Tempered holomorphic solutions of $\mathcal{D}$-modules and tempered ultradistributions

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Outline
Riemann-Hilbert correspondences
Tempered holomorphic solutions in dimension 1
Tempered ultradistributions

Let X be a complex analytic manifold, Z a closed subvariety.

$$\left\{ \begin{array}{c} \text{flat connections locally} \\ \text{free on } X\backslash Z \end{array} \right\} \xrightarrow{holom.solutions} \mathrm{Rep}_f(\pi_1(X \setminus Z)) \ .$$

Hilbert 21^{st} problem asks if the previous morphism is surjective.

The answer is affirmative.

Uniqueness is assured as soon as some regularity conditions on the connections are imposed.

The morphism described above can be generalized in a functor

$$\mathcal{S}ol := R\mathcal{H}om_{\mathcal{D}_X}(\cdot, \mathcal{O}_X) : D^b_{rh}(\mathcal{D}_X)^{op} \longrightarrow D^b_{\mathbb{C}-c}(\mathbb{C}_X)$$

Between '79 and '84, M. Kashiwara proved that $\mathcal{S}ol$ is an equivalence of categories by giving an explicit inverse functor. Such an inverse was based on the functor $T\mathcal{H}om$ which is built using tempered distributions.

Tempered distributions are not a sheaf on X, but they are a sheaf on the subanalytic site X_{sa} relative to X. The open sets of X_{sa} are the subanalytic ones and the coverings are the locally finite ones. Hence tempered distributions were used by M. Kashiwara to define the functor $T\mathcal{H}om$ on the category of $\mathbb{R}$-constructible sheaves on X.

In the '90s, M. Kashiwara and P. Schapira introduced the functor $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O} : \mathbb{R}-c(\mathbb{C}_X) \to \mathrm{Mod}(\mathbb{C}_X)$. They studied the properties of $T\mathcal{H}om$ and $\overset{\mathrm{w}}{\otimes}\mathcal{O}$, proving that they are dual to each other.

The use and the study of functors on the category $\mathbb{R}-c(\mathbb{C}_X)$ led them to consider directly sheaves on the subanalytic site relative to a real analytic manifold M.

In 2001, they defined $\mathcal{O}^t$ and $\mathcal{O}^{\mathrm{w}}$ as complexes of sheaves on X_{sa}.

In 2003, they gave an example which suggests that the use of solutions with values in $\mathcal{O}^t$ can be very usefull in the study of $\mathcal{D}$-modules.

In a recent joint work with N. Honda (Hokkaido University), we realized the space of tempered ultradistributions of Beurling type as a sheaf on the subanalytic site.

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Regular case
Irregular case in dimension 1

An irregular Riemann-Hilbert correspondence already exists in dimension 1 since the '70s due to the ideas of P. Deligne based on the work of M. Hukuhara, A.H.M. Levelt, B. Malgrange, J.-P. Ramis, Y. Sibuya, H. Turrittin ...

At the base there are two main results: the formal Levelt-Turrittin Theorem and the the asymptotic Hukuhara-Turrittin Theorem.

Theorem (Levelt-Turrittin)

Let $X = \mathbb{C}$, $\mathcal{M}$ a localized holonomic $\mathcal{D}_X$-module. There exist $\varphi_1, \dots, \varphi_p \in z^{-1}\mathbb{C}[z^{-1}]$ and regular $\mathcal{D}_X$-modules $\mathcal{R}_1, \dots, \mathcal{R}_p$ such that the germ at 0 of $\mathcal{M}$ is formally isomorphic, up to ramification, to the germ at 0 of

$$\bigoplus_{j=1}^{p} \mathcal{D}_X \cdot \exp(\varphi_j) \otimes \mathcal{R}_j \ .$$

Theorem (Hukuhara-Turrittin)

Let P be a differential operator with holomorphic coefficients in a neighborhood of $0 \in \mathbb{C}$. Then, on any sufficiently small open sector S with vertex at 0, the equation $Pu = 0$ has a basis of holomorphic solutions of the form

$$h(z)\exp(\varphi(z)) \ ,$$

where $\varphi \in z^{-1/l}\mathbb{C}[z^{-1/l}]$ and $C|z|^M \leq |h(z)| \leq (C|z|^M)^{-1}$, for $z \in S$ and some $C, M > 0$.

The exponents φ_j are called the *determinant factors* of P. Analytic continuation induces an isomorphism among the spaces of holomorphic solutions on two different sectors. Such isomorphisms are called the *Stokes coefficients* of P.

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The holomorphic solutions endowed with suitable structure (which takes into account the exponentials and the Stokes coefficients) induce the following equivalence of categories (P. Deligne).

$$\left\{ \begin{array}{c} \text{Germs at 0 of holonomic} \\ \text{localized } \mathcal{D}_X - \text{modules} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \Omega - \text{filtered local systems} \\ \text{of finite rank on } S^1 \times \mathbb{R}_{\geq 0} \end{array} \right\}$$

$$\mathcal{M} \longmapsto \{ \underset{\in \mathbb{Z}_{>0}}{l}, \quad \underset{\in \frac{1}{z}\mathbb{C}[\frac{1}{z}]}{\varphi_j}, \quad \underset{\text{regular}}{\mathcal{R}_j}, \quad St_1, \ldots, St_d \ \}$$

Idea: the main ideas are to add points (S^1) and to put an additional structure on the sheaf of holomorphic solutions keeping track of exponentials and Stokes coefficients.

Limit: it is not a global costruction and the presence of exponentials is known a priori as analytic data.

Tempered holomorphic solutions in dimension 1

The complex of subanalytic sheaves of tempered holomorphic functions $\mathcal{O}^t_{X_{sa}}$ on a complex manifold X is defined as the solutions of the Cauchy-Riemann system in the subanalytic sheaf of tempered distributions.

In dimension 1, $\mathcal{O}^t_{X_{sa}}$ is concentrated in degree 0, i.e. it is a sheaf on the subanalytic site X_{sa}.

If U is a relatively compact subanalytic open subset of $\mathbb{C}$, then

$$\mathcal{O}^t_{X_{sa}}(U) = \left\{ f \in \mathcal{O}_X(U);\ \text{there exist } C, N > 0 \text{ such that } |f(z)| \leq \frac{C}{\mathrm{dist}(z, \partial U)^N} \right\}.$$

Topological description of determinant factors

We use the functor of tempered holomorphic solutions of $\mathcal{D}_X$-modules, defined as

$$\mathcal{S}^t := \mathcal{H}om_{\varrho_!\mathcal{D}_X}(\varrho_! \cdot, \mathcal{O}^t_{X_{sa}}) : \mathrm{Mod}_h(\mathcal{D}_X)^{op} \longrightarrow \mathrm{Mod}(\mathbb{C}_{X_{sa}}) \ .$$

We start by applying such a functor to the objects of the form

$$\bigoplus_{j=1}^{p} \mathcal{D}_X \cdot \exp(\varphi_j) \otimes \mathcal{R}_j \ , \qquad (1)$$

where $\varphi_j \in z^{-1}\mathbb{C}[z^{-1}]$ and $\mathcal{R}_j$ is regular.

For $k \in \mathbb{Z}_{>0}$, we define the category of good models with Katz invariant smaller than k, denoted by GM_k, as the full subcategory of $\mathrm{Mod}(\mathcal{D}_X)$ whose objects are isomorphic to a module of the form (1) where the degree of φ_j is strictly smaller than k.

Proposition (M. '08)

1. *Let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$, $U \subset \mathbb{C}$ a subanalytic relatively compact open set. Then $\exp(\varphi) \in \mathcal{O}^t(U)$ if and only if there exists $A > 0$ such that, for any $z \in U$, $\mathrm{Re}\varphi(z) < A$.*
2. *Let $\varphi_1, \varphi_2 \in z^{-1}\mathbb{C}[z^{-1}]$. The following conditions are equivalent.*
   - (i) *For any $U \subset \mathbb{C}$ subanalytic relatively compact open set, $\exp(\varphi_1) \in \mathcal{O}^t(U)$ if and only if $\exp(\varphi_2) \in \mathcal{O}^t(U)$.*
   - (ii) *There exists $\lambda > 0$ such that $\varphi_1 = \lambda\varphi_2$.*

In particular, the fact that $\exp(\varphi)$ is tempered depends, up to a positive multiplicative constant, on all the coefficients of φ.

Theorem (M. '08)

The functor

$$\mathcal{S}^t(\cdot \otimes \mathcal{D}e^{1/z^k}) : \mathsf{GM}_k \longrightarrow \mathrm{Mod}(\mathbb{C}_{X_{sa}})$$

is fully faithful.

Let us denote by $\mathrm{Mod}(\mathcal{D}_{X,*0})_k$ the category of germs of localized holonomic $\mathcal{D}_X$-modules with Katz invariant strictly smaller than k.

Theorem (M. '08)

Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathrm{Mod}(\mathcal{D}_{X,*0})_k$. The following conditions are equivalent.

(i) $\mathcal{S}^t(\mathcal{M}_1 \otimes \mathcal{D}e^{1/z^k}) \simeq \mathcal{S}^t(\mathcal{M}_2 \otimes \mathcal{D}e^{1/z^k})$.

(ii) *$\mathcal{M}_1$ and $\mathcal{M}_2$ have the same determinant factors and isomorphic sheaves of holomorphic solutions.*

Let us recall the definition of the sheaf $\mathcal{A}^{\leq 0}$ defined on $S^1 \times \mathbb{R}_{\geq 0}$. For $\vartheta \in S^1$,

$$\mathcal{A}^{\leq 0}_{(\vartheta,0)} := \Big\{ f \in \mathcal{O}(S);\ S \text{ an open sector containing } \vartheta \\ \text{there exist } C, N > 0 \text{ such that } |f(z)| \leq \frac{C}{|z|^N} \Big\} .$$

Example: let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$,

$$\varphi(z) = \frac{\varrho e^{i\tau}}{z^n} + \sum_{j=1}^{n-1} \frac{a_j}{z^j} \qquad (\varrho > 0, \tau \in \mathbb{R}) .$$

Then

$$\exp(\varphi) \in \mathcal{A}^{\leq 0}_{(\vartheta,0)} \iff \cos(\tau - n\vartheta) < 0 .$$

Definition

Let $\tau \in \mathbb{R}$. A relatively compact subanalytic open set $U \subset \mathbb{C}$ is said τ-concentrated if U is connected, $0 \in \partial U$ and given an open sector S containing the direction τ, the germ of U is contained in S.

Proposition

Let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ of degree n. There exist $\tau \in \mathbb{R}$ and $U_0, \dots, U_{2n-1}$ relatively compact subanalytic open sets such that

1. *U_j is $\left(\tau + j\frac{\pi}{n}\right)$-concentrated ($j = 0, \dots, 2n-1$),*
2. *for any $j = 0, \dots, 2n-1$, $\exp(\varphi), \exp(-\varphi) \in \mathcal{O}^t(U_j)$.*

Remark

1. The τ-concentrated sets are not open in the topology of $S^1 \times \mathbb{R}_{\geq 0}$.
2. $\nexists \vartheta \in S^1$ such that $\exp(\varphi), \exp(-\varphi) \in \mathcal{A}^{\leq 0}_{(\vartheta,0)}$.

$\mathbb{R}$-constructibility of tempered solutions

Let X be a complex analytic manifold.

Let $D^b_h(\mathcal{D}_X)$ be the bounded derived category of complexes of $\mathcal{D}_X$-modules with holonomic cohomology.

Let $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ be the bounded derived category of complexes of sheaves with $\mathbb{R}$-constructible cohomology. Let $D^b(\mathbb{C}_{X_{sa}})$ be the bounded derived category of subanalytic sheaves.

Definition

An element $F \in D^b(\mathbb{C}_{X_{sa}})$ is said $\mathbb{R}$-constructible on X_{sa} if, for any $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$,

$$\varrho^{-1}\mathrm{R}\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(\varrho_* G, F) \;\in\; D^b_{\mathbb{R}-c}(\mathbb{C}_X)\ .$$

Conjecture (Kashiwara-Schapira '03)

If $\mathcal{M} \in D^b_h(\mathcal{D}_X)$, then $\mathcal{S}ol^t(\mathcal{M}) := \mathrm{R}\mathcal{H}om_{\varrho_! \mathcal{D}_X}\left(\varrho_! \mathcal{M}, \mathcal{O}^t_{X_{sa}}\right)$ is $\mathbb{R}$-constructible on X_{sa}.

Proof of the conjecture in dimension 1

Using subanalytic geometry and classical existence theorems for functional spaces with growth conditions on sufficiently small open sectors (Ramis-Sibuya, Honda ...), we proved the following results.

Proposition

Let $X \subset \mathbb{C}$ be a open neighbourhood of 0, P a linear ordinary differential operator on X. Let U be a relatively compact subanalytic open subset of X.

There exists an open neighbourhood W of 0 and an open covering $\{U_j\}_{j\in J}$ of $U \cap W$ such that, for any $j \in J$,

$$\mathcal{O}^t(U_j) \xrightarrow{P(\cdot)} \mathcal{O}^t(U_j) \ ,$$

is an epimorphism.

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Theorem (M. '07)

Let X be a complex curve, $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module. Then

$$H^1\Big(\mathcal{S}ol^t(\mathcal{M})\Big) \longrightarrow H^1\Big(\mathcal{S}ol(\mathcal{M})\Big)$$

is an isomorphism.

Theorem (M. '07)

Let X be a complex curve, $\mathcal{M} \in D^b_h(\mathcal{D}_X)$. Then $\mathrm{R}\mathcal{H}om_{\varrho_!\mathcal{D}_X}(\varrho_!\mathcal{M}, \mathcal{O}^t_{X_{sa}})$ is $\mathbb{R}$-constructible on X_{sa}.

Tempered ultradistributions

Joint work with N. Honda, Hokkaido University

Let $s > 1$. For U an open set, denote by $\mathscr{D}^{(s)}(U)$ the space of all $\mathscr{C}^\infty$ functions with compact support f in U such that for any $h > 0$ there exists $C_h > 0$ such that for any $\alpha \in \mathbb{N}^n$

$$\sup \left| D^\alpha f(x) \right| \leq C_h h^{|\alpha|} (|\alpha|!)^s \ .$$

The space $\mathscr{D}^{(s)}(U)$ can be endowed with a family of seminorms turning it into a Fréchet space. The dual space of $\mathscr{D}^{(s)}(U)$ is called the *space of ultradistributions of class* (s) *of Beurling type*, it is denoted $\mathcal{D}b^{(s)}(U)$.

Definition

The presheaf $\mathcal{D}b^{(s)t}$ *on* X_{sa} *defined by*

$$U \longmapsto \mathcal{D}b^{(s)t}(U) := \frac{\Gamma(X; \mathcal{D}b^{(s)})}{\Gamma_{X \setminus U}(X; \mathcal{D}b^{(s)})} \ ,$$

is called the presheaf of tempered ultradistributions of class (s).

It can be proved that it is not a sheaf on the subanalytic site.

Let $Z \subset \mathbb{R}^n$ be a closed subanalytic set. We denote by $\Gamma(X, \Gamma_{Z,d}\mathcal{D}b^{(s)})$ the space of ultradistributions u with support in Z such that for any subanalytic stratification $\{Z_\alpha\}$ adapted to Z, there exist $u_\alpha \in \mathcal{D}b^{(s)}(\mathbb{R}^n)$ such that $\mathrm{supp}(u)_\alpha \subset Z_\alpha$ and $u = \sum u_\alpha$.

Definition

Given a subanalytic open set $U \subset X$, we set

$$\mathcal{D}b^{(s)t-d}(U) := \frac{\Gamma(X, \mathcal{D}b^{(s)})}{\Gamma(X, \Gamma_{X \setminus U, d}\mathcal{D}b^{(s)})} \ .$$

Theorem

If X be a real analytic manifold of dimension 2, then $\mathcal{D}b^{(s)t-d}$ is a subanalytic sheaf.

A subanalytic compact connected set $Z \subset \mathbb{R}^n$ is said to be 1-*regular* if it satisfies the following condition. For any $x, y \in Z$ there exist a smooth curve $\gamma \subset Z$ and $C \in \mathbb{R}_{>0}$ such that

$$\mathrm{length}\,\gamma \leq C|x-y| \ .$$

Theorem

Let U be a subanalytic open subset of X such that $X \setminus U$ is 1-regular. Then

$$\begin{aligned} \Gamma(X, \Gamma_{X\setminus U,d}\mathcal{D}b^{(s)}) &\simeq \Gamma(X, \Gamma_{X\setminus U}\mathcal{D}b^{(s)}) \\ \mathcal{D}b^{(s)t-d}(U) &\simeq \mathcal{D}b^{(s)t}(U) \ . \end{aligned}$$

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