On Hilbert schemes of scrolls of genus g

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Algebraic Geometry, D-modules, Foliations and their interactions

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General problem

a. Given $S \subset \mathbb{P}^r$ smooth surface of degree d, sectional genus g, etc.. study properties of the Hilbert scheme Hilb(d, g, r) parametrizing such S's?

b. Existence of "particular" curves on S can influence the behaviour of the component(s) of Hilb(d, g, r) to which S belong?

c. Given S sufficiently general in a component of Hilb(d, g, r), what kind of "limits" S admits (**embedded degenerations**)?

d. Conversely, given a configuration $X = \bigcup_i V_i \subset \mathbb{P}^r$, is it smoothable to an element of Hilb(d, g, r)?

Namely, $\exists \quad \mathcal{X} \to \Delta$ s.t. $\mathcal{X}_0 = X$ and $\mathcal{X}_t = S$, for $t \neq 0$ and $[S]$ general in a component of Hilb(d, g, r)?

e. If $\mathcal{X} \to \Delta$ in **d** actually exists and if we know the combinatorial data of the configuration $\mathcal{X}_0 = X$:

(i) what kind of properties can we deduce for $S = \mathcal{X}_t$, $t \neq 0$?

(ii) what kind of properties can we deduce for Hilb(d, g, r) from the fact that $[X = \mathcal{X}_0]$ is a Hilbert point?

(iii) applications to other parameter spaces of some other "related" geometric objects?

Motivations and inspirations

Classical papers:

[C. Segre]

• Recherches générales sur les courbes et les surfaces réglées algébriques, *Math. Ann.* **34** (1889), 1–25).

[Severi]

• Sulla classificazione delle rigate algebriche. *Rend. Mat. e Appl.*, (5) **2**, (1941). 1–32.

[Zappa]

• Caratterizzazione delle curve di diramazione delle rigate e spezzamento di queste in sistemi di piani, *Rend. Sem. Mat. Univ. Padova*, **13** (1942), 41-56

• Sulla degenerazione delle superficie algebriche in sistemi di piani distinti, con applicazioni allo studio delle rigate, *Atti R. Accad. d'Italia*, **13** (2) (1943), 989-1021

More recent papers:

[Ghione]

• Quelques résultats de Corrado Segre sur les surfaces réglés, *Math. Ann.* **255** (1981), 77–95.

• Un problème du type Brill-Noether pour les fibrés vectoriels, Lecture Notes in Math., **997**, 197–209, Springer, Berlin, 1983.

[Oxbury]

• Varieties of maximal line bundles, *Math. Proc. Camb. Phil. Soc.* **129** (2000), 9–18.

[Fuentes-Garcia, Pedreira]

• Canonical geometrically ruled surfaces, *Math. Nachr.*, **278** (2005), no. 3, 240–257.

• The general special scroll of genus g in $\mathbb{P}^N$. Special scrolls in $\mathbb{P}^3$, *math.AG 0609548* (2006), pp. 13.

Main subject of this talk

Given $d > 0$ and $g \geq 0$ integers, we will give some answers to the previous questions in the case of **scrolls** of degree d, genus g, "sufficiently" general.

Our approach

• [C, C, –, M] "Degenerations of scrolls to union of planes", *Rend. Lincei Mat. Appl.*, **17** (2006), 95-123.

• [C, C, –, M] "Non special scrolls with general moduli", *Rend. Circ. Mat. Palermo*, **57** (2008), 1-31.

• [C, C, –, M] "Brill-Noether theory and non-special scrolls", to appear in *Geom. Dedicata* (2008), pp. 16.

• [C, C, –, M] "Special scrolls with general moduli", *Sub. preprint* (2008).

Notation and general assumptions

From now on:

(1) C smooth, irreducible projective curve of genus $g \geq 0$,

$F \xrightarrow{\rho} C$ **geometrically ruled** surface, i.e.

$$F = \mathbb{P}(\mathcal{F}),$$

$\mathcal{F}$ rank-two vector bundle (equiv. loc. free sheaf) on C

Assume further:

- $\deg(\mathcal{F}) := \deg(\det(\mathcal{F})) = d$;
- $h^0(C, \mathcal{F}) = r + 1$, with $r \geq 3$;
- $|\mathcal{O}_F(1)|$ is b.p.f. and the induced morphism $\Phi : F \to \mathbb{P}^r$ is birational to its image.

Then

$$\Phi(F) := S \subset \mathbb{P}^r$$

is a **scroll** of degree d (sectional) genus g (determined by $(\mathcal{F}, C)$).

Remark: S smooth $\Leftrightarrow$ $\mathcal{F}$ v.a.; otherwise F is its minimal desingularization.

For any $x \in C$, $f_x := \rho^{-1}(x) \cong \mathbb{P}^1$ and $l_x := \Phi(f_x)$ is a line of the **ruling** of S.

(2) For $A \in \text{Pic}(C)$, any $B_1 \in |\mathcal{O}_F(1) \otimes \rho^*(A)| \neq \emptyset$ is a **unisecant curve** of F.

An irreducible unisecant B is called a **section** of F.

$$\updownarrow \qquad 1:1\,\text{correspondence}$$

$$0 \to N \quad \to \mathcal{F} \to \quad L = L_B \to 0.$$

If $B = B_L \subset F$ section, $L \in Pic(C)$, let $\Gamma := \Phi(B) \subset S$.

$\Phi|_B$ birational $\Rightarrow \Gamma$ **section** (or **directrix**) of S.

$\Phi|_B$ $n:1 \Rightarrow \Gamma$ n-**directrix** of S.

(3) Riemann-Roch

$$r + 1 := h^0(\mathcal{O}_F(1)) = h^0(\mathcal{F}) = d - 2g + 2 + h^1$$

$$h^1 := h^1(\mathcal{O}_F(1)) = h^1(\mathcal{F}) = \textbf{ speciality } \text{of the scroll.}$$

S **special** scroll if $h^1 > 0$, **non-special** otherwise.

Since $r \geq 3 \Rightarrow d \geq 2g + 2 - h^1$.

From now on

$$d \geq 2g + 2$$

(necessary bound for linearly normal, non-special scrolls).

Bounds on speciality: Riemann-Roch thm. for $\mathcal{F}$ on C:

$$0 \leq h^1 \leq g$$

$h^1 = g$ cones [Segre - Ghione],

$h^1 = 0$ non-special scrolls.

Any intermediate value $1 \leq h^1 \leq g-1$ can be realized.

Example. Let $g \geq 3$, $d \geq 4g-1$, $1 \leq h^1 \leq g-1$.

$|L|$ b.p.f. with $h^1(L) = h^1$.

N general l.b. of degree $d - deg(L)$.

$deg(L) \leq 2g-2$ and $d \geq 4g-1 \Rightarrow deg(N) \geq 2g+1$ i.e. $|N|$ very ample.

Let $\mathcal{F} = L \oplus N$; then $\mathcal{O}_F(1)$ b.p.f. and $h^1(\mathcal{O}_F(1)) = h^1$. ♣

Remark For large values of h^1, $\mathcal{O}_F(1)$ in general not v.a.

- $h^1 = g-1 \Rightarrow |L| = \mathfrak{g}^1_2 \Rightarrow S$ has a linear 2-directrix
- $h^1 = g-2 \Rightarrow |L| = \mathfrak{g}^1_3$ or $|L| = 2\mathfrak{g}^1_2$ or $|L| = \mathfrak{g}^2_4$. S is smooth only if $|L| = \mathfrak{g}^2_4$ with $g = 3$.

Remark For any section Γ of S,

$$h^1(\mathcal{O}_\Gamma(1)) := \text{speciality of } \Gamma \ \leq h^1.$$

<u>Hilbert schemes of l.n. non-special scrolls</u>

Rational case:

Proposition 1 (Classical). *Let $d \geq 2$ and $r = d + 1$.*

The Hilbert scheme $\mathcal{H}_{d,0}$ parametrizing rational normal scrolls of degree d in $\mathbb{P}^r$ is irreducible, generically smooth.

The general point of $\mathcal{H}_{d,0}$ represents a smooth, <u>balanced</u> scroll.

$dim(\mathcal{H}_{d,0}) = (r + 1)^2 - 7$.

Proof: $S \subset \mathbb{P}^r$ any smooth, rational normal scroll. Consider

$$0 \to T_S \to T_{\mathbb{P}^r}|_S \to \mathcal{N}_{S/\mathbb{P}^r} \to 0.$$

Euler sequence restricted to S + S is a scroll

$$\Downarrow$$

$$h^1(T_{\mathbb{P}^r}|_S) = h^1(\mathcal{N}_{S/\mathbb{P}^r}) = 0$$

so

$$h^0(\mathcal{N}_{S/\mathbb{P}^r}) = h^0(T_{\mathbb{P}^r}|_S) - \chi(T_S) = (r + 1)^2 - 1 - 6.$$

$h^1(\mathcal{N}_{S/\mathbb{P}^r}) = 0 \Rightarrow [S]$ smooth point of the Hilbert scheme of such scrolls. Therefore

$$h^0(\mathcal{N}_{S/\mathbb{P}^r}) = \dim_{[S]}(\mathcal{H}_{d,0}) = \dim(T_{[S]}(\mathcal{H}_{d,0})).$$

Finally, one uses the well-known fact: $S_{a,b}$ degenerates to $S_{h,f}$ $\Leftrightarrow a + b = h + f$ and $|a - b| < |h - f|$ ♣

Remark In particular, for $[S] \in \mathcal{H}_{d,0}$ general, there are ∞^6 projectivities of $\mathbb{P}^r$ fixing S.

Irregular case, i.e. $g \geq 1$:

Theorem 1 [C, C, –, M] *Let $r = d - 2g + 1$, where*

- $d \geq 5$, if $g = 1$

- $d \geq 2g + 4$, if $g \geq 2$.

Then, there exists a unique irreducible component $\mathcal{H}_{d,g}$ of Hilbert scheme of scrolls of degree d and genus g in $\mathbb{P}^r$, whose general point $[S] \in \mathcal{H}_{d,g}$ is a smooth, linearly normal scroll $S \subset \mathbb{P}^r$ (equiv. $h^1(\mathcal{O}_S(1)) = 0$).

Moreover,

(i) $\mathcal{H}_{d,g}$ contains suitable reducible surfaces $[T]$ as smooth points of $\mathcal{H}_{d,g}$;

(ii) $\mathcal{H}_{d,g}$ generically smooth,

$$dim(\mathcal{H}_{d,g}) = h^0(S, \mathcal{N}_{S/\mathbb{P}^r}) = (r+1)^2 + 7(g-1)$$

and

$$h^1(S, \mathcal{N}_{S/\mathbb{P}^r}) = h^2(S, \mathcal{N}_{S/\mathbb{P}^r}) = 0.$$

(iii) $\mathcal{H}_{d,g}$ dominates $\mathcal{M}_g$, i.e. S is with **general moduli**.

Remark. No gaps w.r.t. the initial condition $d \geq 2g + 2$. In other words, the bounds on d in Theorem 1 are sharp.

Indeed, no smooth, linearly normal scrolls in $\mathbb{P}^r$ if either $d = 2g + 2$ and $g \geq 1$ or $d = 2g + 3$ and $g \geq 2$.

Proof: Induction on g + degeneration techniques.

Step 1: $g = 0$, ok from Proposition 1.

Step 2: Let $g \geq 1$. Construct suitable reducible (precisely **Zappatic**) surfaces in $\mathcal{H}_{d,g}$.

Let $[\widetilde{S}] \in \mathcal{H}_{d-2,g-1}$ general $\Rightarrow \widetilde{S} \subset \mathbb{P}^r$, $r = (d-2)-2(g-1)+1 = d-2g+1$.

Let l_1 and l_2 general lines of the ruling of $\widetilde{S}$.

$\langle l_1, l_2 \rangle = \Lambda \cong \mathbb{P}^3$. Let $Q \subset \Lambda$ general quadric through l_1 and l_2

Fact Q is smooth and $\widetilde{S} \cap Q = l_1 \cup l_2$ transverse.

Let

$$T := \widetilde{S} \cup Q$$

reducible surface with g.n.c. $\Rightarrow$

$$\mathrm{Sing}(T) = l_1 \cup l_2 := R.$$

Step 3: **Some invariants of T**

Using e.g.

• [C, C, –, M] "On the geometric genus of reducible surfaces and degenerations of surfaces to unions of planes", Proc. Fano Conference (2004), 277 - 312.

$$g(T) = 0, \quad \chi(\mathcal{O}_T) = 1 - g, \quad p_\omega(T) := h^0(\omega_T) = 0.$$

Step 4: **Some cohomological property of T.**

(a) From

$$0 \to \mathcal{O}_T(1) \to \mathcal{O}_{\widetilde{S}}(1) \oplus \mathcal{O}_Q(1) \to \mathcal{O}_R(1) \to 0$$

one has

$$h^1(\mathcal{O}_T(1)) = 0$$

(b) $\mathcal{N}_T$ and $\mathcal{T}_T$ be the normal and "tangent" sheaf of T in $\mathbb{P}^r$. Then

$$(*) \quad 0 \to \mathcal{T}_T \to \mathcal{T}_{\mathbb{P}^r}|_T \xrightarrow{\tau} \mathcal{N}_T \to T^1 := Coker(\tau) \to 0.$$

[Friedman] T with g.n.c. $\Rightarrow T^1 \cong \mathcal{N}_{R/\widetilde{S}} \otimes \mathcal{N}_{R/Q} \cong \mathcal{O}_R$.

(c) From $T = \widetilde{S} \cup Q$ and $h^1(\mathcal{N}_{\widetilde{S}}) = 0$ by induction

$$\Downarrow$$

$$h^1(\mathcal{N}_T) = h^2(\mathcal{N}_T) = 0 \quad h^0(\mathcal{N}_T) = \chi(\mathcal{N}_T) = (r+1)^2 + 7(g-1)$$

and

$$H^0(\mathcal{N}_T) \stackrel{\alpha}{\twoheadrightarrow} H^0(T^1).$$

<u>Step 5</u>: $h^1(\mathcal{N}_T) = 0 \Rightarrow [T]$ smooth point of the Hilbert scheme of surfaces of degree d and genus g in $\mathbb{P}^r$.

$$\Downarrow$$

$[T]$ belongs to a unique component $\mathcal{H}_{d,g}$ of this Hilbert scheme, of dimension $\chi(\mathcal{N}_T) = h^0(\mathcal{N}_T)$.

α surjective $\Rightarrow$ a general tangent vector to $\mathcal{H}_{d,g}$ at $[T]$ represents an infinitesimal embedded deformation of T which smooths $\Gamma = Sing(T)$.

$$\Downarrow$$

The general point of $\mathcal{H}_{d,g}$ represents a smooth, irreducible surface S degenerating to T.

$$\Downarrow$$

From Step 3 and e.g.

- [C, C, –, M] "On the genus of reducible surfaces and degenerations of surfaces", <u>*Annales Inst. Fourier*</u>, **57** (2007), no. 2, 491-516 (or [Clemens-Schmid])

$$\Downarrow$$

S is necessarily ruled.

From Step 4 - (a) and semi-continuity, $h^1(\mathcal{O}_S(1)) = 0$, i.e. $S \subset \mathbb{P}^r$ is linearly normal.

Step 6: Using

• [C, C, –, M] "On the K^2 of degenerations of surfaces and the multiple point formula", *Annals of Mathematics*, **165** (2007), no. 2, 335-365.

$$S \text{ degenerates to } T \ \Rightarrow \ K_S^2 = 8(1-g)$$

then S is necessarily geometrically ruled.

Adjunction theory implies S is a scroll: otherwise, $0 < d \leq 4(g-1) + K_S^2$, contradicting $K_S^2 = 8(1-g)$.

Step 7: From

$$0 \to \mathcal{T}_S \to \mathcal{T}_{\mathbb{P}^r|S} \to \mathcal{N}_{S/\mathbb{P}^r} \to 0,$$

we get

$$H^0(\mathcal{N}_{S/\mathbb{P}^r}) \twoheadrightarrow H^1(\mathcal{T}_S),$$

(Euler sequence gives $h^1(\mathcal{T}_{\mathbb{P}^r|S}) = 0$).

From $S \xrightarrow{\rho} C$ one has $H^1(\mathcal{T}_S) \twoheadrightarrow H^1(\mathcal{T}_C) \ \Rightarrow$

$$H^0(\mathcal{N}_{S/\mathbb{P}^r}) \twoheadrightarrow H^1(\mathcal{T}_C),$$

i.e. $\mathcal{H}_{d,g}$ dominates $\mathcal{M}_g$.

Step 8: Uniqueness of the component $\mathcal{H}_{d,g}$.

It follows from the previous steps and the **Classical result**:

For any smooth scroll S of degree d, there exists $\varphi : Y := C \times \mathbb{P}^1 \dashrightarrow S$ birational which is the composition of d elementary transformations at d distinct points $\Theta := \{y_1, \ldots, y_d\} \subset Y$ lying on d distinct $\mathbb{P}^1$-fibres of Y. ♣

Remarks. (1) Some previous results partially related: [Arrondo, Pedreira, Sols; 1988] projections of scrolls into $\mathbb{P}^3$ and then study curves in $\mathbb{G}(1,3)$.

(2) With a similar approach, one can reprove **Proposition 1** on $\mathcal{H}_{d,0}$.

(3) From the proof of **Theorem 1**

$$\Downarrow$$

$[S] \in \mathcal{H}_{d,g}$ general degenerates to a reducible surface like T, which is a **Zappatic surface** with **g.n.c.** singularities

(4) Pushing further degeneration techniques and using once again the results in:

• [C, C, −, M] "On the K^2 of degenerations of surfaces and the multiple point formula", _Annals of Mathematics_, **165** (2007), no. 2, 335-365.

• [C, C, −, M] "On the genus of reducible surfaces and degenerations of surfaces", _Annales Inst. Fourier_, **57** (2007), no. 2, 491-516,

we prove that $\mathcal{H}_{d,g}$ contains reducible surfaces which are **union of planes**, with only double lines and further singular points (the so called R_3- and S_4-points, cf. [C, C, −, M] "On the K^2 of degenerations of surfaces and the multiple point formula", _Annals of Mathematics_, **165** (2007), no. 2, 335-365.)

These are examples of **planar Zappatic** surfaces (studied in the above 2 papers.)

Precisely, we prove:

Proposition 2 [C,C,–,M] *$\mathcal{H}_{d,g}$ contains planar Zappatic surfaces $[X_{d,g}]$ such that*

- *if $g = 0$, $X_{d,0}$ has $d-1$ double lines and $d-2$ R_3-points*

- *if $g \geq 1$, $X_{d,g}$ has $d+2g+2$ double lines, $d-2g+2$ R_3-points and $2g-2$ S_4-points.*

In other words, for $g \geq 0$, the general $[S] \in \mathcal{H}_{d,g}$ degenerates to planar Zappatic surfaces $X_{d,g}$ as above.

This in particular answers (improving the expected bound) to Zappa's original questions (1940-50):

Zappa's questions: *If $S \subset \mathbb{P}^r$ is a scroll of genus g, degree $d \geq 3g+2$ and "sufficiently general"*

- *can S degenerate to a union of planes?*

- *If yes, is it possible to have in the limit surface only double lines and E_3-, R_3- and S_4-points?*

Remark. Zappa (1940-50) realized that **g.n.c.** singularities are not sufficient to study embedded degenerations of surfaces, even with general moduli.

Connections with vector bundles

For any $g \geq 1$, for any smooth C of genus g and for any d

$$U_C(d)$$

the moduli space of degree d, rank-two **semistable** vector bundles on C.

Theorem 2 [C, C, $-$, M] *Let $g \geq 1$ and num. assumptions as in* **Thm1**.

Let $[S] \in \mathcal{H}_{d,g}$ be general. Then:

(i) S is determined by $(\mathcal{F}, C)$, where $[C] \in \mathcal{M}_g$ general and $[\mathcal{F}] \in U_C(d)$ general.

(ii) In particular, if $g \geq 2$, $\mathcal{F}$ is **stable** *and, if $G_S :=$ sgr. of projectivities of $\mathbb{P}^r$ fixing S, then $G_S = \{Id\}$.*

Sketch Proof.

Step 1: Small Lemma: *$d \geq 2g$ and $[\mathcal{F}] \in U_C(d)$ general $\Rightarrow$* $h^1(C, \mathcal{F}) = 0$.

Step 2: From Step 1, $[\mathcal{F}] \in U_C(d)$ general $\Rightarrow h^0(\mathcal{F}) = d - 2g + 2 = r + 1$.

We can costruct a morphism

$$\begin{array}{llll} \Psi : & \mathcal{U} & \longrightarrow & Hilb(d, g, r) \\ & (C, \mathcal{F}, \sigma) & \longrightarrow & \sigma(S) \end{array}$$

where

$$[C] \in \mathcal{M}_g, \ \ [\mathcal{F}] \in U_C(d), \ \ \sigma \in PGL(r+1) \ \text{ and } \ S = \Phi(F).$$

S is smooth for $\mathcal{F}$ general.

Step 3: $\mathcal{U}$ irriducible and, from **Theorem 1**, $\Psi(\mathcal{U}) \subseteq \mathcal{H}_{d,g}$.

Semistability is an open condition $\Rightarrow \Psi$ dominant on $\mathcal{H}_{d,g}$.

Step 4: any possible element $Id \neq \tau \in G_S$ induces a non-trivial automorphism of F, contradicting $\mathcal{F}$ stable (so simple) and C with general moduli. ♣

Remarks.

(1) For $g \geq 2$, $\mathcal{U}$ depends on the following parameters:

- $3g-3$ for C;
- $\dim(U_C(d)) = 4g-3$ for $\mathcal{F}$;
- $(r+1)^2 - 1 = \dim(PGL(r+1, \mathbb{C}))$, i.e.

$$\dim(\mathcal{U}) = \dim(\mathcal{H}_{d,g}) = 7g - 7 + (r+1)^2.$$

This gives a **parametric representation** of $\mathcal{H}_{d,g}$

From **Theorem 2** (ii), more precisely Ψ is **birational**.

(2) For $g = 1$, statement (ii) and behaviour of Ψ are more involved (depends on the parity of d).

(3) **Attention**: we are not saying that <u>all</u> smooth scrolls parametrized by $\mathcal{H}_{d,g}$ comes from semistable bundles.

Example. $\mathcal{E}$ any **unstable** bundle on C of degree e. Let A be ample of degree a and consider

$$\mathcal{F} := \mathcal{E} \otimes A^{\otimes k}, \;\; k >> 0.$$

So $\mathcal{F}$ v.a., $h^1(\mathcal{F}) = 0$, thus $(\mathcal{F}, C)$ determines a scroll $[S_{\mathcal{F}}] \in \mathcal{H}_{e+2ka,g}$ coming from $\mathcal{F}$ unstable.

Such scrolls fill up closed sub-loci of $\mathcal{H}_{e+2ka,g}$. ♣

(4) Some previous results partially related: [Pedreira; 1988].

Applications and consequences

Using degeneration techniques as in **Theorem 1** and construction as in **Theorem 2**, we get also the following results:

(1) S is a **general ruled surface** [Ghione, 1981], i.e.

$$\mathrm{Div}_S^{1,m}$$

has the expected dimension e; it is smooth; it is irriducible when $e > 0$.

$$\Downarrow$$

effective existence results of general ruled surfaces (specifying bounds on d), whereas Ghione's existence results were only asymptotic.

(2) Sections of **minimal degree** on S: compute their degree and the dimension of the family.

(3) Enumerative results on $\mathrm{Div}_S^{1,m}$: cardinality (when $e = 0$), index, genus computation, monodromy, etc.....

(4) $M_n(\mathcal{F})$:= Scheme parametrizing sub-line bundles of degree n of $\mathcal{F}$.

Then:

$$M_n(\mathcal{F}) \cong \mathrm{Div}_S^{1,m}$$

where $m + n = d$.

$$\Downarrow$$

• alternative proofs (via proj. geom.) of some results on sub-line bundles of **maximal degree** $\overline{n}$, e.g. [Maruyama], [Lange-Narashiman], [Oxbury].

• affirmative answer to a **Conjecture** of [Oxbury] (2005) in the rank 2 case: connectedness of $M_{\overline{n}}(\mathcal{F})$, on any C, when dim > 0.

(7) Study $W_n(\mathcal{F}) := Im(M_n(\mathcal{F})) \subset \mathrm{Pic}^n(C)$ and some questions on the **Brill-Noether theory** of the line-bundles in $W_n(\mathcal{F})$.

Consequences: we can compute $\dim(|\mathcal{O}_S(\Gamma)|)$, for $[\Gamma] \in \mathrm{Div}_S^{1,m}$ general and for any admissible m.

(8) We show that any irreducible component of $W_n^1(\mathcal{F})$ has the expected dimension.

First possible questions

(1) Is $\mathcal{H}_{d,g}$ the only component of the Hilbert scheme whose general point parametrizes a smooth scroll of degree d and genus g in $\mathbb{P}^r$?

(2) If another such component $\mathcal{Z}$ actually exists,

- $\dim(\mathcal{Z}) = ?$

- $\mathcal{Z}$ generically smooth ?

- general point of $\mathcal{Z}$?

- rank-two vector bundle determining the general point of $\mathcal{Z}$?

- image of $\mathcal{Z}$ via the natural map $\mathcal{Z} \to \mathcal{M}_g$?

All these questions naturally lead to the study of **special scrolls**

Hilbert schemes of l.n. special scrolls

From now on

$$S \subset \mathbb{P}^r$$

smooth scroll of genus g, degree d and speciality h^1, with

$$0 < h^1 < g \quad \text{and} \quad r = d - 2g + 1 + h^1.$$

Main Point: existence of a special section on S.

Proposition 3 [C. Segre, 1889. Revisited C, C, –, M, 2008] *Let $g \geq 3$ and $d \geq 4g - 2$. Let $S \subset \mathbb{P}^r$ be a smooth, linearly normal, special scroll of degree d, genus g and speciality h^1. Then:*

(i) S contains a unique, special section $\Gamma \subset \mathbb{P}^h$ of degree m such that $m - h = g - h^1$. Furthermore:

- *Γ is linearly normally embedded in $\mathbb{P}^h$, i.e. $H^0(\mathcal{F}) \twoheadrightarrow H^0(L_\Gamma)$.*
- *$h^1(\Gamma, \mathcal{O}_\Gamma(H)) = h^1$*
- *Γ curve (different from a line) of minimal degree*
- *Γ unique section with non-positive self-intersection ($\Gamma^2 < 0$).*

(ii) If moreover S has general moduli, then

- *either $g \geq 4h^1$, $h \geq 3$, or*
- *$g = 3$, $h^1 = 1$, $h = 2$.*

Remark (1) Existence results of special sections, with no assumptions on d and g, are given (via completely different techniques) by [Fuentes-Garcia, Pedreira, 2005-2006].

On the other hand, differently from Segre's approach, no information about its uniqueness and its speciality.

(2) Conditions in (ii) follow from Brill-Noether theory and smoothness. Namely

$$\rho(g, h, m) := g - (h + 1)h^1 \geq 0.$$

Corollary 1 *Suppose S determined by a pair $(\mathcal{F}, C)$. Let*

$$(*) \quad 0 \to N \to \mathcal{F} \to L \to 0$$

where L corresponds to the special section Γ.

Then $\mathcal{F}$ is **unstable**. *If, moreover, $d \geq 6g-5$ then $\mathcal{F} = L \oplus N$.*

Proof $\Gamma^2 = 2m - d < 0 \quad \Rightarrow \quad \deg(N) = d - m > \mu(\mathcal{F}) = \frac{d}{2} \quad \Rightarrow \quad \mathcal{F}$ is unstable.

From $(*)$, $[\mathcal{F}] \in \mathrm{Ext}^1(L, N) \cong H^1(C, N \otimes L^\vee)$.

L special and $d \geq 6g-5 \Rightarrow \deg(N \otimes L^\vee) = d - 2m \geq 2g-1 \Rightarrow N \otimes L^\vee$ is non-special $\Rightarrow (*)$ splits. ♣

Lemma 1 *Assume* $\mathrm{Aut}(C) = \{Id\}$ *(in particular, when C has general moduli).*

If $G_S \subset \mathrm{PGL}(r+1, \mathbb{C})$ sub-group of projectivities of $\mathbb{P}^r$ fixing S, then $G_S \cong \mathrm{Aut}(S)$ and

$$\dim(G_S) = \begin{cases} h^0(N \otimes L^\vee) & \textit{if } \mathcal{F} \textit{ indecomposable} \\ h^0(N \otimes L^\vee) + 1 & \textit{if } \mathcal{F} \textit{ decomposable} \end{cases}$$

Proof We want to show the obvious inclusion $G_S \hookrightarrow \mathrm{Aut}(S)$ is an isomorphism.

Let $\sigma \in \mathrm{Aut}(S)$. By **Theorem 3**, $\sigma(\Gamma) = \Gamma$ and since $\mathrm{Aut}(C) = \{Id\}$, σ fixes Γ pointwise.

Now $H \sim \Gamma + \rho^*(N) \Rightarrow \sigma^*(H) = \sigma^*(\Gamma) + \sigma^*(\rho^*(N)) = \Gamma + \rho^*(N) \sim H \Rightarrow \sigma$ is induced by a projective trasformation.

The rest of the claim directly follows from [Maruyama, 1970]. ♣

Remark Different behaviour from $[S] \in \mathcal{H}_{d,g}$. Indeed, from **Theorem 2** we know that for S general non-special, $\mathcal{F}$ is stable and $G_S = \{Id\}$.

Components with general moduli

Let

$$\text{Hilb}(d, g, h^1)$$

the open subset of the Hilbert scheme parametrizing smooth scrolls $S \subset \mathbb{P}^r$ of genus g, degree d and speciality h^1.

Theorem 3 [C, C, –, M] *Let $g \geq 3$ and $d \geq 4g-2$. Let m be any integer such that either*

- $m = 4$, if $g = 3$, $h^1 = 1$, *or*
- $g + 3 - h^1 \leq m \leq \overline{m} := \lfloor \frac{g}{h^1} - 1 \rfloor + g - h^1$, otherwise.

(i) If $h^1 = 1$, Hilb$(d, g, 1)$ *consists of a unique component $\mathcal{H}^{2g-2}_{d,g,1}$ whose general point parametrizes a smooth, linearly normal scroll $S \subset \mathbb{P}^r$ with a canonical curve as the unique special section.*

Furthermore,

(1) $dim(\mathcal{H}^{2g-2}_{d,g,1}) = 7(g-1) + r(r+1)$,

(2) $\mathcal{H}^{2g-2}_{d,g,1}$ *is generically smooth and dominates $\mathcal{M}_g$.*

Moreover, scrolls with $h^1 = 1$, containing a special section of degree $m < 2g-2$ fill up an irreducible subscheme of $\mathcal{H}^{2g-2}_{d,g,1}$ which also dominates $\mathcal{M}_g$ and whose codimension is $2g - 2 - m$.

(ii) If $h^1 \geq 2$ then, for any $g \geq 4h^1$, $d \geq 4g - 2$ and for any m as above, Hilb(d, g, h^1) *contains a unique component $\mathcal{H}^{m}_{d,g,h^1}$ whose general point parametrizes a smooth, linearly normal scroll $S \subset \mathbb{P}^r$, having general moduli whose special section Γ has degree m and speciality h^1.*

Furthermore,

(1) $dim(\mathcal{H}^{m}_{d,g,h^1}) = 7(g-1) + (r+1)(r+1-h^1) + (d-m-g+1)h^1 - (d-2m+g-1)$,

(2) $\mathcal{H}^{m}_{d,g,h^1}$ *is generically smooth.*

Remarks (1) Bounds on m: Since $d \geq 4g-2$, from **Proposition 3**, S contains a unique, special section Γ of speciality h^1.

Γ is the image of C via $|L| = \mathfrak{g}^h_m$.

In order to have C with general moduli, $\rho(g,h,m) = g-(h+1)h^1 \geq 0$

$$\Downarrow$$

$m \leq \overline{m} = \lfloor \frac{g}{h^1} - 1 \rfloor + g - h^1$.

On the other hand, S smooth $\Rightarrow h = m - g + h^1 \geq 2$.

If $h^1 \geq 2$ and the scroll has general moduli, then $h \geq 3$.

If $h^1 = 1$ and $h = 2$, then $m = 4$ and $g = 3$.

(2) Reducibility: $\text{Hilb}(d,g,h^1)$ is reducible as soon as $h^1 \geq 2$,

If moreover $h^1 \geq 3$, it is also not equidimensional.

The component of maximal dimension is

$$\mathcal{H}^{g+3-h^1}_{d,g,h^1}$$

whereas the component of minimal dimension is

$$\mathcal{H}^{\overline{m}}_{d,g,h^1}$$

with $\overline{m}$ as above.

(3) By **Corollary 1**, all smooth scrolls in $\mathcal{H}^m_{d,g,h^1}$, for any $h^1 \geq 1$, correspond to unstable bundles.

To prove **Theorem 3**, no degeneration techniques.

Steps of the Proof:

Step 1: For any m and h, we costruct a morphism

$$\begin{array}{rccl} \Psi_{h,m}: & \mathcal{U}_{h,m} & \longrightarrow & Hilb(d,g,h^1) \\ & (C,L,N,\xi,\sigma) & \longrightarrow & \sigma(S) \end{array}$$

where

$$[C] \in \mathcal{M}_g, \quad L \in W^h_m(C), \quad N \in \mathrm{Pic}^{d-m}(C), \quad \sigma \in PGL(r+1),$$

whereas

$$\xi = \begin{cases} 0 & \text{if } \mathrm{Ext}^1(L,N) = 0 \\ \in \mathbb{P}(\mathrm{Ext}^1(L,N)) & \text{if } \mathrm{Ext}^1(L,N) \neq 0 \end{cases},$$

$$\mathcal{F} = \begin{cases} L \oplus N & \text{if } \xi = 0 \\ \mathcal{F}_\xi & \text{corresponding to } \xi \end{cases}$$

and

$$S = \Phi(F).$$

$\mathcal{H}^m_{d,g,h^1}$ is defined as the closure of the image of $\Psi_{h,m}$.

Step 2: Given $[S] \in \mathcal{H}^m_{d,g,h^1}$ general, C, L and N are uniquely determined.

From Step 2, $\dim {\Psi_{h,m}}^{-1}([S]) = \dim(G_S)$.

Step 3: From **Lemma 1**, we know $\dim(G_S) \Rightarrow$ we know $\dim(\mathcal{H}^m_{d,g,h^1})$.

Step 4: Compute cohomology of $\mathcal{N}_{S/\mathbb{P}^r}$.

C is with general moduli and *Castelnuovo's Lemma* for surjectivity of multiplication maps of sections of suitable H^0's

$$\Downarrow$$

(i) $h^0(S, \mathcal{N}_{S/\mathbb{P}^r}) = 7(g-1) + (r+1)(r+1-h^1) + (d-m-g+1)h^1 - (d-2m+g-1)$;

(ii) $h^1(S, \mathcal{N}_{S/\mathbb{P}^r}) = h^1(d-m-g+1) - (d-2m+g-1)$;

(iii) $h^2(S, \mathcal{N}_{S/\mathbb{P}^r}) = 0$.

Step 5: By comparing $\dim(\mathcal{H}^m_{d,g,h^1})$ in Step 3 and Step 4 (i)

$$\Downarrow$$

$$\dim_{[S]}(\mathcal{H}^m_{d,g,h^1}) = h^0(S, \mathcal{N}_{S/\mathbb{P}^r})$$

$$\Downarrow$$

$$\mathcal{H}^m_{d,g,h^1} \text{ generically smooth.}$$

Step 6: $h^1 = 1$: suppose $L \neq \omega_C$. In particular, $g \geq 4$ and $m < 2g-2$.

$|\omega_C \otimes L^\vee| = \mathfrak{g}^0_{2g-2-m}$, i.e.

$$\omega_C \otimes L^\vee \cong \mathcal{O}_C(p_1 + \cdots + p_{2g-2-m}),$$

where p_j general points on C, $1 \leq j \leq 2g-2-m$, hence

$$L \cong \omega_C(-p_1 - \cdots - p_{2g-2-m}).$$

Fact *Any bundle $\mathcal{G}$ on C such that*

$$0 \to N \to \mathcal{G} \to \omega_C(-p_1 - \cdots - p_{2g-2-m}) \to 0$$

is a degeneration of a vector bundle $\mathcal{F}$ fitting in

$$0 \to N(p_1 + \cdots + p_{2g-2-m}) \to \mathcal{F} \to \omega_C \to 0.$$

$$\Downarrow$$

$\mathcal{H}^m_{d,g,1}$, with $m < 2g-2$, sits in the closure of $\mathcal{H}^{2g-2}_{d,g,h^1}$.

Step 7: $h^1 \geq 2$. From the dimension count in Step 3 and Step 4 (i)

$$\Downarrow$$

$\mathcal{H}^m_{d,g,h^1}$ generically smooth component of Hilb(d, g, h^1). ♣

Remarks (1) Different construction of $\mathcal{H}^{2g-2}_{d,g,1}$ is given by [Fuentes-Garcia, Pedreira, 2006]: they use internal projections from decomposable scrolls.

(2) With similar approach **Theorem 3** holds also for

$$\frac{7}{2}g - h^1 + 1 \leq d \leq 4g - 3 \text{ and } h^1 \geq 2.$$

• Existence of a special section Γ: use [Fuentes-Garcia, Pedreira, 2005-06] istead of [C. Segre];

• Γ is the unique special section, $\Gamma^2 < 0$ and it is the curve (different from a line) of minimal degree: use assumptions on d and some projective geometry arguments;

• computation of $h^i(S, \mathcal{N}_{S/\mathbb{P}^r})$: same approach as in **Theorem 3**; use surjectivity results of suitable multiplication maps of H^0's of [Green] and [Butler], instead of Castelnuovo's lemma.

• the related vector bundles are still **unstable**, as in the Segre's case $d \geq 4g - 2$.

Other components of the Hilbert scheme

Other components with **general moduli**.

Let $s = d - 2g + 1 + k$, with $0 \leq k < h^1$.

Consider the family $\mathcal{Y}^m_{k,h^1}$ whose general element is a general projection to $\mathbb{P}^s$ of the general scroll in $\mathcal{H}^m_{d,g,h^1}$.

Questions With assumptions as above:

- is $\mathcal{Y}^m_{k,h^1}$ contained in $\mathcal{H}^n_{d,g,k}$ for some n, if $k > 0$?
- is $\mathcal{Y}^m_{k,h^1}$ contained in $\mathcal{H}_{d,g}$, if $k = 0$?

Proposition 4 [CCFM] *In the above setting:*

(i) if $k > 0$, $\mathcal{Y}^m_{k,h^1}$ sits in an irreducible component of the Hilbert scheme different from $\mathcal{H}^n_{d,g,k}$, for any n;

(ii) if $h^1 > 1$, $\mathcal{Y}^m_{0,h^1}$ sits in an irreducible component of the Hilbert scheme different from $\mathcal{H}_{d,g}$, for any m;

(iii) $\mathcal{Y}^{2g-2}_{0,1}$ is a divisor inside $\mathcal{H}_{d,g}$, whose general point is a smooth point for $\mathcal{H}_{d,g}$. ♣

Remark In cases (i) and (ii), by construction, we find other components of Hilb(d, g, s) which always dominates $\mathcal{M}_g$.

Components with **special moduli**

$[C] \in \mathcal{M}_g \Rightarrow$ **gonality** of C

$$\gamma := \begin{cases} \frac{g+2}{2} & \text{if } g \text{ even,} \\ \frac{g+3}{2} & \text{se } g \text{ odd,} \end{cases}$$

For any $2 \leq k \leq \gamma$, stratification of $\mathcal{M}_g$

$$\mathcal{M}^1_{g,2} \subset \mathcal{M}^1_{g,3} \subset \ldots \subset \mathcal{M}^1_{g,k} \subset \ldots \subset \mathcal{M}_g,$$

where

$$\mathcal{M}^1_{g,k} := \{[C] \in \mathcal{M}_g |\ C \text{ has a } \mathfrak{g}^1_k\}$$

is the k-**gonal locus**.

[Arbarello-Cornalba, 1981] $\mathcal{M}^1_{g,k}$ irriducible of dimension $2g + 2k - 5$, if $k < \gamma$. Moreover, $[C] \in \mathcal{M}^1_{g,k}$ general has a unique $\mathfrak{g}^1_k$.

Fact $[C] \in \mathcal{M}^1_{g,k}$ general is such that $\mathrm{Aut}(C) = \{Id\}$.

Example Let $g \geq 3$, $d \geq 6g - 5$ and $[C] \in \mathcal{M}^1_{g,k}$ general, with

$$3 \leq k < \gamma.$$

$|A| = \mathfrak{g}^1_k$ on C and $L := \omega_C \otimes A^\vee \Rightarrow m := \deg(L) = 2g-2-k$ and $h^1(L) = 2$.

[Kim,Kim, 2004] if $3k(k-1) \leq 2g - 1$, then L very ample

$N \in Pic^{d-m}(C) \Rightarrow \deg(N) = d + k - 2g + 2 \geq 4g - 4 + k \Rightarrow N$ very-ample and non-special.

$\mathcal{F} := N \oplus L$; not restrictive, since $d \geq 6g - 5$ (**Corollary 1**)

$$\Downarrow$$

$\mathcal{F}$ very-ample, unstable with $h^1(\mathcal{F}) = 2 \Rightarrow S = \Phi(F) \subset \mathbb{P}^{d-2g+3}$.

$\mathrm{Aut}(C) = \{Id\} \Rightarrow$ **Lemma 1** still holds $\Rightarrow \dim(G_S) = \dim(\mathrm{Aut}(S))$ known.

$$\Downarrow$$

We get components $\mathcal{H}_k$ of $\mathrm{Hilb}(d, g, 2)$ such that

- $\dim(\mathcal{H}_k) > \dim(\mathcal{H}_{d,g})$, so different from $\mathcal{H}_{d,g}$,
- $\mathcal{H}_k$ with special moduli. ♣

Remark Other examples with higher speciality: take $|A^{\otimes r}|$ instead of $|A|$. Thus, $L_r := \omega_C \otimes (A^{\otimes r})^\vee$ is of speciality $r + 1$.

Open questions

(1) Classify all the components of the Hilbert scheme of non-special and special scrolls.

(2) What are their images in $\mathcal{M}_g$?

(3) Singular loci ? (we have yet descriptions of some singular points of both $\mathcal{H}_{d,g}$ and $\mathcal{H}^m_{d,g,h^1}$)

(4) **Non-special scrolls**

(i) Brill-Noether loci $W^p_n(\mathcal{F}) \subset \text{Pic}^n(C)$, for $p \geq 1$:

- existence?
- dimension?
- structure?
- Petri's type conjectures?

(ii) Torelli's type results: when $\dim(M_{\overline{n}}(\mathcal{F})) = 0$ we have 2^g sub-line bundles of maximal degree. Therefore, we have

$$U_C(d) \dashrightarrow Sym^{2^g}(\text{Pic}^{\overline{n}}(C)).$$

Can we reconstruct $\mathcal{F}$?

(5) **Special scrolls**

(i) Degenerations ?

(ii) Families of unisecants ?

(iii) Brill-Noether theory ?

(iv) If $d < \frac{7}{2}g - h^1 + 1$, $\mathcal{F}$ can be **stable** even if special ?

Remark (iv) would give existence results of Brill-Noether loci in $U_C(d)$, for C with general moduli, not covered by the results of [Teixidor I Bigas, 2005].