Algebras of Matrix Differential Operators

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Introduction

The study of commuting pairs of ordinary differential operators dates back to the begging of the twentieth century, with the pioneering works of Schur and Burchnall and Chaundy. Let

$$D_2 = \partial^2 + V(x), \qquad D_3 = a_3(x)\partial^3 + a_2(x)\partial^2 + a_1(x)\partial + a_0(x)$$

and now impose the relation $[D_2, D_3] = 0$.

Then we can assume $a_3(x) = 1$ up to scaling, and we also get $a_2(x) = A_2$ and arbitrary constant. Furthermore

$$a_1 = \frac{3}{2}V + A_1, \qquad a_0 = A_2 V + \frac{3}{4}V' + A_0.$$

Finally from the zero order term of $[D_2, D_3]$ we get, after two trivial integrations,

$$V'^2 + 2V^3 + 4A_1V^2 - 2A_{-1}V + A_{-2} = 0.$$

Therefore $V = V(x)$ is an elliptic function, that is a doubly periodic meromorphic function or a degeneration of one, such as a trigonometric or a rational function.

A degeneration yields the simplest example $V = -\frac{2}{x^2}$ and

$$D_2 = \partial^2 - \frac{2}{x^2}, \qquad D_3 = \partial^3 - \frac{3}{x^2}\partial + \frac{3}{x^3},$$

and $D_3^2 = D_2^3$.

If we keep one of the periods finite we may choose $V = -\frac{2}{\sin^2(x)}$, then

$$D_2 = \partial^2 - \frac{2}{sin^2 x}, \qquad D_3 = \partial^3 + \left(1 - \frac{3}{\sin^2 x}\right)\partial + \frac{3\cos x}{\sin^3 x},$$

and $D_3^2 = D_2(D_2 + I)^2$.

The punch line is that only very special choice of $V(x)$ allows for the existence of a differential operator of order three that would commute with one of order two.

Burchnall and Chaundy pointed out that any commuting pair (Q, P) of ordinary differential operators

$$Q = \partial^m + u_2(x)\partial^{m-2} + \cdots + u_m(x), \quad P = \partial^n + v_2(x)\partial^{n-2} + \cdots + u_n(x)$$

satisfy a polynomial relation $F(Q, P) = 0$.

Therefore the eigenvalues of the joint eigenvalue problem

$$Q\psi = z\psi, \qquad P\psi = w\psi$$

satisfy the algebraic relation $F(z,w) = 0$, the spectral curve.

These commuting pairs are classified by a set of algebro-geometric data. This set consist of the spectral curve Γ with a mark point γ_∞, a holomorphic vector bundle E on Γ and some additional data related to the local structure of Γ and E in a neighborhood of γ_∞.

I. M. Krichever 1970, D. Munford 1978, K. Takasaki 2005.

Matrix Orthogonal Polynomials

Let $W = W(x)$ be a weight matrix of size N on the real line. By this we mean a complex $N \times N$-matrix valued integrable function on the interval (a, b) such that $W(x)$ is positive definitive almost everywhere and with finite moments M_n for all n,

$$M_n = \int_a^b x^n W(x)\, dx$$

Let A be the algebra of all $N \times N$ matrices over $\mathbb{C}$, and let $A[x]$ be the algebra of all polynomials in the undetermined x with coefficients in A.

Stieljes 1894, M. G. Krein 1949.

We introduce the following Hermitian sesquilinear form in $A[x]$:

$$(P, Q) = \int_a^b P(x)W(x)Q(x)^* \, dx.$$

$$\begin{aligned}
(aP + bQ, R) &= a(P, R) + b(Q, R), \\
(TP, Q) &= T(P, Q), \\
(P, Q)^* &= (Q, P), \\
(P, P) &\geq 0; \quad \text{if } (P, P) = 0 \text{ then } P = 0.
\end{aligned}$$

In other words we have that $A[x]$ is a left inner product A-module.

Principle of measurable choice (E. A. Azoff)

Let X and Y be complete separable metric spaces and E, a closed σ-compact subset of $X \times Y$. Then $\pi_1(E)$ is a Borel set in X and there exists a Borel function $\phi : \pi_1(E) \to Y$ whose graph is contained in E.

Let $H(N)$ and $U(N)$ denote respectively, the space of all Hermitian $N \times N$ matrices and the unitary group.

Corollary. There is a Borel function $\psi : H(N) \to U(N)$ associating with each Hermitian matrix H, a unitary matrix $\psi(H)$ such that $\psi(H)^* H \psi(H)$ is real diagonal.

Proposition. Let $P = \sum_{0 \leq j \leq n} x^j P_j$ be an A-polynomial of degree n. Then $ker\,(P, P) = \bigcap_{0 \leq j \leq n} ker(P_j^*)$. In particular (P, P) is nonsingular if P_j is nonsingular for some $0 \leq j \leq n$. Moreover $(P, P) = 0$ implies $P = 0$.

Proposition Let $V_n = \{F \in A[x] : \deg F \leq n\}$ for all $n \geq 0$, $V_{-1} = 0$ and $V_{n-1}^{\perp} = \{H \in V_n : (H, F) = 0 \text{ for all } F \in V_{n-1}\}$. Then $V_{n-1}^{\perp}$ is a left free A-module of dimension one and

(i) $V_n = V_{n-1} \oplus V_{n-1}^{\perp}$ for all $n \geq 0$.

(ii) There is a unique monic polynomial P_n in $V_{n-1}^{\perp}$ and it is of degree n for all $n \geq 0$.

Corollary $\{P_n\}_{n\geq 0}$ is the unique sequence of monic orthogonal polynomials in $A[x]$. Any sequence $\{Q_n\}_{n\geq 0}$ of orthogonal polynomials in $A[x]$ is of the form $Q_n = A_n P_n$ where $A_n \in GL_N(\mathbb{C})$. Moreover the sequence $\{P_n\}_{n\geq 0}$ satisfies a three term recursion relation of the form

$$xP_n(x) = A_n P_{n-1}(x) + B_n P_n(x) + P_{n+1}(x)$$

Matrix Differential Operators

A differential operator D could be made to act either on the left or on the right on $A[x]$. If one wants to have matrix weights W that are not direct sums of scalar one and that have matrix polynomials as their eigenfunctions, one should settle for right-hand-side differential operators. We agree now to say that D given by

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \qquad \partial = \frac{d}{dx},$$

acts on $P(x)$ by means of

$$PD = \sum_{i=0}^{s} \partial^i (P)(x) F_i(x).$$

Proposition. Let $W = W(x)$ be a weight matrix of size N and let $\{P_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials in $A[x]$. If

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \qquad \partial = \frac{d}{dx},$$

is a linear right-hand side ordinary differential operator of order s such that

$$P_n D = \Lambda_n P_n \qquad \text{for all} \quad n \geq 0$$

with $\Lambda_n \in A$, then $F_i = F_i(x) \in A[x]$ and $\deg F_i \leq i$. Moreover D is determined by the sequence $\{\Lambda_n\}_{n\geq 0}$.

If

$$F_i(x) = \sum_{j=0}^{i} x^j F_j^i(D),$$

then

$$\Lambda_n = \sum_{i=0}^{s} [n]_i F_i^i(D) \qquad \text{for all} \quad n \geq 0,$$

where

$$[\nu]_i = \nu(\nu-1)\cdots(\nu-i+1), \qquad [\nu]_0 = 1.$$

Let

$$\mathcal{D} = \{D = \sum_{i=0}^{s} \partial^i F_i(x) : F_i \in A[x], \deg F_i \leq i\}.$$

We are ready to introduce the main character of our tale.

Given a sequence of orthogonal polynomials $\{Q_n\}_{n\geq 0}$ we are interested in the following algebra

$$\mathcal{D}(W) = \{D : Q_n D = \Gamma_n(D) Q_n,\ \Gamma_n(D) \in A \text{ for all } n \geq 0\}.$$

Proposition Given a sequence $\{Q_n\}_{n\geq 0}$ of orthogonal polynomials and $D \in \mathcal{D}(W)$ let $\Gamma(D,n) = \Gamma_n(D)$. Then $D \mapsto \Gamma(D,n)$ is a representation of $\mathcal{D}(W)$ into A, for each $n \geq 0$. Moreover the sequence of representations $\{\Gamma_n\}_{n\geq 0}$ separates the elements of $\mathcal{D}(W)$.

The adjoint operation

Proposition. If $D \in \mathcal{D}$ is a right-hand side linear differential operator which satisfies the symmetry condition $(PD, Q) = (P, QD)$ for all $P, Q \in A[x]$, then $D \in \mathcal{D}(W)$.

Theorem. Let $\{P_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials associated to the weight matrix $W = W(x)$. Given $D = \sum_{i=0}^{s} \partial^i F_i \in \mathcal{D}(W)$ let $\widetilde{D} = \sum_{i=0}^{s} \partial^i G_i \in \mathcal{D}$, where the G_i are defined inductively by

(i) $G_0 = (P_0, P_0)\Lambda_0(D)^*(P_0, P_0)^{-1}$, and

(ii) $j!G_j = (P_j, P_j)\Lambda_j(D)^*(P_j, P_j)^{-1}P_j - \sum_{i=0}^{j-1} \partial^i(P_j)G_i$ for $1 \leq i \leq s$.

Then $(PD, Q) = (P, Q\widetilde{D})$ for all $P, Q \in A[x]$.

Corollary. For any $D \in \mathcal{D}(W)$ there exists a unique differential operator $D^* \in \mathcal{D}(W)$ such that $(PD, Q) = (P, QD^*)$ for all $P, Q \in A[x]$. We shall refer to D^* as the adjoint of D. The map $D \mapsto D^*$ is a *-operation in the algebra $\mathcal{D}(W)$, and the orders of D and D^* coincide. Moreover $\mathcal{S}(W)$ is a real form of the space $\mathcal{D}(W)$, i.e.

$$\mathcal{D}(W) = \mathcal{S}(W) \oplus i\mathcal{S}(W).$$

If $\{Q_n\}_{n\geq 0}$ is a sequence of orthogonal polynomials and $\{\Gamma_n\}_{n\geq 0}$ is the corresponding sequence of representations of $\mathcal{D}(W)$, then

$$\Gamma_n(D^*) = (Q_n, Q_n)\Gamma_n(D)^*(Q_n, Q_n)^{-1}$$

for all $D \in \mathcal{D}(W)$. In particular if $\{Q_n\}_{n\geq 0}$ is a sequence of orthonormal polynomials then D is symmetric if and only $\Gamma_n(D)$ is Hermitian for all $n \geq 0$.

Corollary. The representations Λ_n of $\mathcal{D}(W)$ are completely reducible.

We observe that given a weight matrix $W(x)$ the algebra $\mathcal{D}(W)$ is most likely going to be trivial. By integration by parts one finds necessary and sufficient conditions on smooth weights W to have a symmetric second order differential operator. A similar result holds for a symmetric differential operator of any order. Therefore one has, modulo the difficult task of explicitly solving the corresponding system of differential equations, a way of getting $\mathcal{S}(W)$ and hence $\mathcal{D}(W)$.

The ad-conditions

We have a sequence of representations $\{\Lambda_n\}_{n\geq 0}$ of $\mathcal{D}(W)$ into A. In other words we have a homomorphism Λ of $\mathcal{D}(W)$ into the direct product of $\mathbb{N}_0$ copies of A. Moreover Λ is injective. To give a precise description of the range of this homomorphism, recall that our starting point is a weight matrix $W(x)$ on the real line and its unique sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$, together with the three-term recursion relation

$$xP_n(x) = A_nP_{n-1}(x) + B_nP_n(x) + P_{n+1}(x), \quad n \geq 0,$$

where we put $P_{-1}(x) = 0$.

It is convenient to introduce the block tridiagonal matrix L

$$L = \begin{pmatrix} B_0 & I & & \\ A_1 & B_1 & I & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

The recursion relation now takes the form

$$LP = xP \tag{1}$$

where P stands for the vector

$$P(x) = \begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ \vdots \end{pmatrix}$$

Assume that $D \in \mathcal{D}(W)$, i.e.,

$$P_n D = \Lambda_n P_n \quad n \geq 0.$$

If Λ denotes the block diagonal matrix

$$\Lambda = \begin{pmatrix} \Lambda_0 & & \\ & \Lambda_1 & \\ & & \ddots \end{pmatrix}$$

we observe that from (1) we get, for any integer $m \geq 0$,

$$(adL)^m(\Lambda)P = (L - xI)^m \Lambda P.$$

Theorem If $D \in \mathcal{D}(W)$ and Λ is the block diagonal matrix with $\Lambda_n = \Lambda_n(D)$ we have

$$(\mathit{ad}\ L)^{m+1}(\Lambda) = 0 \tag{2}$$

for some m. Conversely, if Λ is a block diagonal matrix satisfying this condition for some $m \geq 0$, then there is a unique differential operator D in $\mathcal{D}(W)$ such that $\Lambda_n = \Lambda_n(D)$ for all $n \geq 0$. Moreover the order of D is equal to the minimum m satisfying (2).

F. A. Grünbaum and T, IEOT, 2007.

Scalar examples

Classical Orthogonal Polynomials

Jacobi:

$$w(x) = (1-x)^{\alpha}(1+x)^{\beta},\ -1 < x < 1;\ \alpha, \beta > -1$$

$\alpha = \beta$ Gegenbauer; $\alpha = \beta = \frac{1}{2}$ Chevishev first kind;

$\alpha = \beta = -\frac{1}{2}$ Chevyshev second kind; $\alpha = \beta = 0$ Legendre

Laguerre:

$$w(x) = x^{\alpha} e^{-x},\ x > 0;\ \alpha > -1$$

Hermite:

$$w(x) = e^{-x^2},\ -\infty < x < \infty$$

$$D(p_n)(x) = a_2(x)p_n''(x) + a_1(x)p_n'(x) = \lambda_n p_n(x)$$

Jacobi:

$$(1-x^2)p_n'' + (\beta - \alpha - (\alpha+\beta+2)x)p_n' = -n(n+\alpha+\beta+1)p_n$$

changing variables: $x = 1 - 2z$

$$z(1-z)p_n'' + (\alpha + 1 - (\alpha+\beta+2)z)p_n' = -n(n+\alpha+\beta+1)p_n$$

Laguerre: $$xp_n'' + (\alpha+1-x)p_n' = -np_n$$

Hermite: $$p_n'' - 2xp_n' = -2np_n$$

L. Miranian, Thesis UC Berkeley, 2005: $\mathcal{D}(w) = C[D]$.

A matrix instructive example

$$W_a(x) = e^{-x^2} e^{Ax} e^{A^*x} \qquad -\infty < x < \infty$$
$$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \qquad a \in \mathbb{C}^\times$$

Rodrigues' formula

$$P_n(x) = (-2)^{-n} e^{x^2} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{|a|^2 n + 2} \end{pmatrix} \left[e^{-x^2} \begin{pmatrix} 1 + |a|^2 x^2 + \frac{|a|^2 n}{2} & ax \\ \bar{a}x & 1 \end{pmatrix} \right]^{(n)}$$
$$\times \begin{pmatrix} 1 & -ax \\ -\bar{a}x & 1 + |a|^2 x^2 \end{pmatrix}$$

(Grünbaum, Durán, IMRN 2004)

If $a = |a|e^{2i\theta}$, then

$$\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} W_a(t) \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = W_{|a|}(t).$$

This implies that $\mathcal{D}(W_a)$ and $\mathcal{D}(W_{|a|})$ are conjugated. Therefore we will assume that $a > 0$. M. Castro and F. A. Grünbaum, (see IMRN, 2006) experimentally, with assistance from symbolic computation, conjectured that $\mathcal{D}(W)$ was generated by the following differential operators:

$$D_1 = -\frac{1}{2}\partial^2 + \partial(xI - aE_{12}) + E_{11}$$

$$D_2 = \partial^2 \, \frac{a^2}{2}(axE_{12} - E_{11}) + \partial \, a(axE_{22} + E_{12} - E_{21}) + 2E_{22}$$

$$D_3 = \partial^2\, a^2 \left(a^2x^2E_{12} + ax(E_{22} - E_{11}) - E_{21}\right)$$
$$+ \partial\, 2a \left(a(a^2+2)xE_{12} + E_{22} - (a^2+1)E_{11}\right) + 2(a^2+2)E_{12}$$

$$D_4 = -\partial^2\, \frac{a^2}{4}E_{12} + \partial\frac{a}{2}(E_{11} - E_{22}) + E_{21}$$

$$\Lambda_1 = nI + E_{11}, \;\; \Lambda_2 = fE_{22}, \;\; \Lambda_3 = f(f + a^2)E_{12}, \;\; \Lambda_4 = E_{21},$$

where $f = a^2n + 2$.

Proposition The set $\{D_1^i, D_1^i D_2, D_1^i D_3, D_1^i D_4 : i \geq 0\}$ is a basis of $\mathcal{D}(W)$, and the multiplication table is

$$D_2D_1 = D_1D_2, \quad D_2D_2 = (a^2D_1 + 2I)D_2, \quad D_2D_3 = 0$$
$$D_2D_4 = (a^2D_1 + 2I)D_4, \quad D_3D_1 = (D_1 - I)D_3$$
$$D_3D_2 = (a^2D_1 + (2 - a^2)I)D_3, \quad D_3D_3 = 0$$
$$D_3D_4 = a^4D_1^2 - a^2D_1D_2 + a^2(4 - a^2)D_1 + (a^2 - 2)D_2 + 2(2 - a^2)I$$
$$D_4D_1 = (D_1 + I)D_4, \quad D_4D_2 = 0$$
$$D_4D_3 = (a^2D_1 + (2 + a^2)I)D_2, \quad D_4D_4 = 0$$

Moreover D_1 and $D_{-1} = D_3 + 4D_4$ generates $\mathcal{D}(W)$.

(T, 2007)

The proof of the proposition starts with the following

Proposition Let $P_n = \sum_{i=0}^{n} x^i B_i^n$ and $D = \sum_{j=0}^{s} \partial^j F_j(t)$, with $F_j(t) = \sum_{i=0}^{j} t^i F_i^j$. Then $D \in \mathcal{D}(W)$ if and only if

$$\sum_{r=0}^{n-m} B_{m+r}^n \Big(\sum_{i=0}^{s-r} [m+r]_{i+r} F_i^{i+r} \Big) - \Big(\sum_{i=0}^{s} [n]_i F_i^i \Big) B_m^n = 0, \qquad (3)$$

for all $0 \le m \le n$, $0 \le n$.

This is an infinite system of linear equations where the unknowns are the matrices F_i^j. In order to simplify this system we take advantage of an involutive automorphism that the algebra $\mathcal{D}(W)$ possesses.

Let

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then

$$W(-t) = TW(t)T.$$

Given $D = \sum_{0 \le i \le s} \partial^i F_i \in \mathcal{D}(W)$ let $\widetilde{D} = \sum_{0 \le i \le s} \partial^i (-1)^i T \check{F}_i T$, where $\check{F}_i(t) = F_i(-t)$. Then $\widetilde{D} \in \mathcal{D}(W)$ and the map $D \mapsto \widetilde{D}$ is an involutive automorphism of the algebra $\mathcal{D}(W)$. Let

$$\mathcal{D}_1(W) = \{D \in \mathcal{D}(W) : \widetilde{D} = D\}, \quad \mathcal{D}_{-1}(W) = \{D \in \mathcal{D}(W) : \widetilde{D} = -D\}.$$

Then

$$\mathcal{D}(W) = \mathcal{D}_1(W) \oplus \mathcal{D}_{-1}(W).$$

Then we are able to find the space of all differential operators in $\mathcal{D}_1(W)$ of order less or equal to two, which turns out to be of dimension three and generated by I, D_1 and D_2. Similarly we determine the space of all differential operators in $\mathcal{D}_{-1}(W)$ of order less or equal to two, which turns out to be of dimension two and generated by D_3 and D_4.

Let $\mathcal{A}(W)$ be the subalgebra of $\mathcal{D}(W)$ generated by all $D \in \mathcal{D}(W)$ of order less or equal to two, and let

$$\mathcal{A}_1(W) = \mathcal{A}(W) \cap \mathcal{D}_1(W) \quad \text{and} \quad A_{-1}(W) = \mathcal{A}(W) \cap \mathcal{D}_{-1}(W).$$

In order to prove that $\mathcal{A}(W) = \mathcal{D}(W)$, let $\mathcal{C}_1$ be equal to the linear space generated by the leading coefficients of all $D \in \mathcal{D}_1(W)$ of order less or equal to two.

The linear space $\mathcal{C}_1$ is a two dimensional subalgebra of $A[t]$. Moreover, for any $F \in \mathcal{C}_1$ and $r \in \mathbb{N}$ there exists $D \in \mathcal{A}_1(W)$ of order $2r$, with leading coefficient F. Then we establish that there is no $D \in \mathcal{D}_1(W)$ of odd order, and that if F is the leading coefficient of a differential operator $D \in \mathcal{D}_1(W)$ then $F \in \mathcal{C}_1$, from where it follows that $\mathcal{A}_1(W) = \mathcal{D}_1(W)$.

Let $\mathcal{D}_n$, $\mathcal{D}_{1,n}$ and $\mathcal{D}_{-1,n}$ be, respectively, the subspaces of $\mathcal{D}(W)$, $\mathcal{D}_1(W)$ and $\mathcal{D}_{-1}(W)$ of all differential operators of order less or equal to n. Similarly let $\mathcal{A}_{1,n}$ and $\mathcal{A}_{-1,n}$ be, respectively, the subspaces of $\mathcal{A}_1(W)$ and $\mathcal{A}_{-1}(W)$ of all differential operators of order less or equal to n.

Then

Theorem For any $r \geq 1$ we have,

(i) $\dim(\mathcal{D}_{1,2r}/\mathcal{D}_{1,2(r-1)}) = 2$,

(ii) $\dim(\mathcal{D}_{-1,2r}/\mathcal{D}_{-1,2(r-1)}) = 2$,

(iii) $\dim(\mathcal{D}_{2r}/\mathcal{D}_{2(r-1)}) = 4$,

(iv) $\dim(\mathcal{A}_{-1,2r}/\mathcal{A}_{-1,2(r-1)}) = 2$,

(v) $\mathcal{A}_{-1}(W) = \mathcal{D}_{-1}(W)$.

Statement (iii) was conjectured by Castro and Grünbaum. At this point since $\mathcal{A}_1(W) = \mathcal{D}_1(W)$ and $\mathcal{A}_{-1}(W) = \mathcal{D}_{-1}(W)$ it follows that the algebras $\mathcal{D}(W)$ and $\mathcal{A}(W)$ coincide. In other words the algebra $\mathcal{D}(W)$ is generated by the subspace $\mathcal{D}_2(W)$.

Now it follows easily that

$$\{D_1^i, D_1^i D_2, D_1^i D_3, D_1^i D_4 : i \geq 0\}$$

is a basis over $\mathbb{C}$ of $\mathcal{D}(W)$.

The element $D_{-1} = D_3 + 4D_4 \in \mathcal{D}_{-1}(W)$, has two nice properties: the set $\{D_1, D_{-1}\}$ generates the algebra $\mathcal{D}(W)$ and D^2_{-1} is a central element. Let $\mathcal{Z} = \mathbb{C}[D^2_{-1}]$ be the polynomial subalgebra of $\mathcal{D}(W)$ generated by the algebraically independent element D^2_{-1}. Then we establish that $\mathcal{D}(W)$ is a free module over $\mathcal{Z}$ of dimension eight. More precisely the set

$$\{I, D_1, D_1^2, D_1^3\} \cup \{D_{-1}, D_1 D_{-1}, D_1^2 D_{-1}, D_{-1} D_1\}$$

is a $\mathcal{Z}$-basis of $\mathcal{D}(W)$.

The algebra $\mathcal{D}(W)$ is also presented by generators and relations: it is generated by two elements E and F, and the relations are

$$
\begin{aligned}
&F^2E - EF^2 = 0,\\
&F^4 - 2a^4F^2E^2 - 8a^2F^2E - 8F^2 + a^8E^4 + 8a^6E^3\\
&- a^4(a^4-24)E^2 - 4a^2(a^4-8)E - 4(a^4-4)I = 0,\\
&- 4a^6E^3 + 2a^2EF^2 - a^2FEF - 24a^4E^2 + 2F^2 + 4a^2(a^4-12)E\\
&+ 8(a^4-4)I = 0,\\
&E^3 + E^2[E,F] - a^4E^2F - a^4E^2[E,F] + a^2(a^2-4)EF\\
&+ a^2(a^2-4)E[E,F] + 2(a^2-2)F + 2(a^2-2)[E,F] = 0,\\
&F^3 - 4a^4EFE - 8a^2(EF+FE) - 16F = 0,\\
&FE^2 + E^2F - 2EFE - F = 0.
\end{aligned}
$$

Then we compute the center of $\mathcal{D}(W)$. Set

$$Z = \left(\tfrac{3}{4}D_{-1}^2 + (a^4 - 12)I\right)D_1 - 6a^2 - a^4 D_1^3.$$

We establish that the center $\mathcal{Z}(W)$ of the algebra $\mathcal{D}(W)$ is generated by D_{-1}^2 and Z, and that it is isomorphic to the affine algebra of the elliptic curve

$$4x^3 - y^2 - 12xy + (a^4 - 36)x^2 - 4(a^4 - 4)y - 24(a^4 - 4)x - 4(a^4 - 4)^2 = 0.$$

A big, and rather blurry, challenge is that of finding the appropriate algebro-geometric objects associated to $\mathcal{D}(W)$ for any weight matrix W, that reduce in the abelian case to a curve and a vector bundle on it. Further study of this example may be instructive in this respect.

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