Abel-Jacobi Maps of Algebraic Cycles

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Abstract. For a smooth quasiprojective variety over $\mathbb{C}$, we give an explicit description of the Bloch cycle class map from the higher Chow groups to Beilinson's absolute Hodge cohomology. We then arrive at an explicit formula for a weight filtered Abel-Jacobi map. This talk is based on earlier joint work with Matt Kerr [Inv. math. 170].

1. Statement

$X/\mathbb{C}$ smooth projective, $\dim X = d$, $Y \subset X$ a NCD. KLM = Kerr - Lewis - Müller-Stach, Compositio Math. 142 (02), 2006).

$$\begin{array}{ccc}
\mathrm{CH}^r_Y(X,m) & \to & H^{2r-m}_{\mathcal{H},Y}(X,\mathbb{Q}(r)) \\
\downarrow & & \downarrow \\
\mathrm{CH}^r(X,m) & \stackrel{KLM}{\to} & H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r)) \\
\downarrow & & \downarrow \\
\mathrm{CH}^r(X\backslash Y,m) & \to & H^{2r-m}_{\mathcal{H}}(X\backslash Y,\mathbb{Q}(r))
\end{array}$$

where $H^{\bullet}_{\mathcal{H}}$ is absolute Hodge cohomology, and fits in the s.e.s.:

$$\mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m-1}(-,\mathbb{Q}(r))\big) \hookrightarrow H^{2r-m}_{\mathcal{H}}(-,\mathbb{Q}(r)) \twoheadrightarrow$$
$$\to \hom_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2r-m}(-,\mathbb{Q}(r))\big).$$

For a $\mathbb{Q}$-MHS H, write

$$\Gamma(H) := \hom_{\mathrm{MHS}}\big(\mathbb{Q}(0), H\big).$$

$$J(H) := \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0), H\big)$$

The Abel-Jacobi map is the induced map $AJ_{r,m}$:

$$\mathrm{CH}^r_{\mathrm{hom}}(-, m) \to J\big(H^{2r-m-1}(-, \mathbb{Q}(r))\big).$$

Example. $m = 0$

$$\mathrm{CH}^r_{\mathrm{hom}}(X, 0) \overset{AJ_{r,0}}{\to} J\big(H^{2r-1}(X, \mathbb{Q}(r))\big)$$

$$\text{(Carlson)}\wr|$$

$$\frac{\{F^{d-r+1}H^{2d-2r+1}(X,\mathbb{C})\}^\vee}{H_{2d-2r+1}(X,\mathbb{Q})}$$

$$\xi \mapsto \big(\omega \in F^{d-r+1}H^{2d-2r+1}(X, \mathbb{C})\big)$$
$$\mapsto \int_{\partial^{-1}\xi} \omega$$

2. Higher Chow groups

(I) *Bloch's higher Chow groups.* Let $W/\mathbb{C}$ a quasiprojective variety. Put $z^r(W)$ = free abelian group generated by subvarieties of codimension r ($= \dim W - r$) in W. Consider the m-simplex:

$$\Delta^m = \operatorname{Spec}\left\{\frac{\mathbb{C}[t_0,\dots,t_m]}{\left(1-\sum_{j=0}^m t_j\right)}\right\} \simeq \mathbb{C}^m.$$

We set

$$z^r(W,m) = \left\{\xi \in z^r(W\times\Delta^m) \;\middle|\; \xi \text{ meets all faces } \{t_{i_1}=\cdots=t_{i_\ell}=0,\ \ell \geq 1\} \text{ properly}\right\}.$$

Note that $z^r(W,0) = z^r(W)$. Now set $\partial_j : z^r(W,m) \to z^r(W,m-1)$, the restriction to j-th face given by $t_j = 0$. The boundary map

$$\partial := \sum_{j=0}^{m} (-1)^j \partial_j :$$
$$z^r(W,m) \to z^r(W,m-1),$$

satisfies $\partial^2 = 0$.

Defn. $\mathrm{CH}^\bullet(W,\bullet)$ = homology of $\{z^\bullet(W,\bullet),\delta\}$.

Put $\mathrm{CH}^r(W) := \mathrm{CH}^r(W,0)$.

(II) *Alternate take: Cubical version.* Let $\square^m := (\mathbb{P}^1 \backslash \{1\})^m$ with coordinates z_i and 2^m codimension one faces obtained by setting $z_i = 0, \infty$, and boundary maps

$$\partial = \sum (-1)^{i-1}(\partial_i^0 - \partial_i^\infty),$$

where $\partial_i^0,\ \partial_i^\infty$ denote the restriction maps to the faces $z_i = 0,\ z_i = \infty$ respectively. The rest of the definition is completely analogous except that one has to divide out degenerate cycles. It is known that both complexes are quasiisomorphic. Put

$$\mathrm{CH}^r(W, m; \mathbb{Q}) := \mathrm{CH}^r(W, m) \otimes \mathbb{Q}.$$

Example. ("Totaro (pre-)cycle")

$X = \mathrm{Pt}$, $a \in \mathbb{C}^{\times}\setminus\{1\}$, $V_2(a) :=$
$\left\{(t, 1{-}t, 1{-}at^{-1}) \,\middle|\, t \in \mathbb{P}^1\right\} \bigcap \square^3$.

One computes $\partial V_2(a) =$

$$\left\{\begin{array}{c} [(1,\infty)-(\infty,1)] \\ \\ -[(1,1-a)-(\infty,1)] \\ \\ +[(a,1-a)-(0,1)] \end{array}\right\} \bigcap \square^2$$

$$= (a, 1-a)$$

Example. $X = \mathbb{P}^2$, with homogeneous coordinates $[z_0, z_1, z_2]$. $\mathbb{P}^1 = \ell_j := V(z_j)$, $j = 0, 1, 2$. Let $P = [0, 0, 1] = \ell_0 \cap \ell_1$, $Q = [1, 0, 0] = \ell_1 \cap \ell_2$, $R = [0, 1, 0] = \ell_0 \cap \ell_2$. Introduce $f_j \in \mathbb{C}(\ell_j)^\times$, where $(f_0) = P - R$, $(f_1) = Q - P$, $(f_2) = R - Q$. Then since

$$\sum_{j=0}^{2} \mathrm{div}(f_j) = 0,$$

and taking graphs,

$$\sum_{j=0}^{2} (f_j, \ell_j) \in \mathrm{CH}^2(\mathbb{P}^2, 1)$$

represents a higher Chow cycle.

Exercise. Show that it is nonzero.

<u>Example</u>. Again $X = \mathbb{P}^2$. Let $C \subset X$ be the nodal rational curve given in affine coordinates by $y^2 = x^3 + x^2$. Let $\tilde{C} \simeq \mathbb{P}^1$ be the normalization of C, with morphism $\pi : \tilde{C} \to C$. Put $P = (0,0) \in C$ (node) and let $R + Q = \pi^{-1}(P)$. Choose $f \in \mathbb{C}(\tilde{C})^\times = \mathbb{C}(C)^\times$, such that $(f)_{\tilde{C}} = R - Q$. Then $\mathrm{div}(f)_C = 0$ and hence $(f, C) \in \mathrm{CH}^2(\mathbb{P}^2, 1)$ defines a higher Chow cycle.

$\mathrm{CH}^r(X\backslash Y, m)$: Recall $X/\mathbb{C}$ a proj mfld of dim d, and $Y = Y_1 \cup \cdots \cup Y_N \subset X$ a NCD. For $t \geq 0 \in \mathbb{Z}$, put $Y^{[t]}$ = disjoint union of t-fold intersections of the Y_j's, with corresponding coskeleton $Y^{[\bullet]}$. Set $Y^{(t)}$ to be the union of t-fold intersections of the Y_j's. Note: $Y^{[t]}$ = desing of $Y^{(t)}$. $Y^{[0]} = Y^{(0)} = X$, $Y^{[1]} = \coprod_1^N Y_j$, $Y^{(1)} = Y$. Propr hypercover: $\cdots \substack{\to \\ \to \\ \to} Y^{[2]} \rightrightarrows Y^{[1]} \to Y$. Descent $\Rightarrow$ can compute homology of Y in terms of $Y^{[\bullet]}$. Arrows are (alternating) Gysin maps (Gy).

If $\{\xi\} \in \mathrm{CH}^r(X\backslash Y, 1)$, then after a moving lemma, get $\overline{\xi} \in z^r(X,1)$ with $\partial\overline{\xi} \in \mathrm{CH}^{r-1}(Y,0)$, which defines a "residue" $\widetilde{\partial\overline{\xi}}$ belonging to a subquotient of $\mathrm{CH}^r(Y^{[1]},0)$. Leads to higher residues on $\mathrm{CH}^r(X\backslash Y, m)$. Corresponding to $Y^{[\bullet]} \to Y^{[0]} := X$ is a third quadrant double complex $E_0^{i,j}(r) := z^{r+i}(Y^{[-i]}, -j)$, $i,\ j \le 0$;

$$\begin{array}{ccc} E_0^{i,j+1} & & \\ \uparrow\partial & & \\ E_0^{i,j} & \xrightarrow{\mathrm{Gy}} & E_0^{i+1,j} \end{array}$$

$\exists$ 2 s.s.'s of the sgle cplx $\mathbf{s}^\bullet E(r)$, $i+j=\bullet$, $D = \partial \pm \mathrm{Gy}$, with E_2-terms:

$${}'E_2^{p,q} := H^p_{\mathrm{Gy}}\big(H^q_\partial(E_0^{\bullet,\bullet}(r))\big)$$

$${}''E_2^{p,q} := H^p_\partial(H^q_{\mathrm{Gy}}\big(E_0^{\bullet,\bullet}(r))\big)$$

The 2nd s.s., together with a quasi-isomorphism (S. Bloch):

$$\frac{z^\bullet(X,*)}{z_Y^\bullet(X,*)} \overset{\text{Restriction}}{-\,-\,--\!\!\longrightarrow} z^\bullet(X\backslash Y,*),$$

$$\Rightarrow H^{-m}(\mathbf{s}^\bullet E(r)) = {}''E_2^{0,-m}$$
$$= \mathrm{CH}^r(X\backslash Y, m).$$

The 1st s.s. gives:

$$E_1^{i,j} = \mathrm{CH}^{r+i}(Y^{[-i]}, -j)$$

$$E_2^{i,j} =$$
$$\frac{\ker\begin{pmatrix}\mathrm{Gy}:\mathrm{CH}^{r+i}(Y^{[-i]},-j)\to\\ \mathrm{CH}^{r+i+1}(Y^{[-i+1]},-j)\end{pmatrix}}{\mathrm{Gy}\big(\mathrm{CH}^{r+i-1}(Y^{[i-1]},-j)\big)}$$

$\exists \uparrow$ wt filtration $\mathrm{Im}\big(\mathrm{CH}^r(X,m) \to \mathrm{CH}^r(X\backslash Y,m)\big) =: \underline{\mathrm{CH}}^r(X\backslash Y,m) =$

$$W_{-m}\mathrm{CH}^r(X\backslash Y,m) \subset \cdots \subset$$

$W_0\mathrm{CH}^r(X\backslash Y,m) = \mathrm{CH}^r(X\backslash Y,m)$.

Geometric interpretation. Let $\{\xi\} \in W_\ell\mathrm{CH}^r(X\backslash Y,m)$. Then $\widetilde{\partial}_R^{\ell+m}(\xi) \in \mathrm{Im}\big(\mathrm{CH}^{r-\ell-m}(Y^{[\ell+m]},-\ell)$ $\to \mathrm{CH}^{r-\ell-m}(Y^{(\ell+m)}\backslash Y^{(\ell+m+1)},-\ell)\big)$.

$E_\infty^{-\ell-m,\ell} =: Gr_W^\ell\mathrm{CH}^r(X\backslash Y,m)$

$$\hookrightarrow \left\{\begin{matrix}\text{A subquotient of}\\ \mathrm{CH}^{r-\ell-m}(Y^{[\ell+m]},-\ell)\end{matrix}\right\}.$$

$[\ell = -m, ..., 0]$

Ex. $Y := Y^{(1)} = Y_1 \cup Y_2 \subset X$, hence $Y^{[1]} = Y_1 \coprod Y_2$, $Y^{(2)} = Y^{[2]} = Y_1 \cap Y_2$. We focus on $\mathrm{CH}^r(U,2)$, with $i+j=-2$ and $U := X \backslash Y$:

$$\begin{array}{ccccc}
z^{r-2}(Y^{[2]},0) & \overset{\mathrm{Gy}}{\hookrightarrow} & z^{r-1}(Y^{[1]},0) & \overset{\mathrm{Gy}}{\to} & z^r(X,0) \\
\partial\uparrow & & \uparrow\partial & & \uparrow\partial \\
z^{r-2}(Y^{[2]},1) & \overset{\mathrm{Gy}}{\to} & z^{r-1}(Y^{[1]},1) & \overset{\mathrm{Gy}}{\to} & z^r(X,1) \\
 & & \uparrow\partial & & \uparrow\partial \\
 & & z^{r-1}(Y^{[1]},2) & \overset{\mathrm{Gy}}{\to} & z^r(X,2)
\end{array}$$

Let $\{\xi\} \in \mathrm{CH}^r(U,2)$. $\exists\overline{\xi} \in z^r(X,2)$ s.t. $\{\overline{\xi}\big|_U\} = \{\xi\}$. If $\partial\overline{\xi} = 0$, then $\{\xi\} \in W_{-2}\mathrm{CH}^r(U,2)$. In general $|\partial\overline{\xi}| \subset Y$. One can lift this to a cycle $\widetilde{\partial\overline{\xi}} \in z^{r-1}(Y^{[1]},1)$. If $\partial\big(\widetilde{\partial\overline{\xi}}\big) = 0$, then $\{\xi\} \in W_{-1}\mathrm{CH}^r(U,2)$. Otherwise $\big\{\partial\big(\widetilde{\partial\overline{\xi}}\big)\big\} \in \mathrm{CH}^{r-2}(Y^{[2]},0)$.

Let $D = \partial \pm \mathrm{Gy}$. The triple

$$\left(\widetilde{\partial(\widetilde{\partial\overline{\xi}})}, \widetilde{\partial\overline{\xi}}, \overline{\xi}\right)$$

defines a D-closed diagonal class in

$$E_0^{-2,0}(r) \bigoplus E_0^{-1,-1}(r) \bigoplus E_0^{0,-2}(r)$$

$$= z^{r-2}(Y^{[2]}, 0) \bigoplus z^{r-1}(Y^{[1]}, 1)$$

$$\bigoplus z^{r}(X, 2),$$

and where modulo D-coboundary,

$$W_{-2},\ W_{-1},\ W_0,$$

correspond respectively to classes of the form

$$(0, 0, *), \quad (0, *, *), \quad (*, *, *).$$

3. Deligne cohomology

$f:(A^\bullet,d)\to(B^\bullet,d)$ a morphism of complexes, then the mapping cone is

$$\mathrm{Cone}\big(A^\bullet \xrightarrow{h} B^\bullet\big)$$

whose differential is given by

$$\big[\mathrm{Cone}\big(A^\bullet \xrightarrow{h} B^\bullet\big)\big]^q := A^{q+1}\oplus B^q,$$
$$\delta(a,b)=(-da,h(a)+db).$$

$\mathcal{D}^\bullet_X$ = sheaf of currents acting on (compactly supported) C^∞ $(2d-\bullet)$-forms. $C^\bullet_X$ = sheaf of (Borel-Moore) $\mathbb{Q}(r)$-coeff chains of real codim $\bullet$.

Defn. Deligne cohomology with $\mathbb{Q}$-coeff is given by $H^k_{\mathcal{D}}(X,\mathbb{Q}(r)) =$

$$H^k\Big(\mathrm{Cone}\Big(C^\bullet_X(X,\mathbb{Q}(r))\oplus F^r\mathcal{D}^\bullet_X(X) \xrightarrow{\epsilon-l} \mathcal{D}^\bullet_X(X)\Big)[-1]\Big).$$

4. The higher Abel-Jacobi map – KLM formula: Smooth projective case

Bloch (and Beilinson via K-theory) constructed cycle class maps

$$\mathrm{cl}^{\mathcal{H}}_{r,m} : \mathrm{CH}^r(X,m) \to H^{2r-m}_{\mathcal{D}}(X,\mathbb{Q}(r)) = H^{2r-m}_{\mathcal{H}}(X,\mathbb{Q}(r)).$$

Deligne cohomology sits in an exact sequence: $J\big(H^{2r-m-1}(X,\mathbb{Q}(r))\big)$

$$\hookrightarrow H^{2r-m}_{\mathcal{D}}(X,\mathbb{Q}(r)) \twoheadrightarrow \Gamma\big(H^{2r-m}(X,\mathbb{Q}(r))\big).$$

Defn. $\mathrm{CH}^r_{\mathrm{hom}}(X,m)$ is the kernel of the composite map $\mathrm{CH}^r(X,m) \to \Gamma\big(H^{2r-m}(X,\mathbb{Q}(r))\big)$.

(ii) From (i), we have an induced map

$$\begin{tikzcd}
 & J\Big(H^{2r-m-1}(X,\mathbb{Q}(r))\Big) \arrow[d, equal, "\wr"] \\
\mathrm{CH}^r_{\mathrm{hom}}(X,m) \arrow[ur, "AJ_{r,m}"] \arrow[dr] & \\
 & \dfrac{F^{d-r+1}H^{2d+m-2r+1}(X,\mathbb{C})^\vee}{H_{2d+m-2r+1}(X,\mathbb{Q}(d-r))},
\end{tikzcd}$$

called the Abel-Jacobi map.

$W \in z^r(X \times \square^m)$ in general position. $\pi_1 : W \to X$, $\pi_2 : W \to \square^m$, and currents:

$$\Omega_\square = \bigwedge_1^m d\log z_j, \ R_\square =$$

$$\Bigg[\int_{Z\backslash z_1^{-1}[-\infty,0]} \log z_1 \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_m}{z_m} \wedge (?)$$

$$-2\pi\mathrm{i} \int_{z_1^{-1}[-\infty,0]\backslash(z_1\times z_2)^{-1}[-\infty,0]^2} \log z_2 \wedge \frac{dz_3}{z_3} \wedge \cdots \wedge \frac{dz_m}{z_m} \wedge (?)$$

$$+\cdots+$$

$$(-2\pi\mathrm{i})^{m-1} \int_{\substack{(z_1\times\cdots\times z_{m-1})^{-1}[-\infty,0]^{m-1} \\ \backslash(z_1\times\cdots\times z_m)^{-1}[-\infty,0]^m}} \log z_m \wedge (?)\Bigg]$$

$$T_\square = (2\pi\mathrm{i})^m \int_{[-\infty,0]^m} (?).$$

$$R_W = \pi_{1,*} \circ \pi_2^* R_\square,$$
$$\Omega_W = \pi_{1,*} \circ \pi_2^* \Omega_\square,$$
$$T_W = \pi_{1,*} \circ \pi_2^* T_\square.$$

In the Deligne homology complex, the differential δ is given by:

$$\delta\big((2\pi \mathrm{i})^{-\dim W}\big(T_W, \Omega_W, R_W\big)\big) = (2\pi \mathrm{i})^{-\dim W}(dT_W, d\Omega_W, T_W - \Omega_W - dR_W\big).$$

[$\dim W = m + d - r$, $d = \dim X$.]

$$\mathcal{M}_{\mathcal{D}}^{\bullet} := \mathrm{Cone}\big\{C_X^{\bullet}(X, \mathbb{Q}(r)) \oplus F^r \mathcal{D}_{X^\infty}^{\bullet}(X) \to \mathcal{D}_{X^\infty}^{\bullet}(X)\big\}[-1].$$

The homology of this complex, at $\bullet = 2r - m$ is precisely the Deligne cohomology $H_{\mathcal{D}}^{2r-m}(X, \mathbb{Q}(r))$.

$$z^r(X, \bullet) \to \mathcal{M}_{\mathcal{D}}^{2r-\bullet}$$
$$W \mapsto (2\pi \mathrm{i})^{-\dim W}\big(T_W, \Omega_W, R_W\big).$$

On the nullhomologous cycle level, with $\{W\} \in \mathrm{CH}^r_{\mathrm{hom}}(X,m)$ now a cycle with $R_{\partial W} = 0$, one has $\gamma := \{\pi_2^{-1}[-\infty, 0]^m\} \cap W = \partial\zeta$, and the AJ map is given by $R_W + (-2\pi \mathrm{i})^m \int_\zeta$. More explicitly, using the cubical complex, the formula for the AJ map

$$AJ_{r,m} : \mathrm{CH}^k_{\mathrm{hom}}(X,m) \to J^{r,m}(X)$$

$$:= \frac{\{F^{d-r+1}H^{2d-2r+m+1}(X,\mathbb{C})\}^\vee}{H_{2d-2r+m+1}(X,\mathbb{Q}(d-r))},$$

for $\omega \in F^{d-r+1}H^{2d-2r+m+1}(X,\mathbb{C})$:

$$\frac{1}{(2\pi\mathrm{i})^{d-r+m}}\times$$

$$\Big[\int_{W\setminus\{W\cap\pi_2^{-1}([-\infty,0]\times\square^{m-1})\}} \pi_2^*\Big((\log z_1)\bigwedge_2^m d\log z_j\Big)\wedge\pi_1^*(\omega)$$

$$-(2\pi\mathrm{i})\int_{\substack{\{W\cap\pi_2^{-1}([-\infty,0]\times\square^{m-1})\}\\ \setminus\{W\cap\pi_2^*([-\infty,0]^2\times\square^{m-2})\}}} \pi_2^*\Big((\log z_2))\bigwedge_3^m d\log z_j\Big)\wedge\pi_1^*(\omega)$$

$$+\cdots+$$

$$(-2\pi\mathrm{i})^{m-1}\int_{\substack{\{W\cap\pi_2^{-1}([-\infty,0]^{m-1}\times\square^1)\}\\ \setminus\{W\cap\pi_2^{-1}([-\infty,0]^m)\}}} \pi_2^*(\log z_m)\wedge\pi_1^*(\omega)$$

$$+\Big\{(-2\pi\mathrm{i})^m\int_\zeta\pi_1^*(\omega)\Big\}\Big].$$

5. Quasiprojective case

$H^i_{\mathcal{H}}(X\backslash Y, \mathbb{Q}(j))$: Recall $Y = Y_1 \cup \cdots \cup Y_N$ a NCD in a smooth projective $X/\mathbb{C}$, $\dim X = d$, with coskeleton $Y^{[\bullet]}$. For $i \leq 0$:

$$\mathcal{D}(r)^{i,j} = \mathcal{D}^{2r+2i+j}\big(Y^{[-i]}\big)$$

$$C(r)^{i,j} = C^{2r+2i+j}\big(Y^{[-i]}, \mathbb{Q}(r+i)\big)$$

with diff'ls Gy, d ($D = d \pm \mathrm{Gy}$).

$$\mathbf{s}^{\bullet}\mathcal{D}(r) = \bigoplus_i \mathcal{D}(r)^{i,\bullet-i}$$

$$= \bigoplus_i \mathcal{D}^{2r+i+\bullet}\big(Y^{[-i]}\big),$$

$$\mathbf{s}^{\bullet}C(r) = \bigoplus_i C(r)^{i,\bullet-i}$$

$$= \bigoplus_i C^{2r+i+\bullet}\big(Y^{[-i]}, \mathbb{Q}(r+i)\big),$$

($i+j = \bullet$) the associated single complexes with differential D.

Consider $\downarrow$ "weight" filtration

$$'W^\ell\big(\mathbf{s}^\bullet\mathcal{D}(r)\big) := \bigoplus_{i\geq\ell} \mathcal{D}(r)^{i,\bullet-i}$$

$$'W^\ell\big(\mathbf{s}^\bullet C(r)\big) := \bigoplus_{i\geq\ell} C(r)^{i,\bullet-i}$$

Put

$$\mathbb{Q}^\bullet_{\mathcal{H}}(r) := \mathrm{Cone}\Big\{\widehat{W}_0\mathbf{s}^\bullet C(r)$$

$$\bigoplus \widehat{W}_0F^0\mathbf{s}^\bullet\mathcal{D}(r) \to \bigoplus \widehat{W}_0\mathbf{s}^\bullet\mathcal{D}(r)\Big\}[-1],$$

where $\widehat{W}_0\mathbf{s}^k =$

$$\ker\left(D : {}'W^k\mathbf{s}^k(r) \to \frac{\mathbf{s}^{k+1}(r)}{'W^{k+1}\mathbf{s}^{k+1}(r)}\right),$$

and accordingly put

$$H^{2r+\ell}_{\mathcal{H}}(X\backslash Y, \mathbb{Q}(r)) := H^\ell\big(\mathbb{Q}^\bullet_{\mathcal{H}}(r)\big).$$

$\mathbb{Q}^{\bullet}_{\mathcal{H}}(r) :=$ the single complex $\mathbf{s}^{\bullet}\mathcal{H}(r)$ assoc to the double cplx $\mathcal{H}_0^{i,j}(r) :=$

$$\begin{cases} 0 & \text{if } j > 1 \\ \ker d \subset \mathcal{D}^{2(r+i)}(Y^{[-i]}) & \text{if } j = 1 \\ \ker d \subset C^{2(r+i)}(Y^{[-i]}, \mathbb{Q}(r+i)) & \\ \bigoplus \ker d \subset F^0\mathcal{D}^{2(r+i)}(Y^{[-i]}) & \\ \bigoplus \mathcal{D}^{2r+2i-1}(Y^{[-i]}) & \text{if } j = 0 \\ \mathcal{M}_{\mathcal{D}}^{2r+2i+j}(Y^{[-i]}, \mathbb{Q}(r+i)) & \text{if } j < 0 \end{cases}$$

Recall that $\mathcal{Z}_0^{i,j}(r) := z^{r+i}(Y^{[-i]}, -j)$. The KLM map $\mathcal{Z}_0^{i,j}(r) \to \mathcal{H}_0^{i,j}(r)$, induces a morphism of double complexes. At the infinity level, we have: $\mathcal{H}_\infty^{-\ell-m,\ell}(r) \simeq Gr_W^\ell H_{\mathcal{H}}^{2r-m}(X\backslash Y, \mathbb{Q}(r))$,

$$Gr_W^\ell \mathrm{CH}^r(X\backslash Y, m) \to$$

$Gr_W^\ell H_{\mathcal{H}}^{2r-m}(X\backslash Y, \mathbb{Q}(r)), \ell = -m, \ldots, 0.$

Consider the s.e.s.:

$$\underbrace{0 \to W_{\ell-1} \to W_0 \to Gr_W^{0,\ell-1} \to 0}_{\text{applied to } H^{2r-m-1}(X\backslash Y, \mathbb{Q}(r))}$$

Put $\Xi_\ell :=$ Image of

$$\begin{pmatrix} \Gamma\big(Gr_W^{0,\ell-1} H^{2r-m-1}(X\backslash Y, \mathbb{Q}(r))\big) \\ \downarrow \\ J\big(Gr_W^{\ell-1} H^{2r-m-1}(X\backslash Y, \mathbb{Q}(r))\big) \end{pmatrix}.$$

Prop. $\mathcal{H}_\infty^{-m-\ell,\ell}(r) \simeq$

$$\frac{J\big(Gr_W^{\ell-1} H^{2r-m-1}(X\backslash Y, \mathbb{Q}(r))\big)}{\Xi_\ell},$$

for $-m \leq \ell < 0$. For $\ell = 0$, $\exists$ s.e.s.:

$$\frac{J\big(Gr_W^{-1} H^{2r-m-1}(X\backslash Y, \mathbb{Q}(r))\big)}{\Xi_0} \hookrightarrow$$

$$\mathcal{H}_\infty^{-m,0}(r) \twoheadrightarrow \Gamma\big(H^{2r-m}(X\backslash Y, \mathbb{Q}(r))\big).$$

6. The Abel-Jacobi map

Define $\mathrm{CH}^r_{\mathrm{hom}}(X\backslash Y, m; \mathbb{Q}) :=$

$$\ker\left\{\mathrm{CH}^r(X\backslash Y, m; \mathbb{Q}) \to \Gamma\big(H^{2r-m}(X\backslash Y, \mathbb{Q}(r))\big)\right\},$$

with induced Abel-Jacobi map

$$AJ_{r,m} : \mathrm{CH}^r_{\mathrm{hom}}(X\backslash Y, m; \mathbb{Q}) \to J\big(H^{2r-m-1}(X\backslash Y, \mathbb{Q}(r))\big),$$

and corresponding $Gr^\ell AJ$:

$$Gr^\ell_W \mathrm{CH}^r_{\mathrm{hom}}(X\backslash Y, m; \mathbb{Q}) \\ \downarrow \\ \frac{J\big(Gr^{\ell-1}_W H^{2r-m-1}(X\backslash Y, \mathbb{Q}(r))\big)}{\Xi_\ell},$$

for $-m \le \ell \le 0$. This is evaluated using the KLM formula for smooth projective varieties.

<u>Example</u>. We will work out the case $\mathrm{CH}^r(X\backslash Y,1) \to H_{\mathcal{H}}^{2r-1}(X\backslash Y,\mathbb{Q}(r))$, where $Y = Y_1 \cup Y_2$. In this case

$$Y^{[1]} = Y_1 \coprod Y_2$$

$$Y^{[2]} = Y^{(2)} = Y_1 \cap Y_2$$

The weight filtration on $\mathrm{CH}^r(X\backslash Y,1)$ is 2-step, viz.,

$$\mathrm{CH}^r(X\backslash Y,1) = W_0\mathrm{CH}^r(X\backslash Y,1)$$

$$\supset W_{-1}\mathrm{CH}^r(X,1) = Gr_W^{-1}\mathrm{CH}^r(X\backslash Y,1)$$

$$=: \underline{\mathrm{CH}}^r(X\backslash Y,1) = \mathcal{Z}_\infty^{0,-1}(r).$$

We calculate the highest weight situation first. In this case we have ${}'E_\infty^{-1,0} = {}'E_2^{-1,0} =: \mathcal{Z}_\infty^{-1,0}(r) =$

$$Gr_W^0\mathrm{CH}^r(X\backslash Y,1) =$$

$$\frac{\ker \mathrm{Gy} : \mathrm{CH}^{r-1}(Y^{[1]}) \to \mathrm{CH}^r(X)}{\mathrm{Gy}(\mathrm{CH}^{r-2}(Y^{[2]}))}.$$

We have a diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \uparrow \\
 & & \Gamma\Big(H^{2r-1}(X\backslash Y, \mathbb{Q}(r))\Big) \\
 & [\,]\nearrow & \uparrow \\
\mathcal{Z}_{\infty}^{-1,0}(r) & \rightarrow & \mathcal{H}_{\infty}^{-1,0}(r) \\
 & AJ\searrow & \uparrow \\
 & & \dfrac{J\Big(Gr_W^{-1} H^{2r-2}(X\backslash Y, \mathbb{Q}(r))\Big)}{\Xi_0} \\
 & & \uparrow \\
 & & 0
\end{array}$$

where for $\xi \in \mathrm{CH}^r(X\backslash Y, 1; \mathbb{Q})$, the map $AJ(\xi)$ is only defined if $\xi \in \mathrm{CH}^r_{\mathrm{hom}}(X\backslash Y, 1; \mathbb{Q})$, i.e. $[\xi] = 0$.

We 'calculate' $\mathcal{Z}_\infty^{-1,0} \to \mathcal{H}_\infty^{-1,0}(r)$. Put $W_i := W_i H^{2r-1}(X \backslash Y, \mathbb{Q}(r))$, and consider the s.e.s.

$$0 \to W_{-1} \to W_0 \to Gr_W^0 \to 0.$$

We have $\Gamma(W_{-1}) = 0$, $\Gamma(W_0) = \Gamma\big(H^{2r-1}(X\backslash Y, \mathbb{Q}(r))\big)$, hence $\Gamma\big(H^{2r-1}(X\backslash Y, \mathbb{Q}(r))\big) \hookrightarrow \Gamma(Gr_W^0)$.

$$Gr_W^0 H^{2r-1}(X\backslash Y, \mathbb{Q}(r)) \simeq$$
$$(*) \quad \frac{H^{2r-2}_{\ker \mathrm{Gy}}(Y^{[1]}, \mathbb{Q}(r-1))}{\mathrm{Gy}(H^{2r-4}(Y^{[2]}, \mathbb{Q}(r-2)))}.$$

Let $\xi \in z^r(X\backslash Y, 1)$ represent $\{\xi\} \in \mathrm{CH}^r(X\backslash Y, 1)$, with $\overline{\xi} \in z^r(X, 1)$. $\underset{\sim}{\partial \xi} = 0$ on $X\backslash Y \underset{\sim}{\Rightarrow} |\partial\overline{\xi}| \subset Y$. $\exists$ lift $\partial\overline{\xi}$ on $Y^{[1]}$, with $\big[\partial\overline{\xi}\big] \in (*)$, agreeing $\underset{\sim}{\text{with}}$ $[\xi] \in \Gamma\big(H^{2r-1}(X\backslash Y, \mathbb{Q}(r))\big)$. If $\big[\partial\overline{\xi}\big] \neq 0$, then have 'detected' ξ.

Assume $[\widetilde{\partial\overline{\xi}}] = 0$, hence looking at

$$AJ(\xi) \in \frac{J\big(Gr_W^{-1} H^{2r-2}(X\backslash Y, \mathbb{Q}(r))\big)}{\Xi_0}.$$

Note that $Gr_W^{-1} H^{2r-2}(X\backslash Y, \mathbb{Q}(r)) \simeq$ cohomology: $H^{2r-5}(Y^{[2]}, \mathbb{Q}(r-2)) \xrightarrow{\mathrm{Gy}} H^{2r-3}(Y^{[1]}, \mathbb{Q}(r-1)) \xrightarrow{\mathrm{Gy}} H^{2r-1}(X, \mathbb{Q}(r))$ hence the dual is the cohomology of

$$H^{2d-2r+1}(X, \mathbb{Q}(d-r)) \xrightarrow{\mathrm{Gy}^*} H^{2d-2r+1}(Y^{[1]}, \mathbb{Q}(d-r)) \xrightarrow{\mathrm{Gy}^*} H^{2d-2r+1}(Y^{[2]}, \mathbb{Q}(d-r))$$

Apply F^{d-r+1} after $\otimes\mathbb{C}$. For $\omega \in F^{d-r+1}H^{2d-2r+1}(Y^{[1]}, \mathbb{C})$, $\mathrm{Gy}^*(\omega) = d\eta$ on $Y^{[2]}$, (η having Hodge type F^{d-r+1}), we compute $AJ(\xi)(\omega)$.

Thus we can assume given $[\gamma] \in H^{r-2,r-2}(Y^{[2]}, \mathbb{Q}(r-2))$, s.t. $\big[\widetilde{\partial\overline{\xi}}\big] = 2\pi\mathrm{iGy}[\gamma] \in \underset{\sim}{H}^{2r-2}(Y^{[1]}, \mathbb{Q}(r-1))$. In this case $\widetilde{\partial\overline{\xi}} - 2\pi\mathrm{iGy}(\gamma) \sim_{\mathrm{hom}} 0$ on $Y^{[1]}$, and therefore

$$\widetilde{\partial\overline{\xi}} - 2\pi\mathrm{iGy}(\gamma) = \partial\zeta,$$

bounds an integral (real) chain ζ. One shows that

$$AJ(\xi)(\omega) = \int_{\zeta} \omega + 2\pi\mathrm{i} \int_{\gamma} \eta.$$

Note that if γ is algebraic, then by Hodge type of η, we have

$$\int_{\gamma} \eta = 0.$$

This AJ map calculation is well defined modulo "periods".

Note that $\Xi_0 \simeq$ Image of $\big(H^{r-2,r-2}_{\ker \mathrm{Gy}}(Y^{[2]}, \mathbb{Q}(r-2)) \to J(\cdots)\big)$.

If $\widetilde{\partial\overline{\xi}}$ determines a nonzero value in

$$\frac{J\left(\frac{\ker \mathrm{Gy}: H^{2r-3}(Y^{[1]},\mathbb{Q}(r-1))\to H^{2r-1}(X,\mathbb{Q}(r))}{\mathrm{Gy}(H^{2r-5}(Y^{[2]},\mathbb{Q}(r-2)))}\right)}{\Xi_0},$$

then again we have detected ξ. But suppose this value is zero. Then there is still no guarantee that ξ belongs to the lower weight filtration. However, if we assume strict compatibility of the weight filtrations for both the Chow groups and absolute Hodge cohomology, then we can now attempt to detect ξ cohomologically on the lowest weight $W_{-1}\mathrm{CH}^r(X\backslash Y, 1)$.

Specifically we can assume (after possibly modifying $\overline{\xi}$ by a cycle supported on Y) that $\partial\overline{\xi} = 0$ on X. Then $\xi \in W_{-1}\mathrm{CH}^r(X\backslash Y,1) = \underline{\mathrm{CH}}^r(X\backslash Y,1)$, and the formula, which again will involve additional "periods" Ξ_{-1}:

$$\underline{\mathrm{CH}}^r(X\backslash Y,1) \to \frac{J\left(\frac{H^{2r-2}(X,\mathbb{Q}(r))}{H_Y^{2r-2}(X,\mathbb{Q}(r))}\right)}{\Xi_{-1}},$$

and which now involves the pure HS

$$\frac{H^{2r-2}(X,\mathbb{Q}(r))}{H_Y^{2r-2}(X,\mathbb{Q}(r))} \simeq$$

$$\ker\Big\{H^{2d-2r+2}(X,\mathbb{Q}(d-r))$$
$$\to H^{2d-2r+2}(Y,\mathbb{Q}(d-r))\Big\}^\vee,$$

is given by the KLM formula.

7. Arithmetic Normal Functions

Consider a smooth projective $X/\mathbb{C}$, and $\xi \in \mathrm{CH}^r(X, m; \mathbb{Q})$. One can spread out X and ξ in the form:

$$\rho : \mathcal{X} \to \mathcal{S},$$

where ρ is a smooth and morphism of smooth quasiprojective varieties over $\overline{\mathbb{Q}}$, $\tilde{\xi} \in \mathrm{CH}^r(\mathcal{X}/\overline{\mathbb{Q}}, m; \mathbb{Q})$, where if η is the generic point of $\mathcal{S}/\overline{\mathbb{Q}}$, then wrt a suitable embedding $\overline{\mathbb{Q}}(\eta) \hookrightarrow \mathbb{C}$, $X = \mathcal{X} \times \mathbb{C}$ and $\xi = \xi_\eta$. The resulting cohomological datum associated to ρ amounts to an arithmetic VHS.

$\exists$ a cycle map $\mathrm{CH}^r(\mathcal{X}, m; \mathbb{Q})$
$$\to \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(\mathcal{X})}\big(\mathbb{Q}_{\mathcal{X}}(0), \mathbb{Q}_{\mathcal{X}}(r)\big).$$
The Leray spectral sequence for ρ degenerates at $E_2^{p,q} :=$
$$\mathrm{Ext}^{p}_{\mathrm{MHM}(\mathcal{S})}\big((\mathbb{Q}_{\mathcal{S}}(0), R^q\rho_*\mathbb{Q}_{\mathcal{X}}(r)\big),$$
$$(p+q = 2r-m)$$
$$\Rightarrow \mathrm{Ext}^{2r-m}_{\mathrm{MHM}(\mathcal{X})}\big(\mathbb{Q}_{\mathcal{X}}(0), \mathbb{Q}_{\mathcal{X}}(r)\big).$$
Further, the Leray spectral sequence associated to $\mathcal{S}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$, together with $\mathrm{MHM}(\mathrm{Spec}(\mathbb{C})) = \mathrm{MHS}$, leads us to the s.e.s.:
$$0 \to J\big(H^{p-1}(\mathcal{S}, R^q\rho_*\mathbb{Q}(r))\big) \to$$
$$\mathrm{Ext}^{p}_{\mathrm{MHM}(\mathcal{S})}\big((\mathbb{Q}_{\mathcal{S}}(0), R^q\rho_*\mathbb{Q}_{\mathcal{X}}(r)\big) \to$$
$$\Gamma\big(H^{\nu}(\mathcal{S}, R^q\rho_*\mathbb{Q}(r))\big) \to 0.$$

If we set $p = \nu$, $\Rightarrow q = 2r - m - \nu$, we can rewrite the aforementioned s.e.s. in the form: $\underline{E}_{\infty}^{\nu,2r-m-\nu}(\rho) \hookrightarrow E_{\infty}^{\nu,2r-m-\nu}(\rho) \twoheadrightarrow \underline{\underline{E}}_{\infty}^{\nu,2r-m-\nu}(\rho)$.
$\exists$ induced $\{\mathcal{F}^{\nu}\mathrm{CH}^{r}(\mathcal{X}, m; \mathbb{Q})\}_{\nu\geq 0}$ s.t. $Gr_{\mathcal{F}}^{\nu}\mathrm{CH}^{r}(\mathcal{X}, m; \mathbb{Q}) \hookrightarrow E_{\infty}^{\nu,2r-m-\nu}(\rho)$.
A limit process leads to a candidate B-B filtration

$$\{F^{\nu}\mathrm{CH}^{r}(X/\mathbb{C}, m; \mathbb{Q})\}_{\nu\geq 0},$$

which in fact stabilizes $F^{r+1} = F^{r+2} = \cdots$ (and is conjecturally zero). One approach to attempt to describe this filtration explicitly is in terms of arithmetic normal functions.

Assume $\mathcal{S}$ is affine, $V \subset \mathcal{S}(\mathbb{C})$ a smooth closed affine, $\dim V = \nu - 1$.

$$\begin{array}{ccc} \mathcal{X}_V & \hookrightarrow & \mathcal{X} \\ \rho_V \downarrow & & \downarrow \rho \\ V & \hookrightarrow & \mathcal{S} \end{array}$$

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \underline{E}_\infty^{\nu,2r-m-\nu}(\rho) & \to & \underline{E}_\infty^{\nu,2r-m-\nu}(\rho_V) \\ \downarrow & & \downarrow \\ E_\infty^{\nu,2r-m-\nu}(\rho) & \to & E_\infty^{\nu,2r-m-\nu}(\rho) \\ \downarrow & & \downarrow \\ \underline{E}_\infty^{\nu,2r-m-\nu}(\rho) & \xrightarrow{0} & \underline{E}_\infty^{\nu,2r-m-\nu}(\rho_V) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

By varying $V \in \mathcal{S}(\mathbb{C})$, a class $\xi \in \mathcal{F}^\nu \mathrm{CH}^r(\mathcal{X}, m; \mathbb{Q})$ induces a normal function, which we call an arithmetic normal function.

Questions. (i) Can one characterize the B-B filtration in terms of arithmetic normal functions?

(ii) By choosing V sufficiently general, can one characterize the B-B filtration in terms of the corresponding Abel-Jacobi map?