Exceptional planar webs and their associated (exceptional) surfaces

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29 juillet 2008

Plan of the talk

1. INTRODUCTION TO WEB GEOMETRY
   1.1 *webs, abelian relation, rank*
   1.2 *examples : algebraic webs, exceptional webs*
   1.3 *algebraization, Chern's problem*

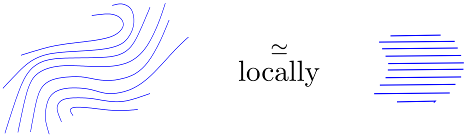
2. TOOLS AND RESULTS
   2.1 *webs with an infinitesimal symetry*
   2.2 *characterization of maximal rank webs*
   2.3 *CDQL webs*

3. FROM WEBS TO PROJECTIVE SURFACES
   3.1 *the « canonical map » of a web*
   3.2 *Segre's principal directions and generalizations*
   3.3 *exceptional projective surfaces*

Web geometry : first definitions

M : a $\mathbb{C}$-manifold $\qquad (M = (\mathbb{C}^2, 0))$

Definition : Foliation $\mathcal{F}$



Definition : 1. A completely decomposable (CD) d-web is

$$W_d = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \cdots \boxtimes \mathcal{F}_d$$

where $\mathcal{F}_1, \ldots, \mathcal{F}_d$ are foliations in general position on M.

2. d-web W_d : defined by gluing local CD d-webs.

Web geometry

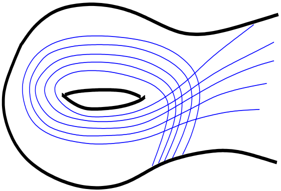


Fig.: A non decomposable 2-web

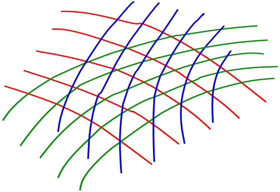


Fig.: A planar 3-web

Web geometry : a classification problem

Definition : W_d and W'_d are *equivalent* if it exists φ invertible s.t.

$$\varphi^*(W'_d) = W_d$$

Web geometry :

Classification of webs up to equivalence.

Classically : equivalence = local analytic equivalence

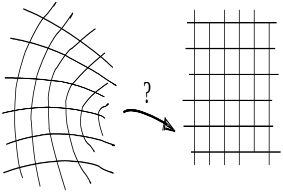


FIG.: The local geometry of a 2-web is trivial in dimension 2

Claim : The local geometry of a planar 3-web is rich

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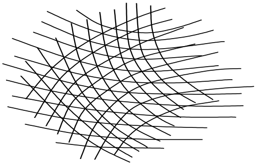

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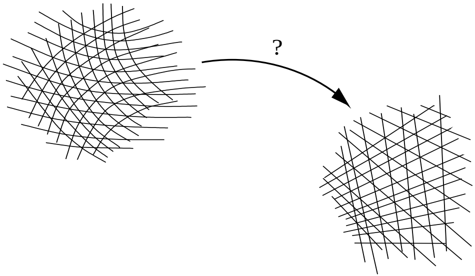

FIG.: Problem of linearization

Web geometry : a classification problem

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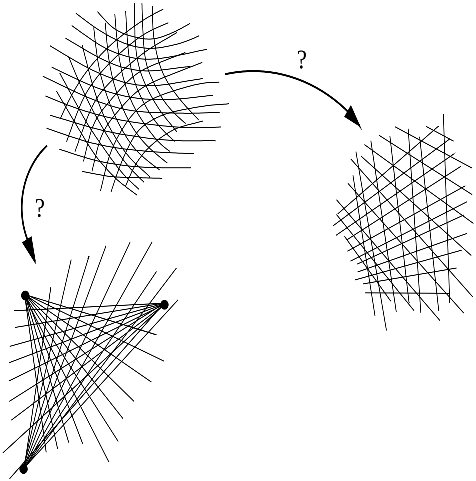
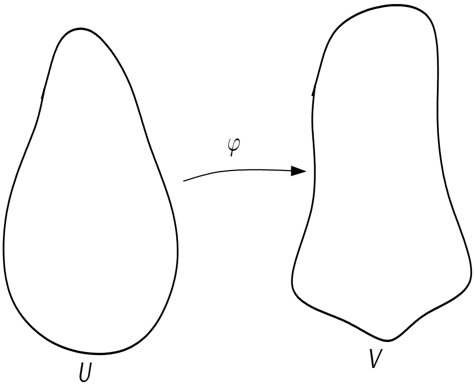


Webs : examples

Example 1 : planar 3-web associated to $\varphi \in Diff(\mathbb{C}^2, 0)$



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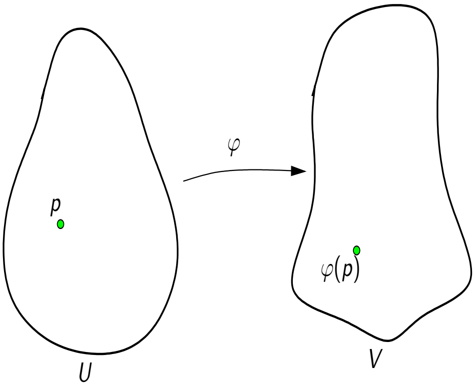

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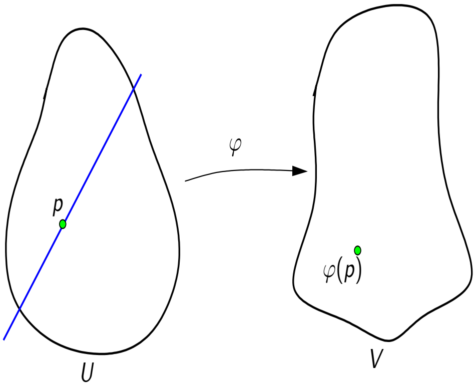

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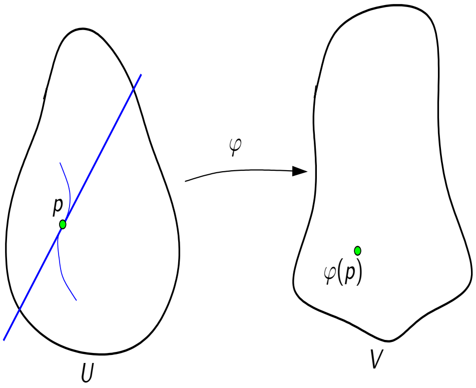

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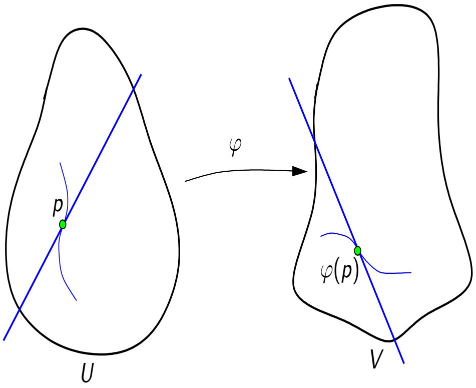

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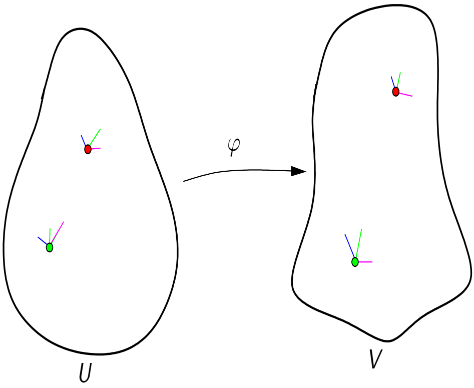

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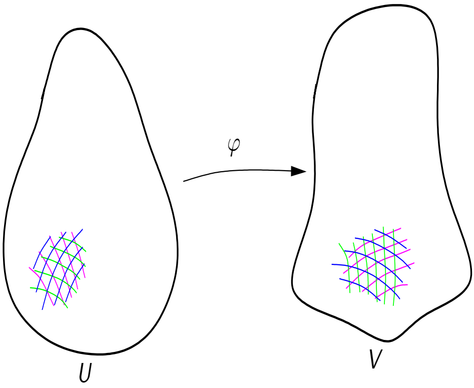
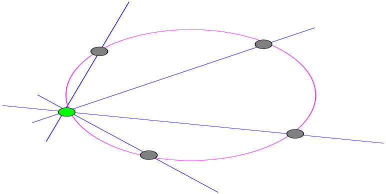


Example 2 : Bol's web $\mathcal{B}$

1. $$\mathcal{B} = \ M_{0,5} \overset{\longrightarrow}{\longrightarrow} M_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

2. 

3. $$\mathcal{B} = W\left(x, y, \frac{x}{y}, \frac{1-x}{1-y}, \frac{x(1-y)}{y(1-x)}\right)$$

Webs : examples

Example 3 : Algebraic web

Reduced algebraic curve $C \subset \mathbb{P}^2, \ \deg(C) = d$

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Linear *algebraic d-web* W_C on $\check{\mathbb{P}}^2$

Webs : examples

Example 3 : Algebraic web

$$\text{Reduced algebraic curve } C \subset \mathbb{P}^2,\ \deg(C) = d$$

$$\Big\downarrow \ \textit{projective duality}\ \check{}$$

$$\text{Linear } \textit{algebraic d-web } W_C \text{ on } \check{\mathbb{P}}^2$$

Remark :

$$\begin{array}{c} V^r \subset \mathbb{P}^{n+r} \text{ algebraic} \\ \deg(V^r) = d \end{array} \ \overset{\check{}}{\longrightarrow} \ \text{algebraic } d\text{-web of codim } r \text{ on } G_n(\mathbb{P}^{n+r})$$

Algebraic webs

$C \subset \mathbb{P}^2$: reduced algebraic curve

$$\deg(C) = d$$

Algebraic webs

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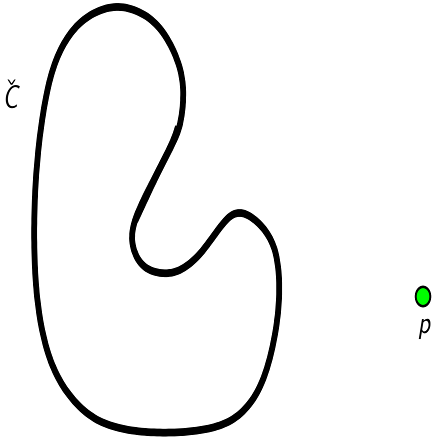

Algebraic webs

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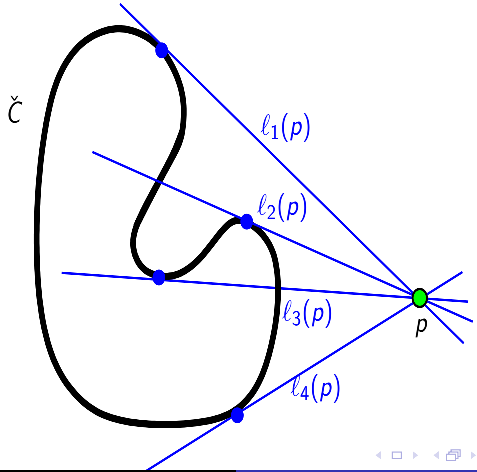

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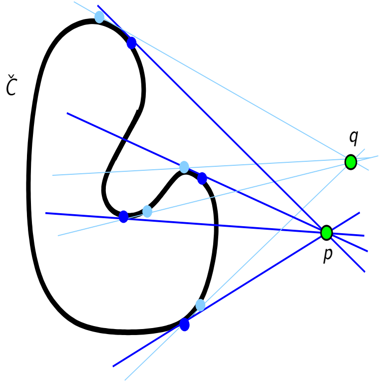


Definition : the *algebraic web* W_C is the linear d-web on $\check{\mathbb{P}}^2$ formed by the tangents lines to $\check{C}$.

A real picture of an algebraic web

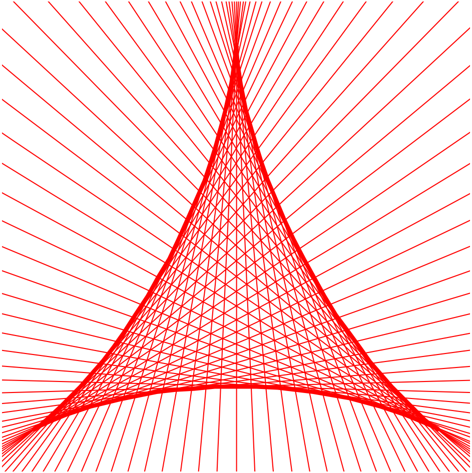


FIG.: Algebraic 3-web formed by the tangents of a hypocycloid

Another picture of an algebraic web

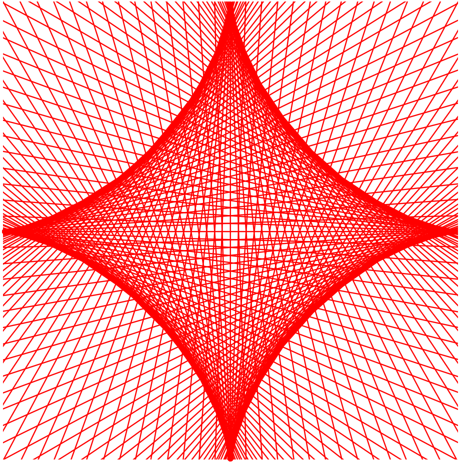
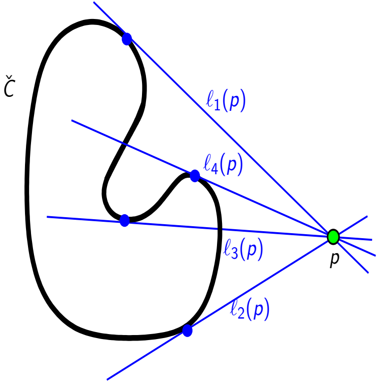


Fig.: Algebraic 4-web formed by the tangents of an astroid

Algebraic webs



Remark : the maps

$$\check{\mathbb{P}}^2 \ni p \longmapsto \ell_i(p) \in C$$

are (local) first integral for the algebraic web W_C.

Abelian relations and rank

$C \subset \mathbb{P}^2$ algebraic curve of degree d : $W_C = W(\ell_1, \ldots, \ell_d)$.

Abel's Theorem :

$$\sum_{i=1}^{d} F_i\big(\ell_i(p)\big) = 0 \quad \Longleftrightarrow \quad \exists\, \omega \in H^0(C, \omega_C^1) \ \text{ s.t. } \ F_i(\bullet) = \int^{\bullet} \omega.$$

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Generalization to webs :

Let $U_i : (\mathbb{C}^2, 0) \to \mathbb{C}$ are first integrals of $W_d = W(U_1, \ldots, U_d)$.

Definitions :

1. Abelian relation (AR)

$$F_1(U_1) + \cdots + F_d(U_d) = 0$$

2. $\mathbb{C}$-vector space $\mathcal{A}(W_d) = \Big\langle AR's\ of\ W_d \Big\rangle$

3. Rank of W_d : $rk(W_d) = \dim \mathcal{A}(W_d)$

Properties of abelian relations and of the rank

Proposition : If $W_d = W_C$ with $deg(C) = d$, then

1. $\mathcal{A}(W_C) \simeq H^0(C, \omega_C^1)$
2. $rk(W_C) = h^0(\omega_C^1) = p_a(C) = \frac{(d-1)(d-2)}{2}$

Bol's Theorem : For any d-web W_d

$$rk(W_d) \leq \frac{(d-1)(d-2)}{2} =: \pi_d \quad \text{(Bol's bound)}$$

Definition : W_d is of *maximal rank* if $rk(W_d) = \pi_d$

Example : Algebraic webs are examples of webs of maximal rank

Webs of maximal rank : algebraization

Theorem (Lie, Poincaré) :

$$rk(W_4) = \pi_4 = 3 \quad \Longrightarrow \quad W_4 \text{ is algebraizable}$$

Theorem (Blaschke-Howe, Griffiths) : For any $n \geq 2$:

W_d : $1-$codimensional
linear d-web on $U \subset \mathbb{C}^n$ $\Longrightarrow$ W_d is algebraizable
with one complete AR

Webs of maximal rank : algebraization

Theorem (Bol, (Chern-Griffiths), Trépreau) : For $n > 2$:

$$\left\| \begin{array}{l} W_d :\ 1-\text{codimensional} \\ d\text{-web on } U \subset \mathbb{C}^n \\ \text{of maximal rank } \pi(d,n) \end{array} \right. \implies W_d \text{ is algebraizable}$$

Work in progress (Pirio-Trépreau) : Algebraization of maximal rank webs of codimension $r > 1$

Philosophy : Sufficiently many AR $\implies$ algebraization

Webs of maximal rank : exceptional webs

Fact : for a planar 5-web W_5

$$rk(W_5) = \pi_5 = 6 \quad \not\Rightarrow \quad W_5 \text{ is algebraizable}$$

Bol's counterexample :

$$\mathcal{B} = W\left(x, y, \frac{x}{y}, \frac{1-x}{1-y}, \frac{x(1-y)}{y(1-x)}\right) \quad \text{is}$$

1. not linearizable
2. of maximal rank $\pi_5 = 6$

Definition :

$$\text{Exceptional web} = \begin{cases} \text{non-algebraizable web} \\ \text{of maximal rank} \end{cases}$$

Rogers dilogarithm

$$\mathbf{D}(x) := \mathbf{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x) - \frac{\pi^2}{6}$$

satisfies Abel's 5-terms relation **(Ab)** :

$$\mathbf{D}(x) - \mathbf{D}(y) - \mathbf{D}\Big(\frac{x}{y}\Big) - \mathbf{D}\left(\frac{1-y}{1-x}\right) + \mathbf{D}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$$

Abelian relations of Bol's web

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Thus **(Ab)** is an abelian relation for

$$\mathcal{B} = W\left(x, y, \frac{x}{y}, \frac{1-y}{1-x}, \frac{x(1-y)}{y(1-x)}\right)$$

$\Longrightarrow$ Bol's web is exceptional (Bol, 1936).

Abelian relations of Bol's web

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Chern's Problem : Classify all exceptional webs

Why is Chern's problem interesting

Reason 1 : Chern-Griffiths (1981)

> *(...) we cannot refrain from mentionning what we consider to be the fundamental problem on the subject, which is to determine the maximum rank non-linearizable webs. The strong conditions must imply that there are not many.*

Reason 2 : Abel's 5-terms relation **(Ab)** appears in

- hyperbolic geometry
- algebraic K-theory
- etc...

$\Longrightarrow$ Abelian relations of exceptional webs could be related to others domains in mathematics.

Exceptional webs and algebraic geometry

Credo :

Exceptional planar webs are analogs of plane algebraic curves.

Such webs exist :

- First new example : Pirio, Robert (2001)
- A continuous family : Pirio-Trépreau (2003)
- ...

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   2.3 *CDQL webs*

Webs with an infinitesimal symmetry

Joint work with D. Marìn and J.-V. Pereira

W_d with an infinitesimal symmetry $X \quad \Rightarrow \quad \mathcal{L}_X \in End\big(\mathcal{A}(W_d)\big)$

Theorem :

> $X =$ infinitesimal symmetry of W_d s.t. $\mathcal{F}_X \notin W_d$
>
> $$rk(W_d \boxtimes \mathcal{F}_X) = rk(W_d) + (d-1)$$

Corollary :

> For any $d \geq 5$, it exists a continuous family of exceptional d-webs on $\mathbb{P}^2$ of the form
>
> $$W_C \boxtimes \mathcal{F}_X$$

« Proof » : for any d, there are continuous families of algebraic curves $C \subset \mathbb{P}^2$ invariant by a linear action of $\mathbb{C}^*$ on $\mathbb{P}^2$

Characterization of webs of maximal rank

Blaschke-Dubourdieu :

$$W_3 \text{ on } U \subset \mathbb{C}^2 \quad \rightsquigarrow \quad \text{curvature } K(W_3) \in \Omega^2_U$$

Remark : for φ invertible

$$K\left(\varphi^* W_3\right) = \varphi^*\Big(K(W_3)\Big)$$

Theorem :

W_3 has maximal rank $\iff$ W_3 is « flat »

i.e. $rk(W_3) = \pi_3 = 1$ $\quad$ *i.e.* $K(W_3) \equiv 0$

Characterization of webs of maximal rank

Generalization to planar d-webs $\left(\begin{array}{l}\text{Pantazi-Mihăileanu}\\ \text{Hénaut-Ripoll}\end{array}\right)$

$$W_d \text{ on } U \subset \mathbb{C}^2 \quad \rightsquigarrow \quad \begin{array}{l} (E, \nabla_{W_d}) \text{ fiber bundle on } U \\ \text{of rank } \pi_d = \frac{1}{2}(d-1)(d-2) \\ \text{such that } \mathcal{A}(W_d) \hookrightarrow Ker\big(\nabla_{W_d}\big) \end{array}$$

Theorem (Pantazi-Hénaut) :

$$\begin{array}{l} W_d \text{ has maximal rank} \\ \textit{i.e. } rk(W_d) = \pi_d \end{array} \iff \begin{array}{l} (E, \nabla_{W_d}) \text{ is flat} \\ \textit{i.e. } K(\nabla_{W_d}) \equiv 0 \end{array}$$

Characterization of webs of maximal rank

Definition : the curvature of W_d is

$$K(W_d) := \sum_{W_3 < W_d} K(W_3) \in \Omega^2_U$$

Corollary (Mihăileanu) :

W_d has maximal rank *i.e.* $rk(W_d) = \pi_d$	$\Longrightarrow$	W_d is « flat » *i.e.* $K(W_d) \equiv 0$

Remark : if W_d is a global d-web on a surface S with singular set $\Delta = \Delta(W_d) \subset S$, one can also define its curvature

$$K(W_d) \in \Omega^2_S(\star\Delta)$$

CDQL webs

Joint work with J.-V. Pereira

Remark : Bol's web $\mathcal{B} \quad = \quad$ 4 pencils of lines + a pencil of conics

Definition : *CDQL web* = Completely Decomposable QuasiLinear web

- CDQL web on $\mathbb{P}^2 \quad = \quad$ pencils of lines + a non-linear foliation
- CDQL web on $T \quad = \quad$ pencils of lines + a non-linear foliation

CDQL version of Chern's Pbm : Classify exceptional CDQL webs

Exceptional CDQL webs on $\mathbb{P}^2$: classification

Theorem (Pereira-Pirio) :

> Up to projective equivalence, there are
> 4 infinite families and
> 13 sporadic examples
> of exceptional CDQL webs on $\mathbb{P}^2$

The four infinite families are (with $k \geq 1$)

- $A_I^k = [(dx^k - dy^k)\, d(xy)]$
- $A_{II}^k = [dx\, dy\, (dx^k - dy^k)\, d(xy)]$
- $A_{III}^k = [(xdy - ydx)(dx^k - dy^k)\, d(xy)]$
- $A_{IV}^k = [dx\, dy\, (xdy - ydx)\, (dx^k - dy^k)\, d(xy)]$

Exceptional CDQL webs on $\mathbb{P}^2$: classification

Among the sporadic examples, 7 are invariant by homotheties. These are (with $\xi = \exp(2i\pi/3)$)

- $A_5^a = [\,dx\,dy\,d(x/y)\,d(x+y)] \boxtimes [d(xy(x+y))]$
- $A_5^b = [\,dx\,dy\,d(x/y)\,d(x+y)\,] \boxtimes [d(xy/(x+y))]$
- $A_5^c = [\,dx\,dy\,d(x/y)\,d(x+y)\,] \boxtimes [d(\frac{x^2+xy+y^2}{xy(x+y)})]$
- $A_5^d = [dx\,dy\,d(x+y)\,d(x+\xi\,y)] \boxtimes [d(xy(x+y)(x+\xi\,y))]$
- $A_6^a = [dx\,dy\,d(x+y)\,d(x+\xi\,y)\,d(\frac{x}{y})] \boxtimes [d(xy(x+y)(x+\xi\,y))]$
- $A_6^b = [dx\,dy\,(dx^3+dy^3)] \boxtimes [d(x^3+y^3)]$
- $A_7^a = [dx\,dy\,(dx^3+dy^3)\,d(x/y)] \boxtimes [d(x^3+y^3)]$.

Exceptional CDQL webs on $\mathbb{P}^2$: classification

Among the last sporadic examples, there are Bol's web and his brothers

- $\mathcal{B} = [dx\, dy\, d(\frac{x}{1-y})\, d(\frac{y}{1-x})\,] \boxtimes \left[d(\frac{xy}{(1-x)(1-y)})\right]$
- $\mathcal{B}_6 = \mathcal{B} \boxtimes [d(x+y)]$
- $\mathcal{B}_7 = \mathcal{B}_6 \boxtimes [d(x/y)]$
- $\mathcal{B}_8 = \mathcal{B}_7 \boxtimes [d(\frac{1-x}{1-y})]$

Exceptional CDQL webs on $\mathbb{P}^2$: classification

The last two sporadic examples are

- $H_5 = \big[(dx^3 + dy^3)\, d(x/y)\big] \boxtimes \mathcal{H}_{esse}$
- $H_{10} = \big[(dx^3 + dy^3)\, \prod_{i=0}^{2} d(\frac{y-\xi^i}{x})\, \prod_{i=0}^{2} d(\frac{x-\xi^i}{y})\big] \boxtimes \mathcal{H}_{esse}$

where $\mathcal{H}_{esse}$ designates Hesse's pencil of cubics

$$\mathcal{H}_{esse} = \left[d\left(\frac{xy}{1 + x^3 + y^3}\right)\right]$$

Corollary (Pereira-Pirio) :

> Up to isogenies, there are
>
> — 1 continuous family (for $d = 5$)
>
> — 3 sporadic examples (one for each $d \in \{5, 6, 7\}$)
>
> of exceptional CDQL d-webs on 2-dimensional complex tori

Classification on $\mathbb{P}^2$: « proof »

0.
- $W_{d+1} = L_{p_1} \boxtimes \cdots L_{p_d} \boxtimes \mathcal{F}$ with $\begin{cases} p_1, \ldots, p_d \in \mathbb{P}^2 \\ \mathcal{F} \in Fol(\mathbb{P}^2) \end{cases}$
- singular set of $W_{d+1} =: \Delta \subset \mathbb{P}^2$

1. $\left[\begin{array}{c} W_{d+1} \text{ has} \\ \text{max. rank} \end{array} \right] \overset{Mihaileanu}{\Longrightarrow} K(W_{d+1}) \equiv 0 \Leftrightarrow \left[\begin{array}{c} K(W_{d+1}) \\ \text{holom. on } \mathbb{P}^2 \end{array} \right]$

$$\Longrightarrow \left[\begin{array}{c} K(W_{d+1}) \text{ is holom. over} \\ \text{the generic point of } C \subset \Delta \end{array} \right] =(1)$$

2. $(1) \Longrightarrow \left[\begin{array}{c} \text{constraints on} \\ \mathcal{P} = \{p_1, \ldots, p_d\} \\ \mathcal{F} \in Fol(\mathbb{P}^2) \end{array} \right] \Longrightarrow \begin{cases} deg(\mathcal{F}) \leq 4 \\ \mathcal{P} \subset \ell_{\mathcal{F}} \cup Sing(\mathcal{F}) \\ \quad \ldots\ldots\ldots \end{cases}$

3. Case by case classification depending on $deg(\mathcal{F}) \in \{1, 2, 3, 4\}$ $\square$

Plan of the talk

Canonical map : algebraic curves

C : smooth algebraic curve of genus g

$$\left\langle \omega^\lambda \,\middle|\, \lambda = 1, \ldots g \right\rangle = H^0(C, \Omega^1_C)$$

Let z be a local coordinate on C. Thus

$$\omega^\lambda(z) = f^\lambda(z)\, dz$$

Definition : the canonical map of C is

$$\begin{aligned} \varphi_{|\Omega^1_C|} : C &\longrightarrow \mathbb{P}^{g-1} \\ z &\longmapsto \left[f^1(z) : \cdots : f^g(z) \right] \end{aligned}$$

Canonical map : webs

$W_d = W(U_1, \ldots, U_d)$ d-web with FI $U_i : (\mathbb{C}^2, 0) \to C_i = Im(U_i)$.

Assume $\pi = rk(W_d) > 0$

$$\Big\langle \textstyle\sum_{i=1}^d F_i^\lambda(U_i) = 0 \mid \lambda = 1, \ldots, \pi \Big\rangle = \mathcal{A}(W_d)$$
$$\Updownarrow$$
$$\Big\langle \textstyle\sum_{i=1}^d f_i^\lambda(U_i) dU_i = 0 \mid \lambda = 1, \ldots, \pi \Big\rangle = \mathcal{A}(W_d)$$

Definition : 1. the ith canonical map of W_d is

$$\begin{aligned} \varphi_i : C_i &\longrightarrow \mathbb{P}^{\pi-1} \\ u_i &\longmapsto \big[f_i^1(u_i) : \cdots : f_i^\pi(u_i)\big] \end{aligned}$$

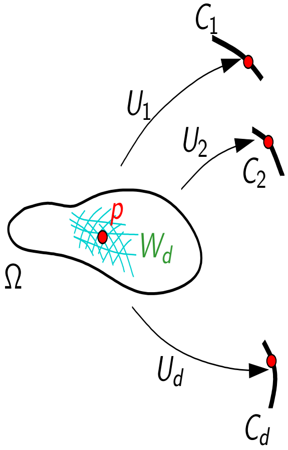
2. the canonical map of W_d is $\varphi_{W_d} = (\varphi_i) : \sqcup_i C_i \to \mathbb{P}^{\pi-1}$

Canonical curve of webs of maximal rank

Algebraic curves	Exceptional webs
$C' = Im\ \varphi_{\lvert\Omega^1_C\rvert} \subset \mathbb{P}^{\pi_d - 1}$	$C'_{W_d} = \sqcup_i C'_i \subset \mathbb{P}^{\pi_d - 1}$ $C'_i = Im\ \varphi_i$
C' non-degenerate	C'_{W_d} non-degenerate
C' algebraic	C'_{W_d} algebraic or transcendent
$deg(C) = 2g - 2$	?

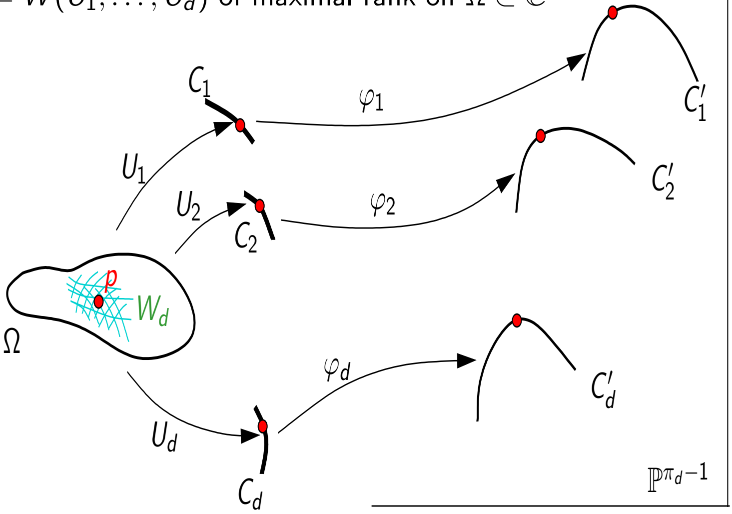
Canonical curves of webs of maximal rank

$W_d = W(U_1, \ldots, U_d)$ of maximal rank on $\Omega \subset \mathbb{C}^2$

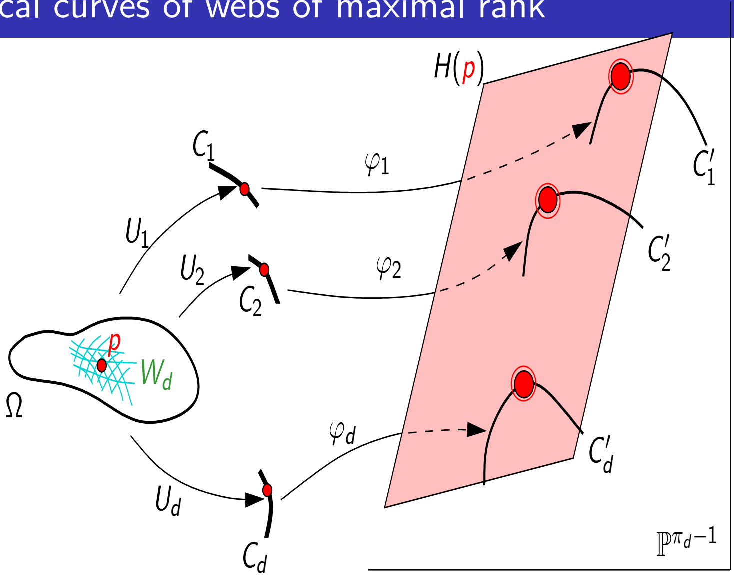


Canonical curves of webs of maximal rank

$W_d = W(U_1, \ldots, U_d)$ of maximal rank on $\Omega \subset \mathbb{C}^2$



Canonical curves of webs of maximal rank



Canonical curves of webs of maximal rank

Proposition : Let $p \in \Omega$: the osculating spaces

$$Osc^{d-3}_{C'_i}\Big(\varphi_1\big(U_1(p)\big)\Big) \qquad i = 1, \ldots, d$$

span a hyperplane $H(p) \in \check{\mathbb{P}}^{\pi_d - 1}$

Proposition :

1. : The map $H_{W_d} : \begin{array}{ccc} \Omega & \rightarrow & \check{\mathbb{P}}^{\pi_d-1} \\ p & \mapsto & H(p) \end{array}$ parametrizes a surface

$$S_{W_d} \subset \check{\mathbb{P}}^{\pi_d - 1}$$

with regular osculation of order d

2. : W_d is algebraizable $\iff$ $S_{W_d} = \mathcal{V}_d$ (Veronese surface)

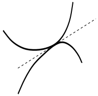
3. : $H_{W_d *}\Big(W_d\Big)$ is canonically defined on S_{W_d}

Projective surface $S \subset \check{\mathbb{P}}^{\pi_d - 1}$ with regular osculation of order d

Proposition : (Segre $d = 5$)

Let $s \in S$. Counted with multiplicities, there are exactly 5 **principal hyperplanes** $K_1, \ldots, K_5 \in \mathbb{P}^5$ such that

$$K_i \cap S \quad = \quad \text{tacnode in } s$$



$\Longrightarrow$ 5 **principal directions** tangent to S at s

$\Longrightarrow$ Segre's 5-web on S (invariantly attached to S)

Proposition (Segre d=5) :
S totally umbilic $\Longleftrightarrow$ $S \subset$ Veronese surface $\mathcal{V}_2 \subset \mathbb{P}^5$

Exceptional 5-web $W_5 \Rightarrow \begin{cases} H_{W_5} : \Omega \to \check{\mathbb{P}}^5 \\ S_{W_5} = Im \ (H_{W_5}) \end{cases}$

Proposition :

$$\text{Segre's 5-web of } S_{W_5} \quad = \quad H_{W_5 *}(W_5)$$

$\Longrightarrow S_{W_5}$ determines W_5

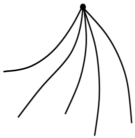
Definition : Notion of exceptional surface $S \subset \mathbb{P}^{\pi_d - 1}$

Examples of exceptional surfaces

Proposition : the following 5-web is exceptional

$$W\Big(x, y, x+y, x-y, x^2+y^2\Big)$$

Canonical curve :



Associated exceptional surface :

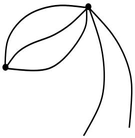
$$H = \Big[1 \,:\, x^2+y^2 \,:\, x^2y^2 \,:\, (x^2+y^2)^2 \,:\, x^3(5y^2-x^2) \,:\, y^3(5x^2-y^2)\Big]$$

Examples of exceptional surfaces

Proposition : the following 5-web is exceptional

$$W\left(x, y, x+y, x-y, x^2-y^2\right)$$

Canonical curve :



Associated exceptional surface :

$$H = \left[1 \,:\, x^2 \,:\, y^2 \,:\, x^3 \,:\, y^3 \,:\, \left(x^2-y^2\right)^2\right]$$