CURVES OF GENUS TEN ON K3 SURFACES

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Introduction. Let C denote a smooth complete algebraic curve and L a line bundle on C. There is a natural map, called the Wahl or Gaussian map,

$$\Phi_L : \wedge^2 H^0(C, L) \to H^0(C, \Omega^1_C \otimes L^{\otimes 2})$$

which sends $s \wedge t$ to $s\ dt - t\ ds$. J. Wahl made the striking observation that if C is embeddable in a K3 surface then Φ_L is not onto for $L = \Omega^1_C$ ([W], Thm. 5.9); this raises the natural problem of studying the stratification of the moduli space of curves $\mathcal{M}_g$ by the rank of the Wahl map $\Phi(C) = \Phi_{\Omega^1_C}$. Roughly speaking, our main theorem says that the closure of the locus of curves of genus 10 which lie on a $K3$ is equal to the locus where $\Phi(C)$ fails to be surjective.

In order to state the theorem precisely and explain what is special about the case of genus 10, we need to introduce some spaces. Let $\mathcal{F}_g$ be the moduli space of $K3$ surfaces with a polarization of genus g, $\mathcal{P}_g$ the union, over all $S \in \mathcal{F}_g$ of the linear series $|\mathcal{O}_S(1)|$. Let $\mathcal{K}$ be the closure of the image of the natural rational map $\mu : \mathcal{P}_g \to \mathcal{M}_g$. As the dimension of $\mathcal{P}_g$ is 29 and the dimension of $\mathcal{M}_g$ is $3g - 3$, one might naively expect μ to be dominant for $g \leq 10$ and finite onto its image for $g \geq 11$. These expectations hold for $g \leq 9$ ([M], Thm. 6.1) and for odd $g \geq 11$ and even $g \geq 20$ ([M-M], Thm. 1), but for $g = 10$, Mukai showed that μ is not dominant ([M], Thm. 0.7). This exceptional behavior is due to the fact that the general $K3$ surface of genus 10 is a codimension 3 plane section of a certain 5-fold, so that when a curve lies on a general $K3$, it in fact lies on a 3-dimensional family of them. One of our first tasks is to show that $\mathcal{K}$, the closure of the image of μ, is a divisor when $g = 10$.

Over the open subset $\mathcal{M}^o_{10}$ of $\mathcal{M}_{10}$ of curves without automorphisms we have the relative Wahl map; let $\mathcal{W}^o$ denote its degeneracy locus and $\mathcal{W}$ the closure of $\mathcal{W}^o$ in $\mathcal{M}_{10}$. It is a

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theorem of Ciliberto-Harris-Miranda [C-H-M] that $\mathcal{W}$ is a divisor (i.e. the Wahl map does not degenerate everywhere), and by Wahl's theorem $\mathcal{K} \leq \mathcal{W}$. Our result can then be stated as follows.

Theorem. *We have an equality of divisors*

$$\mathcal{W} = 4\mathcal{K}.$$

Moreover, for the general curve C of genus 10 which can be embedded in a $K3$ surface, the codimension of the image of the Wahl map $\Phi(C)$ is 4.

It is worth remarking that *a priori* not every curve of genus 10 on a $K3$ appears in $\mathcal{K}$: the variety $\mathcal{P}$ consists of pairs (S, C) where $\mathcal{O}_S(C)$ is indivisible in $\mathrm{Pic}(S)$. But by Wahl's theorem, every curve on a $K3$ has a degenerate Wahl map, so by the theorem defines a point of $\mathcal{K}$. It would be interesting to see explicitly a family of curves polarizing $K3$s of genus 10 degenerating, for instance, to a plane sextic (which polarizes a $K3$ of genus 2).

We also note that Voisin proved ([V] Prop. 3.3) that the corank of $\Phi(C)$ is at most 3 for a genus 10 curve satisfying certain hypothesis (3.1) i), ii) and iii) (*loc. cit.*). These hypotheses hold for a general curve, and i) and iii) hold for a general curve on a $K3$. It follows that ii) does not hold for any curve of genus 10 on a $K3$, answering negatively [V] (4.13)a) for $g = 10$.

To prove the theorem we first study the cohomology of a certain 5-fold X, which is a homogeneous space for the exceptional Lie group G_2, using a theorem of Bott as in [M]. This allows us to show, in §2, that $\mathcal{K}$ is a divisor and that for every C which is a smooth codimension 4 plane section of X, the corank of $\Phi(C)$ is 4. This establishes the inequality of divisors $\mathcal{W} \geq 4\mathcal{K}$. In §3, we compute the classes of the divisors $\mathcal{W}$ and $\mathcal{K}$ and find that $\mathcal{W}$ is linearly equivalent to $4\mathcal{K}$. The desired equality of divisors then follows.

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§1. The cohomology of the 5-fold X. One of the main tools in our analysis will be the cohomology groups of a certain homogeneous variety X used by Mukai [M] to study the moduli space of $K3$ surfaces of genus 10. To recall the definition, let $\mathbf{g}$ be the complex semisimple Lie algebra attached to the exceptional root system G_2, let G be the corresponding simply connected Lie group, and let $\rho : G \to \mathrm{Aut}(\mathbf{g})$ be the adjoint representation. If $v \in \mathbf{g}$ is a highest weight vector for ρ, then $X = \rho(G)v$ is the orbit of v. Equivalently, if $P \subseteq G$ is the maximal parabolic subgroup of G associated to the longer of the two roots in a system of simple roots for $\mathbf{g}$, then $X \cong G/P$. The homogeneous variety X has dimension

5 and is naturally embedded in $\mathbf{P}(\mathbf{g})$ as a subvariety of degree 18; its canonical bundle is isomorphic to $\mathcal{O}(-3)$ ([M], p. 363). Mukai shows that the general $K3$ surface of genus 10 is a codimension 3 plane section of X and any abstract isomorphism between two such $K3$s is realized by the action of G on the Grassmannian of codimension 3 planes in $\mathbf{P}(\mathbf{g})$ ([M], Thm. 0.2).

Recall that homogeneous vector bundles on X are in one to one correspondence with finite dimensional linear representations of P. For example, if $\{\alpha_1, \alpha_2\}$ is a basis for the root system G_2 with α_1 the shorter root, so that P is the subgroup corresponding to the subalgebra whose roots are all of the negative roots together with α_1, then the tangent bundle to $X = G/P$ corresponds to the (reducible) representation of P with highest weight $w_1 = 3\alpha_1 + 2\alpha_2$. It has an irreducible rank 4 subbundle corresponding to the representation of P with highest weight $\alpha_2 + 3\alpha_1$ and the quotient is isomorphic to $\mathcal{O}_X(1)$, corresponding to the irreducible representation of P with highest weight w_1. Similarly N_X, the normal bundle of X in $\mathbf{P}(\mathbf{g})$, has a composition series with quotients of rank 1, 3 and 4 corresponding to irreducible representations with highest weights 0, $4\alpha_1 + 2\alpha_2$, and $6\alpha_1 + 3\alpha_2$ respectively.

Now a theorem of Bott ([B]; see also [M], 1.6) asserts that when E is an irreducible homogeneous vector bundle on a compact homogeneous variety $X = G/P$, at most one of the cohomology groups $H^i(X, E)$ is non-zero, and when non-zero, the group is an irreducible G-module. Moreover, he gives a recipe for calculating the index of the non-vanishing cohomology group. Application of this result to the X considered above, which we leave as a pleasant exercise for the reader (compare [M], §1), yields the following result.

Lemma 1.1.

(1) *We have* $h^0(X, T_X(-1)) = 0$ *and* $H^0(X, T_X) \cong \mathbf{g}$ *as a* G*-module. Moreover,* $h^i(X, T_X(-i)) = h^i(X, T_X(-i-1)) = 0$ *for* $i = 1, 2, 3, 4$.

(2) *We have* $H^0(X, N_X(-1)) \cong \mathbf{g}$ *as a* G*-module and* $h^i(X, N_X(-i-1)) = 0$ *for* $i = 1, \dots, 4$*. Also,* $h^i(X, N_X(-i-2) = 0$ *for* $i = 0, \dots, 4$.

Now suppose that S is a smooth codimension 3 plane section of X and that C is a smooth hyperplane section of S; then S is a $K3$ surface and C is a canonically embedded curve of genus 10. Using Koszul resolutions of $\mathcal{O}_S$ and $\mathcal{O}_C$ as $\mathcal{O}_X$-modules, one easily checks the following assertions.

Lemma 1.2.

(1) $h^0(S, N_S(-1)) = 14$.

(2) $h^0(C, T_X(-1)|_C) = 0$ *and* $h^0(C, T_X|_C) = 14$.

(3) $h^0(C, N_C(-2)) = 0$ *and* $h^0(C, N_C(-1)) = 14$.

(Here N_C and N_S are the normal bundles to C and S in the projective spaces they span in $\mathbf{P}(\mathbf{g})$; the last part also uses the standard isomorphism $N_X|_C \cong N_C$.)

§**2. The corank of the Wahl map.** We retain the notations of the introduction.

Proposition 2.1. *Suppose S is a general $K3$ surface of genus 10. Then $h^1(S, T_S(-1)) = 3$ and $h^2(S, T_S(-1)) = 1$.*

Proof: Consider the exact sequence

$$0 \to T_S(-1) \to T_{\mathbf{P}}(-1)|_S \to N_S(-1) \to 0$$

where $S \subseteq \mathbf{P} = \mathbf{P}^{10}$ is the given embedding. The long exact sequence of cohomology yields

$$0 \to H^0(S, T_{\mathbf{P}}(-1)|_S) \to H^0(S, N_S(-1)) \to H^1(S, T_S(-1)) \to H^1(S, T_{\mathbf{P}}(-1)|_S).$$

But the Euler sequence for $T_{\mathbf{P}}|_S$ implies that $h^0(T_{\mathbf{P}}(-1)|_S) = 11$ and $h^1(T_{\mathbf{P}}(-1)|_S) = 0$. Indeed, we have

$$\begin{aligned} 0 \to H^0(S, \mathcal{O}_S)^{11} \to H^0(S, T_{\mathbf{P}}(-1)|_S) \to H^1(S, \mathcal{O}_S(-1)) \to \\ H^1(S, \mathcal{O}_S)^{11} \to H^1(S, T_{\mathbf{P}}(-1)|_S) \to H^2(S, \mathcal{O}_S(-1)) \to H^2(S, \mathcal{O}_S)^{11} \end{aligned}$$

with $H^1(S, \mathcal{O}_S) = 0$ (S is a $K3$) and $H^1(S, \mathcal{O}_S(-1)) = 0$ ([K], Thm. 2.5); moreover, the map $H^2(S, \mathcal{O}_S(-1)) \to H^2(S, \mathcal{O}_S)^{11}$ is injective by duality and the projective normality of S ([Ma], Prop. 2). By Lemma 1.2, $h^0(S, N_S(-1)) = 14$, so $h^1(S, T_S(-1)) = 3$. As $h^0(S, T_S(-1)) = 0$, Riemann-Roch implies $h^2(S, T_S(-1)) = 1$.

Proposition 2.2. *The locus $\mathcal{K} \subseteq \mathcal{M}_{10}$ is a divisor.*

Proof: First we need some deformation theory. Generally, given a smooth complete curve C in a smooth complete surface S, we have the tangent sheaf T_S of S, the tangent sheaf T_C of C and the restriction $T_S|_C = T_S \otimes \mathcal{O}_C$. Extending the latter two sheaves by 0 on S, we can define a coherent sheaf F on S as the fiber product

$$\begin{array}{ccc} F & \longrightarrow & T_C \\ \downarrow & & \downarrow \\ T_S & \longrightarrow & T_S|_C. \end{array}$$

The sheaf F is locally free of rank 2 and sits in exact sequences

$$0 \to T_S(-C) \to F \to T_C \to 0 \tag{2.3}$$

and

$$0 \to F \to T_S \to N_{C|S} \to 0. \tag{2.4}$$

It is easy to check that the space of first order deformations of the pair $C \subseteq S$ is isomorphic to $H^1(S, F)$.

Returning to the case where S is a general $K3$ of genus 10 and C is a smooth plane section of C, the long exact cohomology sequence of (2.3) gives

$$0 \to H^1(S, T_S(-C)) \to H^1(S, F) \to H^1(C, T_C) \to H^2(S, T_S(-C)) \to H^2(S, F) \to 0$$

and by Proposition 2.1, $h^2(S, T_S(-C)) = 1$. But $H^1(S, F) \to H^1(C, T_C)$ cannot be surjective as the locus of curves on $K3$s has codimension at least one in $\mathcal{M}_{10}$. Thus $h^2(S, F) = 0$, $h^1(S, F) = 29$ and the codimension of the image of $H^1(S, F) \to H^1(C, T_C)$ is exactly 1. But this last map is the differential of the map μ of the Introduction, so the image of μ actually fills out a divisor.

Remark 2.5. Let $\mu : \mathcal{P} \to \mathcal{M}_{10}$ be the rational moduli map as in the Introduction. If $\mathcal{K}$ is the closure of the image of μ and N is the normal bundle of $\mathcal{K}$ in $\mathcal{M}_{10}$ then it follows from the long exact cohomology sequence of (2.4) and the analysis above that the fiber at $(C, S) \in \mathcal{P}$ (for C a curve in the K3 surface S) of the bundle $\mu^*(N)$ is the one dimensional vector space $H^2(S, T_S(-C))$.

Proposition 2.6. *If C is a smooth codimension 4 plane section of X, then Corank $\Phi(C) = 4$. For every C in $\mathcal{K}$, Corank $\Phi(C) \geq 4$.*

Proof: By [B-E-L] (2.11), Corank $\Phi(C) = h^0(C, N_C(-1)) - g$ where N_C is the normal bundle to C in its canonical embedding. But by Lemma 1.2, $h^0(C, N_C(-1)) = 14$ for a smooth codimension 4 plane section of X. The second assertion follows by semi-continuity.

Remarks 2.7. a) If C is any smooth codimension 4 plane section of X then the Clifford index of C is at least 3: if Cliff$(C) \leq 2$, C is either hyperelliptic, trigonal, or a degeneration of a smooth plane sextic and in all these cases, the corank of $\Phi(C)$ is strictly greater than 4.

b) It is possible to give (at least) two other proofs of the inequality Corank $\Phi(C) \geq 4$: if C

has Cliff$(C) \geq 3$, it follows from results in [B-E-L] that $h^0(N_C(-2)) = 0$ where N_C is the normal bundle to C in its canonical embedding. On the other hand, a smooth codimension 4 plane section C of X is clearly 4-extendable, so applying a theorem of Zak (described in [B-E-L]) and [B-E-L], 2.11, we find Corank $\Phi(C) \geq 4$.
c) For a third proof, let C be a smooth codimension 4 plane section of X and consider the commutative diagram

$$\begin{array}{ccccc}
\bigwedge^2 H^0(X, \mathcal{O}_X(1)) & \xrightarrow{a} & H^0(X, \Omega^1_X(2)) & & \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} & & H^0(C, \Omega^1_X(2)|_C). \\
\bigwedge^2 H^0(C, \mathcal{O}_C(1)) & \xrightarrow[d]{} & H^0(C, \mathcal{O}_C(3)) & &
\end{array}$$

Here the horizontal maps are the Wahl maps for $\mathcal{O}(1)$ and the other maps are the natural restrictions. Now b is clearly surjective, so the image of $d = \Phi(C)$ is contained in the image of f. We claim that f has corank 4: the exact sequence of cohomology of $0 \to N^*_{C|X}(2) \to \Omega^1_X(2)|_C \to \Omega^1_C(2) \to 0$ gives

$$H^0(C, \Omega^1_X(2)|_C) \to H^0(C, \Omega^1_C(2)) \to H^1(C, N^*_{C|X}(2)) \to H^1(C, \Omega^1_X(2)|_C)$$

and the claim follows by observing that $h^1(N^*_{C|X}(2)) = h^1(\mathcal{O}_C(-1)^{\oplus 4}(2)) = 4$ and that $H^1(\Omega^1_X(2)|_C) = H^0(T_X(-1)|_C)^* = 0$ (Lemma 1.2).

Corollary 2.8. *We have an inequality of divisors $\mathcal{W} \geq 4\mathcal{K}$.*

PROOF: Let $\mathcal{M} = \mathcal{M}^o_{10}$ denote the moduli space of smooth automorphism-free genus 10 curves over the complex numbers, $\pi : \mathcal{C} \to \mathcal{M}$ the universal curve, $\omega = \Omega^1_{\mathcal{C}|\mathcal{M}}$ the sheaf of relative differentials and $\lambda = \det(\pi_*(\omega)) \in \mathrm{Pic}(\mathcal{M})$. We have the relative Wahl map

$$\Phi : \wedge^2 \pi_*(\omega) \to \pi_*(\omega^{\otimes 3})$$

which is a map of bundles of rank 45; let $\mathcal{W}$ denote its degeneracy locus. By [C-H-M] the support of $\mathcal{W}$ is a proper subvariety of $\mathcal{M}$ and hence $\mathcal{W}$ is a divisor.

By proposition 2.6, the universal Wahl map Φ has corank at least 4 at each point of $\mathcal{K}$. It follows that $\det(\Phi)$ vanishes to order at least 4 along $\mathcal{K}$. Indeed, take a small arc $\{C_t\}$ crossing $\mathcal{K}$ transversally at a general point $C_0 \in \mathcal{K}$ and apply the following observation: if $\{M_t\}$ is a one parameter family of square matrices then $\mathrm{ord}_{t=0}\det(M_t) \geq \dim\ker(M_0)$; this is easily seen by diagonalizing the matrix $\{M_t\}$ over the discrete valuation ring of convergent power series in t.

§3. The classes of $\mathcal{W}$ and $\mathcal{K}$. We continue to use the notations of the Introduction and §2. For divisors D and E, linear equivalence will be denoted $D \sim E$. If L is a line bundle, we write $D \sim L$ to mean that the line bundles $\mathcal{O}(D)$ and L are isomorphic. We will show that $\mathcal{W} \sim 28\lambda$ and that $\mathcal{K} \sim 7\lambda$. The divisor $\mathcal{W} - 4\mathcal{K}$ is then linearly equivalent to zero and by Corollary 2.8 it is effective. But in the variety $\mathcal{M} = \mathcal{M}_{10}^{o}$ the only effective divisor D linearly equivalent to zero is $D = 0$: since $\mathcal{M}$ has a projective compactification with boundary of codimension 2, if D were not zero, there would exist a complete curve $T \subset \mathcal{M}$ not contained in D and intersecting D; since $D \sim 0$ we have $D.T = \deg(\mathcal{O}(D)|_T) = 0$, a contradiction. It follows that $\mathcal{W} = 4\mathcal{K}$.

Proposition 3.1. $\mathcal{W} \sim 28\lambda$.

PROOF: Since $\mathcal{W}$ is the divisor of zeros of the section $\det(\Phi)$, $\mathcal{W}$ belongs to the class $c_1(\pi_*(\omega^{\otimes 3})) - c_1(\bigwedge^2 \pi_*(\omega))$. From [Mu], 5.10, $c_1(\pi_*(\omega^{\otimes 3})) \sim 37\lambda$. By the splitting principle if E is a bundle of rank r then $c_1(\bigwedge^2 E) = (r-1)c_1(E)$, so $c_1(\bigwedge^2 \pi_*(\omega)) \sim 9\lambda$ and the result follows.

Computing the class of $\mathcal{K}$ will require some more preparation. We start with some enumerative formulas. If $f : X \to B$ is a flat family of curves, where X and B are smooth complete and $\dim(B) = 1$, it follows from the Leray spectral sequence that $\chi(X, \mathcal{O}_X) = \chi(B, \mathcal{O}_B) - \chi(B, R^1 f_* \mathcal{O}_X)$. Applying Riemann-Roch and duality to $E = R^1 f_* \mathcal{O}_X$, we obtain $\chi(E) = \deg(E) + \mathrm{rk}(E)\chi(\mathcal{O}_B)$ and $R^1 f_* \mathcal{O}_X = (f_* \omega_{X|B})^*$ so

$$\deg(\lambda_{X|B}) = \chi(X, \mathcal{O}_X) - \chi(B, \mathcal{O}_B)\chi(C, \mathcal{O}_C)$$

where we write $\lambda_{X|B}$ for $\det(f_* \omega_{X|B})$ and where C is a general fiber of f.

For example, if $C \subset S$ is a smooth curve on a smooth surface which moves in a pencil, consider $f : \tilde{S} \to \mathbf{P}^1$ where $\tilde{S}$ is the blow-up of S at the base locus of the pencil. Then $\deg(\lambda_f) = \chi(\tilde{S}, \mathcal{O}_{\tilde{S}}) - 1 + g_C = \chi(S, \mathcal{O}_S) - 1 + g_C$ since χ is a birational invariant. In particular, if S is a $K3$ surface,

$$\deg(\lambda_f) = 1 + g_C. \tag{3.2}$$

If C is a very ample smooth curve on a smooth complete surface S, let $\mathcal{D} \subset |C|$ denote the discriminant hypersurface, consisting of singular members of the complete linear system $|C|$. If we consider a general (Lefschetz) pencil in $|C|$ and apply the Leray spectral sequence to the constant sheaf $\mathbf{C}$ this time, we may count the number of singular fibers and obtain (see [G-H], pp. 508-510 for details) $\deg(\mathcal{D}) = 4(g_C - 1) + C^2 + \chi_{top}(S)$. In particular, if S is a $K3$ surface,

$$\deg(\mathcal{D}) = 6(g_C + 3). \tag{3.3}$$

Lemma 3.4. *If S is a general $K3$ surface of genus 10, then*
a) only finitely many smooth curves C in the linear series $|\mathcal{O}_S(1)|$ have automorphisms.
b) the linear series $|\mathcal{O}_S(1)|$ contains at most a 2 dimensional family of curves with a single node and with automorphisms.
c) S carries a Lefschetz pencil consisting entirely of curves without automorphisms.

PROOF: a) Consider a 19 dimensional family $\mathcal{F}$ of $K3$ surfaces of genus 10 in $\mathbf{P}^{10}$ which dominates $\mathcal{F}_{10}$ (see, e.g., [M] for a construction) and let $\mathcal{P}$ be the canonical $\mathbf{P}^{10}$ bundle over $\mathcal{F}$ (whose fiber at S is $|\mathcal{O}_S(1)|$). Let k be the dimension, for a general S in $\mathcal{F}$, of the subset of $|\mathcal{O}_S(1)|$ representing smooth curves with nontrivial automorphisms. We want to show that $k \leq 0$. By the definition of k there exists a subvariety $\mathcal{A} \subset \mathcal{P}$ of dimension $19 + k$ consisting of smooth curves with automorphisms, such that $\mathcal{A}$ dominates $\mathcal{F}$. Let $\mu : \mathcal{A} \to \mathcal{M}_{10}$ be the moduli map.

As S is general, its Picard group is isomorphic to $\mathbf{Z}$, generated by $\mathcal{O}_S(C)$. It then follows immediately from the main theorem of [G-L] that S contains no n-gonal curves for $n \leq 5$. But the largest component of curves with automorphisms in $\mathcal{M}_{10}$ which are not of this type has dimension 16 and consists of curves with an involution such that the quotient has genus 3. Thus the fibers of μ are at least $k + 3$-dimensional.

On the other hand the dimension of the fibers of μ is constant in a linear series $|\mathcal{O}_S(1)|$ and generically this dimension is 3 (as follows from the proof of Proposition 2.2). Thus $k \leq 0$ as was to be shown.

b) The argument in this case is similar, except that we work in $\Delta_0 \subsetneq \mathcal{M}_{10}$, the boundary component of $\mathcal{M}_{10}$ representing curves of arithmetic genus 10 with one node. Here the locus of curves with non-trivial automorphisms has dimension 17, consisting of hyperelliptic curves of (geometric) genus 9 with two points conjugate under the involution identified. We find $k \leq 2$. (Perhaps a more refined analysis would improve this estimate.)

c) This is an immediate consequence of a) and b).

Proposition 3.5. $\mathcal{K} \sim 7\lambda$

PROOF: Fix a general $S \in \mathcal{F}_{10}$, and let $C \subset S$ be a smooth genus 10 curve. Consider a general Lefschetz pencil $\ell \subset |C|$. By Lemma 3.4 $\mu(\ell) \subset \bar{\mathcal{M}}$, where $\bar{\mathcal{M}}$ is the moduli space of stable genus 10 curves without automorphisms. The Picard group of the smooth variety $\bar{\mathcal{M}}$ is freely generated by λ and the classes of the divisors $\Delta_0, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ where for $i > 0$, Δ_i consists of stable curves with a node that separates the curve into components of genus i and $10 - i$, and Δ_0 is the divisor of stable curves with a singular irreducible component (as follows from [A-C] §4 and [C] §1.3).

Denote $\bar{\mathcal{K}}$ the closure of $\mathcal{K}$ in $\bar{\mathcal{M}}$. Then we have a relation

$$\bar{\mathcal{K}} \sim a.\lambda - b_0.\Delta_0 - b_2.\Delta_2 - b_3.\Delta_3 - b_4.\Delta_4 - b_5.\Delta_5 \tag{3.6}$$

with $a, b_i \in \mathbf{Z}$. Now we pull-back (3.6) to ℓ in order to determine a. Since the surface S is general, its Picard group is generated by the class of C and then there are no reducible curves in $|C|$. This implies that $\Delta_i.\ell = 0$ for $i > 0$ (notice that since ℓ is general its singular members have only nodes as singularities). From (3.3), $\Delta_0.\ell = 78$ (notice that $\tilde{S}$, the blow-up of S along the base locus of the pencil ℓ, is smooth and hence $\mu(\ell)$ is transverse to Δ_0) and from (3.2) we obtain $\lambda.\ell = 11$.

To find $\bar{\mathcal{K}}.\ell = \deg \mu^*(N_{\bar{\mathcal{K}}|\bar{\mathcal{M}}})|_\ell$, we need to compute the degree of the line bundle over ℓ with fiber $H^2(S, T_S(-C))$ for $C \in \ell$ (Remark 2.5). More precisely, suppose ℓ is spanned by $C_0 = \{s_0 = 0\}$ and $C_1 = \{s_1 = 0\}$ for $s_0, s_1 \in H^0(S, L)$ (we write $L = \mathcal{O}_S(C)$). We have a diagram

$$\begin{array}{ccc} \tilde{S} \subset S \times \mathbf{P}^1 & \xrightarrow{\ g\ } & S \\ \Big\downarrow{\scriptstyle f} & & \\ \mathbf{P}^1 & & \end{array}$$

and $\tilde{S} = \{(x, t_0, t_1) | t_0.s_0(x) + t_1.s_1(x) = 0\} \subset S \times \mathbf{P}^1$ is the zero set of a section of $f^*\mathcal{O}_{\mathbf{P}^1}(1) \otimes g^*L$. Then

$$\begin{aligned} \bar{\mathcal{K}}.\ell &= \deg R^2 f_*(T_{S\times\mathbf{P}^1|\mathbf{P}^1}(-\tilde{S})) \\ &= \deg R^2 f_*(g^*T_S \otimes g^*(L^*) \otimes f^*\mathcal{O}_{\mathbf{P}^1}(-1) \\ &= \deg R^2 f_*(g^*(T_S \otimes L^*)) \otimes \mathcal{O}_{\mathbf{P}^1}(-1) \end{aligned}$$

which equals (by base change and cohomology) $\deg H^2(S, T_S \otimes L^*) \otimes \mathcal{O}_{\mathbf{P}^1}(-1) = -1$.

Combining these results we obtain the relation

$$-1 = 11a - 78b_0. \tag{3.7}$$

The integral solutions to this equation are $a = 7 + 78k$, $b_0 = 1 + 11k$ for $k \in \mathbf{Z}$. We know (2.8) that $\mathcal{W} \geq 4\mathcal{K}$ and (3.1) that $\mathcal{W} \sim 28\lambda$. Hence $0 \leq a \leq 7$ and so $k = 0$, $a = 7$, as desired.

As explained at the beginning of this section, the linear equivalence $\mathcal{W} \sim 4\mathcal{K}$ together with the inequality $\mathcal{W} \geq 4\mathcal{K}$ implies $\mathcal{W} = 4\mathcal{K}$; this completes the proof of the main theorem.

REMARK 3.8. Note that our computation of the class of $\mathcal{K}$ in Pic($\mathcal{M}$) uses the inequality $a \leq 7$ (coming from Corollary 2.8 and Proposition 3.1) and the equality 3.7, together with the fact that the coefficients a and b_0 in 3.7 are *integral*. This integrality is why we work in the smooth variety $\mathcal{M}^o_{10}$. A more traditional approach, which we were unable to carry out, would proceed by writing down several pencils of genus 10 curves, computing their intersections with $\bar{\mathcal{K}}$, λ, and the Δ_i, and then solving the resulting system of linear equations over $\mathbf{Q}$.

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