ON PRYM MODULI SPACES IN LOW GENUS, TALK AT DAGFO2008 IN BUENOS AIRES

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1. Introduction

The purpose of this talk is to discuss the unirationality problem of the Prym moduli space

$$\mathcal{R}_g$$

for very low values of g. $\mathcal{R}_g$ is the moduli space of connected étale double coverings

$$\pi : \tilde{C} \to C,$$

where C is a compact, connected Riemann surface of genus g. Let me recall that the datum of π is equivalent to the datum of a non trivial element of order two

$$\eta \in Pic^0(C).$$

I will always denote the induced fixed-point-free involution on $\tilde{C}$ as

$$i : \tilde{C} \to \tilde{C}.$$

Therefore $\mathcal{R}_g$ is also the moduli space of pairs (C, η). More in general one can pose the question of what is the Kodaira dimension of $\mathcal{R}_g$ and, for low values if g, whether $\mathcal{R}_g$ has one of the following properties:
Kodaira dimension $-\infty$, uniruledness, rational connectedness, unirationality, rationality.
A completely analogous problem can be posed for

$$\mathcal{M}_g$$

and the most interesting case to be considered after $\mathcal{M}_g$ is perhaps the case of $\mathcal{R}_g$. Both cases are still open and very much interesting for low values of g.
In the case of $\mathcal{M}_g$ there is a classical result of Eisenbud-Harris-Mumford:

Theorem 1.1. *$\mathcal{M}_g$ is of general type for $g \geq 24$.*

Recently this result has been ameliored by Farkas:

Theorem 1.2. *$\mathcal{M}_g$ is of general type also for $g = 22, 23$.*

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Moreover there has been a lot of work, due to Farkas and Farkas-Popa, on the slope conjecture of Morrison-Harris, which has been disproved for infinitely many values of g.
A corollary to the slope conjecture was $kod(\mathcal{M}_g) = -\infty$ for $g \leq 22$.
For very low values of g there are classical and recent result proving that:
-$\mathcal{M}_g$ is unirational for $g \leq 10$
-$\mathcal{M}_g$ is unirational for $g = 11, 12, 13, 14$ (Chamg-Ran, Sernesi, Chang-Ran, Verra)
-$\mathcal{M}_{15}$ is rationally connected (Bruno-Verra)
-$\mathcal{M}_{16}$ is uniruled (Chang-Ran $kod(\mathcal{M}_{16}) = -\infty$ + recent birational geometry).
The case of $\mathcal{R}_g$ has a similar more recent story: of course, due to the previous results on $\mathcal{M}_g$, it follows that $\mathcal{R}_g$ is of general type for $g \geq 22$. Indeed the natural forgetful map:

$$f : \mathcal{R}_g \to \mathcal{M}_g$$

sending the moduli point of (C, η) to the moduli point of C, is finite (of degree $2^{2g} - 1$. In particular one has $kod(\mathcal{R}_g) \geq kod(\mathcal{M}_g)$.
A very recent result of Farkas and Ludwig tells in addition that

Theorem 1.3. *$\mathcal{R}_g$ is of general type for $g > 13$, with the possible exception of $g = 15$.*

Moreover, with the same methods, one has

Theorem 1.4. *(1) $\mathcal{R}_{15}$ has Kodaira dimension ≥ 1.*
(2) $\mathcal{R}_7$ has Kodaira dimension $-\infty$.

In this talk I want to produce a somehow general geometric description of some universal Prym Brill-Noether locus

$$\mathcal{R}^2_g,$$

which dominates $\mathcal{R}_g$. Using some more geometry produced from this description I can show the following

Theorem 1.5. *$\mathcal{R}^2_g$, and hence $\mathcal{R}_g$, is unirational for $g \leq 7$.*

A further remark to be possibly eploited in the future is the following:

Theorem 1.6. *$\mathcal{R}^2_8$, and hence $\mathcal{R}_8$, is uniruled.*

To conclude this introduction let me recall that the unirational of $\mathcal{R}_g$ was known, by various independent methods, for $g \leq 6$:
- $g \leq 4$ the rationality is known (Dolgachev, Catanese $g \leq 3$), (Catanese $g = 4$).
- $g = 5$ (Clemens, Izadi-Lo Giudice-Sankaran, Verra).
- $g = 6$ (Donagi, Verra).

2. Basic reminds on Pryms

Before of continuing let me recall some well known facts on the Prym variety associated to an étale docuble cover

$$\pi : \tilde{C} \to C$$

defined by η. The Norm map

$$Nm : Pic^d(\tilde{C}) \to Pic^d(C)$$

is just the map sending $\mathcal{O}_{\tilde{C}}(\sum x_i)$ to $\mathcal{O}_C(\sum x_i)$. Nm is surjective and each of its fibres cosists of two disjoint copies of the same abelian variety

$$Prym(C, \eta)$$

of dimension $g-1$. This is known as the Prym variety of π or of (C, η). The theory of Brill-Noether is available for curves $\tilde{C}$, even if they are not general in moduli. Putting

$$d = 2g - 2$$

we have

$$Nm^{-1}(\omega_C) = P^+ \cup P^-$$

where

$$P^+ = \{\tilde{L} \ / \ Nm(\tilde{L}) \cong \omega_C \ , \ h^0(\tilde{L}) \ is \ even\}$$

and

$$P^- = \{\tilde{L} \in Pic^{2g-2}(\tilde{C}) \ / \ Nm(\tilde{L}) \cong \omega_C \ , \ h^0(\tilde{L}) \ is \ odd\}.$$

Moreover let

$$P^r = \{\tilde{L} \in P^+ \cup P^- \ / \ h^0(\tilde{L}) = 0 \ mod \ r+1 \ and \ h^0(\tilde{L}) \geq r+1\}.$$

P^r has a natural structure of scheme and it is known as the r-th Prym Brill-Noether locus. One has

$$P^0 = P^+ \ , \ P^1 = \ twice \ a \ principal \ polarization \ on \ P^+ \ \}.$$

Let $\tilde{L} \in P^r$ then $\tilde{L} \otimes i^*\tilde{L} \cong \omega_{\tilde{C}}$. The Petri map

$$\mu : H^0(\tilde{L}) \otimes H^0(i^*\tilde{L}) \to H^0(\omega_{\tilde{C}})$$

can be composed with the natural projection $h \to h - i^*h$ onto the -1 eigenspace of i^*. This composition is by definition the Prym-Petri map

$$\mu^- : H^0(\tilde{L}) \otimes H^0(i^*\tilde{L}) \to H^0(\omega_{\tilde{C}})^- = \pi^* H^0(\omega_C).$$

The main property is that

$$T_{P^+ \cup P^-, \tilde{L}} = (Im \ \mu^-)^\perp.$$

Moreover

Theorem 2.1. *For a general $\pi : \tilde{C} \to C$ the Prym-Petri map is always injective.*

In particular it follows that

$$codim\ P^r = \binom{r+1}{2}$$

and also that P^r is smooth for a general C and connected if its dimension is non zero. The general $\tilde{L} \in P^r$ satisfies $h^0(\tilde{L}) = r+1$.

Definition 2.1. *The universal r-th Prym-Brill-Noether locus is the moduli space of triples*

$$(C, \eta, \tilde{L})$$

suh that (C,η) defines a point of $\mathcal{R}_g$ and $\tilde{L} \in P^r$. it will be denoted as

$$\mathcal{R}_g^r.$$

Note that P^2 is always a codimension three subscheme of P^-: I am specially interested to this Prym-Brill Noether locus. I want to show that

Theorem 2.2. *The universal Prym Brill-Noether locus $\mathcal{R}_g^r$ is unirational for $r = 2$ and $g \leq 7$.*

3. Hypersurfaces with a quasi-étale double covering

Definition 3.1. *A quasi-etale double covering $s : \tilde{D} \to D$ is a double covering of an integral variety D which is étale in codimension one.*

We will be specially interested to the following case:

D is a hypersurface through a canonical curve C of genus g not intersecting the branch locus of s.

Actually there is no problem in replacing the canonical model of C by another projective model: this is also useful. Nevertheless I prefer to fix the ideas only to the case of a canonical (hence non hyperelliptic) curve. Another severe restriction is that I will only consider the case

$$deg\ D = 3.$$

However this is enough for my purposes. Actually I can simply start from a cubic

$$D = \{det(A) = 0\} \subset \mathbf{P}^{g-1},$$

where $A = (a_{ij})$ is a symmetric 3×3 matrix of linear forms. Of course we have the conic bundle fibration

$$\Gamma \subset \mathbf{P}^{g-1} \times \mathbf{P}^{2*} \quad \text{(dual for simplicity of further notations)}$$

of equation $(z_0, z_1, z_2)A^t(z_0, z_1, z_2) = 0$. This is uniquely defined up to projective equivalence. We have a commutative diagram

$$\begin{array}{ccccc} D & = & D & \longrightarrow & \mathbf{P}^{g-1} \\ \uparrow & & {\scriptstyle \lambda/D}\downarrow & & {\scriptstyle \lambda}\downarrow \\ \tilde{D} & \longrightarrow & D^{5+} & \longrightarrow & \mathbf{P}^{5+} \\ \downarrow & & {\scriptstyle s}\uparrow & & \uparrow \\ \mathbf{P}^2 \times \mathbf{P}^2 & = & \mathbf{P}^2 \times \mathbf{P}^2 & \longrightarrow & \mathbf{P}^8 \end{array}$$

where $\tilde{D}$ is the fibre product of s and λ/D.

Proposition 3.1. *Let $D \subset \mathbf{P}^{g-1}$ be defined by the determinant of a symmetric 3×3 matrix of linear forms $A = (a_{ij})$ as above. Assume that :*
(1) Sing $D = \{x \in D \ / \ rk\ A(x) \leq 1\}$,
(2) The linear space $Sing_3(D)$ has codimension ≥ 4 in $\mathbf{P}^{g-1}$.
Then there exist exactly one quasi-etale double covering of D and such a covering is reconstructed from D as in the previous diagram.

For $g = 3$ the étale double covering is not unique: they are prametrized by non trivial order two elements of $Pic^0(D)$. The same for cones over plane cubics: assumption (2) excludes this case.

So far we start with a canonical curve

$$C \subset \mathbf{P}^{g-1}$$

and I am looking for cubic hypersurfaces as above containing C.
Let η be a non trivial two torsion element of $Pic^0(C)$ and let $\pi : \tilde{C} \to C$ the induced étale double covering. Consider a general $\tilde{L} \in P^2$. Then $h^0(\tilde{L}) = 3$ and the Petri map

$$\mu : H^0(\tilde{L}) \otimes H^0(i^*\tilde{L}) \to H^0(\omega_{\tilde{C}})$$

defines an embedding (provided C and $\tilde{L}$ are sufficiently general)

$$\tilde{C} \subset \mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8.$$

The latter inclusion is the Segre embedding. The former one is defined by the product map $f \times f \cdot i$, where $f : \tilde{C} \to \mathbf{P}^2 = \mathbf{P}H^0(\tilde{L})^*$ is the morphism associated to $\tilde{L}$. We can arrange things so that

$$i = \iota/\tilde{C},$$

where ι is the projectivized involution $a \otimes b \to b \otimes a$. For its projectivized eigenspaces we have

$$\mathbf{P}^{2-} = \mathbf{P}V^- \quad \text{and} \quad \mathbf{P}^{5+} = \mathbf{P}\mathrm{V}^+,$$

where V^-, V^+ are the subspaces in $H^0(\omega_{\tilde{C}})$ of antisymmetric and symmetric tensors.

Let $s : \mathbf{P}^8 \to \mathbf{P}^-$ be the linear projection of center $\mathbf{P}^{2-}$, then s factors through ι. Moreover

$$D^+ = s(\mathbf{P}^2 \times \mathbf{P}^2)$$

is a cubic with equation $det(a_{ij})$, a symmetric determinant of order three of linear forms. The map

$$s : \mathbf{P}^2 \times \mathbf{P}^2 \to D^+$$

is a quasi étale double covering: its branch locus is the Veronese surface $SingD^+$.
D^+ contains $s(\tilde{C})$ which is a copy of C. More precisely $s : \tilde{C} \to s(\tilde{C})$ is the map π.

Remark 3.1. Both the curves $\tilde{C}$ and $s(\tilde{C})$ are embedded by a linear subsystem of the canonical system, respectively by

$$Im(\mu) \subset H^0(\omega_{\tilde{C}}) \text{ and } \mathrm{Im}(\mu^+) \subset \mathrm{H}^0(\omega_{\mathrm{C}}).$$

Here $H^0(\omega_C)$ is identified via π^* to $H^0(\omega_{\tilde{C}})^+$ and $\mu^+ = \mu / V^+$. Note that $\mathbf{P}^{5+} = \mathbf{P}V^{+*}$.

Dualizing μ and μ^+ we obtain two linear projections

$$\tilde{\lambda} : \mathbf{P}^{2g-2} \to \mathbf{P}^8 \; and \; \lambda : \mathbf{P}^{g-1} \to \mathbf{P}^{5+}.$$

Let $\tilde{k} : \tilde{C} \to \mathbf{P}^{2g-2}$ and $k : C \to \mathbf{P}^{g-1}$ be the canonical embeddings of $\tilde{C}$ and C. It is easy to deduce that the following diagram commutes

$$\begin{array}{ccccccc}
\tilde{C} & \xrightarrow{\tilde{k}} & \mathbf{P}^{2g-2} & \longrightarrow & \mathbf{P}^{g-1} & \xleftarrow{\kappa} & C \\
 & & \tilde{\lambda}\downarrow & & \lambda\downarrow & & \\
 & & \mathbf{P}^8 & \longrightarrow & \mathbf{P}^{5+} & & \\
 & & \uparrow & & \uparrow & & \\
\tilde{C} & \longrightarrow & \mathbf{P}^2 \times \mathbf{P}^2 & \xrightarrow{s} & D^+ & \longleftarrow & s(\tilde{C})
\end{array}$$

Note that the linear projection has an image which is the linear span

$$\Lambda = < s(\tilde{C} > .$$

For a general $\pi : \tilde{C} \to C$ we expect that μ^+ has maximal rank and we assume this property. In particular $\Lambda = \mathbf{P}^{5+}$ for $g \geq 6$. The conclusion is as follows:
the pull-back of D^+ by λ is a cubic hypersurface

$$D \subset \mathbf{P}^{g-1}$$

containing the canonical model of C. D is a cone over $\lambda \cdot \mathbf{D}^+$ of equation $det(a_{ij}) = 0$. D is endowed with a unique quasi-étale double covering, under the assumptions of the previous proposition,

$$\tilde{s} : \tilde{D} \to D,$$

where $\tilde{D}$ is a cone over $s^{-1}(\Lambda) \cdot \mathbf{P}^2 \times \mathbf{P}^2$.

Assume now that a canonical curve C of genus g is in $D - Sing(D)$. Then (1) From the quasi étale double cover σ we can reconstruct a curve $\tilde{C} \subset \tilde{D}$. Projecting from C in D^+ from the vertex of C and taking its pull-back by s, we obtain a curve

$$\tilde{C} \subset \mathbf{P}^2 \times \mathbf{P}^2$$

and an étale double covering $\pi : \tilde{C} \to C$.
(2) In addition we have a pair of line bundles $\tilde{L} = \mathcal{O}_{\tilde{C}(1,0}$ and $i^*\tilde{L} = \mathcal{O}_{\tilde{C}}(0,1)$. Notice also that $\mathcal{O}_{\tilde{C}}(1,1) \cong \omega_{\tilde{C}}$ so that $Nm(\tilde{L}) = \omega_C$ and $deg\tilde{L} = 2g-2$. In particular $h^0(\tilde{L}) \geq 3$. If the equality holds then

$$\tilde{L} \in P^2,$$

where P^2 is the Prym Brill-Noether locus of order 2 associated to π.
Roughly speaking the basic conclusion is the following

Theorem 3.2. *Let $C \subset \mathbf{P}^{g-1}$ be a canonical curve. Fix a non trivial order two element η and consider the corresponding Prym Brill-Noether locus P^2. Then*

$$P^2/ < i^* > \cong \mathcal{D}_\eta$$

where $\mathcal{D}_\eta$ is an irreducible component of the family of symmetric determinantal cubic hypersurfaces containing C.

The birational map is of course the map $D \to (\tilde{L}, i^*\tilde{L})$. Let us see two examples: we recall that P^2 has cohomology class $\Xi^3/3$ in P^-, where Ξ is a principal polarization.

Example 3.1. $g = 4$ (Catanese). $\Xi^3/3$ is the class of two points. Hence there is exactly one pair $\tilde{L}$, $i^*\tilde{L}$ for each η. The linear space Λ is the canonical space of C. The construction yelds a 4-nodal Cayley cubic surface

$$D = \Lambda \cap D^+.$$

The linear system $\mid \mathcal{O}_D(2) \mid$ dominates $\mathcal{R}_4$, the rationality of $\mathcal{R}_4$ can be shown: see Catanese.
$g = 5$ Fixing η the family $\mathcal{D}_\eta$ is a curve: its elements are cubic threefolds singular along a rational normal quartic curve. This curves turns out to be a copy of C!

4. Application to genus 7

The most new application is in genus 7: we start with the moduli space

$$\mathcal{R}_7^2$$

of pairs $(\pi : \tilde{C} \to C, \tilde{L})$ such that π is a connected étale double covering of a smooth, irreducible curve of genus 7. $\tilde{L}$ si a line bundle of degree 12 on $\tilde{C}$ such that $dim \ \mid \tilde{L} \mid$ is even and at least 2. We will always assume that the previous triple is sufficiently general. Then, applying our basic construction, the multiplication

$$\mu : H^0(\tilde{L}) \otimes H^0(i^*\tilde{L}) \to H^0(\omega_{\tilde{C}})$$

induces an embedding
$$\tilde{C} \subset \mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$$
where the latter inclusion is the Segre embedding and $i = \iota/\tilde{C}$. μ is the Petri map:

Proposition 4.1. *In genus $g \geq 6$, μ is injective for a general triple as above.*

Let $s : \mathbf{P}^8 \to \mathbf{P}^5$ be the projection of center $\mathbf{P}^-$. Then $D = s(\mathbf{P}^2 \times \mathbf{P}^2)$ is the standard symmetric cubic determinant of $\mathbf{P}^5$. Note that $s(\tilde{C}) \subset D$ is the canonical model of C projected from one point. For simplicity of notations we put
$$s(\tilde{C}) := C.$$

Proposition 4.2. *If C is general then:*
(1) C is contained in a smooth complete intersection X of 3 quadrics: $X = Q_1 \cap Q_2 \cap Q_3$.
(2) X is not contained in D.

So far we have constructed a complete intersection
$$D \cap Q_1 \cap Q_2 \cap Q_3 = C \cup \overline{C}.$$

Proposition 4.3. *For a general triple as above $\overline{C}$ is a smooth, irreducible curve.*

Now we want to analyse in detail the properties of X and $\overline{C}$.

Theorem 4.4. *X contains two disjoint, smooth conics B_1 and B_2, moreover*
$$C \in \mid H + B_1 + B_2 \mid \ , \ \overline{C} \in \mid 2H - B_1 - B_2 \mid .$$

Proof. Note that C is not linearly normal by definition and that $h^1(\mathcal{I}_C(1)) = 1$. Then
$$0 \to \mathcal{I}_S(1) \to \mathcal{I}_C(1) \to \mathcal{O}_S(H - C) \to 0$$
yelds, via the associated long exact sequence, $h^1(\mathcal{O}_S(H - C)) = 1$. Since $(H - C)^2 = -4$, Riemann-Roch implies that $h^0(C - H)) = 1$. It is easy to conclude, excluding $degB_i$ odd. □

Theorem 4.5. *The curve $\overline{C}$ has degree 12, genus 7 and the following special properties:*
(1) B_i is a 6-secant conic to $\overline{C}$,
(2) the image of $\overline{C}$ via the projection of center $< B_i >$ is a plane sextic with 3 nodes.
(3) $\overline{C}$ is not quadratically normal: $h^0(\mathcal{I}_{\overline{C}}(2)) = 4$.

Proof. Note that $\overline{C}^2 = 12$ and $H\overline{C} = 12$. To see that B_i is 6-secant to $\overline{C}$ just observe that $B_i\overline{C} = 6$. Projecting the plane $< B_i >$ the image of $\overline{C}$ is a plane sextic. Finally $h^0(\mathcal{I}_{\overline{C}}(2)) = 4$ because X is a complete intersection and $\overline{C} \sim 2H - B_1 - B_2$. □

Let

$$B$$

be one of the two conics: B_1 or B_2. Since $h^0(\mathcal{I}_{\overline{C}}(2)) = 4$ there exists exactly one net of quadrics

$$N \subset \mid \mathcal{I}_{\overline{C}}(2) \mid$$

whose base locus is

$$\Pi \cup Y$$

where Π is the plane spanned by B. This follows because a quadric through $\overline{C}$ also contains the 6-secant conic B. Hence the Kernel of the restriction $H^0(\mathcal{I}_{\overline{C}}(2)) \to H^0(\mathcal{O}_\Pi(2))$ is 3-dimensional.

Proposition 4.6. *Y is a smooth, rational surface of degree seven. It is not contained in D and*

$$D \cdot Y = \overline{C} + F$$

where F is a smooth, irreducible curve of genus 4 and degree 9.

Theorem 4.7. *(1) The scheme $F \cdot \Pi$ is an effective divisor f of degree 3 on F.*
(2) $< f >$ is a line and $\mathcal{O}_F(1) \cong \omega_F(f)$.

Proof. Recall that $E_Y := Y \cdot < B >$ is a plane cubic. Of course it contains f and $b = \overline{C} \cdot \Pi$. On the other the cubic $E_D := D \cdot < B >$ also contains f and b. Since b is in a conic it follows that f is in a line. Projecting from it we obtain the canonical model of F, hence $\mathcal{O}_F(1) \cong \omega_F(f)$. □

The embedding in D endows F with an étale double covering

$$\pi_F : \tilde{F} \to F,$$

where

$$\tilde{F} = s^{-1}(F) \subset \mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$$

$\tilde{F}$ is a curve of genus 7 and degree 18. In the $\mathbf{P}^2$ of hyperplane sections P of D such that $s^*P = P_1 + P_2$, we can consider the irreducible curve parametrizing those P which contain $< f >$. This family defines a decomposition

$$\tilde{f} = f_1 + f_2, \quad \mathrm{with\ f_i = P_i \cdot \tilde{F}}$$

So we can define

$$\tilde{M}_i := \mathcal{O}_{\tilde{F}}(P_i - f_i) \ (i = 1, 2)$$

and, by the theorem,

$$Nm(\tilde{M}) \cong \omega_F.$$

So far we have reconstructed from the point $(\pi : \tilde{C} \to C, \tilde{L})$ of $\mathcal{R}^2_7$ the following data:

- *an étale double covering: $\pi_F : \tilde{F} \to F$ of a genus 4 curve F,*
- *an effective divisor $\tilde{f}$ of degree 3 on $\tilde{F}$,*
- *a line bundle $\tilde{M}$ such that $Nm\ \tilde{M} \cong \omega_F$,*

- a plane Π *containing the trisecant line* $< f >$ *in the projective model defined by* $\omega_F(f)$*,*

Theorem 4.8. *The previous data are sufficient to reconstruct the curve* $\overline{C}$*.*

After we have $\overline{C}$ the curve C, as well as π and $\tilde{L}, i^*\tilde{L}$, are obtained from the complete interswection

$$D \cap Q_1 \cap Q_2 \cap Q_3 = C + \overline{C}$$

where Q_1, Q_2, Q_3 define a net of quadrics through $\overline{C}$ that is a plane in the web $\mid \mathcal{I}_{\overline{C}}(2) \mid = 3$.

Theorem 4.9. *Let* $\mathcal{R}$ *be the moduli space of data:*
(1) $\pi_F : \tilde{F} \to F$*, an étale double cover*
(2) $\tilde{M} \in Pic^6(\tilde{F})$ *such that* $Nm\tilde{M} cong \omega_F$ *and* $h^0(\tilde{M}) = 1$*,*
(3) $\tilde{f}$*, an effective divisor of degree three,*
(4) a plane Π *through the line* $< f >$ *in the embedding of* F *by* $\omega_F(f)$ *where* $\pi_{F*}\tilde{f} = f$*,*
(5) a net of quadrics through $\overline{C}$*, the curve constructed as above from data (1) - (4).*
Then $\mathcal{R}$ *dominates the moduli space* $\mathcal{R}^2_7$*.*

Let us count parameters: 9 for étale double coverings π_F, 3 for the line bundles considered (if they have exactly one global section), 3 for the divisors f and $\tilde{f}$, 3 for a plane through the line $< f >$, 3 for a net of quadrics in the web of quadrics containing $\overline{C}$. The total is

$$21 = dim\ \mathcal{R}^2_7!!$$

5. The unirationality of $\mathcal{R}$

We start with the easy rationality result for $\mathcal{R}_4$: let

$$S \subset \mathbf{P}^3$$

be a symmetric cubic determinant of maximal rank, that is Cayley 4-nodal cubic surface. Then

$$\mid \mathcal{O}_S(2) \mid$$

naturally dominates $\mathcal{R}_4$ via our usual construction. Let

$$\tilde{S} \subset \mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$$

be the pull-back of S by s. Then

$$\tilde{S} = \Lambda \cdot \mathbf{P}^2 \times \mathbf{P}^2,$$

where Λ is a general space of dimension 6 passing through $\mathbf{P}^{2-}$.
$\tilde{S}$ is a sextic Del Pezzo surface endowed with an involution with 4 fixed points: $\iota/\tilde{S}$. On $\tilde{S}$ we consider the linear system of curves

$$\mid \mathcal{O}_{\tilde{S}}(2) \mid^+ = \pi^* \mid \mathcal{O}_S(2) \mid := \mid \tilde{F} \mid .$$

These curves are just the pull-back by s of quadratic sections of S. The line bundles we want on a curve $\tilde{F}$ of this linear system are of the type

$$\tilde{M} = \mathcal{O}_{\tilde{F}}(x_1 + \cdots + x_6)$$

where

$$s(x_1) + \cdots + s(x_6) = S \cap A$$

where A is a conic in $\mathbf{P}^3$. In other words we are lookng to 0-dimensional subschemes z of $\tilde{S}$ having length 6 and such that

$$s_* z = S \cap Q \cap P$$

where Q is a quadric and P is a plane. In particular z is contained in a curve

$$\tilde{E} = s^* E \in \mid \mathcal{O}_S(1) \mid .$$

As a divisor on $\tilde{E}$, z defines a line bundle of degree 6 $\mathcal{O}_{\tilde{E}}(z)$ such that

$$Nm\mathcal{O}_{\tilde{E}}(z) \otimes s^*\mathcal{O}_E(-1) \cong \mathcal{O}_E.$$

Since the Kernel of

$$NmPic^0(\tilde{E}) \to Pic^0(E)$$

is $\mathbf{Z}_2$, there is a unique such a line bundle $\mathcal{O}_{\tilde{E}}(z)$ different from $s^*\mathcal{O}_E(1)$. The conclusion is the following

Proposition 5.1. *For each smooth $\tilde{E} \in \mid \mathcal{O}_{\tilde{S}}(1) \mid^+$ there exists exactly one linear system*

$$\mid z \mid$$

if divisors of degree 6 such that $s_* z$ is contained in a conic section of $E = s(\tilde{E})$ and z is not in $\mid s^*\mathcal{O}_E(1) \mid$.

Corollary 5.2. *Let $\mathcal{Z}$ be the family of 0-dimensional schemes z as above then $\ddagger$ is a $\mathbf{P}^5$-bundle over $\mathbf{P}^3$.*

Let

$$\tilde{S}[3]$$

be the Hilbert scheme of 3 points in $\tilde{S}$, for each pair

$$(z, t) \in \mathcal{Z} \times S[3]$$

we consider

$$\mid I_{z+t}(2) \mid^+ \subset \mid \tilde{F} \mid = \mid \mathcal{O}_{\tilde{S}}(2) \mid^+ .$$

This is a pencil: actually it is the pull-back of a pencil of quadrics passing through the conic c_z defined by the push-down $s_* z$ and through $s_* t$.

Proposition 5.3. *The incidence correspondence parametrizing triples*

$$(z, t, \tilde{F}') \in \mathcal{Z} \times \tilde{S}[3] \times \mid \tilde{F} \mid \ / \ z + t \subset \tilde{F}'\}$$

is a $\mathbf{P}^1$-bundle on $\mathcal{Z} \times S[3]$.

We denote such a rational 15-dimensional variety as

$$\mathbb{F}.$$

Since a general $\pi : \tilde{F} \to F$ is represented by an embedding

$$\tilde{F} \in \tilde{S}$$

as a quadratic secion which is a +1 eigenvector of $\iota/\tilde{S}$, it is clear that $\mathbf{P}$ dominates the family of triples

$$(\pi : \tilde{F} \to F, \tilde{M}, \tilde{f})$$

such that $Nm\tilde{M} \cong \omega_F$, $h^0(\tilde{M}) = 1$, $\tilde{f} \in \tilde{F}[3]$. On $\mathbb{F}$ we construct a $\mathbf{P}^3$-bundle as follows: let

$$\mathcal{V}$$

be the vector bundle on $\mathbb{F}$ with fibre

$$H^0(\omega_F(s_{*t})^*$$

at $(z, t, \tilde{F})$. We can consider the universal family

$$\mathcal{U} \subset \mathbb{F} \times S$$

and its natural embedding

$$\mathcal{U} \subset \mathcal{V}.$$

For each $(z, t, \tilde{F})$ the divisor t spans a line in the embedding $F \subset \mathbf{P}\mathcal{V}_{(z,t,\tilde{F})}$. The $\mathbf{P}^3$-bundle we consider is the family of planes

$$\Pi \supset < t > .$$

We denote such a projective bundle as

$$\mathbb{P}.$$

It parametrizes 4-tuples

$$(z, t, \tilde{F}, \Pi)$$

as above. We know that $\mathbb{P}$ is also the parameter space for a family of curves

$$\overline{C} \subset \mathbf{P}^5$$

of degree 12 birational to plane sextics with three nodes. For each $(z, t, \tilde{F}, \Pi)$ we have indeed a cubic

$$D \subset \mathbf{P}H^0(\omega_F(s_*t))^*$$

defined by the pair of line bundles

$$\mathcal{O}_{\tilde{F}}(z + t) \ , \ \mathcal{O}_{\tilde{F}}(\iota^* z + \iota^* t).$$

Moreover there is a unique net of quadrics N passing through F and such that the base locus is

$$\Pi \cup Y.$$

Finally

$$D \cdot Y = \overline{C} + F,$$

where $\overline{C}$ is the required curve. Let

$$\overline{\mathcal{C}} \subset \mathbf{P}\mathcal{V}$$

be the corresponding universal family of curves. These curves are not quadratically normal and $h^0(\mathcal{I}_{\overline{C}}(2)) = 4$. Let

$$\mathbb{C}$$

be the projective bundle with fibre $\mid \mathcal{I}_{\overline{C}}(2) \mid$ at $\overline{C}$. Then $\mathbb{C}$ maps onto the moduli space

$$\mathcal{R}^2_7.$$

Indeed a point of $\mathbb{C}$ uniquely defines, in particular, the symmetric determinantal cubic D and a net of quadrics through $\overline{C}$ generated say by Q_1, Q_2, Q_3. Then

$$D \cap Q_1 \cap Q_2 \cap Q_3 = \overline{C} + C$$

and C is a curve of genus 7 and degree 12 which is the linear projection from one point of the canonical space. D, using also z, defines $\pi : \tilde{C} \to C$ and $\tilde{L}$. The map is dominant because we started with this construction. Conclusion

Theorem 5.4. *$\mathcal{R}^2_7$ is unirational.*

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