Temperate holomorphic solutions and regularity of holonomic D-modules on curves

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Strategy

1. Definition of the functor of tempered cohomology and its relation with tempered holomorphic functions
2. Definition of subanalytic site
3. Definition of the category of ind-sheaves
4. Motivation and definition of regularity for Ind-sheaves.

X: Complex analytic manifold

$\mathcal{D}_X$: Sheaf of rings of holomorphic differential operators of finite order on X

$\mathcal{M}$: Coherent $\mathcal{D}_X$-module

$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$: Complex of holomorphic solutions of the differential system $\mathcal{M}$

$F \in D^b_{\mathbb{R}-c}(\mathbf{k}_X)$

$R\mathcal{H}om(F, \mathcal{O}_X)$: complex of generalized functions associated with F

This complex allow us to treat various situations:

- M real analytic manifold complexified by X: $R\mathcal{H}om(\mathrm{D}'(\mathbb{C}_M), \mathcal{O}_X) = \mathcal{B}_X$ (Sato's hyperfunctions)
- Z complex analytic hypersurface of X: $R\mathcal{H}om(\mathbb{C}_Z[-1], \mathcal{O}_X)$ sheaf of holomorphic functions with singularities on Z

However, the complex $R\mathcal{H}om(F, \mathcal{O}_X)$ does not allow us to treat objects with temperate growth such as Schwartz's distributions or meromorphic functions.

To study such cases, we have to replace it by the complex

$$thom(F, \mathcal{O}_X)$$

of temperate cohomology.

Subanalytic sets

X real analytic manifold

Definition

(i) A subset $A \subset X$ is said to be semi-analytic if each $x \in A$ has a neighborhood U such that:

$$A \cap U = \bigcup_{i \in \mathrm{I}} \bigcap_{j \in \mathrm{J}} X_{ij},$$

where I,J are finite sets and either $X_{ij} = \{y \in U; f_{ij}(y) > 0\}$ or $X_{ij} = \{y \in U; f_{ij}(y) = 0\}$, for some analytic function f_{ij}.

$\mathbb{R}$-constructible sheaves

(ii) A subset $A \subset X$ is said to be subanalytic if each $x \in A$ has a neighborhood U such that:

$$A \cap U = \pi(A'),$$

where π denotes the projection $X \times Y \to X$, Y a real analytic manifold and $A' \subset X \times Y$ a relatively compact semi-analytic subset.

Definition

A complex of sheaves $F \in D^b(\mathbf{k}_X)$ is said to be $\mathbb{R}$-**constructible** if there exists a locally finite covering $X = \cup_{i \in I} X_i$ by subanalytic sets such that the sheaves $H^j(F)|_{X_i}$ are **locally constant** for all $j \in \mathbb{Z}, i \in I$.

Tempered distributions

M: real analytic manifold

$\mathcal{D}b_M$: sheaf of distributions of M

Definition

A distribution u defined in an open subset U of M is called tempered if there exist a distribution w defined on M such that:

$$u = w|_U.$$

Functor of temperate cohomology

Definition

Let $F \in D^b_{\mathbb{R}-c}(\mathbf{k}_M)$. We denote by $thom(F, \mathcal{D}b_M)$ the subsheaf of $\mathcal{H}om(F, \mathcal{D}b_M)$ defined as follows:

$$\Gamma(U; thom(F, \mathcal{D}b_M)) = \{\varphi = (\varphi_V)_{V \subset U} \in \Gamma(U; \mathcal{H}om(F, \mathcal{D}b_M));$$

$$\forall V \subset\subset U, \forall s \in F(V), \varphi_V(s) \text{is } \textbf{tempered}\}$$

Remark: When $F = \mathbb{C}_M$ we recover the sheaf $\mathcal{D}b_M$!

Functor of temperate cohomology

X complex manifold
$\overline{X}$ complex conjugate of X (this is $(X, \mathcal{O}_{\overline{X}})$)

Definition

Let $F \in D^b_{\mathbb{R}-c}(\mathbf{k}_X)$. We define

$$thom(F, \mathcal{O}_X) := R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, thom(F, \mathcal{D}b_X)).$$

Tempered functions

Definition

Let U be an open subset of a real analytic manifold X. One says $f \in \mathcal{C}_X^\infty(U)$ has polynomial growth at $p \in X$ if, for a local coordinate system $(x_1, \ldots, x_n)$ around p, there exist a compact neighborhood K of p and $N \in \mathbb{N}$ such that:

$$\sup_{x \in K \cap U} (\mathrm{dist}(x, K \setminus U))^N |f(x)| < \infty.$$

We say that f is tempered at p if all its derivatives have polynomial growth at p. We say that f is tempered if it is tempered at any point.

Tempered functions

We denote by $\mathcal{C}_X^{\infty,t}(U)$ the subspace of $\mathcal{C}_X^{\infty}(U)$ consisting of tempered functions.

The property of being temperate is not local!

Example

Let $X = \mathbb{C} \setminus \{0\}$ and U be an open subset of X.

One has $\exp(1/z) \in \mathcal{C}_X^{\infty}(U)$ and it may be proved that:

$$\exp(1/z) \in \mathcal{C}_X^{\infty,t}(U) \text{ if and only if } U \subset X \setminus \overline{B_\varepsilon(\varepsilon, 0)}, \text{ for some } \varepsilon > 0.$$

$$U = \{z \in X; 0 < arg(z) < \frac{\pi}{4}, |z| < 1\}$$

$$U_n = U \setminus \overline{B_{\frac{1}{n}}\left(\frac{1}{n}, 0\right)}, \ n \in \mathbb{N}$$

$$U = \cup_{n \in \mathbb{N}} U_n$$

$$\Downarrow$$

- $\exp(1/z) \in \mathcal{C}_X^{\infty,t}(U_n)$, for all $n \in \mathbb{N}$
- $\exp(1/z) \notin \mathcal{C}_X^{\infty,t}(U)$

$$\downarrow$$

The presheaf $U \in \mathrm{Op}(X) \mapsto \mathcal{C}_X^{\infty,t}(U)$ is not a sheaf!

Sites

*To overcome this difficulty we introduce a **Grothendieck topology** on X!*

A Grothendieck topology is not a topology, but an axiomatization of the notion of covering, defined on a category, where the objects play the role of open sets.

A category endowed with a Grothendieck topology is called a **site**.

Example

X topological space
$\mathrm{Op}(X)$: category of the open subsets of X
$\forall U \in \mathrm{Op}(X),\ \mathrm{Cov}(U) = \{(U_i)_i; U_i \in \mathrm{Op}(X), \cup_i U_i = U\}$

The Subanalytic Site

X real analytic manifold
$\mathrm{Op}(X_{sa})$: category of the open subanalytic subsets of X
$U \in \mathrm{Op}(X_{sa})$
$\mathrm{Cov}(U) = \{\{U_i\}_i; U_i \in \mathrm{Op}(X_{sa})$, for all compact subset $K \subset X$, there exists a finite subfamily which covers $U \cap K\}$.

$$\Downarrow$$

One denotes by X_{sa} the site defined by this topology and by $\mathrm{Mod}(\mathbf{k}_{X_{sa}})$ the category of sheaves of $\mathbf{k}$-vector spaces on X_{sa}.

The Subanalytic Site

Example

The presheaves

$$U \in \mathrm{Op}(X_{sa}) \mapsto \mathcal{C}_X^{\infty,t}(U),$$

$$U \in \mathrm{Op}(X_{sa}) \mapsto \mathcal{D}b_X^t(U)$$

and when X is a (one-dimensional) complex manifold

$$U \in \mathrm{Op}(X_{sa}) \mapsto \mathcal{O}_X^t(U),$$

are sheaves on X_{sa}.

Temperate holomorphic functions and the functor *thom*

For each $F \in D^b_{\mathbb{R}-c}(\mathbf{k}_X)$:

$$thom(F, \mathcal{O}_X) \simeq R\mathcal{H}om(F, \mathcal{O}^t_X).$$

Temperate holomorphic solutions of $\mathcal{D}_X$-modules

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We denote:

$$Sol(\mathcal{M}) := R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$$

$$\mathcal{S}ol^t(\mathcal{M}) := R\mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{M}, \mathcal{O}^t_X) \in D^b(\mathbf{k}_{X_{sa}}).$$

Temperate holomorphic solutions of $\mathcal{D}_X$-modules

When $\mathcal{M}$ is regular holonomic one has:

$$\mathcal{S}ol^t(\mathcal{M}) \simeq \mathcal{S}ol(\mathcal{M}).$$

So the holomorphic solutions of regular holonomic $\mathcal{D}_X$-modules are all tempered!

And what about irregular holonomic $\mathcal{D}_X$-modules?

Irregular Holonomic $\mathcal{D}_X$-modules

An example: $X = \mathbb{C}$

$$\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X (z^2 \partial_z + 1)$$

$$\mathcal{S} := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) = \mathbb{C}_{X \setminus \{0\}} \exp(1/z)$$

$$\mathcal{S}^t := H^0(Sol^t(\mathcal{M})) \simeq \mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{M}, \mathcal{O}_X^t)$$

Proposition(Kashiwara-Schapira)

One has the following isomorphism:

$$\varinjlim_{\varepsilon > 0} \rho_* \mathbb{C}_{X \setminus \overline{B_\varepsilon(\varepsilon, 0)}} \xrightarrow{\sim} \mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{M}, \mathcal{O}_X^t).$$

Irregular Holonomic $\mathcal{D}_X$-modules in dimension one

Let us now study the tempered holomorphic solutions of $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$, for some $P \in \mathcal{D}_X$ of the form

$$P = z^N \partial_z^m + \sum_{k=0}^{m-1} a_k(z)\partial_z^k,$$

with an irregular singularity at the origin.

Let $u \in \mathcal{O}_X(U)$. One has

$$(z^N \partial_z^m + \sum_{k=0}^{m-1} a_k(z)\partial_z^k)u = 0$$

if and only if $v = (u, u', u'', ..., u^{(m-1)}) \in \mathcal{O}_X(U)^m$ is solution of the system

Irregular Holonomic $\mathcal{D}_X$-modules in dimension one

$$\begin{cases} z^N \partial_z v_1 = z^N v_2 \\ z^N \partial_z v_2 = z^N v_3 \\ \dots \\ z^N \partial_z v_m + \sum_{k=0}^{m-1} a_k(z) v_{k+1} = 0 \end{cases}$$

Hence,

$$(z^N \partial_z^m + \sum_{k=0}^{m-1} a_k(z) \partial_z^k) u = 0, u \in \mathcal{O}_X(U) \;\Leftrightarrow (z^N \partial_z I_m + A(z)) v = 0,$$

where $v = (u, u', u'', \dots, u^{(m-1)}) \in \mathcal{O}_X^m(U)$, $m, N \in \mathbb{N}$, I_m is the identity matrix of order m and $A \in \mathrm{M}_m(\mathcal{O}_X(U))$.

From now on we denote by P the system

$$P = z^N \partial_z I_m + A(z),$$

and we reduce to the case where $\mathcal{M} = \mathcal{D}_X^m / \mathcal{D}_X^m P$, so that

$$\mathcal{S} \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X^m / \mathcal{D}_X^m P, \mathcal{O}_X),$$

and

$$\mathcal{S}^t \simeq \mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! (\mathcal{D}_X^m / \mathcal{D}_X^m P), \mathcal{O}_X^t).$$

Irregular Holonomic $\mathcal{D}_X$-modules in dimension one

Theorem (Hukuhara-Turrittin)

There exist $l \in \mathbb{N}$, a diagonal matrix $\Lambda(z) \in \mathrm{M}_m(z^{-1/l}\mathbb{C}[z^{-1/l}])$ and, for any θ, there exist an open sector $S_\theta = S(\theta_0, \theta_1, R)$ $(\theta_0 < \theta < \theta_1)$ and $F_\theta \in \mathrm{GL}_m(\mathcal{O}_X^t(S_\theta))$, such that:

- $F_\theta^{-1} \in \mathrm{GL}_m(\mathcal{O}_X^t(S_\theta))$,
- the m-columns of the matrix $F_\theta(z)\exp(-\Lambda(z))$ are $\mathbb{C}$-linearly independent holomorphic solutions of the system

$$(z^N \partial_z I_m + A(z))u = 0.$$

Irregular Holonomic $\mathcal{D}_X$-modules in dimension one

$$u = (u_1, \ldots, u_m) \in \mathcal{O}_X(S)^m, \ (z^N \partial_z I_m + A(z))u = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = C_1 \begin{bmatrix} f_{11} \exp(-\Lambda_1) \\ \vdots \\ f_{m1} \exp(-\Lambda_1) \end{bmatrix} + \ldots + C_m \begin{bmatrix} f_{1m} \exp(-\Lambda_m) \\ \vdots \\ f_{mm} \exp(-\Lambda_m) \end{bmatrix},$$

$C_1, \ldots, C_m \in \mathbb{C}$.

Irregular Holonomic $\mathcal{D}_X$-modules in dimension one

$$
\begin{array}{ccc}
 & Sol(\cdot) & \\
P = z^2\partial_z + 1 & \rightarrow & C\exp(1/z), C \in \mathbb{C} \\
P = z^N\partial_z I_m + A(z) & \rightarrow & CF_\theta \exp(-\Lambda(z)), C \in \mathbb{C}^m
\end{array}
$$

Irregular Holonomic $\mathcal{D}_X$-modules in dimension one

Corollary (AM)

Let $V \in \mathrm{Op}^c(X_{sa})$ such that P has a fundamental solution $F(z)\exp(-\Lambda(z))$ on V. Then,

$$\Gamma(V;\mathcal{S}^t) \simeq \mathbb{C}^{n(V)},$$

where $n(V) = \#\{j \in \{1,\dots,m\}; \exp(-\Lambda_j(z))|_V \in \mathcal{O}_X^t(V)\}$.

Irregular Holonomic $\mathcal{D}_X$-modules in dimension one

Lemma (M)

Let S be an open sector of amplitude smaller than 2π, $l \in \mathbb{N}$, $p \in z^{-1/l}\mathbb{C}[z^{-1/l}]$ and $V \in \mathrm{Op}^c(X_{sa})$, with $V \subset S$ and $0 \in \partial V$. Then:

$$\exp(p(z)) \in \mathcal{O}^t_X(V)$$

if and only if

$$\exists A > 0, \mathrm{Re}(p(z)) < A, \forall z \in V.$$

Proposition (AM)

There exist:

- an open sector S, with amplitude smaller than 2π and radius $R > 0$,
- a non-empty subset I of $\{1, ..., m\}$,

such that:

$$\forall j \in I,\ V \in \mathrm{Op}^c(X_{sa}),\ V \subset S\ (\exists A > 0:\ \mathrm{Re}(-\Lambda_j(z)) < A, \forall z \in V)$$

if and only if

$$V \subset S_\delta := \{z \in S; |z| > \delta\}, \text{ for some } \delta > 0$$

Moreover,

$$\forall j \in \{1, ..., m\} \setminus \mathrm{I},\ V \in \mathrm{Op}^c(X_{sa}),\ V \subset S$$

$$\exists A > 0\ s.t.\ \mathrm{Re}(-\Lambda_j(z)) < A, \forall z \in V$$

Theorem (AM)

Let $\mathcal{M} = \mathcal{D}_X^m / \mathcal{D}_X^m P$, $P = z^N \partial_z I_m + A(z)$. Then there exist an open sector S and a positive integer n such that:

$$\varinjlim_{R>\delta>0} \mathbb{C}^n_{S_\delta} \oplus \mathbb{C}^{m-n}_S \xrightarrow{\sim} \mathcal{S}^t \otimes \mathbb{C}_S.$$

Ind-Sheaves

Every $F \in \mathrm{Mod}(\mathbf{k}_{X_{sa}})$ can be written as:

$$F \simeq \varinjlim_i \rho_* F_i,$$

where $F_i \in \mathrm{Mod}^c_{\mathbb{R}-c}(\mathbf{k}_X)$ and ρ_* is the natural functor

$$\rho_* : \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_{X_{sa}}).$$

We denote by $\mathrm{I}(\mathbf{k}_X)$ the abelian category of ind-sheaves on X, that is, the category of ind-objects

$$\text{“}\varinjlim_i\text{”} F_i,$$

for sheaves $F_i \in \mathrm{Mod}^c(\mathbf{k}_X)$.

Ind-Sheaves

$$\begin{array}{ccccc} \mathrm{Mod}^c_{\mathbb{R}-c}(\mathbf{k}_X) & & & & \\ \downarrow & & & & \\ \mathrm{Mod}(\mathbf{k}_X) & \xrightarrow{\rho_*} & \mathrm{Mod}(\mathbf{k}_{X_{sa}}) & \hookrightarrow & \mathrm{I}(\mathbf{k}_X) \\ & & & & \\ F & & \longmapsto & & \text{“}\varinjlim_{U \subset\subset X}\text{”} F_U \end{array}$$

Microsupport

The notion of microsupport was introduced for classical sheaves in 1982 by M. Kashiwara and P. Schapira.

This notion appeared in the course of the study of analytical singularities of the solutions of systems of differential equations and describes the directions of non propagation of this solutions.

More precisely, the notion of microsupport appeared as an answer to the following problem:

Given a system of differential equations on a manifold X, or equivalently, a $\mathcal{D}_X$-module $\mathcal{M}$, when do the solutions of $\mathcal{M}$ defined in an open set Ω propagate through the boundary of Ω?

Microsupport

More precisely,

Definition (KS)

Let X be a real manifold and $F \in D^b(\mathbf{k}_X)$, the microsupport of F, denoted by $SS(F)$, is a closed conic subset of T^*X whose complementary is the set of points $p \in T^*X$ for which there exists a conic open neighborhood U such that:

$$R\Gamma_{\{\varphi \geq 0\}}(F)_x = 0,$$

for every C^1-real function φ, with $\varphi(x) = 0$ and $(x; d\varphi(x)) \in U$.

Microsupport

$\mathcal{M}$ coherent $\mathcal{D}_X$-module

$F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$

$$R\Gamma_{\{\varphi\geq 0\}}(F)_x \to F_x \to R\Gamma_{\{\varphi<0\}}(F)_x \xrightarrow{+1} .$$

Microsupport for ind-sheaves

To generalize the definition of microsupport to ind-sheaves we need the functor

$$J : D^b(\mathrm{I}(\mathbf{k}_X)) \to \mathrm{Ind}(D^b(\mathrm{Mod}^c(\mathbf{k}_X)))$$

defined by the equality

$$J(F)(G) = \mathrm{Hom}_{D^b(\mathrm{I}(\mathbf{k}_X))}(G, F),$$

for every $F \in D^b(\mathrm{I}(\mathbf{k}_X))$ and $G \in D^b(\mathrm{Mod}^c(\mathbf{k}_X))$.

In particular, for every $F \in D^b(\mathrm{I}(\mathbf{k}_X))$ one has

$$J(F) \simeq \text{“}\varinjlim\text{”} J(F_i),$$

for $F_i \in D^b(\mathrm{Mod}^c(\mathbf{k}_X))$.

Microsupport for ind-sheaves

Definition (KS)

Let $F \in D^b(\mathrm{I}(\mathbf{k}_X))$. The microsupport of F, denoted by $SS(F)$, is defined as the complementary of the set of points $p \in T^*X$ such that the following condition is satisfied:

- There exist a conic open neighborhood U of p in T^*X,
- a functor $I \to D^b(\mathbf{k}_X)$; $i \mapsto F_i$ and F' isomorphic to F in a neighborhood of $\pi(p)$ such that $J(F') \simeq$ "$\varinjlim$"$J(F_i)$ and
- $SS(F_i) \cap U = \emptyset$, for all $i \in I$.

Functorial properties of microsupport

Proposition (AM)

Let $F \in D^b(\mathrm{I}(\mathbf{k}_X))$ and $G \in D^b(\mathrm{I}(\mathbf{k}_Y))$. Then:

$$SS(F \boxtimes G) \subset SS(F) \times SS(G).$$

Proposition (AM)

Let M be a closed submanifold of X and let j denote the embedding $M \hookrightarrow X$. Let $G \in D^b(\mathrm{I}(\mathbf{k}_M))$. Then,

$$SS(Rj_* G) = j_\pi j_d^{-1}(SS(G)).$$

Functorial properties of microsupport

Proposition (AM)

Let Y, X be real analytic manifolds, $f\colon Y \to X$ be a smooth morphism and $F \in D^b(\mathrm{I}(\mathbf{k}_X))$. Then

$$SS(f^{-1}F) \subset f_d f_\pi^{-1}(SS(F)).$$

Proposition (AM)

Let $F \in D^b(\mathrm{I}(\mathbf{k}_X))$ and $G \in D^b(\mathrm{I}(\mathbf{k}_Y))$. Then:

$$SS(R\mathcal{I}hom(q_2^{-1}G, q_1^{!}F)) \subset SS(F) \times SS(G)^a.$$

Regularity for ind-sheaves

Given $F \in D^b(\mathrm{I}(\mathbf{k}_X))$ such that $J(F) \simeq$ "$\varinjlim$"$J(F_i)$, $F_i \in D^b(\mathbf{k}_X)$, we don't have in general

$$SS(F_i) \subset SS(F), \text{ for all i},$$

not even locally!

Regularity for ind-sheaves

Definition (KS)

Let $F \in D^b(\mathrm{I}(\mathbf{k}_X))$, $\Lambda \subset T^*X$ be a locally closed conic subset and $p \in T^*X$. We say that *F is regular along Λ at p* if there exist:

- an open neighborhood U of p with $\Lambda \cap U$ closed in U
- F' isomorphic to F in a neighborhood of $\pi(p)$
- a small and filtrant category I and a functor $I \to D^{[a,b]}(\mathbf{k}_X); i \mapsto F_i$ such that:

$$J(F') \simeq \text{“}\varinjlim\text{”} J(F_i) \text{ and } SS(F_i) \cap U \subset \Lambda.$$

Otherwise, we say that F is irregular along Λ at p.

(K-S)-Conjecture

(K-S)-Conjecture

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then $\mathcal{M}$ is regular holonomic if and only if $Sol^t(\mathcal{M})$ is regular.