q) = \text{ flexes of } V_q.$$

Since Z is an irreducible variety, the monodromy associated to the finite map f acts transitively on the flexes of V_q and we conclude, as in [**E-H,3**] Proposition (2.3), that

(6.2) *by varying the Z-aspect we obtain monodromy that acts transitively on the limits of n-Weierstrass points on Z.*

Let $\bar{\mathcal{M}}_g^0$ denote the open subvariety of $\bar{\mathcal{M}}_g$ parametrizing automorphism-free stable curves having a family of limit n-canonical series of dimension at most one, and let $\pi : \bar{\mathcal{C}}_g^0 \to \bar{\mathcal{M}}_g^0$ denote the universal family. Also, let $\rho : \tilde{\mathcal{M}}_g \to \bar{\mathcal{M}}_g^0$ be the family of limit n-canonical series on the fibers of π (see (4.5)); a typical point of $\tilde{\mathcal{M}}_g$ is (X, L) where L is a limit n-canonical series on X. ρ is birational with fibers of dimension at most one. Let

$$\tilde{\pi} : \tilde{\mathcal{C}}_g \to \tilde{\mathcal{M}}_g$$

be the pull-back of π.
Assume $g \geq 5$; if $\Delta_i = \{(Y \cup_p Z), g_Z = i\}$ (with $1 < i < g/2$) is one of the boundary components of $\bar{\mathcal{M}}_g^0$ then

$$\rho^{-1}(\Delta_i) = \tilde{\Delta}_i \cup A_i \cup B_i$$

where

$$\begin{aligned}
\tilde{\Delta}_i &= \text{closure } \{(Y \cup_p Z, L), p \notin W^n(Y) \cup W^n(Z)\} \\
A_i &= \text{closure } \{(Y \cup_p Z, L), p \in W^n(Y),\ p \notin W^n(Z)\} \\
B_i &= \text{closure } \{(Y \cup_p Z, L), p \notin W^n(Y),\ p \in W^n(Z)\}.
\end{aligned}$$

The divisor $\tilde{\pi}^{-1}(A_i)$ breaks into two components $A_i' = \{(Y \cup_p Z, L, q),\ q \in Z\}$ and $A_i'' = \{(Y \cup_p Z, L, q),\ q \in Y\}$, with similar definitions for $B_i',\ B_i'',\ \tilde{\Delta}_i',\ \tilde{\Delta}_i''$.

Let $\tilde{\mathcal{W}}_g^n$ denote the closure in $\tilde{\mathcal{C}}_g$ of the divisor of n-Weierstrass points of smooth curves. It suffices to show that $\tilde{\mathcal{W}}_g^n$ is irreducible, and we shall do this by showing that the monodromy of $\tilde{\mathcal{W}}_g^n \to \tilde{\mathcal{M}}_g$ (see [**H**]) is transitive. By (6.2), the monodromy of

$$A_i' \cap \tilde{\mathcal{W}}_g^n \to A_i$$

is transitive and hence there is an irreducible divisor $\mathcal{V} \subset \tilde{\mathcal{W}}_g^n$ of relative degree at least $i(2n-1)^2(g-1)^2+1$ such that $A_i' \cap \mathcal{V} = A_i' \cap \tilde{\mathcal{W}}_g^n$, and it follows that

$$\tilde{\Delta}_i' \cap \mathcal{V} = \tilde{\Delta}_i' \cap \tilde{\mathcal{W}}_g^n. \tag{6.3}$$

For $g \geq 7, n \geq 2$ or $g = 5, n \geq 3$ we can repeat the argument with A_i replaced by B_i to conclude that there exists an irreducible divisor $\mathcal{U} \subset \tilde{\mathcal{W}}_g^n$ of relative degree at least $(g-i)(2n-1)^2(g-1)^2+1$ such that

$$\tilde{\Delta}_i'' \cap \mathcal{U} = \tilde{\Delta}_i'' \cap \tilde{\mathcal{W}}_g^n. \tag{6.4}$$

If $\mathcal{U} \neq \mathcal{V}$ then we get the contradiction $\deg \mathcal{U} + \deg \mathcal{V} > \deg \tilde{\mathcal{W}}_g^n$. Then $\mathcal{U} = \mathcal{V}$ and it follows from (6.3) and (6.4) that $\deg \mathcal{V} = \deg \tilde{\mathcal{W}}_g^n$, which implies that $\mathcal{V} = \tilde{\mathcal{W}}_g^n$ and hence $\tilde{\mathcal{W}}_g^n$ is irreducible.

For $g = 6$, $\pi^{-1}(\Delta_3) \subset \bar{\mathcal{C}}_6$ is irreducible and $A_3 = B_3$ (using notation similar to the one introduced above). We deduce the existence of $\mathcal{V}$ as before, and since monodromy interchanges the components of a curve in Δ_3, we conclude that $\mathcal{W}_6^n$ is irreducible for $n \geq 2$.

For $g = 4$, the same argument works for $n \geq 3$, using Δ_2.

For the remaining cases $g = 5, (n = 2), g = 4, (n = 2)$ and $g = 3, (n \geq 3)$ we modify the previous proof as follows (in fact, the argument below covers all cases, but we think the reasoning is more clear by allowing this repetition).

Let $X = Y \cup_p Z$ with Y smooth of genus $g-1$ and Z smooth of genus 1, and suppose that $p \in W^n(Y)$ is ordinary (see §5). Let $f : \mathcal{X} \to \mathcal{B}$ be the Kuranishi family of X, $\Delta \subset \mathcal{B}$ the divisor of singular curves and $f^{-1}(\Delta) = \Delta' + \Delta''$ where $\Delta' \supset Z$. Also, let $\rho : \tilde{\mathcal{B}} \to \mathcal{B}$ be the family of n-canonical series on the fibers of f (Proposition (4.5)). Exactly as above, we deduce the existence of an irreducible divisor $\mathcal{V} \subset W^n(f) \subset \mathcal{X}$ of relative degree at least $(g-1)(2n-1)^2+1$ such that

$$\Delta' \cap \mathcal{V} = \Delta' \cap W^n(f). \tag{6.5}$$

Let $\bar{\mathcal{V}}$ be the Zariski closure in $\bar{\mathcal{C}}_g$ of the image of $\mathcal{V}$.
We claim that $\bar{\mathcal{V}} = \bar{\mathcal{W}}_g^n$, and hence $\mathcal{W}_g^n$ is irreducible. To see this, let X_0 be $\mathbb{P}^1$ with g elliptic tails $E_1, \ldots, E_g$ joined at $p_1, \ldots, p_g \in \mathbb{P}^1$ respectively. By Proposition (4.6) and §5, in a limit series situation with central fiber X_0 the n-Weierstrass points specialize on E_i to the $(2n-1)(g-1)$-torsion points away from p_i, and on $\mathbb{P}^1$ to g points $q_1, \ldots, q_g$ different from the p_i's. By (6.5) the pull-back of $\bar{\mathcal{V}}$ to the Kuranishi family of X_0 contains the torsion points of some E_i and, by monodromy, it contains the torsion points of all E_i's. Hence, deg $\bar{\mathcal{V}} \geq$ deg $\mathcal{W}_g^n - g$ and $\bar{\mathcal{W}}_g^n - \bar{\mathcal{V}}$ is an effective divisor of relative degree at most g. But there is no rationally defined effective divisor of degree at most g on curves of genus g (see [**Ci**]) and hence $\bar{\mathcal{V}} = \bar{\mathcal{W}}_g^n$, as claimed.

(ii) Follows from the case $d = 4$ of Proposition (6.6) below. **Q.E.D.**

If $X \subset \mathbb{P}^2$ is a smooth plane curve of degree d and $n \in \mathbb{N}$, we shall denote $F^n(X)$ the set of flexes for $\mathcal{O}_X(n)$. We have $F^1(X) \subset F^2(X)$ and $S(X) = F^2(X) - F^1(X)$ is classically known as the set of *sextatic points* of X. We want to show that the sextatic points of smooth plane curves form an irreducible family.
Let S_d denote the vector space of homogeneous polynomials of degree d in three variables, $\mathcal{B}_d \subset \mathbb{P}(S_d)$ the open set parametrizing smooth plane curves, and

$$\begin{array}{ccc} \mathcal{X}_d = \{(X,p) \in \mathcal{B}_d \times \mathbb{P}^2,\ p \in X\} & \xrightarrow{\ \eta\ } & \mathbb{P}^2 \\ \pi \big\downarrow & & \\ \mathcal{B}_d & & \end{array}$$

the universal family of smooth plane curves of degree d. If $L_n = \eta^* \mathcal{O}_{\mathbb{P}^2}(n)$ then we define the divisors $\mathcal{F}^n \subset \mathcal{X}_d$ as the degeneracy loci of the maps of vector bundles of same rank $\pi^*\pi_*(L_n) \to P^\rho(L_n)$, where P^ρ means principal parts of order $\rho = h^0(X, \mathcal{O}_X(n))$.

(6.6) **Proposition:** $\mathcal{F}^1 \subset \mathcal{F}^2$ and $\mathcal{S} := \mathcal{F}^2 - \mathcal{F}^1$ is irreducible.

Proof:

Consider

$$\begin{array}{ccc} \mathcal{I} = \{(X,C,p) \in \mathcal{B}_d \times \mathbb{P}(S_2) \times \mathbb{P}^2,\ (X,C)_p \geq 6\} & \xrightarrow{\beta} & \mathbb{P}(S_2) \times \mathbb{P}^2 \\ \alpha \big\downarrow & & \\ \mathcal{X}_d & & \end{array}$$

where $(X,C)_p$ is the intersection multiplicity of X and C at p. First of all, we have $\mathcal{F}^2 = \alpha(\mathcal{I})$. If C is a smooth conic and $p \in C$ then $\beta^{-1}(C,p)$ is open in a linear system of dimension $d(d+3)/2-6$; then, $\mathcal{I}' = \beta^{-1}\{(C,p),\ p \in C,\ C \text{ smooth conic}\}$ is an irreducible component of $\mathcal{I}$ and $\mathcal{S} = \alpha(\mathcal{I}')$ is irreducible.

If $C = 2L$ is a double line and $p \in C$ then $\beta^{-1}(C,p)$ has dimension $d(d+3)/2-3$ and $\mathcal{F}^1 = \alpha(\beta^{-1}\{(2L,p),\ L \text{ line},\ p \in L\})$ is another component of $\mathcal{F}^2$.

We claim that there are no other components of $\mathcal{F}^2$, and to see this we analize the other strata of the space of pointed conics. Suppose that $C = L \cup L'$ where L and L' are distinct lines and $p \in L - L'$. Then $(X,C)_p \geq 6$ is equivalent to $(X,L)_p \geq 6$; this imposes six conditions on X and since our stratum is five dimensional it does not contribute a component of $\mathcal{F}^2$ (notice that since $\mathcal{F}^2$ is determinantal all its components are hypersurfaces). If $C = L \cup L'$ with $L \neq L'$ and $p \in L \cap L'$ then $(X,C)_p \geq 6$ if and only if $(X,L)_p = k$, $(X,L')_p \geq 6-k$, for some $k \geq 1$. For each value of k we have at least five conditions on X and (C,p) moves in a four dimensional family, so again we have no new component of $\mathcal{F}^2$. **Q.E.D.**

(6.7) **Remark:** Let X be a smooth hyperelliptic curve of genus g, realized as a double cover $\pi : X \to \mathbb{P}^1$, with branch points $p_1, \dots, p_{2g+2} \in \mathbb{P}^1$ and involution σ. From Example (2.1)(ii), X has $N = (2n-1)^2 g(g-1)^2 - g(g+1)^2$ n-Weierstrass points that are not 1-Weierstrass points. The set $W^n(X) - W^1(X)$ is clearly σ-invariant and we obtain a $PSL(2,\mathbb{C})$-equivariant rational map

$$h : P^{2g+2} \to P^{N/2}$$

sending $p_1 + \dots + p_{2g+2}$ to $\pi(W^n(X) - W^1(X))$, where we let $P^m = \mathrm{Sym}^m(\mathbb{P}^1) \cong \mathbb{P}^m$.

Consider the case $g = 2$ and let's analyze the limit of the n-Weierstrass points as a smooth curve degenerates to $X = E \cup_p E'$ where E and E' are elliptic curves joined at the point p. By Proposition (4.3) (iii), X has a unique limit n-canonical series with E-aspect

$$V_E = H^0(E, \mathcal{O}_E((2n-2)p)) + \mathbb{C}.\eta^n \subset H^0(E, \mathcal{O}_E(2np))$$

and similarly for E'. The vanishing sequence of V_E at p is $(0,2,3,\dots,2n-2,2n)$ and hence this limit series is refined. The weight at p is $2n-1$ and by the Plücker formula on E, there are $(2n-1)^2$ flexes of V_E away from p. In a degeneration, half of the total number $2(2n-1)^2$ of n-Weierstrass points go to E, the other half goes to E', and none of them goes to p.
Embed $E \hookrightarrow \mathbb{P}^{2n-1}$ by the linear system $|2np|$ and let T be the tangent line to E at p. The hyperplane $\eta^n = 0$ does not contain p and cuts T at a point p' (notice that η is defined up to an additive constant, but if we make a different choice the point p' is the same); we remark that the point p' is canonical, it depends only on E and $p \in E$. The linear system V_E is then the linear system cut by hyperplanes through p' and the $(2n-1)^2$ points we are investigating are the points $r \in E$ such that the osculating hyperplane H_r contains p'.
If $q \in E$ is one of the three 2-torsion points with respect to p, let f be a rational function such that $(f) = 2q - 2p$. We may take f as our choice for η and we see that q is a flex for V_E with weight three (this corresponds to the fact that q is a limit of 1-Weierstrass points). Notice that this also gives another description of the point p': the osculating hyperplanes at the 2-torsion points all cut T at the same point p'.
The other $(2n-1)^2 - 9$ flexes of V_E are points canonically associated to the pointed elliptic curve (E,p), and our lack of control over these points is the reason why we didn't consider the case $g = 2$ in Theorem (6.1).

(6.8) **Remark:** In the proof of Theorem (6.1) we considered a curve X consisting of $Y = \mathbb{P}^1$ with g elliptic tails $E_1,\dots,E_g$ attached at points $p_1,\dots,p_g$ in general position. The Y-aspect of the unique limit n-canonical series on X (Proposition (4.6)) has flexes away from the p_i's with total weight g. This defines a $PSL(2,\mathbb{C})$-equivariant rational map

$$\phi_n : P^g \to P^g$$

with notation as in (6.7). Unfortunately, our understanding of these maps is far from being complete. This seems to be an obstruction for determining, among other things, the monodromy group of $\mathcal{W}_g^n \to \mathcal{M}_g$. The maps ϕ_n are more explicitely described in [**Li**].

§7 Class of $\bar{\mathcal{W}}_g^n$

Let $\mathcal{M}_g$ denote the moduli space of genus g curves over the complex numbers, $\mathcal{C}_g$ the moduli space of pointed curves, $\bar{\mathcal{M}}_g$ the moduli space of stable curves, $\bar{\mathcal{C}}_g$ the moduli space of pointed stable curves, $\pi : \mathcal{C}_g \to \mathcal{M}_g$ and $\pi : \bar{\mathcal{C}}_g \to \bar{\mathcal{M}}_g$ the natural morphisms (see [**K**]).

Let

$$\mathcal{W}_g^n = \{(X,p)/p \text{ is a } n\text{-Weierstrass point of } X\} \subset \mathcal{C}_g$$

be the divisor of n-Weierstrass points and $\bar{\mathcal{W}}_g^n \subset \bar{\mathcal{C}}_g$ its closure. We shall determine the class $[\bar{\mathcal{W}}_g^n] \in \mathrm{Pic}(\bar{\mathcal{C}}_g) \otimes \mathbb{Q}$ as a linear combination of the standard basis $\{\omega, \lambda, \delta_0, \delta_1, \ldots, \delta_{g-1}\}$.

(7.1) **Theorem:** For $g \geq 2$ and $n \geq 2$, the following relation holds in $\mathrm{Pic}(\bar{\mathcal{C}}_g) \otimes \mathbb{Q}$

$$[\bar{\mathcal{W}}_g^n] = (2n-1)^2\binom{g}{2}\omega - (6n^2-6n+1)\lambda + \binom{n}{2}\delta_0 - \delta_{g-1} - (2n-1)^2\sum_{i=1}^{g-2}\binom{g-i}{2}\delta_i.$$

Proof:

Consider a family $f: X \to B$ of stable curves of genus g, where B, X and the general fiber of f are smooth; let $\Delta_i' \subset B$ be the divisor parametrizing singular fibers with a singular point of type i for $i = 0, 1, \ldots, [g/2]$, let $f^*(\Delta_i') = \Delta_i \cup \Delta_{g-i}$, where a point in Δ_i belongs to a piece of genus i of the fiber through it, and let S denote the locus of singular points of fibers of f. We have a natural map of rank $r+1$ locally free sheaves on $X - S$

$$\tau : f^* f_*(\omega^{\otimes n}) \to P^{r+1}(\omega^{\otimes n})$$

where $\omega = \omega_{X|B}$ is the relative dualizing sheaf, P^{r+1} are principal parts of order $r+1$ and, as before, $r+1 = (2n-1)(g-1)$. If $\mathcal{W}_g^n(f)$ denotes the divisor of n-Weierstrass points on the smooth fibers of f then we have the equality of divisors on X

$$(7.2) \qquad (\det\ \tau) = \overline{\mathcal{W}_g^n(f)} + \sum_{i=0}^{g-1} \mathrm{ord}_{\Delta_i}(\det\ \tau)\ \Delta_i$$

and the equality in linear equivalence

$$(7.3) \qquad [\det\ \tau] = c_1 P^{r+1}(\omega^{\otimes n}) - c_1 f^* f_*(\omega^{\otimes n})$$

To compute the Chern classes in (7.3) consider the exact sequences of sheaves on $X - S$

$$0 \to F \otimes \mathrm{Sym}^{k-1}(\omega) \to P^k(F) \to P^{k-1}(F) \to 0$$

where F is a locally free sheaf on $X - S$. It follows that

$$c_1 P^m(F) = \sum_{k=1}^{m} c_1(F) + (k-1)c_1(\omega) = mc_1(F) + \binom{m}{2} c_1(\omega)$$

and then,

$$c_1 P^{r+1}(\omega^{\otimes n}) = (2n-1)^2 \binom{g}{2} c_1(\omega). \tag{7.4}$$

On the other hand, it follows from the theorem of Grothendieck-Riemann-Roch (see [**M1**] Theorem (5.10)) that

$$c_1 f^* f_*(\omega^{\otimes n}) = (6n^2 - 6n + 1)c_1(\lambda_f) - \binom{n}{2} \delta_f \tag{7.5}$$

where $\lambda_f := \det\ f^* f_*(\omega)$ and δ_f is the class of $\Delta = \sum_{i=0}^{g-1} \Delta_i$.

To determine $\mathrm{ord}_{\Delta_i}(\det\ \tau)$ is a local question and we can assume that $B \subset \mathbb{C}$ is a disk and the central fiber X_0 is a general curve of arithmetic genus g with one node. First, we claim that

$$\mathrm{ord}_{\Delta_0}(\det\ \tau) = 0. \tag{7.6}$$

To see this, we observe that since the formation of τ commutes with base change (see [**G**]), if $\det\ \tau = 0$ on the irreducible X_0 then there would be a nonzero section of $\omega_{X|B}^{\otimes n}|_{X_0} = \omega_{X_0}^{\otimes n}$ vanishing on X_0, which is absurd.

Next, we claim that

$$\begin{aligned} \mathrm{ord}_{\Delta_i}(\det\ \tau) &= \binom{n}{2} + (2n-1)^2 \binom{g-i}{2}, \qquad 1 \le i \le g-2 \\ &= \binom{n}{2} + 1, \qquad i = g-1 \end{aligned} \tag{7.7}$$

To see this, let $X_0 = Y \cup_p Z$ where Y and Z are general curves of genus i and $g-i$ respectively. Denote $a_0, a_1, \ldots, a_s$ the vanishing sequence of $\omega_{X|B}^{\otimes n}|_Z = \omega_Z^{\otimes n}(np)$ at p, and let $\{\eta_0, \ldots, \eta_s\}$ be sections realizing this vanishing sequence. By Corollary (4.4) there are no n-Weierstrass points specializing to p and then η_j extends to a section of $\omega_{X|B}^{\otimes n}$ vanishing to order a_j along Y. Then det τ vanishes along Y to order at least $\sum_{j=0}^{s} a_j$, and by the argument of [**Cu**] Proposition (2.0.8), this is the exact order of vanishing. Now, for $g-i>1$ we have $a_j = j,\ j = 0,\dots,s$ and when $g-i=1$, $a_j = j,\ j=0,\dots,s-1,\ a_s = s+1$. A little arithmetic proves (7.7), and combining (7.2)-(7.7) we obtain (7.1). **Q.E.D.**

(7.8) **Remark:** Applying Hurwitz formula to $\pi : \bar{\mathcal{W}}_g^n \to \bar{\mathcal{M}}_g$ one may compute the class of the closure of the divisor of curves with special n-Weierstrass points, modulo the coefficient of Δ_0. Also, combining (6.1) with Proposition (5.1) of [**Cu**] one obtains formulas for the classes of the effective divisors $\pi_*(\bar{\mathcal{W}}_g^n \cap \bar{\mathcal{W}}_g^m)$.

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