On the algebraic integrability of a plane foliation

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The problem

Problem (Poincaré):

To decide whether a complex polynomial differential equation on the plane of first order and degree 1 is algebraically integrable.

Modern terminology

To decide whether an algebraic foliation $\mathcal{F}$ with singularities on the projective plane (over an algebraically closed field of characteristic zero) admits a rational first integral.

- Poincaré's observation: It is enough to find an upper bound for the degree of the integral. Afterwards, we "only need" to perform purely algebraic computations.
- Poincaré studied a particular case within the one where the singularities of the foliation are non-degenerated.

More general problem: is it possible to give a bound of the degree of an integral algebraic invariant curve of a given foliation $\mathcal{F}$?

Returning to the original problem (algebraic integrability)

Main objective

To provide an algorithm to decide, under certain condition, whether an algebraic plane foliation $\mathcal{F}$ has a rational first integral, and to compute it in the affirmative case.

(C. Galindo, F. Monserrat, Algebraic integrability of foliations of the plane, *J. Differential Equations* (2006), 611—632)

Related reference:

(J.V. Pereira, Vector fields, invariant varieties and linear systems, *Ann. Inst. Fourier* **51(5)** (2001), 1385—1405) provides a computational criteria for the existence of rational first integrals of a given degree using the so-called "extactic curves".

Singular algebraic foliations of $\mathbb{P}^2$

A (singular) foliation $\mathcal{F}$ on $\mathbb{P}^2 (:= \mathbb{P}^2_k)$ of degree r is given, up to a scalar factor, by means of a homogeneous 1-form

$$\Omega = AdX + BdY + CdZ,$$

where A, B and C are homogeneous polynomials of degree $r + 1$ without common factors satisfying the Euler's condition $XA + YB + ZC = 0$.

Seidenberg's resolution of singularities (1968)

There exists a (minimal) sequence of point blowing-ups

$$X_{n+1} \xrightarrow{\pi_n} X_n \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 := \mathbb{P}^2,$$

such that the strict transform $\mathcal{F}_{n+1}$ of $\mathcal{F}$ on X_{n+1} has only certain type of singularities which cannot be removed by blowing-up, the so-called simple singularities.

$\mathcal{K}_{\mathcal{F}} = \{p_i\}_{i=1}^{n}$: set of centers of the blowing-ups (configuration of infinitely near points).

Some notation related with “configurations”

Let $\mathcal{K} = \{p_i\}_{i=1}^n$ be a configuration.

$$X_{n+1} \xrightarrow{\pi_n} X_n \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 := \mathbb{P}^2,$$

where $X_{i+1} = Bl_{p_i}(X_i)$.

X_{n+1}: sky of $\mathcal{K}$.

We identify two configurations with $\mathbb{P}^2$-isomorphic skies.

E_{p_i}: the exceptional divisor appearing in the blowing-up π_i.

$\tilde{E}_{p_i}$: strict transform of E_{p_i} on X_{n+1}.

$E_{p_i}^*$: total transform of E_{p_i} on X_{n+1}.

p_i es proximate to p_j if p_i belongs to the strict transform on X_i of the divisor E_{p_j}.

Dicritical points and divisors

Let $\mathcal{F}$ be a plane foliation with associated configuration $\mathcal{K}_{\mathcal{F}} = \{p_i\}_{i=1}^n$.

- A divisor E_{p_i} is dicritical if it is not invariant by $\mathcal{F}_{i+1}$.
- A point p_i is dicritical if there exists a dicritical exceptional divisor E_{p_j} which maps to p_i.

$\mathcal{B}_{\mathcal{F}} := \{p_i \in \mathcal{K}_{\mathcal{F}} \mid p_i \text{ is dicritical } \}$ (It is a configuration)

$\mathcal{N}_{\mathcal{F}} := \{p_i \in \mathcal{B}_{\mathcal{F}} \mid E_{p_i} \text{ is not a dicritical exceptional divisor}\}$

Rational first integral and associated pencil

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Definition

A plane foliation $\mathcal{F}$ has a *rational first integral* if there exists a rational function R of $\mathbb{P}^2$ such that $dR \wedge \Omega = 0$.

The existence of rational first integral is equivalent to the following facts:

- $\mathcal{F}$ has infinitely many integral algebraic invariant curves.
- All the leaves of $\mathcal{F}$ are restrictions of algebraic invariant curves.
- There exists a unique *irreducible* pencil of plane curves $\mathcal{P}_{\mathcal{F}} := \langle F, G \rangle \subseteq \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$, $d \geq 1$, such that the integral algebraic invariant curves of $\mathcal{F}$ are exactly the integral components of the curves in $\mathcal{P}_{\mathcal{F}}$. F/G is a rational first integral on $\mathcal{F}$.

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Configuration of base points of the pencil $\mathcal{P}_{\mathcal{F}}$

Assume that $\mathcal{F}$ has a rational first integral

Associated irreducible pencil: $< F, G >$.

$\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^2}$: ideal sheaf supported at the base points of $\mathcal{P}_{\mathcal{F}}$ and such that $\mathcal{I}_p = (F_p, G_p)$ for each such a point p, where F_p and G_p are the natural images of F and G in $\mathcal{O}_{\mathbb{P}^2,p}$. There exists a sequence of blow-ups centered at closed points

$$X_{m+1} \xrightarrow{\pi_m} X_m \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 := \mathbb{P}^2 \qquad (1)$$

such that $\mathcal{I}\mathcal{O}_{X_{m+1}}$ becomes an invertible sheaf of X_{m+1}.

$\mathcal{C}_{\mathcal{F}}$: Set of centers of the blow-ups that appear in a minimal sequence with this property.

Proposition

If $\mathcal{F}$ is a foliation on $\mathbb{P}^2$ with a rational first integral, then

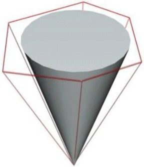
$$\mathcal{C}_{\mathcal{F}} = \mathcal{B}_{\mathcal{F}}.$$

Convex cones associated with the surface $Z_{\mathcal{F}}$

- $Z_{\mathcal{F}}$:= Sky of the configuration $\mathcal{B}_{\mathcal{F}}$ (with cardinality m).
- $A(Z_{\mathcal{F}}) := \mathrm{Pic}(Z_{\mathcal{F}}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{m+1}$.
- $NE(Z_{\mathcal{F}})$:= cone of curves (or Mori cone) of $Z_{\mathcal{F}}$= Convex cone of $A(Z_{\mathcal{F}})$ spanned by the images of the effective classes in $\mathrm{Pic}(Z_{\mathcal{F}})$.

Intersection form extends by linearity: $A(Z_{\mathcal{F}}) \times A(Z_{\mathcal{F}}) \to \mathbb{R}$.

- $Q(Z_{\mathcal{F}}) := \{x \in A(Z_{\mathcal{F}}) \mid x^2 \geq 0 \text{ and } x \cdot [H] \geq 0\} \subseteq \overline{NE}(Z_{\mathcal{F}})$, where H is an ample divisor.



The divisor $D_{\mathcal{F}}$ associated with $\mathcal{F}$

$B = \{[L^*]\} \cup \{[E_q^*]\}_{q \in \mathcal{B}_{\mathcal{F}}}$ is a $\mathbb{Z}$-basis (respect., $\mathbb{R}$-basis) of $\mathrm{Pic}(Z_{\mathcal{F}})$ (respect., $A(Z_{\mathcal{F}})$), L being a general line of $\mathbb{P}^2$.

Assuming that $\mathcal{F}$ has r.f.i.

$$D_{\mathcal{F}} := dL^* - \sum_{q \in \mathcal{B}_{\mathcal{F}}} r_q E_q^*,$$

d: degree of the curves in $\mathcal{P}_{\mathcal{F}}$ (degree of the r.f.i.)

r_q: multiplicity at q of the strict transform of a general curve of $\mathcal{P}_{\mathcal{F}}$ on the surface that contains q.

Proposition

$$\mathbb{P}H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(D_{\mathcal{F}})) = \mathcal{P}_{\mathcal{F}}.$$

$\pi_{\mathcal{F}} : Z_{\mathcal{F}} \to \mathbb{P}^2 \Longrightarrow$ Composition of the blow-ups of points in $\mathcal{B}_{\mathcal{F}}$.

If we know the divisor $D_{\mathcal{F}}$ then we can compute the rational first integral by solving a system of linear equations.

A distinguished face of $NE(Z_{\mathcal{F}})$

$|D_{\mathcal{F}}|$ is base-point-free $\Rightarrow D_{\mathcal{F}}$ is nef and $D_{\mathcal{F}}^2 = 0$.

$NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ is a face of the cone $NE(Z_{\mathcal{F}})$, not necessarily of codimension 1.

Theorem

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ with a rational first integral. Let C be a curve on $Z_{\mathcal{F}}$.

(a) $[C] \in NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ if and only if

$$C = D + E$$

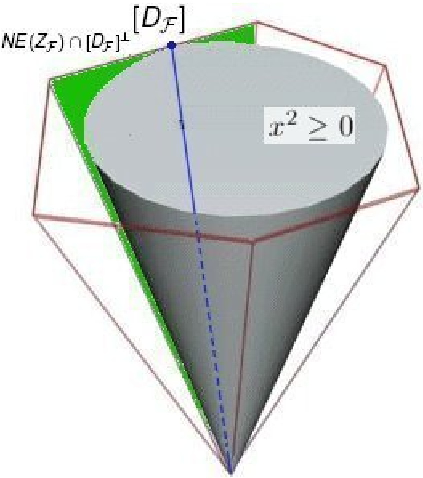
where

- E is a sum (may be empty) of strict transforms of non-dicritical exceptional divisors.
- either $D = 0$, or D is the strict transform on $Z_{\mathcal{F}}$ of an invariant curve.

(b) If $[C] \in NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ then $C^2 \leq 0$. Moreover, in this case:

$$C^2 = 0 \Leftrightarrow [C] = r[D_{\mathcal{F}}] \text{ with } r \in \mathbb{Q}^+.$$

Geometric interpretation



$Q(Z_{\mathcal{F}}) = \{x \in A(Z_{\mathcal{F}}) \mid x^2 \geq 0 \text{ and } x \cdot [H] \geq 0\} \subseteq \overline{NE}(Z_{\mathcal{F}})$, where H is an ample divisor.

CONDITIONS DISCARDING THE EXISTENCE OF RATIONAL FIRST INTEGRAL

Corollary

If one of the following two condition is satisfied, then a foliation $\mathcal{F}$ has not a rational first integral:

1. There exists an invariant curve, C, such that $\tilde{C}^2 > 0$.
2. There exist two invariant curves, C_1 and C_2, such that $\tilde{C}_1^2 = 0$ and $\tilde{C}_1 \cdot \tilde{C}_2 \neq 0$.

Independent systems of algebraic solutions

Let $\mathcal{F}$ be an arbitrary foliation of $\mathbb{P}^2$.

Definition

An independent system of algebraic solutions of $\mathcal{F}$ is a set of reduced and irreducible algebraic invariant curves of $\mathcal{F}$, $S = \{C_1, C_2, \ldots, C_s\}$, such that

- s is the number of dicritical exceptional divisors appearing in the minimal resolution of the singularities of $\mathcal{F}$, that is, $\mathrm{card}(\mathcal{B}_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}})$.
- $\tilde{C}_i^2 \leq 0 \;\forall\, i$,
- the set of classes

$$\mathcal{A}_S := \{[\tilde{C}_1], [\tilde{C}_2], \ldots, [\tilde{C}_s]\} \cup \{[\tilde{E}_q]\}_{q \in \mathcal{N}_{\mathcal{F}}} \subseteq A(Z_{\mathcal{F}})$$

  is $\mathbb{R}$-linearly independent.

OBSERVATIONS:

$\mathrm{Card}\, \mathcal{A}_S = \dim A(Z_{\mathcal{F}}) - 1$.

If $\mathcal{F}$ has a rational first integral, the existence of a independent system of algebraic solutions is equivalent to say that the face $NE(Z_{\mathcal{F}}) \cap [D_{\mathcal{F}}]^{\perp}$ has codimension 1.

The divisor $T_{\mathcal{F},S}$

Assume that a foliation $\mathcal{F}$ (not necessarily with rational first integral) admits an independent system of algebraic solutions $S = \{C_1, C_2, \ldots, C_s\}$.

The set $\mathcal{A}_S := \{[\tilde{C}_1], [\tilde{C}_2], \ldots, [\tilde{C}_s]\} \cup \{[\tilde{E}_q]\}_{q \in \mathcal{N}_{\mathcal{F}}}$ spans a hyperplane in the vector space $A(Z_{\mathcal{F}}) \cong \mathbb{R}^{m+1}$.

One can compute its equation with coordinates $(x_0, \ldots, x_m)$ in the basis $\{L^*, E^*_{q_1}, \ldots, E^*_{q_m}\}$:

$$\sum_{i=0}^{m} \delta_i x_i = 0,$$

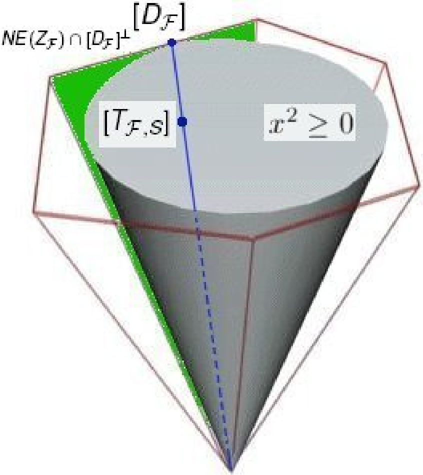
such that $\delta_i \in \mathbb{Z}$, $\delta_0 > 0$ and $\gcd(\delta_0, \ldots, \delta_m) = 1$.

$$T_{\mathcal{F},S} := \delta_0 L^* - \sum_{j=1}^{m} \delta_j E^*_{q_j},$$

Using orthogonality with respect to the intersection product, the hyperplane generated by $\mathcal{A}_S$ is $[T_{\mathcal{F},S}]^{\perp}$.

Relation between the divisors $T_{\mathcal{F},S}$ and $D_{\mathcal{F}}$

If $\mathcal{F}$ has a rational first integral then $NE(Z_{\mathcal{F}}) \cap [T_{\mathcal{F},S}]^{\perp}$ is the face generated by the classes of invariant curves on $Z_{\mathcal{F}}$, $T_{\mathcal{F},S}^2 = 0$, $[T_{\mathcal{F},S}]$ is the "primitive" element of the ray generated by $[D_{\mathcal{F}}]$ and, therefore, $D_{\mathcal{F}} = \alpha T_{\mathcal{F},S}$, with $\alpha \in \mathbb{Z}_+$.



A FIRST RESULT FOR THE EXISTENCE AND COMPUTATION OF A RATIONAL FIRST INTEGRAL

Theorem

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ with a rational first integral. Assume that

1. $\mathcal{F}$ admits an independent system of algebraic solutions $S = \{C_i\}_{i=1}^s$ and set
$$[T_{\mathcal{F},S}] = \sum_{i=1}^{s} \alpha_i [\tilde{C}_i] + \sum_{q \in \mathcal{N}_{\mathcal{F}}} \beta_q [\tilde{E}_q]$$
the decomposition of $[T_{\mathcal{F},S}]$ as a linear combination of the classes in $\mathcal{A}_S$.
2. α_i $(1 \leq i \leq s)$ and β_q $(q \in \mathcal{N}_{\mathcal{F}})$ are strictly positive.

Then $D_{\mathcal{F}} = \delta_{\mathcal{F}} T_{\mathcal{F},S}$ where r is the minimum positive integer such that $r\alpha_i \in \mathbb{Z}$ $\forall$ i and

$$\delta_{\mathcal{F}} := \frac{r(\deg(\mathcal{F}) + 2 - \sum_{i=1}^{s} \deg(C_i))}{\gcd(\sum_{i=1}^{s} r\alpha_i \deg(C_i), \deg(\mathcal{F}) + 2 - \sum_{i=1}^{s} \deg(C_i))}.$$

How to use it: Given a foliation $\mathcal{F}$ satisfying the above Conditions 1 and 2, compute a basis of the vector space $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\delta_{\mathcal{F}} T_{\mathcal{F},S}))$. If its dimension is $\neq 2$ then $\mathcal{F}$ is not algebraically integrable. Otherwise, check if $\Omega \wedge d(F/G) = 0$, where Ω is a differential 1-form defining $\mathcal{F}$ and $\{F, G\}$ is a basis of the above vector space.

Example

Let $\mathcal{F}$ be the foliation on $\mathbb{P}^2$ defined by the projective 1-form $\Omega = AdX + BdY + CdZ$, where:

$$A = -3X^2Y^3 + 9X^2Y^2Z - 9X^2YZ^2 + 3X^2Z^3,$$
$$B = 3X^3Y^2 - 6X^3YZ - 5Y^4Z + 3X^3Z^2 \text{ and}$$
$$C = -3X^3Y^2 + 5Y^5 + 6X^3YZ - 3X^3Z^2.$$

$\mathcal{K}_{\mathcal{F}} = \{q_i\}_{i=1}^{19} \cup \{q_i\}_{i=20}^{23}$ (union of 2 chains with the following additional proximity relations: q_4 is proximate to q_2, q_{22} to q_{20} and q_{23} to q_{21}).

Unique dicritical exceptional divisor: $E_{q_{19}}$. Then: $\mathcal{B}_{\mathcal{F}} = \{q_i\}_{i=1}^{19}$ and $\mathcal{N}_{\mathcal{F}} = \{q_i\}_{i=1}^{18}$.

Independent system of algebraic solutions: $S = \{C\}$, where C is the line defined by the equation $Y - Z = 0$. $\mathcal{A}_S = \{[\tilde{C}]\} \cup \{[\tilde{E}_{q_j}]\}_{j=1}^{18}$.

$$T_{\mathcal{F},S} = 5L^* - 2E^*_{q_1} - 2E^*_{q_2} - \sum_{i=3}^{19} E^*_{q_i}$$

$$[T_{\mathcal{F},S}] = 5[\tilde{C}] + 3[\tilde{E}_{q_1}] + 6[\tilde{E}_{q_2}] + 10[\tilde{E}_{q_3}] + \sum_{i=4}^{18}(19-i)[\tilde{E}_{q_i}]$$

Conditions 1 and 2 are satisfied, $\delta_{\mathcal{F}} = 1$ and

$$H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(T_{\mathcal{F},S})) = \langle F_1, F_2 \rangle,$$

where $F_1 = Y^5 - X^3Y^2 + 2X^3YZ - X^3Z^2$ and $F_2 = (Y-Z)^5$. One can check that F_1/F_2 is a rational first integral of $\mathcal{F}$.

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ALGORITHM TO DECIDE ALGEBRAIC INTEGRABILITY

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ that admits an independent system of algebraic solutions S. Consider the conditions:

(1) $T^2_{\mathcal{F},S} \neq 0$.

(2) The decomposition of the class $[T_{\mathcal{F},S}]$ as a linear combination of those in the set $\mathcal{A}_S$ contains all the classes in $\mathcal{A}_S$ with positive coefficients.

(3) $\Sigma(\mathcal{F}, S) := \{\lambda \in \mathbb{Z}_+ \mid h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\lambda T_{\mathcal{F},S})) \geq 2\} \neq \emptyset$

Algorithm 1

Input: A projective 1-form Ω defining $\mathcal{F}$, $\mathcal{B}_{\mathcal{F}}$, $\mathcal{N}_{\mathcal{F}}$ and an independent system of algebraic solutions S satisfying at least one of the above conditions (1), (2) and (3).
Output: A rational first integral for $\mathcal{F}$, or "0" if there is no such first integral.

1. If (1) holds, then return "0".
2. If Condition (2) is satisfied and $h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\delta_{\mathcal{F}} T_{\mathcal{F},S})) \neq 2$, then return "0".
3. Consider the minimum $\alpha_{\mathcal{F}}$ of the set $\Sigma(\mathcal{F}, S)$.
4. If $h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\alpha_{\mathcal{F}} T_{\mathcal{F},S})) > 2$, then return "0".
5. Take a basis $\{F, G\}$ of $H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\alpha_{\mathcal{F}} T_{\mathcal{F},S}))$ and check the equality $d(F/G) \wedge \Omega = 0$. If it is satisfied, then return F/G. Else, return "0".

Problems: When does an independent system of algebraic solutions exist? In this case, how to compute one of them? When does Property (3) hold?

Example: A foliation which does not admit an independent system of algebraic solutions

$\mathcal{F}$: Foliation on the projective plane over the complex numbers with rational first integral

$$\frac{XZ^2 + 3YZ^2 - Y^3}{YZ^2 + 3XZ^2 - X^3}.$$

The configuration of dicritical points $\mathcal{B}_{\mathcal{F}}$ has 9 points, all in $\mathbb{P}^2$, and therefore, $\mathcal{N}_{\mathcal{F}} = \emptyset$.

$D_{\mathcal{F}} = 3L^* - \sum_{q \in \mathcal{B}_{\mathcal{F}}} E_q^*$.

$D_{\mathcal{F}} \cdot \tilde{C} = 0$ for whichever invariant curve C. This forces to the following fact:

The non-general algebraic solutions are among the lines passing through three points in $\mathcal{B}_{\mathcal{F}}$ and the irreducible conics passing through six points in $\mathcal{B}_{\mathcal{F}}$.

Simple computations show that these are, exactly, the 8 curves given by the equations:

$$X - Y = 0;\ X + Y = 0;\ 2X + (\sqrt{5} + 3)Y = 0;\ -2X + (\sqrt{5} - 3)Y = 0;$$

$$X^2 - XY + Y^2 - 4Z^2 = 0;\ X^2 + XY + Y^2 - 2Z^2 = 0;$$
$$2X^2 + (\sqrt{5} - 3)XY - (3\sqrt{5} - 7)Y^2 + (8\sqrt{5} - 24)Z^2 = 0;$$
$$-2X^2 + (\sqrt{5} + 3)XY - (3\sqrt{5} + 7)Y^2 + (8\sqrt{5} + 24)Z^2 = 0.$$

Hence, $\mathcal{F}$ does not admit an independent system of algebraic solutions.

CONDITION IMPLYING THOSE OF ALGORITHM 1

Theorem

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ such that $NE(Z_{\mathcal{F}})$ is polyhedral. Then:

- If $\mathcal{F}$ has a rational first integral then it admits an independent system of algebraic solutions S such that $\tilde{C}^2 < 0$ for all $C \in S$.
- If S is an independent system of algebraic solutions with this property and Conditions (1) and (2) of Algorithm 1 do not hold, then Condition (3) is true (that is, the set $\Sigma(\mathcal{F}, S)$ is not empty).

CONSEQUENCE: Assume $NE(Z_{\mathcal{F}})$ polyhedral:

(i) If we are able to prove that $\mathcal{F}$ has not an independent system of algebraic solutions then $\mathcal{F}$ is not algebraically integrable.

(ii) If we are able to find an independent system of algebraic solutions, then we can apply Algorithm 1 to decide if $\mathcal{F}$ is algebraically integrable.

ALGORITHM TO DECIDE THE EXISTENCE (AND COMPUTE) OF AN INDEPENDENT SYSTEM OF ALGEBRAIC SOLUTIONS

Algorithm 2

Input: A projective 1-form Ω defining $\mathcal{F}$, $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$.
Output: Either "0" ($\mathcal{F}$ has no rational first integral) or an indep. system of algebraic solutions

1. Define $V := \mathrm{con}(\{[\tilde{E}_q]\}_{q\in\mathcal{B}_{\mathcal{F}}})$, $G := \emptyset$ and let Γ be the set of divisors $C = dL^* - \sum_{q\in\mathcal{B}_{\mathcal{F}}} e_q E_q^*$ satisfying the following conditions:
   (a) $d > 0$ and $0 \leq e_q \leq d$ for all $q \in \mathcal{B}_{\mathcal{F}}$.
   (b) $C \cdot \tilde{E}_q \geq 0$ for all $q \in \mathcal{B}_{\mathcal{F}}$.
   (c) Either $C^2 = K_{Z_{\mathcal{F}}} \cdot C = -1$, or $C^2 < 0$ and $K_{Z_{\mathcal{F}}} \cdot C \geq 0$.
2. Pick $D \in \Gamma$ such that $D \cdot L^*$ is minimal.
3. If D satisfies the conditions
   (a) $[D] \notin V$,
   (b) $h^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(D)) = 1$,
   (c) $[D] = [\tilde{Q}]$, where Q is the divisor of zeros of a global section of $\pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(D)$,

   then set $V := \mathrm{con}(V \cup \{[D]\})$. If, in addition, Q is an invariant curve of $\mathcal{F}$, no curve in G is a component of Q and $\{[\tilde{R}] \mid R \in G\} \cup \{[D]\} \cup \{[\tilde{E}_q]\}_{q\in\mathcal{N}_{\mathcal{F}}}$ is a $\mathbb{R}$-linearly independent system of $A(Z_{\mathcal{F}})$, then set $G := G \cup \{Q\}$.
4. Let $\Gamma := \Gamma \setminus \{D\}$.
5. Repeat the steps 2, 3 and 4 while the following two conditions are satisfied:
   (a) $\mathrm{card}(G) < \mathrm{card}(\mathcal{B}_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}})$, where card stands for cardinality.
   (b) There exists $x \in V^{\vee}$ such that $x^2 < 0$.
6. If $\mathrm{card}(G) < \mathrm{card}(\mathcal{B}_{\mathcal{F}} \setminus \mathcal{N}_{\mathcal{F}})$, then return "0". Else, return G.

When is $NE(Z_{\mathcal{F}})$ polyhedral?

- $\mathcal{B}_{\mathcal{F}}$ has less than 9 points.
- $\mathcal{F}$ is given by $d(H/Z^d)$, where H is an homogeneous polynomial of degree d that defines a curve with a unique branch at the line of infinity (A. Campillo, O. Piltant and A. Reguera, Cones of curves and of line bundles on surfaces associated with curves having one place at infinity, *Proc. London Math. Soc.* **84** (2002), 559—580.).
- $\mathcal{B}_{\mathcal{F}}$ is a toric configuration.
- $\mathcal{B}_{\mathcal{F}}$ is a P-sufficient configuration (C. Galindo and F. Monserrat, The cone of curves associated to a plane configuration, *Comm. Math. Helv.* **80** (2005), 75—93.).

P-suficient configurations

Given a configuration $\mathcal{K} = \{p_1, p_2, \ldots, p_n\}$:

- Square symmetric matrix: $G_{\mathcal{K}} = (g_{ij})_{1 \leq i,j \leq n}$:

$$g_{ij} := -9D_i \cdot D_j - (K_{Z_{\mathcal{K}}} \cdot D_i)(K_{Z_{\mathcal{K}}} \cdot D_j),$$

  where $D_i = \sum_{j=1}^{n} m_{ij} E_j^*$ and $\mathbf{P}_{\mathcal{K}}^{-1} = (m_{ij})_{1, \leq i,j \leq n}$.

  Proximity matrix: $\mathbf{P}_{\mathcal{K}} = (q_{ij})_{1 \leq i,j \leq n}$ such that

$$q_{ij} = \begin{cases} 1 & \text{if } \; i = j \\ -1 & \text{if } \; i \to j \\ 0 & \text{otherwise} \end{cases}$$

- $\mathcal{K}$ is *P-sufficient* if $\mathbf{x} G_{\mathcal{K}} \mathbf{x}^t > 0$ $\forall \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}$ such that $x_i \geq 0 \; \forall i$.

P-sufficient configurations

- Using a criterion given in (J. W. Gaddum, Linear inequalities and quadratic forms, *Pacific J. Math.* **8** (1958), 411—414) it is posible to decide whether a configuration is P-sufficient. It consists of checking the non-emptiness of certain sets defined by linear inequalities.
- When the configuration $\mathcal{K}$ is a *chain*, a very simple to verify criterion can be given: $\mathcal{K}$ *is P-sufficient iff the last entry of the matrix* $G_{\mathcal{K}}$ *is positive*.

Proposition

Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ such that the configuration $\mathcal{B}_{\mathcal{F}}$ is P-sufficient. Then:

(a) The cone $NE(Z_{\mathcal{F}})$ is polyhedral.

(b) If S is an independent system of algebraic solutions obtained by calling Algorithm 2 and $\mathcal{F}$ has a rational first integral, then

$$D_{\mathcal{F}} = T_{\mathcal{F},S}.$$

Example

Let $\mathcal{F}$ be a foliation as above defined by the 1-form $\Omega = AdX + BdY + CdZ$, where

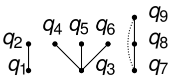
$$A = X^4Y^3Z + 5X^3Y^4Z + 9X^2Y^5Z + 7XY^6Z + 2Y^7Z + X^4Z^4 - X^3YZ^4,$$

$$B = -3X^5Y^2Z - 13X^4Y^3Z - 21X^3Y^4Z - 15X^2Y^5Z - 4XY^6Z + 2X^4Z^4 \text{ and}$$

$$C = 2X^5Y^3 + 8X^4Y^4 + 12X^3Y^5 + 8X^2Y^6 + 2XY^7 - X^5Z^3 - X^4YZ^3.$$

$\mathcal{K}_{\mathcal{F}} = \mathcal{B}_{\mathcal{F}} = \{q_i\}_{i=1}^{9}$ and $\mathcal{N}_{\mathcal{F}} = \{q_1, q_3, q_7, q_8\}$.

Proximity graph:



$\mathcal{B}_{\mathcal{F}}$ is a P-sufficient configuration.

Output of Algorithm 2: Independent set of algebraic solutions $S = \{X = 0, X + Y = 0, Z = 0, XY + Y^2 + XZ = 0, jXY + jY^2 + XZ\}$, where j is a primitive cubic root of unity.

$$T_{\mathcal{F},S} = 6L^* - 3\sum_{i=1}^{3} E^*_{q_i} - \sum_{i=4}^{6} E^*_{q_i} - 2E^*_{q_7} - \sum_{i=8}^{9} E^*_{q_i}.$$

$$\mathbb{P}H^0(\mathbb{P}^2, \pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(T_{\mathcal{F},G})) = \langle F_1 = (X+Y)^2X^2Z^2, F_2 = (X+Y)^3Y^3 + X^3Z^3\rangle.$$

The equality $d(F_1/F_2) \wedge \Omega = 0$ shows that F_1/F_2 is a rational first integral of $\mathcal{F}$.

Bound on the number of dicritical singularities for the non-degenerated case

Theorem

Let $\mathcal{F}$ be a degree r non-degenerated foliation of $\mathbb{P}^2$ which has a rational first integral. Then

$$r + 1 \leq n,$$

where n is the number of dicritical singularities of $\mathcal{F}$.

(C. Galindo, F. Monserrat, On the characterization of algebraically integrable plane foliations, *to appear in Trans. Amer. Math. Soc.*)