STABILITY OF HOLOMORPHIC FOLIATIONS WITH SPLIT TANGENT SHEAF

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ABSTRACT. We show that the set of singular holomorphic foliations on projective spaces with split tangent sheaf and *good* singular set is open in the space of holomorphic foliations. We also give a cohomological criterion for the rigidity of holomorphic foliations induced by group actions and prove the existence of rigid codimension one foliations of degree $n-1$ on $\mathbb{P}^n$ for every $n \geq 3$.

CONTENTS

1. INTRODUCTION AND STATEMENT OF RESULTS

Our main object of study in this article is the geometry of the spaces of singular holomorphic distributions and singular holomorphic foliations on complex projective spaces.

Loosely speaking, a codimension q singular holomorphic distribution on $\mathbb{P}^n$ is a holomorphic field of $(n-q)$-planes on the complement of a Zariski closed set of codimension at least two. When this plane field is involutive we have a singular holomorphic foliation. The most basic projective invariant that one can attach to a distribution or to a foliation is its degree. The degree is defined as the degree of the tangency of the distribution or foliation with a generic $\mathbb{P}^q$ linearly embedded in $\mathbb{P}^n$.

In this work we will denote by $\mathcal{D}_q(n,d)$ and $\mathcal{F}_q(n,d)$ the spaces of distributions and of foliations on $\mathbb{P}^n$ of codimension q and degree d. These spaces turn out to be quasi-projective varieties and we will give new information about the irreducible components of $\mathcal{F}_q(n,d)$ and $\mathcal{D}_q(n,d)$ for arbitrary $n \geq 3$ and $1 \leq q \leq n-1$.

Precise definitions of the relevant concepts will be given latter in the Introduction. Before that we would like to recall some known results and share our main motivation with the reader.

1.1. **Known Results.** The systematic study of the irreducible components of $\mathcal{F}_1(n, d)$ seems to have been initiated by Jouanolou in [14] where the irreducible components of $\mathcal{F}_1(n, 0)$ and $\mathcal{F}_1(n, 1)$ where classified for all $n \geq 3$. The classification of the irreducible components of $\mathcal{F}_1(n, 2)$ was achieved by Cerveau and Lins Neto in [8]. Besides the classification in low degrees, a few infinite families of irreducible components of the space of codimension one foliations on projective spaces are known:

  (a) rational [11];   (b) logarithmic [2];
  (c) linear pull-back [4];   (d) generic pull-back [7];
  (e) associated to the affine Lie algebra [3].

Unlike the codimension one case, where there is a growing literature, we are aware of just one result about the irreducible components of $\mathcal{F}_q(n, d)$ when $q \geq 2$: the classification of the irreducible components of $\mathcal{F}_q(n, 0)$ given in [6, Proposition 3.1]. We point out that for $d \geq 0$ and $q \geq 2$ no irreducible components of $\mathcal{F}_q(n, d)$ were known so far. Althought [16, Theorem A] solves a similar problem for degree one codimension q foliations on $\mathbb{C}^n$, i.e., codimension q foliations on $\mathbb{C}^n$ induced by linear q-forms.

1.2. **Motivation: Foliations induced by Group Actions.** The key question behind the developments here presented was motivated by a conjecture of Cerveau and Deserti made in the recent monograph [6]. There, after classifying the codimension one foliations of degree 3 on $\mathbb{P}^4$ induced by Lie subalgebras of $\mathfrak{aut}(\mathbb{P}^4) \cong \mathfrak{sl}(5, \mathbb{C})$, they conjecture that one of these foliations, more precisely the one that admits

$$\frac{\left(z_0 z_4^3 - (2z_1 z_3 + z_2^2) z_4^2 + 2 z_2 z_3^2 z_4 - \left(\frac{z_3^4}{2} \right) \right)^3}{\left(z_1 z_4^2 - z_2 z_3 z_4 + \frac{z_3^3}{3} \right)^4}$$

as rational first integral is rigid, that is, there exists an irreducible component of $\mathcal{F}_1(4, 3)$ whose generic element is projectively equivalent to the one induced by the levels of the rational function above (*cf.* introduction of *loc.cit.*). With this conjecture in mind we were naturally lead to the following:

Question 1. *Under which conditions a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{sl}(n + 1, \mathbb{C})$ induces a rigid foliation of $\mathbb{P}^n$?*

If we assume that $\mathcal{F}$ is a foliation induced by a subalgebra $\mathfrak{g} \subset \mathfrak{aut}(\mathbb{P}^n) \cong \mathfrak{sl}(n + 1, \mathbb{C})$ and that the integration of $\mathfrak{g}$ induces an action which is locally free on the complement of an algebraic subvariety $\Sigma \subset \mathbb{P}^n$ of codimension at least two then the tangent sheaf of $\mathcal{F}$ is trivial. Thus, at least on this case, we see that Question 1 is strictly related to

Question 2. *Under which conditions a deformation of a foliation $\mathcal{F}$ with trivial tangent sheaf still has trivial tangent sheaf?*

After a careful study of the myriad of examples presented in [6] we realized that good candidates for sufficient conditions for codimension q foliations in Question 2

are:

$$(1.1) \qquad \begin{cases} \text{codim sing}(d\omega) \geq 3 & \text{when} \quad q = 1 \\ \text{codim sing}(\omega) \geq 3 & \text{when} \quad q \geq 2 \end{cases}$$

where ω is a homogeneous q-form defining $\mathcal{F}$.

At this point, in order to keep the prose intelligible, it is clear that we need to be more precise about some basic notions.

1.3. Basic Definitions. A **singular holomorphic distribution** $\mathcal{D}$ of dimension p on a projective space $\mathbb{P}^n$ is a rational section of $\mathbb{G}_p(T\mathbb{P}^n)$, where $\mathbb{G}_p(T\mathbb{P}^n)$ is the Grassmann bundle of p-planes in $T\mathbb{P}^n$. The distribution $\mathcal{D}$ can also be dually presented as a rational section of $\mathbb{G}_q(T^*\mathbb{P}^n)$, where q is the codimension of $\mathcal{D}$, i.e., $p + q = n$.

If we consider the standard embedding of $\mathbb{G}_q(T^*\mathbb{P}^n)$ in $\mathbb{P}(\Omega_{\mathbb{P}^n}^q)$ then the rational section defining $\mathcal{D}$ can be interpreted as the projectivization of the image of a rational q-form ω. If $(\omega)_0$ is the divisorial part of the zero scheme of ω, $(\omega)_\infty$ is the divisor of poles of ω and we set

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}\big((\omega)_\infty - (\omega)_0\big)$$

then the rational section defining $\mathcal{D}$ is the projectivization of the image of some $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q \otimes \mathcal{L})$ with zero (or singular) set of codimension at least two.

The **singular set** of $\mathcal{D}$, denoted by $\mathrm{sing}(\mathcal{D})$, is the zero set of the twisted q-form ω. Notice that by definition $\mathrm{sing}(\mathcal{D})$ has always codimension at least two. The **degree** of $\mathcal{D}$, denoted by $\deg(\mathcal{D})$, is by definition the degree of the zero locus of $i^*\omega$, where $i : \mathbb{P}^q \to \mathbb{P}^n$ is a linear embedding of a generic q-plane. Since $\Omega_{\mathbb{P}^q}^q = \mathcal{O}_{\mathbb{P}^q}(-q-1)$ it follows at once that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(\deg(\mathcal{D}) + q + 1)$.

If $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q \otimes \mathcal{L})$ and $q > 1$ then in general the projectivization of the graph of ω is not contained in $\mathbb{G}_q(T^*\mathbb{P}^n)$. It will be the case (cf. for instance [12, 16]) if, and only if, ω satisfies (pointwise) the Plücker conditions.

It is well known that the vector space $\mathrm{H}^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q \otimes \mathcal{L})$ can be canonically identified with the vector space of q-forms on $\mathbb{C}^{n+1}$ with homogeneous coefficients of degree $d+1$ whose contraction with the radial (or Euler) vector field $R = \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$ is identically zero, cf. [14]. Taking advantage of this identification the Plücker equations can be written on $\mathbb{C}^{n+1}$ as

$$(1.2) \qquad (i_v \omega) \wedge \omega = 0 \quad \text{for every } v \in \bigwedge^{q-1} \mathbb{C}^{n+1}.$$

When $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(d+q+1)$ satisfies (1.2) then the kernel of the morphism of sheaves

$$\begin{aligned} T\mathbb{P}^n &\longrightarrow \Omega_{\mathbb{P}^n}^{q-1}(d+q+1) \\ v &\longmapsto i_v\omega \end{aligned}$$

defined by contraction with ω has generic rank q and is a sheaf called the **tangent sheaf** of $\mathcal{D}$, denoted in this work by $T\mathcal{D}$. Alternatively one could define a codimension q distribution on $\mathbb{P}^n$ as a rank $n - q$ saturated subsheaf of $T\mathbb{P}^n$, where saturated means that the corresponding cokernel is torsion-free. Both definitions turn out to be equivalent and for our purposes the definition in terms of twisted differential forms satisfying Plücker equations will be more manageable.

A **codimension** q **singular holomorphic foliation** $\mathcal{F}$ on $\mathbb{P}^n$ is a codimension q distribution $\mathcal{F}$ with tangent sheaf closed under Lie bracket, i.e., $[T\mathcal{F}, T\mathcal{F}] \subset T\mathcal{F}$. If $\omega \in \mathrm{H}^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q \otimes \mathcal{L})$ is a q-form defining $\mathcal{F}$ then ω satisfies (1.2) and the involutiveness of $T\mathcal{F}$ is equivalent to, *cf.* [16, proposition 1.2.2],

$$(1.3) \qquad (i_v \omega) \wedge d\omega = 0 \quad \text{for every } v \in \bigwedge^{q-1} \mathbb{C}^{n+1}.$$

Of course ω and $d\omega$ on the formula above are homogeneous differential forms on $\mathbb{C}^{n+1}$.

We will denote $\mathcal{D}_q(n, d)$ (resp. $\mathcal{F}_q(n, d)$) the quasi-projective subvariety of $\mathbb{P}\mathrm{H}^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(d + q + 1))$ whose points parametrize the degree d and codimension q distributions (resp. foliations) on $\mathbb{P}^n$, i.e.,

$$
\begin{aligned}
\mathcal{D}_q(n, d) &= \{[\omega] \mid \omega \text{ satisfies } (1.2) &&\text{and} &&\text{codim sing}(\omega) \geq 2\}; \\
\mathcal{F}_q(n, d) &= \{[\omega] \mid \omega \text{ satisfies } (1.2), (1.3) &&\text{and} &&\text{codim sing}(\omega) \geq 2\}.
\end{aligned}
$$

In words: $\mathcal{D}_q(n, d)$ (resp. $\mathcal{F}_q(n, d)$) is the space of codimension q singular holomorphic distributions (resp. foliations) of degree d on $\mathbb{P}^n$.

1.4. **Main Results.** Our first main result says that the conditions (1.1) are indeed sufficient for the stability of trivial tangent sheaf for codimension one foliations. In fact we are able to settle the more general:

Theorem 1. *Let $n \geq 3$, $d \geq 0$ be integers and $\mathcal{F} = [\omega] \in \mathcal{F}_1(n, d)$ be a singular holomorphic foliation on $\mathbb{P}^n$. If* codim sing$(d\omega) \geq 3$ *and*

$$T\mathcal{F} \cong \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n}(e_i), \quad e_i \in \mathbb{Z},$$

then there exists a Zariski-open neighborhood $\mathcal{U} \subset \mathcal{F}_1(n, d)$ of $\mathcal{F}$ such that $T\mathcal{F}' \cong \oplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^n}(e_i)$ for every $\mathcal{F}' \in \mathcal{U}$.

In dimension 3 a variant of the above Theorem appears as the first step of the proof of [3, Theorem 1]. There the argumentation is based on the deformation theory of holomorphic vector bundles and a detailed understanding of germs of integrable 1-forms ω on $\mathbb{C}^3$ such that $d\omega$ has an isolated singularity. Unfortunately such a detailed understanding is not available in higher dimensions or codimensions. Our proof, based on infinitesimal techniques, is completely different and works uniformly in all dimensions.

Our method also works for codimension q distributions when $q \geq 2$, and yields the following result.

Theorem 2. *Let $n \geq 4$, $q \geq 2$, $d \geq 0$ be integers and $\mathcal{D} \in \mathcal{D}_q(n, d)$ be a singular holomorphic distribution on $\mathbb{P}^n$. If* codim sing$(\mathcal{D}) \geq 3$ *and*

$$T\mathcal{D} \cong \bigoplus_{i=1}^{n-q} \mathcal{O}_{\mathbb{P}^n}(e_i), \quad e_i \in \mathbb{Z},$$

then there exists a Zariski-open neighborhood $\mathcal{U} \subset \mathcal{D}_q(n, d)$ of $\mathcal{D}$ such that $T\mathcal{D}' \cong \oplus_{i=1}^{n-q} \mathcal{O}_{\mathbb{P}^n}(e_i)$ for every $\mathcal{D}' \in \mathcal{U}$.

In §5 we present two immediate consequences of Theorems 1 and 2. The first one is a generalization of a well-known result of Camacho and Lins Neto about the linear pull-back of foliations. The second one is a generalization of a result of Calvo-Andrade, Cerveau, Lins Neto and Giraldo about the stability of foliations associated to affine Lie algebras.

Combining Theorems 1 and 2 with Richardson's results about the rigidity of subalgebras of Lie algebras we are able to give an answer to Question 1. It comes in the form of our third main result.

Theorem 3. *Let $\mathcal{F}$ be a codimension q foliation on $\mathbb{P}^n$ induced by a Lie subalgebra $\mathfrak{g} \subset \mathfrak{sl}(n + 1, \mathbb{C})$ whose corresponding action is locally free outside a codimension 2 analytic subset of $\mathbb{P}^n$. Suppose that $\mathcal{F}$ satisfies the hypothesis of Theorem 1 (for $q = 1$) or Theorem 2 (for $q \geq 2$). If $\mathrm{H}^1\left(\mathfrak{g}, \mathfrak{sl}(n + 1, \mathbb{C})/\mathfrak{g}\right) = 0$ then $\mathcal{F}$ is rigid, i.e., the closure of the orbit of such foliation under the automorphism group of $\mathbb{P}^n$ is an irreducible component of $\mathcal{F}_q(n, n - q)$.*

On the one hand, Theorem 3 together with well-known vanishing results for the cohomology of semi-simple Lie algebras yields a handful of new rigid foliations of codimension at least two on projective spaces. On the other hand these general vanishing results are ineffective in the codimension one case: we prove in proposition 6.5 that all codimension one foliations induced by generically locally free actions of semi-simple groups are not rigid. Albeit we are able to prove that for every $n \geq 3$ there exists a codimension one rigid foliation on $\mathbb{P}^n$ of degree $n - 1$ induced by a meta-abelian subalgebra of $\mathfrak{aut}(\mathbb{P}^n)$.

Theorem 4. *For every $n \geq 3$ the codimension one foliation on $\mathbb{P}^n$ induced by the Lie subalgebra of $\mathfrak{aut}(\mathbb{P}^n)$ generated by*

$$X = \sum_{i=0}^{n} (n - 2i) z_i \frac{\partial z}{\partial z_i} \quad and \quad Y_k = \sum_{i=0}^{n-k} z_{i+k} \frac{\partial z}{\partial z_i}, \quad k = 1 \ldots n - 2$$

is rigid.

We point out that when $n = 3$ the rigidity of the foliation in Theorem 4 was established in [8]. For $n = 4$ the Theorem gives a positive answer to the Cerveau-Deserti conjecture mentioned in §1.2.

We also found two other examples of rigid codimension one foliations induced by group actions: one in $\mathbb{P}^6$ and the other in $\mathbb{P}^7$, *cf.* Table 1 below. In contrast with the Lie algebras presented in Theorem 4 for these examples the first derived algebras are not abelian but just nilpotent.

1.5. **New Irreducible Components.** Throughout the text the reader will find several new irreducible components of the spaces of foliations. We summarize in the table below all the rigid foliations associated to Lie subalgebras of $\mathfrak{aut}(\mathbb{P}^n)$ that appear in this work.

All the omitted Lie brackets that can't be deduced by anti-symmetry from the specified ones are understood to be zero. We emphasize that Corollary 5.1 implies that the foliations obtained from the ones in the Table by a generic linear pull-back are also rigid.

Besides the rigid foliations and associated irreducible components of $\mathcal{F}_q(n, d)$ presented above we found an infinite family of irreducible components whose generic

q	n	degree	Lie Algebra	Reference
$q \geq 1$	$2+q$	2	$\mathfrak{aff}(\mathbb{C})$	Ex. 6.8
$q \geq 2$	$3+q$	3	$\mathfrak{sl}(2,\mathbb{C})$	Ex. 6.6
$q \geq 2$	$n \gg 0$	$n-q$	semi-simple	Ex. 6.7
1	$n \geq 3$	$n-1$	$< X, Y_1, \ldots, Y_{n-2} >$ satisfying $[X,Y_i] = -2iY_i$	Thm 4
1	6	5	$< X, Y_1, \ldots, Y_4 >$ satisfying $[X,Y_i] = -2iY_i$, $[Y_1,Y_j] = Y_{j+1}$	Prop. 6.9
1	7	6	$< X, Y_1, \ldots, Y_5 >$ satisfying $[X,Y_i] = -2iY_i$, $[Y_1,Y_j] = Y_{j+1}$, $[Y_2,Y_3] = -\frac{5}{2}Y_5$	Prop. 6.9

TABLE 1. Rigid Foliations in $\mathcal{F}_q(n,d)$

element is the linear pull-back of foliations by curves (*cf.* Example 5.2) and an infinite family of irreducible components whose generic element is a foliation induced by an abelian action (*cf.* Example 6.2). Both families generalize to arbitrary codimension previously known examples of codimension one.

2. PRELIMINARIES

2.1. **Calculus on $\mathbb{C}^{n+1}$.** For vector fields X, Y on $\mathbb{C}^{n+1}$ we recall that the interior product and Lie derivative satisfy the following useful relations

$$(2.1) \qquad [L_X, i_Y] = i_{[X,Y]}; \quad [L_X, L_Y] = L_{[X,Y]}; \quad L_X\Omega = \mathrm{div}(X)\Omega;$$

where Ω denotes the euclidean volume form in $\mathbb{C}^{n+1}$, i.e., $\Omega = dx_0 \wedge \cdots \wedge dx_n$. For instance they imply that

$$(2.2) \qquad \mathrm{div}([X,Y]) = X(\mathrm{div}(Y)) - Y(\mathrm{div}(X)).$$

If ω denotes a degree d homogeneous p-form, i.e. the coefficients of ω are homogeneous polynomials of degree d, then

$$L_R\omega = (d+p)\omega.$$

In particular if ω is annihilated by the radial vector field then

$$(2.3) \qquad i_R d\omega = (d+p)\omega.$$

If X is a degree d homogeneous vector field then

$$[X,R] = (1-d)X.$$

The next lemma is a kind of dual version of formula (2.3) for integrable distributions and will be used in the proof of Theorem 1.

Lemma 2.1. *Let $X_1, \ldots, X_q$ be homogeneous polynomial vector fields on $\mathbb{C}^{n+1}$ such that the $(n-q)$-form $\eta = i_{X_1} \cdots i_{X_q} i_R \Omega$ is integrable and has singular set of codimension ≥ 2. Then there exist homogeneous polynomial vector fields $\widetilde{X}_1, \ldots, \widetilde{X}_q$ such that*

(1) $\eta = i_{\widetilde{X}_1} \cdots i_{\widetilde{X}_q} i_R \Omega$;

(2) $d\eta = (-1)^q \left(n + 1 - q + \sum_i \deg(X_i) \right) i_{\widetilde{X}_1} \cdots i_{\widetilde{X}_q} \Omega$;

(3) $\deg(X_i) = \deg(\widetilde{X}_i)$ *for every* i.

Proof. Let $X_{q+1} = R$. Since η is integrable,

$$[X_i, X_j] = \sum_{l=1}^{q+1} a_{ij}^l X_l$$

for some rational functions a_{ij}^l. Under our hypothesis the rational functions a_{ij}^l are regular everywhere, i.e., homogeneous polynomials. In fact

$$[X_i, X_j] \wedge X_1 \cdots \wedge \widehat{X_l} \wedge \cdots \wedge X_q \wedge R = \pm a_{ij}^l X_1 \cdots \wedge X_l \wedge \cdots \wedge X_q \wedge R\,,$$

and since, by hypothesis, the zero set of $X_1 \cdots \wedge X_l \wedge \cdots \wedge X_q \wedge R$ does not have divisorial components a_{ij}^l is a polynomial.

The identities (2.1) imply that

$$\begin{aligned} d\eta &= \left(L_{X_1} - i_{X_1}d\right) i_{X_2} \cdots i_{X_q} i_R \Omega \\ &= \left(i_{[X_1, X_2]} - i_{X_2} L_{X_1} - i_{X_1} d i_{X_2}\right) i_{X_3} \cdots i_{X_q} i_R \Omega \,. \end{aligned}$$

Using similar manipulations we can proceed by induction on q to deduce that $d\eta$ is equal to

$$(-1)^q \left(n + 1 - q + \sum_i \deg(X_i) \right) i_{X_1} \cdots i_{X_q} \Omega + \sum_i \lambda_i i_{X_1} \cdots \widehat{i_{X_i}} \cdots i_{X_q} i_R \, \Omega\,,$$

where the λ_i are homogeneous polynomials of degree $\deg(X_i) - 1$. If we set

$$\widetilde{X}_i = X_i + \frac{\lambda_i R}{(-1)^q \left(n + 1 - q + \sum_i \deg(X_i)\right)}$$

then the lemma follows. $\qquad\square$

2.2. The Tangent Sheaf of Foliations.

Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$, induced by a twisted q-form ω. As in the Introduction the tangent sheaf of $\mathcal{F}$, denoted by $T\mathcal{F}$, is the coherent subsheaf of $T\mathbb{P}^n$ generated by the germs of vector fields annihilating ω.

In general $T\mathcal{F}$ is not locally free. For instance, let $\mathcal{F}$ be the codimension one foliation of $\mathbb{C}^3$ induced by df, where $f : \mathbb{C}^3 \to \mathbb{C}$ is the function $f(x, y, z) = x^2 + y^2 + z^2$. Clearly $T\mathcal{F}$ is a locally free subsheaf of $T\mathbb{C}^3$ outside the origin of $\mathbb{C}^3$ since at these points f is a local submersion. Nevertheless, at the origin of $\mathbb{C}^3$, $T\mathcal{F}$ is not locally free.

More generally, for codimension one foliations if $T\mathcal{F}$ is locally free in a neighborhood U of a point p then the singular scheme of $\mathcal{F}$ on U is defined by the $(n-1)$-minors of a $n \times (n-1)$ matrix. In particular it is either empty or has codimension 2.

We say that the tangent sheaf of $\mathcal{F}$ **splits** if

$$T\mathcal{F} = \bigoplus_{i=1}^{n-q} \mathcal{O}_{\mathbb{P}^n}(e_i),$$

for some integers e_i. Note that the inclusion of $T\mathcal{F}$ in $T\mathbb{P}^n$ induces sections $X_i \in \mathrm{H}^0(\mathbb{P}^n, T\mathbb{P}^n(-e_i))$ for $i = 1 \ldots n - q$. It follows from the Euler sequence that these sections are defined by homogeneous vector fields of degree $1 - e_i \geq 0$ on $\mathbb{C}^{n+1}$,

which we still denote by X_i. The foliation $\mathcal{F}$ is induced by the homogeneous q-form on $\mathbb{C}^{n+1}$

$$\omega = i_{X_1} \cdots i_{X_{n-q}} i_R \Omega \,.$$

2.3. The singular set of $d\omega$ and Kupka Singularities. Another key hypothesis for our results for codimension one foliations is that codim $\mathrm{sing}(d\omega) \geq 3$. We will now explain the *geometrical meaning* of this hypothesis.

If ω_0 is a germ of integrable holomorphic q-form on $(\mathbb{C}^n, 0)$ such that $\omega_0(0) = 0$ then 0 is called a Kupka singularity of ω_0 if $d\omega_0(0) \neq 0$. The local structure of codimension one foliations in a neighborhood of Kupka singularities is fairly simple: the germ of foliation is the pull-back of a germ of foliation on $(\mathbb{C}^2, 0)$, *cf.* [16, proposition 1.3.1] and references therewithin. As a side remark we point out that the result proved in *loc. cit.* also holds for integrable q-forms ω: if $d\omega(0) \neq 0$ then the germ of codimension q foliation induced by ω is the pull-back of a germ of foliation on $(\mathbb{C}^{q+1}, 0)$.

If $u \in \mathcal{O}_0^n$ is a unit then $d(u\omega_0) = du \wedge \omega_0 + u\,d\omega_0$. Thus the singular set of $d\omega_0$ is in principle distinct from the singular set of $d(u\omega_0)$, i.e., it is not an invariant of $\mathcal{F}_0$, the foliation induced by ω_0. Although

$$\mathcal{B}(\mathcal{F}_0) = \mathrm{sing}(\omega_0) \cap \mathrm{sing}(d\omega_0) = \mathrm{sing}(u\omega_0) \cap \mathrm{sing}(d(u\omega_0)) \,,$$

is a invariant of $\mathcal{F}_0$ which we will call the non-Kupka singular set of $\mathcal{F}_0$.

It is easy to verify that for ω homogeneous 1-form on $\mathbb{C}^{n+1}$ inducing a foliation $\mathcal{F}$ of $\mathbb{P}^n$ that

$$\mathcal{B}(\mathcal{F}) = \mathrm{sing}(d\omega) \,.$$

In other words our hypothesis is on the codimension of the *non-Kupka* singular set of $\mathcal{F}$.

3. Division Lemmata

Lemma 3.1. *Let* $X_1, \ldots, X_{n-1}$ *be holomorphic vector fields on* $\mathbb{C}^{n+1}$ *and* $\Theta \in \Omega^2(\mathbb{C}^{n+1})$ *be the 2-form given by* $\Theta = i_{X_1} \cdots i_{X_{n-1}} \Omega$. *Suppose that* codim $\mathrm{sing}(\Theta) \geq 3$. *If* $\eta \in \Omega^2(\mathbb{C}^{n+1})$ *is such that*

$$\Theta \wedge \eta = 0$$

then there exist holomorphic vector fields $\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}$ *such that*

$$\eta = \sum_{i=1}^{n-1} i_{X_1} \cdots i_{X_{i-1}} i_{\widetilde{X}_i} i_{X_{i+1}} \cdots i_{X_{n-1}} \Omega.$$

Proof. This follows from the dual version of the main result in [19]; see also [15, Proposition (1.1) with $q = 3$]. $\qquad\square$

This Lemma is the case $q = 2$ of Lemma 3.2 below (notice that when $q = 2$ the Plücker relations below are $\Theta \wedge \Theta = 0$ and the tangent space of the Grassmannian is given by $\Theta \wedge \eta = 0$).

Lemma 3.2. *Let* $X_1, \ldots, X_{n+1-q}$ *be holomorphic vector fields on* $\mathbb{C}^{n+1}$ *and* $\Theta \in \Omega^q(\mathbb{C}^{n+1})$ *be the q-form given by* $\Theta = i_{X_1} \cdots i_{X_{n+1-q}} \Omega$. *Suppose that* codim $\mathrm{sing}(\Theta) \geq 3$. *If* $\eta \in \Omega^q(\mathbb{C}^{n+1})$ *is such that*

$$i_v(\eta) \wedge \Theta + i_v(\Theta) \wedge \eta = 0$$

for all $v \in \wedge^{q-1}T(\mathbb{C}^{n+1})$, then there exist holomorphic vector fields $\widetilde{X}_1, \ldots, \widetilde{X}_{n+1-q}$ such that

$$\eta = \sum_{i=1}^{n+1-q} i_{X_1} \cdots i_{X_{i-1}} i_{\widetilde{X}_i} i_{X_{i+1}} \cdots i_{X_{n+1-q}} \Omega.$$

Proof. Denote $V = \mathbb{C}^{n+1}$ and $\mathbf{G} = \text{Grass}(V, q)$ the Grassmannian of linear subspaces of V of codimension q (i. e. dimension $n + 1 - q$). The Plücker embedding

$$\wp : \mathbf{G} \to \mathbb{P}\bigwedge^{q}(V^*)$$

gives an isomorphism of $\mathbf{G}$ with the subvariety of decomposable q-linear forms. It is well known (see for example [12]) that a q-linear form $\theta \in \wedge^q(V^*)$ is decomposable if and only if

$$(i_v \theta) \wedge \theta = 0$$

for every $v \in \wedge^{q-1}V$. It is also known that these equations, the well known Plücker quadrics, generate the ideal of $\wp(\mathbf{G})$. Therefore, the tangent space of $\wp(\mathbf{G})$ at a point θ may be described as the set of q-linear forms η such that

$$(i_v \eta) \wedge \theta + (i_v \theta) \wedge \eta = 0$$

for every $v \in \wedge^{q-1}V$.

Let us consider the standard q-multilinear map

$$\mu : (V^*)^q \to \bigwedge^{q}(V^*)$$

defined by $\mu(u_1, \ldots, u_q) = u_1 \wedge \cdots \wedge u_q$.

If $(V^*)_0^q = (V^*)^q - \mu^{-1}(0)$ is the open set consisting of linearly independent vectors then $\mathbf{G}$ is the quotient of $(V^*)_0^q$ by the general linear group $\text{GL}(V)$ and the Plücker embedding is the quotient map (i. e. projectivization) of μ. Hence the tangent space of $\wp(\mathbf{G})$ at a point $\theta = \mu(u_1, \ldots, u_q)$ coincides with the image of the derivative of μ at $u = (u_1, \ldots, u_q)$, which is given by

$$d\mu(u).(\widetilde{u}) = \sum_{i=1}^{q} u_1 \wedge \cdots \wedge u_{i-1} \wedge \widetilde{u}_i \wedge u_{i+1} \wedge \cdots \wedge u_q$$

for $\widetilde{u} = (\widetilde{u}_1, \ldots, \widetilde{u}_q) \in (V^*)^q$.

Contraction with $\Omega = dx_0 \wedge \cdots \wedge dx_n$ induces an isomorphism $\wedge^{n+1-q}(V) \cong \wedge^q(V^*)$. Therefore $\wp(\mathbf{G})$ is also the projective image of the multilinear map

$$\nu : (V)^{n+1-q} \to \bigwedge^{q}(V^*)$$

defined by $\nu(v_1, \ldots, v_{n+1-q}) = i_{v_1} \cdots i_{v_{n+1-q}} \Omega$. Hence, the tangent space of $\wp(\mathbf{G})$ at a point θ has a third description as the image of the derivative of ν

$$d\nu(v).(\widetilde{v}) = \sum_{i=1}^{n+1-q} i_{v_1} \cdots i_{v_{i-1}} i_{\widetilde{v}_i} i_{v_{i+1}} \cdots i_{v_{n+1-q}} \Omega$$

for $v = (v_1, \ldots, v_{n+1-q}) \in V_0^{n+1-q}$ and $\widetilde{v} = (\widetilde{v}_1, \ldots, \widetilde{v}_{n+1-q}) \in V^{n+1-q}$.

Returning to the statement that we are proving, let us remark that a differential q-form in $\mathbb{C}^{n+1}$ may be considered as a map $\mathbb{C}^{n+1} \to \wedge^q(\mathbb{C}^{n+1})^*$. Our hypothesis

about Θ implies that $\Theta(x)$ is decomposable for all $x \in \mathbb{C}^{n+1}$ and then Θ induces a regular map

$$\bar{\Theta} : U \to \mathbf{G}$$

where $U = \mathbb{C}^{n+1} - \text{sing}(\Theta)$

The hypothesis on η means that $\eta(x)$ belongs to the tangent space to $\mathbf{G}$ at $\bar{\Theta}(x)$ for all $x \in U$. By the last characterization of the tangent space of $\mathbf{G}$, there exists an open cover $U = \cup_\alpha U_\alpha$ and holomorphic vector fields $\widetilde{X}_1^\alpha, \ldots, \widetilde{X}_{n+1-q}^\alpha$ on U_α such that

$$\eta|_{U_\alpha} = \sum_{i=1}^{n+1-q} i_{X_1} \cdots i_{X_{i-1}} i_{\widetilde{X}_i^\alpha} i_{X_{i+1}} \cdots i_{X_{n+1-q}} \Omega.$$

In $U_\alpha \cap U_\beta$ we have

$$0 = \eta|_{U_\alpha} - \eta|_{U_\beta} = \sum_{i=1}^{n+1-q} i_{X_1} \cdots i_{X_{i-1}} i_{(\widetilde{X}_i^\alpha - \widetilde{X}_i^\beta)} i_{X_{i+1}} \cdots i_{X_{n+1-q}} \Omega.$$

Then, for each $j = 1, \ldots, n+1-q$

$$i_{X_j}(\eta|_{U_\alpha} - \eta|_{U_\beta}) = \pm i_{(\widetilde{X}_j^\alpha - \widetilde{X}_j^\beta)} \Theta = 0$$

Therefore $\widetilde{X}_j^\alpha - \widetilde{X}_j^\beta$ is a linear combination of $X_1, \ldots, X_{n+1-q}$ with holomorphic coefficients. If X denotes the matrix with columns $X_1, \ldots, X_{n+1-q}$ (and similar notation for $\widetilde{X}^\alpha$), then there exists a matrix $A^{\alpha\beta}$ with coefficients holomorphic functions in $U_\alpha \cap U_\beta$ such that

$$\widetilde{X}^\alpha - \widetilde{X}^\beta = A^{\alpha\beta} X$$

The collection $\{A^{\alpha\beta}\}_{\alpha\beta}$ is a 1-cocycle and defines an element of

$$\mathrm{H}^1\left(U, \mathcal{O}_{\mathbb{C}^{n+1}}^{(n+1)(n+1)}\right) = \mathrm{H}^1\left(U, \mathcal{O}_{\mathbb{C}^{n+1}}\right)^{(n+1)(n+1)}.$$

The hypothesis codim $\text{sing}(\Theta) \geq 3$ implies that $H^1(U, \mathcal{O}_{\mathbb{C}^{n+1}}) = 0$ (see [13, pg. 133]). Hence, after a refinement of the open cover, we may write $A^{\alpha\beta} = A^\alpha - A^\beta$ where A^α is a holomorphic matrix in U_α. Then

$$\widetilde{X}^\alpha - A^\alpha X = \widetilde{X}^\beta - A^\beta X$$

in $U_\alpha \cap U_\beta$ and hence the columns of these matrices define the required holomorphic vector fields in U. To conclude we apply Hartog's extension Theorem to extend these vector fields to $\mathbb{C}^n$. $\qquad\square$

Remark 3.3. If the vector fields $X_1, \ldots, X_{n+1-q}$ are homogeneous we can take the vector fields $\tilde{X}_1, \ldots, \tilde{X}_{n+1-q}$ homogeneous and satisfying $\deg(\tilde{X}_i) = \deg(X_i)$ for all $i = 1 \ldots n+1-q$. In fact, if we replace $\tilde{X}_i$ by its homogeneous component of degree $\deg(X_i)$ then we still have that

$$\eta = \sum_{i=1}^{n+1-q} i_{X_1} \cdots i_{X_{i-1}} i_{\tilde{X}_i} i_{X_{i+1}} \cdots i_{X_{n+1-q}} \Omega.$$

4. Foliations with Split Tangent Sheaf

In this section we will prove Theorems 1 and 2.

Proof of Theorem 1. Let $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d+2))$ be a saturated integrable twisted 1-form on $\mathbb{P}^n$ and $\mathcal{F}$ the induced foliation. If $T\mathcal{F}$ splits then there exists a collection $X_1, X_2, \ldots, X_{n-1}, R$ of homogeneous vector fields in involution such that

$$\omega = i_{X_1} \cdots i_{X_{n-1}} i_R \Omega.$$

Using lemma 2.1 we can also assume that

$$d\omega = \lambda \cdot i_{X_1} \cdots i_{X_{n-1}} \Omega,$$

for some $\lambda \in \mathbb{C}^*$.

If $T_\omega = T_\omega \mathcal{F}_1(n,d)$ denotes the Zariski tangent space of the scheme $\mathcal{F}_1(n,d)$ at the point ω then $\eta \in T_\omega \mathcal{F}_1(n,d)$ if, and only if,

$$(\omega + \epsilon \eta) \wedge (d\omega + \epsilon d\eta) = 0 \quad \mathrm{mod}\ \epsilon^2.$$

It follows that $\eta \in T_\omega$ if, and only if, $\omega \wedge d\eta + \eta \wedge d\omega = 0$. Note that $\omega \wedge d\eta + \eta \wedge d\omega$ is annihilated by the radial vector field. Then from (2.3) we conclude that

$$\omega \wedge d\eta + \eta \wedge d\omega = 0 \quad \Longleftrightarrow \quad d\omega \wedge d\eta = 0.$$

Thus we can apply Lemma 3.1 to $d\eta$ and assure the existence of a collection $\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}$ of homogeneous vector fields such that

$$d\eta = \sum_{i=1}^{n-1} i_{X_1} \cdots i_{X_{i-1}} i_{\widetilde{X}_i} i_{X_{i+1}} \cdots i_{X_{n-1}} \Omega$$

Since $i_R \eta = 0$ it follows from (2.3) that

$$\eta = \sum_{i=1}^{n-1} i_{X_1} \cdots i_{X_{i-1}} i_{\widetilde{X}_i} i_{X_{i+1}} \cdots i_{X_{n-1}} i_R \Omega.$$

Let $d_i = \deg(X_i) = \deg(\widetilde{X}_i)$ and denote by $\mathcal{X}(d_i)$ the $\mathbb{C}$-vector space of degree d_i homogeneous polynomial vector fields on $\mathbb{C}^{n+1}$. Consider the alternate multi-linear map

$$\Psi : \bigoplus_{i=1}^{n-1} \mathcal{X}(d_i) \quad \longrightarrow \quad H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d+2))$$

$$(Y_1, \ldots, Y_{n-1}) \quad \mapsto \quad i_{Y_1} \cdots i_{Y_{n-1}} i_R \Omega.$$

The derivative of Ψ at $Y = (Y_1, \ldots, Y_{n-1})$ is

$$d\Psi(Y)(Z_1, \ldots, Z_{n-1}) = \sum_{i=1}^{n-1} i_{Y_1} \cdots i_{Y_{i-1}} i_{Z_i} i_{Y_{i+1}} \cdots i_{Y_{n-1}} i_R \Omega.$$

It is now clear that the every $\eta \in T_\omega \mathcal{F}_1(n,d)$ is contained in the image of $d\Psi(X_1, \ldots, X_{n-1})$. This is sufficient to assure that the image of Ψ contains an open neighborhood of ω in $\mathcal{F}_1(n,d)$. $\qquad \square$

We will see in the proof of Proposition 6.5 that the hypothesis on the singular set of $d\omega$ is indeed necessary.

The proof of Theorem 2 is analogous to the proof of Theorem 1; we highlight the unique difference.

Proof of Theorem 2. If $T_\omega = T_\omega \mathcal{D}_q(n, d)$ stands for the Zariski tangent space of the scheme $\mathcal{D}_q(n, d)$ at the point ω then $\eta \in T_\omega \mathcal{D}_q(n, d)$ if, and only if, $i_v(\eta) \wedge \Theta + i_v(\Theta) \wedge \eta = 0$ for all $v \in \wedge^{q-1} T(\mathbb{C}^{n+1})$. Now we apply the division lemma 3.2 and conclude as in the proof of Theorem 1. $\qquad\square$

5. Two Applications

5.1. **Linear Pull-backs.** Our first application is a generalization of a well-known result by Camacho and Lins Neto which says that the pull-back of generic degree d foliations of $\mathbb{P}^2$ under generic linear projections form an irreducible component of $\mathcal{F}_1(n, d)$ for every $n \geq 3$, see [4]. More precisely we prove the

Corollary 5.1. *Let $\mathcal{C}$ be an irreducible component of $\mathcal{F}_q(n, d)$. If the generic element of $\mathcal{C}$ satisfies the hypothesis of Theorem 1 (for $q = 1$) or Theorem 2 (for $q \geq 2$) then for every integer $m \geq 1$ there exists an irreducible component of $\mathcal{F}_q(n + m, d)$ such that the generic element is the pull-back under a generic linear projection of a generic element of $\mathcal{C}$.*

Proof. Let $\mathcal{G} \in \mathcal{F}_q(n, d)$ be a foliation whose tangent sheaf splits, i.e.,

$$T\mathcal{G} = \bigoplus_{j=1}^{n-q} \mathcal{O}_{\mathbb{P}^n}(e_j).$$

Suppose also that $\mathcal{G}$ is induced by a 1-form ω with codim sing$(d\omega) \geq 3$ when $q = 1$ or codim sing$(\omega) \geq 3$ when $q \geq 2$. If $\mathcal{F} \in \mathcal{F}_q(n + m, d)$ is the pull-back of $\mathcal{G}$ under a generic linear projection $\pi : \mathbb{P}^{n+m} \dashrightarrow \mathbb{P}^n$ then

$$T\mathcal{F} = (\bigoplus_{j=1}^{n-q} \mathcal{O}_{\mathbb{P}^{n+q}}(e_j)) \oplus \mathcal{O}_{\mathbb{P}^{n+q}}(1)^q.$$

Moreover the codimensions of the singular set of ω, resp. $d\omega$, and $\pi^*\omega$, resp. $\pi^* d\omega$, are the same. From Theorems 1 and 2 it is sufficient to prove that every foliation $\mathcal{F}' \in \mathcal{F}_q(n + m, d)$ with $T\mathcal{F}' = T\mathcal{F}$ is the pull-back of a foliation $\mathcal{G}' \in \mathcal{F}_q(n, d)$ under a linear projection.

From the splitting type of $\mathcal{F}'$ we see that it is induced by an i-form ω' that may be written as

$$\omega' = i_{X_1} \cdots i_{X_{n-q}} i_{Z_1} \cdots i_{Z_q} i_R \Omega,$$

where the X_j are homogeneous vector fields of degree $1 - e_j$ and the Z_j are constant vector fields. In suitable coordinate system $(z_0, \ldots, z_{n+m})$ of $\mathbb{C}^{n+m+1}$ we can write $Z_j = \frac{\partial}{\partial z_{n+j}}$. It follows that the fibers of the linear projection $\pi'(z_0, \ldots, z_{n+m}) = (z_0, \ldots, z_n)$ are everywhere tangent to the leaves of $\mathcal{F}'$. In particular the leaves of $\mathcal{F}'$ are dense open sets of cones over the center of projection. Thus there exists $\mathcal{G}' \in \mathcal{F}_q(n, d)$ such that $\mathcal{F}' = \pi'^* \mathcal{G}'$. For more details the reader may consult [9, lemma 2.2]. $\qquad\square$

As an immediate corollary we obtain irreducible components of $\mathcal{F}_q(n, d)$ for arbitrary $q \geq 1$, $n \geq q + 2$ and $d \geq 0$.

Example 5.2. *The pull-back to $\mathbb{P}^n$ under linear projections of degree d foliations by curves on $\mathbb{P}^{q+1}$ fill out irreducible components of $\mathcal{F}_q(n, d)$.*

Among these irreducible components the only ones that have appeared before in the literature are the ones with $q = 1$ or $d = 0$.

5.2. Foliations associated to Affine Lie Algebras.

Foliations induced by representations of the affine Lie algebra in one variable into the Lie algebra of polynomial vector fields on $\mathbb{C}^n$, $n \geq 4$, with homogeneous generators with the usual hypothesis on the singular set also fill out components of $\mathcal{F}_q(n, d)$. More precisely, we prove the

Corollary 1. *Let $\mathcal{F}$ be a codimension q foliation of $\mathbb{P}^{2+q}$ given by a q-form $\omega = i_X i_Y i_R \Omega$ where Ω is the euclidean volume form on $\mathbb{C}^{3+q}$ and X, Y are homogeneous vector fields of degree 1 and $e \geq 2$ satisfying the relation*

$$[X, Y] = Y .$$

If $\operatorname{codim sing}(d\omega) \geq 3$ (for $q = 1$) or $\operatorname{codim sing}(\omega) \geq 3$ (for $q \geq 2$) then any foliation $\mathcal{F}'$ sufficiently close to $\mathcal{F}$ is induced by a q-form $\omega' = i_{X'} i_{Y'} i_R \Omega$ where X', Y' are homogeneous vector fields of degree 1 and e satisfying

$$[X', Y'] = Y'.$$

Proof. Let ω' be an integrable q-form sufficiently close to ω. It follows from Theorem 1 (when $q = 1$) and Theorem 2 (when $q \geq 2$) that there exist homogeneous vector fields X' and Y' such that $\omega' = i_{X'} i_{Y'} i_R \Omega$.

From the integrability of ω' one deduces that

$$[X', Y'] = aX' + \lambda Y + bR ,$$

for suitable $\lambda \in \mathbb{C}$ and $a, b \in S_{e-1}$. Here S_{e-1} denotes the space of homogeneous polynomials of degree $e - 1$. Notice that since we can take X' and Y' sufficiently close to X and Y and similarly λ sufficiently close to 1. Moreover, after replacing X' by $\lambda^{-1} X'$, we can assume that $\lambda = 1$.

Since, for arbitrary $\mu \in \mathbb{C}$,

$$i_{X'} i_{Y'} i_R \Omega = i_{X' + \mu R} i_{Y'} i_R \Omega$$

we can suppose that the linear map

$$
\begin{aligned}
T : S_{e-1} &\rightarrow S_{e-1} \\
f &\mapsto X'(f) - f
\end{aligned}
$$

is invertible.

If we set $X'' = X'$ and $Y'' = Y' - T^{-1}(a)X' - T^{-1}(b)R$ then

$$
\begin{aligned}
[X'', Y''] &= aX' + Y' + bR - X'(T^{-1}(a))X' - X'(T^{-1}(b))R \\
&= (a - X'(T^{-1}(a))X' + Y' + (b - X'(T^{-1}(b))R \\
&= Y'' .
\end{aligned}
$$

Notice that $i_{X''} i_{Y''} i_R \Omega = i_{X'} i_{Y'} i_R \Omega$ to conclude the proof of the corollary. $\square$

When $q = 1$ the result below appeared in a slightly different form in [3]. For $q \geq 2$ it is new. Using the corollary above it is possible to obtain irreducible components of $\mathcal{F}_q(q + 2, d)$, for every q, d, adapting the constructions presented in [3].

6. Foliations induced by Group Actions

To recall the basic definitions on foliations induced by group actions, let $\mathfrak{g} \subset \mathfrak{sl}(n+1, \mathbb{C})$ be a Lie subalgebra of dimension $n-q$. Since $\mathrm{H}^0(\mathbb{P}^n, T\mathbb{P}^n) = \mathfrak{sl}(n+1, \mathbb{C})$ is the Lie algebra of $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PSL}(n+1, \mathbb{C})$, we may view $\bigwedge^{n-q} \mathfrak{g}$ as a one-dimensional linear subspace of $\bigwedge^{n-q} \mathrm{H}^0(\mathbb{P}^n, T\mathbb{P}^n)$.

By duality we obtain a twisted integrable q-form

$$\omega(\mathfrak{g}) \in \mathrm{H}^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(n+1)).$$

When $\omega(\mathfrak{g}) \neq 0$, the leaves of the foliation $\mathcal{F}(\mathfrak{g})$ induced by $\omega(\mathfrak{g})$ are tangent to the orbits of $\exp(\mathfrak{g})$, the connected (but not necessarily closed) subgroup of $\mathrm{Aut}(\mathbb{P}^n)$ with Lie algebra $\mathfrak{g}$. Moreover the action of $\exp(\mathfrak{g})$ on $\mathbb{P}^n$ is locally free outside $\mathrm{sing}(\omega(\mathfrak{g}))$. Notice that

$$\deg(\omega(\mathfrak{g})) = n - q = \dim(\mathfrak{g})$$

When $\omega(\mathfrak{g}) \neq 0$ and its singular set has no divisorial components then, clearly, $T\mathcal{F}(\mathfrak{g}) = \mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^n}$. In particular, as an immediate consequence of Theorems 1 and 2 we obtain the

Corollary 6.1. *Let $\mathfrak{g} \subset \mathfrak{sl}(n+1, \mathbb{C})$ be a Lie subalgebra of dimension $n-q$ such that $\omega(\mathfrak{g}) \neq 0$. If $\omega(\mathfrak{g})$ satisfies the hypothesis of Theorem 1 (for $q = 1$) or Theorem 2 (for $q \geq 2$) then for every foliation $\mathcal{F}'$ sufficiently close to $\mathcal{F}(\mathfrak{g})$ there exists a Lie subalgebra $\mathfrak{g}' \subset \mathfrak{sl}(n+1, \mathbb{C})$ such that $\mathcal{F}' = \mathcal{F}(\mathfrak{g}')$.*

Corollary 6.1 provides a way to translate results from the representation theory of Lie algebras to results about foliations. Let us consider a simple example.

Example 6.2 (Diagonal algebras)**.** Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{sl}(n+1, \mathbb{C})$ of dimension $n-q$ such that every $g \in \mathfrak{g}$ has all eigenvalues with multiplicity one. In particular, by an elementary classical result on Lie algebras, $\mathfrak{g}$ is diagonal in a suitable system of coordinates. Moreover from the choice of $\mathfrak{g}$ this property is clearly stable: every $\mathfrak{g}'$ with generators sufficiently close to the generators of $\mathfrak{g}$ is also diagonalizable.

If we identify $\mathfrak{sl}(n+1, \mathbb{C})$ with the Lie algebra of linear homogeneous vector fields on $\mathbb{C}^{n+1}$ with zero divergence then we can write

$$\mathfrak{g} = \langle X_1, \ldots, X_{n-q} \rangle$$

where

$$X_i = \sum_{j=0}^{n} \lambda_{ij} z_j \frac{\partial}{\partial z_j}$$

for suitable complex numbers λ_{ij}. Consequently, we can associate to $\mathfrak{g}$ the $\mathcal{L}$-foliation $\mathcal{F}(\mathfrak{g}) \in \mathcal{F}_q(n, n-q)$ induced by the q-form

$$\omega(\mathfrak{g}) = i_{X_1} \cdots i_{X_{n-q}} i_R \Omega.$$

When $q = 1$ a straightforward computation shows that

$$d\omega(\mathfrak{g}) = \pm i_{X_1} \cdots i_{X_{n-1}} \Omega.$$

In particular the singular set of $d\omega(\mathfrak{g})$ is the union of the sets $\{z_i = z_j = z_k = 0\}$ where i, j, k are pairwise distinct integers in $\{0, \ldots, n\}$ and hence it has codimension 3. For $q \geq 2$ the singular set of $\omega(\mathfrak{g})$ is defined by similar conditions and has codimension q.

We can apply Corollary 6.1 to conclude the existence of irreducible components of $\mathcal{F}_q(n, n - q)$ associated to the diagonal algebras. When $q = 1$ these are the well-known logarithmic components on $\mathbb{P}^n$ with poles in $n + 1$ hyperplanes, cf. [6, Corollary 1.19]. Of course we can use Corollary 5.1 to obtain the known result that logarithmic 1-forms with poles on $r \leq n + 1$ hyperplanes fill out an irreducible component. $\qquad\square$

Another immediate consequence of Theorems 1 and 2 is the Theorem 3.

Theorem 6.3 (Theorem 3 of the Introduction). *Let $\mathcal{F}$ be a codimension q foliation on $\mathbb{P}^n$ induced by a Lie subalgebra $\mathfrak{g} \subset \mathfrak{sl}(n + 1, \mathbb{C})$ whose corresponding action is locally free outside a codimension 2 analytic subset of $\mathbb{P}^n$. Suppose that $\mathcal{F}$ satisfies the hypothesis of Theorem 1 (for $q = 1$) or Theorem 2 (for $q \geq 2$). If $\mathrm{H}^1(\mathfrak{g}, \mathfrak{sl}(n + 1, \mathbb{C})/\mathfrak{g}) = 0$ then $\mathcal{F}$ is rigid, i.e., the closure of the orbit of such foliation under the automorphism group of $\mathbb{P}^n$ is an irreducible component of $\mathcal{F}_q(n, n - q)$.*

Proof. The main result of [18] says that a subalgebra $\mathfrak{g} \subset \mathfrak{sl}(n + 1, \mathbb{C})$ for which $\mathrm{H}^1(\mathfrak{g}, \mathfrak{sl}(n + 1, \mathbb{C})/\mathfrak{g}) = 0$ is rigid. The Theorem follows at once combining this result with Corollary 6.1 combined with [18]. $\qquad\square$

6.1. **Foliations induced by Semi-Simple Lie Groups.** We now turn our attention to foliations associated to semi-simple Lie subalgebras of $\mathfrak{sl}(n + 1, \mathbb{C})$.

Corollary 6.4. *Let $\mathfrak{g} \subset \mathfrak{sl}(n + 1, \mathbb{C})$ be a semisimple Lie subalgebra of dimension $n - q$, $q \geq 2$, such that $\omega(\mathfrak{g})$ is non-zero. If $\omega(\mathfrak{g})$ satisfies the hypothesis of Theorem 2 then $\mathcal{F}(\mathfrak{g})$ is rigid.*

Proof. If $\mathfrak{g}$ is semisimple then for any finite dimensional $\mathfrak{g}$-module M one has $\mathrm{H}^1(\mathfrak{g}, V) = 0$, see [1, Ex. 1.b., Chapter I, paragraph 6, page 102]. Taking $M = \mathfrak{sl}(n + 1, \mathbb{C})/\mathfrak{g}$, the result follows from Theorem 3. $\qquad\square$

The reader may wonder why we have not stated the Corollary 6.4 for codimension one foliations since the proof works also in this case. The reason is very simple: the hypothesis of Theorem 1 are never satisfied. More precisely we have the

Proposition 6.5. *If $\mathfrak{g} \subset \mathfrak{sl}(n + 1, \mathbb{C})$ is a semisimple Lie subalgebra of dimension $n - 1$ such that $\omega(\mathfrak{g}) \neq 0$ then* codim sing$(d(\omega(\mathfrak{g})) \leq 2$.

Proof. It follows from [6, Thm 1.22] that $\mathcal{F}(\mathfrak{g})$ admits a rational integral $F : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$. From Stein's factorization Theorem we can assume that the generic fiber of F is irreducible and that such F is unique up to right composition with elements in $\mathrm{Aut}(\mathbb{P}^1)$. The argument used to prove the above mentioned result shows that every fiber of F is irreducible. The point is that for every $X \in \mathfrak{g}$ and every $\mathcal{F}(\mathfrak{g})$-invariant hypersurface $\{P = 0\}$

$$X(P) = 0 .$$

Otherwise it would exist a non-trivial morphism of Lie algebras $\mu_X : \mathfrak{g} \to \mathbb{C}$, cf. loc. cit. for more details. If one of the fibers of F admits a prime decomposition of the form $P_1^{n_1} \cdots P_l^{n_l}$ with $l \geq 2$ then

$$X \left(\frac{P_1^{\deg(P_2)}}{P_2^{\deg(P_1)}} \right) = 0 ,$$

contradicting the unicity of F.

We recall that a classical Theorem of Halphen says that a pencil on $\mathbb{P}^n$ with irreducible generic element has at most two multiple elements, cf. [17]. Since every fiber of F is irreducible it follows that F has at most two non-reduced fibers. In particular, after composing with an automorphism of $\mathbb{P}^1$, we can assume that F is of the form

$$F = \frac{G^{\deg(H)}}{H^{\deg(G)}}$$

and every fiber of F distinct from $F^{-1}(0)$ and $F^{-1}(\infty)$ is reduced and irreducible. It follows that $\omega(\mathfrak{g})$ is a complex multiple of the 1-form

$$\deg(H)H\,dG - \deg(G)G\,dH \ .$$

Notice that $\deg(H) + \deg(G) - 2 = \deg(\mathcal{F}(\mathfrak{g})) = n - 1$. If we take H' and G' homogeneous polynomials, arbitrarily close to H and G respectively, that cut out smooth hypersurfaces intersecting transversely on $\mathbb{P}^n$ then it follows from [10] that the 1-forms $\deg(H')H'dG' - \deg(G')G'dH'$ do have isolated singularities. In particular the tangent sheaf of the induced foliations is not locally free. It follows from Theorem 1 that $\operatorname{codim}\operatorname{sing}(d(\omega(\mathfrak{g})) \leq 2$. $\qquad\square$

The next example shows that in the case of higher codimension we do have foliations satisfying the hypothesis of Corollary 6.4.

Example 6.6 (Exceptional component of $\mathcal{F}_q(q+3,3)$). *If $q \geq 2$ then there exists an irreducible component of $\mathcal{F}_q(q + 3, 3)$ such that the generic element is conjugate by an automorphism of $\mathbb{P}^{q+3}$ to the foliation induced by the natural action of $\mathrm{PSL}(2, \mathbb{C})$ on $\mathrm{Sym}^{q+3}\mathbb{P}^1 \cong \mathbb{P}^{q+3}$.*

Proof. For each natural number $r = q + 3$ let us consider the action of $\mathrm{PSL}(2, \mathbb{C})$ on the projective space $\mathbb{P}S^r(\mathbb{C}^2)^* = \mathbb{P}^r$ of binary forms of degree r. Let $\mathcal{F}_r$ be the three-dimensional foliation on $\mathbb{P}^r$ induced by this action.

A positive divisor D on $\mathbb{P}^1$ has finite stabilizer if, and only if, its support contains at least three points. Hence, the singular set of $\mathcal{F}_r$ is the union of the two-dimensional varieties $S_m = \{mp + (r - m)q, \ p, q \in \mathbb{P}^1\}$ for $0 \leq m \leq r/2$. Hence, it follows from Corollary 6.4 that for $r \geq 5$, equivalently $q \geq 2$, the foliation $\mathcal{F}_r$ is rigid. $\qquad\square$

The example above is in fact a particular case of the more general

Example 6.7. *Let $G = SL(n, \mathbb{C})$ and consider the natural action on the m-th symmetric power $V = S^m(\mathbb{C}^n)$. More generally, let G be a classical simple Lie group of dimension d and let V be a finite direct sum of irreducible representations, for instance, symmetric or alternating powers of the standard representation. In most of these cases the hypothesis above on stabilizers is satisfied and hence one obtains irreducible components of $\mathcal{F}_{n-d}(n, d)$ corresponding to these rigid foliations.*

Proof. The hypothesis on the stabilizers implies that $\omega(\mathfrak{g})$ satisfies the hypothesis of Theorem 2. Hence the statement follows from Corollary 6.4. $\qquad\square$

6.2. An Infinite Family of Rigid Foliations. Since the core of Theorem 4 proof consists in stablishing the vanishing of a certain cohomology group we will briefly recall the definition of Lie algebra cohomology thinking on reader's ease.

If $\mathfrak{g}$ is a Lie algebra and M is a $\mathfrak{g}$-module then the cohomology groups $\mathrm{H}^*(\mathfrak{g}, M)$ are defined as the cohomology of the complex $(C^*(\mathfrak{g}, M), d)$ in which the n-cochains $f \in C^n(\mathfrak{g}, M)$ are multilinear antisymmetric maps

$$f : \underbrace{\mathfrak{g} \times \ldots \times \mathfrak{g}}_{n \text{ times}} \longrightarrow M$$

and the coboundary is given by

$$df(v_0, \ldots, v_n) = \sum_{i=0}^{n} (-1)^i [v_i, f(v_0, \ldots, \hat{v}_i, \ldots, v_n)] +$$

$$+ \sum_{i<j} (-1)^{i+j} f([v_i, v_j], v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n).$$

Proof of Theorem 4. We are considering the foliation $\mathcal{F} \in \mathcal{F}_1(n, n-1)$, $n \geq 3$, induced by the subalgebra $\mathfrak{g} \subset \mathfrak{sl}(n+1, \mathbb{C})$ generated by

$$X = \sum_{i=0}^{n} (n-2i) z_i \frac{\partial z}{\partial z_i},$$

$$Y_k = \sum_{i=0}^{n-k} z_{i+k} \frac{\partial z}{\partial z_i}, \quad k = 1, \ldots, n-2.$$

Notice that

$$[X, Y_i] = -2iY_i, \text{ when } i = 1, \ldots, n-2, \quad \text{and} \quad [Y_i, Y_j] = 0 \text{ for arbitrary } i, j.$$

Using (2.1) of §2.1 to compute $d\omega$ like in lemma 2.1 we verify that

$$d\omega = (-1)^{n-2} i_Z i_{Y_1} \cdots i_{Y_{n-2}} \Omega,$$

where $\Omega = dz_0 \wedge \cdots \wedge dz_n$ and $Z = (n+1)X - (n-1)(n-2)R$.

The singular locus of $d\omega$ is thus defined by the vanishing of the $(n-1) \times (n-1)$ minors of the $(n-1) \times (n+1)$ matrix

$$\begin{pmatrix} \lambda_0 z_0 & \lambda_1 z_1 & \cdots & \lambda_{n-1} z_{n-3} & \lambda_{n-2} z_{n-2} & \lambda_{n-1} z_{n-1} & \lambda_n z_n \\ z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n & 0 \\ z_2 & z_3 & \cdots & z_{n-1} & z_n & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ z_{n-2} & z_{n-1} & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\lambda_i = (n+1)(n-2i) - (n-1)(n-2)$.

Observe that λ_n, λ_{n-1} and λ_{n-2} are all different from zero when $n \geq 3$. Thus omitting the first two columns of the matrix above we see that z_n^{n-1} appears in the ideal generated by the $(n-1) \times (n-1)$ minors. Therefore, set theoretically, $\mathrm{sing}(d\omega) \subset \{z_n = 0\}$. If we set $z_n = 0$ and omit the first and the last column we see that $\mathrm{sing}(d\omega) \subset \{z_n = 0\} \cap \{z_{n-1} = 0\}$. Analogously after omitting the last two columns and setting $z_n = z_{n-1} = 0$ we conclude that $\mathrm{sing}(d\omega) \subset \{z_n = 0\} \cap \{z_{n-1} = 0\} \cap \{z_{n-2} = 0\}$ for every $n \geq 3$. In particular $\mathrm{codim}\,\mathrm{sing}(d\omega) \geq 3$ when $n \geq 3$.

We have just verified that to prove the rigidity of $\mathcal{F}$ we can apply Theorem 3 once we know that $H^1(\mathfrak{g}, M) = 0$ where M is the $\mathfrak{g}$-module $\mathfrak{sl}(n+1, \mathbb{C})/\mathfrak{g}$. The remaining part of the proof will be devoted to establish the vanishing of this cohomology group.

Observe that $ad(X) : M \to M$ is semi-simple and that

$$M = \bigoplus_{i=-n}^{n} M_{2i}$$

where M_i is the $ad(X)$-eigenspace corresponding to the eigenvalue i.

Let $f \in C^1(\mathfrak{g}, M)$ be a cocycle, i.e., $df = 0$. We can assume that $f(X) \in M_0$. Indeed, if $f(X) \notin M_0$ then we have just to replace f by $f - dv$ for a suitable $v \in C^0(\mathfrak{g}, M)$.

Remark that $df(X, Y_1) = 0$ implies

(6.1) $$[X, f(Y_1)] = -2f(Y_1) + [Y_1, f(X)].$$

If we write

$$f(Y_1) = \sum_i v_i, \quad \text{where } v_i \in M_i,$$

then (6.1) can be rewritten as

$$\begin{cases} ad(X)(v_i) + 2v_i &= 0 \qquad \text{when} \quad i \neq -2, \\ ad(X)(v_{-2}) + 2v_{-2} &= ad(Y_1)(f(X)). \end{cases}$$

It is now clear that

$$[Y_1, f(X)] = 0 \quad \text{and} \quad f(Y_1) \in M_{-2}.$$

Since $ad(Y_1)$ maps M_0 isomorphically to M_{-2} we can assume that $f(X) = f(Y_1) = 0$. As before we have just to replace f by $f - dv$ for a suitable $v \in C^0(\mathfrak{g}, M)$.

Since $f(X) = 0$ the identity $df(X, Y_i) = 0$ holds and consequently $f(Y_i) \in M_{-2i}$ for all $i \in \{2, \ldots, n-2\}$.

Let now $F : \mathfrak{g} \to \mathfrak{sl}(n+1, \mathbb{C})$ be a a linear map lifting $f : \mathfrak{g} \to M$ such that the image of F is contained in a vector subspace $\overline{M}$ of $\mathfrak{sl}(n+1, \mathbb{C})$ satisfying

$$ad(X)(\overline{M}) \subset \overline{M} \quad \text{and} \quad \mathfrak{sl}(n+1, \mathbb{C}) = \overline{M} \oplus \mathfrak{g}.$$

The existence of $\overline{M}$ follows at once from the fact that $ad(X) : \mathfrak{sl}(n+1, \mathbb{C}) \to \mathfrak{sl}(n+1, \mathbb{C})$ is semi-simple.

If we set $F(Y_k) = B_k$, $k = 2, \ldots, n-2$, then $f(Y_k) \in M_{-2k}$ implies that

$$B_k = \sum_{i=0}^{n-k} b_i^{(k)} z_{i+k} \frac{\partial}{\partial z_i}, \quad \forall k \in \{2, \ldots, n-2\}.$$

We claim that $B_{n-2} = 0$. In fact $df(Y_1, Y_{n-2}) = 0$ implies that

$$[Y_1, B_{n-2}] = \sum_i^1 \left(b_i^{(n-2)} - b_{i+1}^{(n-2)}\right) z_{i+n-1} \frac{\partial}{\partial z_i} = 0 \qquad \text{mod } \mathfrak{g}.$$

Thus B_{n-2} must be a complex multiple of Y_{n-2}. Since $Y_{n-2} \notin \overline{M}$ the claim follows.

To conclude the proof of the Theorem we will show that $B_k = 0$ for every $k \in \{2, \ldots, n-2\}$. Clearly it suffices to settle that

(a) If $B_{n-k} = 0$ then $B_k = 0$;
(b) If $B_k = 0$ then $B_{n-(k+1)} = 0$.

To prove that (a) holds first observe that $df(Y_1, Y_k) = 0$ implies that there exists $\lambda_k \in \mathbb{C}$ such that $[Y_1, B_k] = \lambda_k Y_{k+1}$. In more explicit terms

$$\sum_{i=0}^{n-(k+1)} \left(b_i^{(k)} - b_{i+1}^{(k)} \right) z_{i+k+1} \frac{\partial}{\partial z_i} = \lambda_k Y_{k+1} .$$

Thus the sequence $\{ b_i^{(k)} \}_{i=0}^{n-k}$ is an arithmetic progression with step $-\lambda_k$.

Since $B_{n-k} = 0$ and $df(Y_k, Y_{n-k}) = 0$ then $[Y_{n-k}, B_k] = b_0^{(k)} - b_{n-k}^{(k)} z_n \frac{\partial}{\partial z_0} = 0$. Therefore $\lambda_k = 0$ and consequently B_k is a complex multiple of Y_k. Since $Y_k \notin \overline{M}$ we conclude that $B_k = 0$. Assertion (a) follows.

To prove (b) we will proceed similarly. On the one hand $df(Y_1, Y_{n-k+1}) = 0$ implies that the sequence $\{ b_i^{(n-(k+1))} \}_{i=0}^k$ is an arithmetic progression with step $-\lambda_{k+1}$. On the other hand $B_k = 0$ implies that $df(Y_k, Y_{n-(k+1)}) = [Y_k, B_{n-(k+1)}] = 0$. Since

$$[Y_k, B_{n-(k+1)}] = \sum_{i=0}^1 \left(b_i^{(n-(k+1))} - b_{i+k}^{(n-(k+1))} \right) z_{i+n-1} \frac{\partial}{\partial z_i}$$

we obtain that $\lambda_{n-(k+1)} = 0$ and that $B_{n-(k+1)}$ is a complex multiple of $Y_{n-(k+1)}$. Since $Y_{n-(k+1)} \notin \overline{M}$ the assertion (b) follows and so does the Theorem. $\square$

6.3. **Two other Rigid Foliations.** The reader will notice that with minor modifications the proof of Theorem 4 also shows that the Lie subalgebras $\mathfrak{g}(n, r) \subset \mathfrak{sl}(n+1, \mathbb{C})$ generated by

$$X = \sum_{i=0}^n (n-2i) z_i \frac{\partial z}{\partial z_i} ,$$

$$Y_k = \sum_{i=0}^{n-k} z_{i+k} \frac{\partial z}{\partial z_i} , \quad k = 1 \ldots \mathbf{n} - \mathbf{r} ,$$

satisfy $\mathrm{H}^1(\mathfrak{g}(n, r), \mathfrak{sl}(n+1, \mathbb{C})/\mathfrak{g}(n, r)) = 0$ for every $r \in \{2, \ldots, n-1\}$. The rigidity of the corresponding foliations will follows from Theorem 3 once we verify that the singular set has codimension at least 3. When the algebra above has dimension two $(r = n - 1)$ we are in a particularly interesting situation described in the example below.

Example 6.8 (Exceptional component of $\mathcal{F}_q(q+2, 2)$). *If $q \geq 2$ then there exists an irreducible component of $\mathcal{F}_q(q+2, 2)$ such that the generic element is conjugate by an automorphism of $\mathbb{P}^{q+2}$ to the foliation induced by the natural action of $\mathrm{Aff}(\mathbb{C})$ on $\mathrm{Sym}^{q+2}\mathbb{P}^1 \cong \mathbb{P}^{q+2}$.*

Proof. Let $q \geq 2$ and $\mathcal{F}_q$ be the foliation of $\mathrm{Sym}^{q+2}\mathbb{P}^1 \cong \mathbb{P}(\mathbb{C}[x,y]_{q+2}) \cong \mathbb{P}^{q+2}$ induced by the natural action of the following subgroup of $\mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{Aut}(\mathbb{P}^1)$:

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^*, b \in \mathbb{C} \right\} .$$

A positive divisor D on $\mathbb{P}^1$ has finite stabilizer if, and only if, its support contains at least two points of $\mathbb{P}^1 - \{\infty\} = \{[x : y] \in \mathbb{P}^1 | y \neq 0\}$. Therefore the generic orbit has dimension two and the singular set of $\mathcal{F}_q$ is the union of the one-dimensional varieties

$$C_m = \{(q-m)\infty + mp \, ; \, p \in \mathbb{P}^1\} \quad \text{for} \quad 0 \leq m \leq q .$$

Observe that C_q is the Veronese curve of degree of q in $\mathbb{P}^q$ and C_m is a Veronese curve of degree m in the osculating $\mathbb{P}^m$ to C_q at the point ∞. In particular the singular set of $\mathcal{F}_q$ has codimension at least 3.

To verify the rigidity of $\mathcal{F}_r$ we first give explicit expressions for the vector fields on $\mathbb{P}^r$ inducing $\mathcal{F}_r$. Since

$$
\begin{array}{rcll}
((1+\epsilon)x)^i((1+\epsilon)^{-1}y)^j & = & x^i y^j + \epsilon(i-j)x^i y^j & \mod \epsilon^2 \\
(x+\epsilon)^i y^j & = & x^i y^j + \epsilon i x^{i-1} y^{j+1} & \mod \epsilon^2
\end{array}
$$

it follows that on the basis $< z_i = x^{q-i} y^i >$ of $\mathbb{C}_q[x,y]$ the tangent sheaf of $\mathcal{F}_q$ is generated by the vector fields

$$
X = \sum_{i=0}^{q+2}(q+2-2i)z_i\frac{\partial}{\partial z_i} \quad \text{and} \quad Y = \sum_{i=0}^{q+1}(k-i)z_{i+1}\frac{\partial}{\partial z_i}.
$$

After a change of coordinates of the form $(z_0,\ldots,z_{q+2}) \mapsto (\lambda_0 z_0,\ldots,\lambda_{q+2}z_{q+2})$, where $(\lambda_0,\ldots,\lambda_{q+2}) \in \mathbb{C}^{q+3}$, we can assume that

$$
X = \sum_{i=0}^{q+2}(k-2i)z_i\frac{\partial}{\partial z_i} \quad \text{and} \quad Y = \sum_{i=0}^{q+1}z_{i+1}\frac{\partial}{\partial z_i}.
$$

Thus the corresponding algebra is isomorphic to $\mathfrak{g}(q+2,q+1)$ and the rigidity follows from Theorem 3. $\qquad\square$

We can thus interpret the foliations obtained in Theorem 4 as *extensions* of the foliations on $\mathbb{P}^n = \mathrm{Sym}^n\mathbb{P}^1$ induced by the natural action of $\mathrm{Aff}(\mathbb{C})$. Here, by an extension of a foliation $\mathcal{F}$ we mean a foliation $\mathcal{G}$ such that $T\mathcal{F} \subset T\mathcal{G}$. Since the codimension one foliations in question are rigid it is therefore natural to wonder if these extensions are unique. Below we present some examples in dimensions 6 and 7 showing that this is not the case. As we will see they also correspond to rigid foliations.

To construct the examples we will take $X, Y_1, Y_{n-2} \in \mathfrak{sl}(n+1,\mathbb{C})$ as in Theorem 4 and will look for $Y_2, Y_3, \ldots, Y_{n-3} \in \mathfrak{sl}(n+1,\mathbb{C})$ such that for every $k \in \{2,\ldots,n-3\}$ the following relations holds:

(6.2) $\qquad$ (a) $ad(X)(Y_k) = -2kY_k$ $\quad$ and $\quad$ (b) $ad(Y_1)(Y_k) = -Y_{k+1}$.

From (6.2.a) it follows that Y_k must be of the form

$$
Y_k = \sum_{i=0}^{n-k} b_i^{(k)} z_{i+k}\frac{\partial}{\partial z_i} \quad \text{for some } b_i^{(k)} \in \mathbb{C}.
$$

The relations (6.2.b) imply that $b_{i+1}^{(k)} = b_i^{(k+1)} + b_i^{(k)}$ for every $i \in \{0,\ldots,n-k-1\}$ and $k \in \{2,\ldots,n-3\}$. It is then an amusing exercise to deduce that

(6.3) $\qquad b_i^{(n-k)} = \sum_{l=0}^{k-2}\binom{i}{l}b_0^{(n-k+l)} \quad \forall k \in \{3,\ldots,n-2\}$.

The equations quoted in (6.2) together with Jacobi's relation implies that $[Y_i, Y_j]$ is an eigenvector of $ad(X)$ with eigenvalue $-2(i+j)$. Thus if the vector subspace of $\mathfrak{sl}(n+1,\mathbb{C})$ spanned by $X, Y_1,\ldots,Y_{n-2}$ is a Lie subalgebra then Y_i also satisfies the relations

$$
[Y_{n-k}, Y_k] = 0 \quad \text{and} \quad [Y_{n-k-1}, Y_k] = 0.
$$

Now notice that

$$(6.4) \qquad \begin{cases} [Y_{n-k}, Y_k] = 0 & \implies b_{i+k}^{(n-k)} b_i^{(k)} - b_i^{(n-k)} b_{i+n-k}^{(k)} = 0, \\ [Y_{n-k}, Y_{k-1}] = 0 & \implies b_{i+k-1}^{(n-k)} b_i^{(k-1)} - b_i^{(n-k)} b_{i+n-k}^{(k-1)} = 0. \end{cases}$$

The solutions of the system defined through the equations (6.3, 6.4) are completely determined by the values of $b_0^{(k)}$ where k ranges from 2 to $n-3$. For $n \in \{5, 6, 7, 8\}$ we carried out in detail the calculations. We summarize below the results:

n = 5. There are no solutions.

n = 6. There is only one solution. Namely $\left(b_0^{(2)}, b_0^{(3)} \right) = \left(\frac{9}{8}, -\frac{3}{2} \right)$. This solution corresponds indeed to a Lie algebra since $[Y_2, Y_3] = 0$ thanks to (6.4). We will denote the corresponding subalgebra by $\mathfrak{g}_6$.

n = 7. There is only one solution. Namely

$$\left(b_0^{(2)}, b_0^{(3)}, b_0^{(4)} \right) = \left(\frac{\sqrt{3}}{2}, 1 - \sqrt{3}, \frac{-3 + \sqrt{3}}{2} \right)$$

The only bracket whose vanishing is not imposed by (6.4) is $[Y_2, Y_3]$. It turns out that

$$[Y_2, Y_3] = \frac{5}{2} Y_5.$$

We will denote the corresponding subalgebra by $\mathfrak{g}_7$.

n = 8. Here we have two possibilities for $\left(b_0^{(2)}, b_0^{(3)}, b_0^{(4)}, b_0^{(5)} \right)$. Namely

$$\left(\frac{45 + 15\sqrt{265}}{256}, \frac{-15 + 5\sqrt{265}}{64}, \frac{35 - \sqrt{265}}{32}, -\frac{3}{2} \right) \text{ and } (0, 0, -1, 0).$$

In both cases we have that $[Y_2, Y_3]$ is not a complex multiple of Y_5. Thus they do not correspond to Lie subalgebras.

Proposition 6.9. *The foliations* $\mathcal{F}_k = \mathcal{F}(\mathfrak{g}_k) \in \mathcal{F}_1(k, k-1)$, $k = 6, 7$, *are rigid.*

Proof. Arguing as in the proof of Theorem 4 we can verify that $\mathrm{codim\,sing}(d\omega(\mathfrak{g}_k)) \geq 3$ for $k = 6, 7$. Instead of computing the relevant cohomology groups we will prove the rigidity of $\mathcal{F}_6$ and $\mathcal{F}_7$ by a more elementary argument.

Corollary 6.1 implies that every foliation $\mathcal{F} = [\omega]$ sufficiently close to $\mathcal{F}_k$ is induced by a $\mathfrak{g} \subset \mathfrak{sl}(k+1, \mathbb{C})$. Moreover we can also assume that $\mathrm{codim\,sing}(d\omega) \geq 3$.

Notice that $\mathfrak{h}_k = [\mathfrak{g}_k, \mathfrak{g}_k]$ has codimension one in $\mathfrak{g}_k$. By semi-continuity it follows that $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$ has either codimension one or zero in $\mathfrak{g}$. Since $\mathrm{codim\,sing}(d\omega) \geq 3$ it follows from Proposition 6.5 that $\mathfrak{h}$ has indeed codimension one in $\mathfrak{g}$.

Let $X' \in \mathfrak{g} - \mathfrak{h}$ be sufficiently close to $X \in \mathfrak{g}_k$. Since $ad(X) : \mathfrak{h}_k \to \mathfrak{h}_k$ is semi-simple we distinct eigenvalues the same holds for $ad(X') : \mathfrak{h} \to \mathfrak{h}$. Thus there exists $Z_1 \in \mathfrak{h}$ such that $[X', Z_1]$ is a multiple of Z_1 and Z_1 is a deformation of Y_1. It follows from Example 6.8 that after a change of coordinates we suppose that $X' = X$ and $Z_1 = Y_1$. In particular the eigenvalues of $ad(X') : \mathfrak{h} \to \mathfrak{h}$ are integers and by continuity they are equal to $-2, -4, -6, -8$ for $k = 6$ and $-2, -4, -6, -8, -10$ for $k = 7$. Let $Z_1, Z_2, \ldots, Z_{k-2} \in \mathfrak{h}_k$ be the corresponding eigenvectors.

Now from Jacobi's identity we deduce that

$$[X, [Z_i, Z_j]] = -2(i+j)Z_{i+j} \implies [Z_1, Z_j] = \lambda_j Z_{j+1}, \quad j = 2, \ldots, k-2.$$

Consequently after replacing Y_j by a complex multiple for $j = 2, \ldots, k - 2$ we can assume that $Z_{k-2} = Y_{k-2}$ and that $[Z_1, Z_j] = Z_{j+1}$ for $j = 2, \ldots, k - 3$. The proposition follows from the calculations made before its statement. $\square$

If $\mathfrak{g} \subset \mathfrak{sl}(n+1, \mathbb{C})$ is a rigid Lie subalgebra then, in general, we cannot guarantee that $\mathrm{H}^1(\mathfrak{g}, \mathfrak{sl}(n+1, \mathbb{C})/\mathfrak{g}) = 0$. The point is that it may happen that the variety of Lie subalgebras is non-reduced at $\mathfrak{g}$. This does not happen in the examples that we studied and we are not aware of any concrete example. Although R. Carles constructed several examples of rigid Lie algebras (of dimension at least 9) where the variety of Lie algebras is non-reduced, see for instance [5] and references there within. Thus it its natural to expect that the $\mathcal{F}_q(n, d)$ are non-reduced in general. It would be interesting to construct examples of irreducible components which are everywhere non-reduced.

Another intriguing fact is that up to now all the known irreducible components of $\mathcal{F}_q(n, d)$ are unirational varieties. It would be interesting to know if this is a general fact or if it is just a testimony of our limited knowledge about the irreducible components of the space of holomorphic foliations on projective spaces.

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