Infinitesimal variations of Hodge structure at infinity

E. Cattani[1] J. Fernandez[2]

[1]Department of Mathematics & Statistics
University of Massachusetts

[2]Instituto Balseiro
Universidad Nacional de Cuyo

Algebraic Geometry, D-modules and Foliations
Buenos Aires, July 2008

The Hodge decomposition

- X smooth projective variety, $\dim_{\mathbb{C}}(X) = n$.

The Hodge decomposition

- X smooth projective variety, $\dim_{\mathbb{C}}(X) = n$.
- As a real manifold, for all k:
  - $H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C}$,
  - $Q : H^k(X, \mathbb{R}) \times H^k(X, \mathbb{R}) \to \mathbb{R}$ non-degenerate bilinear form (Hodge-Riemann).

The Hodge decomposition

- X smooth projective variety, $\dim_{\mathbb{C}}(X) = n$.
- As a real manifold, for all k:
  - $H^k(X,\mathbb{C}) = H^k(X,\mathbb{R}) \otimes \mathbb{C}$,
  - $Q : H^k(X,\mathbb{R}) \times H^k(X,\mathbb{R}) \to \mathbb{R}$ non-degenerate bilinear form (Hodge-Riemann).
- Hodge decomposition:

$$H^k(X,\mathbb{C}) = \bigoplus_{p=0}^{k} H^{k-p,p}(X) \tag{1}$$

The Hodge decomposition

- X smooth projective variety, $\dim_{\mathbb{C}}(X) = n$.
- As a real manifold, for all k:
  - $H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C}$,
  - $Q : H^k(X, \mathbb{R}) \times H^k(X, \mathbb{R}) \to \mathbb{R}$ non-degenerate bilinear form (Hodge-Riemann).
- Hodge decomposition:

$$H^k(X, \mathbb{C}) = \bigoplus_{p=0}^{k} H^{k-p,p}(X) \qquad (1)$$

- such that:
  - $\overline{H^{k-p,p}(X)} = H^{p,k-p}(X)$

The Hodge decomposition

- X smooth projective variety, $\dim_{\mathbb{C}}(X) = n$.
- As a real manifold, for all k:
  - $H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C}$,
  - $Q : H^k(X, \mathbb{R}) \times H^k(X, \mathbb{R}) \to \mathbb{R}$ non-degenerate bilinear form (Hodge-Riemann).
- Hodge decomposition:

$$H^k(X, \mathbb{C}) = \bigoplus_{p=0}^{k} H^{k-p,p}(X) \qquad (1)$$

- such that:
  - $\overline{H^{k-p,p}(X)} = H^{p,k-p}(X)$
  - (1) is polarized by Q (induced by the intersection pairing on X).

The Hodge decomposition

- X smooth projective variety, $\dim_{\mathbb{C}}(X) = n$.
- As a real manifold, for all k:
  - $H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C}$,
  - $Q : H^k(X, \mathbb{R}) \times H^k(X, \mathbb{R}) \to \mathbb{R}$ non-degenerate bilinear form (Hodge-Riemann).
- Hodge decomposition:

$$H^k(X, \mathbb{C}) = \bigoplus_{p=0}^{k} H^{k-p,p}(X) \tag{1}$$

- such that:
  - $\overline{H^{k-p,p}(X)} = H^{p,k-p}(X)$
  - (1) is polarized by Q (induced by the intersection pairing on X).
- *Technicality:* must restrict to the primitive cohomology $H^k(X, \mathbb{C})_0$.

Hodge structure

- Abstract version of the previous setup.

Hodge structure

- Abstract version of the previous setup.
- A *Hodge structure* of weight k is a direct sum decomposition of the complexification of a fixed vector space H.

Hodge structure

- Abstract version of the previous setup.
- A *Hodge structure* of weight k is a direct sum decomposition of the complexification of a fixed vector space H.
  -
    $$H \otimes \mathbb{C} = \bigoplus_{p=0}^{k} H^{k-p,p}$$

Hodge structure

- Abstract version of the previous setup.
- A *Hodge structure* of weight k is a direct sum decomposition of the complexification of a fixed vector space H.
  - $$H \otimes \mathbb{C} = \bigoplus_{p=0}^{k} H^{k-p,p}$$
  - $$\overline{H^{k-p,p}} = H^{p,k-p} \quad \text{for all} \quad p$$

Hodge structure

- Abstract version of the previous setup.
- A *Hodge structure* of weight k is a direct sum decomposition of the complexification of a fixed vector space H.
  - $$H \otimes \mathbb{C} = \bigoplus_{p=0}^{k} H^{k-p,p}$$
  - $$\overline{H^{k-p,p}} = H^{p,k-p} \quad \text{for all} \quad p$$
- If the decomposition is Q-orthogonal and some positivities hold it is a *polarized Hodge structure*.

Hodge structure

- Abstract version of the previous setup.
- A *Hodge structure* of weight k is a direct sum decomposition of the complexification of a fixed vector space H.
  - $$H \otimes \mathbb{C} = \bigoplus_{p=0}^{k} H^{k-p,p}$$
  - $$\overline{H^{k-p,p}} = H^{p,k-p} \quad \text{for all} \quad p$$
- If the decomposition is Q-orthogonal and some positivities hold it is a *polarized Hodge structure*.
  - $$Q(H^{p,k-p}, H^{p',k-p'}) = 0 \quad \text{unless} \quad p' = k - p$$

Hodge structure

- Abstract version of the previous setup.
- A *Hodge structure* of weight k is a direct sum decomposition of the complexification of a fixed vector space H.
  - $$H \otimes \mathbb{C} = \bigoplus_{p=0}^{k} H^{k-p,p}$$
  - $$\overline{H^{k-p,p}} = H^{p,k-p} \quad \text{for all} \quad p$$
- If the decomposition is Q-orthogonal and some positivities hold it is a *polarized Hodge structure*.
  - $$Q(H^{p,k-p}, H^{p',k-p'}) = 0 \quad \text{unless} \quad p' = k-p$$
  - $$i^{2p-k} Q(\alpha, \overline{\alpha}) > 0 \quad \text{for all} \quad \alpha \in H^{p,k-p} - \{0\}$$

Classifying space of polarized Hodge structures

- Given H, Q, k, and $h^{p,k-p} \in \mathbb{N} \cup \{0\}$

Classifying space of polarized Hodge structures

- Given H, Q, k, and $h^{p,k-p} \in \mathbb{N} \cup \{0\}$
- D is the space of splittings of $H \otimes \mathbb{C}$ satisfying the conditions of the previous slide.

Classifying space of polarized Hodge structures

- Given H, Q, k, and $h^{p,k-p} \in \mathbb{N} \cup \{0\}$
- D is the space of splittings of $H \otimes \mathbb{C}$ satisfying the conditions of the previous slide.
- $\check{D}$ is the space of splittings of $H \otimes \mathbb{C}$ satisfying only the orthogonality condition.

Classifying space of polarized Hodge structures

- Given H, Q, k, and $h^{p,k-p} \in \mathbb{N} \cup \{0\}$
- D is the space of splittings of $H \otimes \mathbb{C}$ satisfying the conditions of the previous slide.
- $\check{D}$ is the space of splittings of $H \otimes \mathbb{C}$ satisfying only the orthogonality condition.
- $D \subset \check{D} \subset \prod_p Gr(h^{p,k-p}, H \otimes \mathbb{C})$

Classifying space of polarized Hodge structures

- Given H, Q, k, and $h^{p,k-p} \in \mathbb{N} \cup \{0\}$
- D is the space of splittings of $H \otimes \mathbb{C}$ satisfying the conditions of the previous slide.
- $\check{D}$ is the space of splittings of $H \otimes \mathbb{C}$ satisfying only the orthogonality condition.
- $D \subset \check{D} \subset \prod_p Gr(h^{p,k-p}, H \otimes \mathbb{C})$
- D and $\check{D}$ are complex manifolds

Classifying space of polarized Hodge structures

- Given H, Q, k, and $h^{p,k-p} \in \mathbb{N} \cup \{0\}$
- D is the space of splittings of $H \otimes \mathbb{C}$ satisfying the conditions of the previous slide.
- $\check{D}$ is the space of splittings of $H \otimes \mathbb{C}$ satisfying only the orthogonality condition.
- $D \subset \check{D} \subset \prod_p Gr(h^{p,k-p}, H \otimes \mathbb{C})$
- D and $\check{D}$ are complex manifolds
- $$\check{D} = G_{\mathbb{C}}/P \quad \text{and} \quad D = G/B$$

  where $G_{\mathbb{C}} := O(H \otimes \mathbb{C}, Q)$ and $G = O(H, Q)$.

Classifying space of polarized Hodge structures

- $F^* \in \check{D}$ defines filtration of $\mathfrak{g}_{\mathbb{C}}$ by

$$F^s\mathfrak{g}_{\mathbb{C}} := \{X \in \mathfrak{g}_{\mathbb{C}} : X(F^a) \subset F^{s+a} \text{ for all } a\}.$$

Classifying space of polarized Hodge structures

- $F^* \in \check{D}$ defines filtration of $\mathfrak{g}_{\mathbb{C}}$ by

$$F^s\mathfrak{g}_{\mathbb{C}} := \{X \in \mathfrak{g}_{\mathbb{C}} : X(F^a) \subset F^{s+a} \text{ for all } a\}.$$

- If $F^* \in D$ the filtration $F^*\mathfrak{g}_{\mathbb{C}}$ becomes a Hodge structure of weight 0 on $\mathfrak{g}$ with grading:

$$\mathfrak{g}_{\mathbb{C}}^{s,-s} := \{X \in \mathfrak{g}_{\mathbb{C}} : X(H^{a,k-a}) \subset H^{a+s,k-a-s} \text{ for all } a\}.$$

Classifying space of polarized Hodge structures

- $F^* \in \check{D}$ defines filtration of $\mathfrak{g}_\mathbb{C}$ by

$$F^s\mathfrak{g}_\mathbb{C} := \{X \in \mathfrak{g}_\mathbb{C} : X(F^a) \subset F^{s+a} \text{ for all } a\}.$$

- If $F^* \in D$ the filtration $F^*\mathfrak{g}_\mathbb{C}$ becomes a Hodge structure of weight 0 on $\mathfrak{g}$ with grading:

$$\mathfrak{g}_\mathbb{C}^{s,-s} := \{X \in \mathfrak{g}_\mathbb{C} : X(H^{a,k-a}) \subset H^{a+s,k-a-s} \text{ for all } a\}.$$

- $$(T\check{D})_{F^*} \simeq \mathfrak{g}_\mathbb{C}/F^0\mathfrak{g}_\mathbb{C} \quad \text{and} \quad (TD)_{F^*} = \oplus_{a<0}\mathfrak{g}^{a,-a}.$$

Classifying space of polarized Hodge structures

- $F^* \in \check{D}$ defines filtration of $\mathfrak{g}_{\mathbb{C}}$ by

$$F^s\mathfrak{g}_{\mathbb{C}} := \{X \in \mathfrak{g}_{\mathbb{C}} : X(F^a) \subset F^{s+a} \text{ for all } a\}.$$

- If $F^* \in D$ the filtration $F^*\mathfrak{g}_{\mathbb{C}}$ becomes a Hodge structure of weight 0 on $\mathfrak{g}$ with grading:

$$\mathfrak{g}_{\mathbb{C}}^{s,-s} := \{X \in \mathfrak{g}_{\mathbb{C}} : X(H^{a,k-a}) \subset H^{a+s,k-a-s} \text{ for all } a\}.$$

- $$(T\check{D})_{F^*} \simeq \mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} \quad \text{and} \quad (TD)_{F^*} = \oplus_{a<0}\mathfrak{g}^{a,-a}.$$

- *Horizontal bundle*: $(T_h\check{D})_{F^*} \subset (T\check{D})_{F^*}$

$$(T_h\check{D})_{F^*} = F^{-1}\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} \quad \text{and} \quad (TD)_{F^*} = \mathfrak{g}_{\mathbb{C}}^{-1,1}$$

Families of smooth projective varieties

- U open (simply connected).

Families of smooth projective varieties

- U open (simply connected).
- $\mathcal{X} \to U$ family of smooth projective varieties.

Families of smooth projective varieties

- U open (simply connected).
- $\mathcal{X} \to U$ family of smooth projective varieties.

$$\begin{tikzcd} \mathcal{X} \arrow[d] \\ U \end{tikzcd}$$

Families of smooth projective varieties

- U open (simply connected).
- $\mathcal{X} \to U$ family of smooth projective varieties.

$$\begin{array}{c} H^k(\mathcal{X}_u, \mathbb{C}) = \oplus_p H^{k-p,p}(\mathcal{X}_u) \\ \downarrow \\ U \end{array}$$

Families of smooth projective varieties

- U open (simply connected).
- $\mathcal{X} \to U$ family of smooth projective varieties.

$$\begin{tikzcd}
H^k(\mathcal{X}_u, \mathbb{C}) = \oplus_p H^{k-p,p}(\mathcal{X}_u) \arrow[d] \arrow[dr] & \\
U \arrow[r, "\Phi", "\text{Period mapping}"'] & D
\end{tikzcd}$$

Families of smooth projective varieties

- U open (simply connected).
- $\mathcal{X} \to U$ family of smooth projective varieties.

$$H^k(\mathcal{X}_u, \mathbb{C}) = \oplus_p H^{k-p,p}(\mathcal{X}_u)$$

$$\begin{tikzcd} H^k(\mathcal{X}_u, \mathbb{C}) = \oplus_p H^{k-p,p}(\mathcal{X}_u) \arrow[d] \arrow[dr] & \\ U \arrow[r, "\Phi", "\text{Period mapping}"'] & D \end{tikzcd}$$

$$u \longmapsto \Phi(u)$$

Families of smooth projective varieties

- U open (simply connected).
- $\mathcal{X} \to U$ family of smooth projective varieties.

$$\begin{tikzcd}
H^k(\mathcal{X}_u, \mathbb{C}) = \oplus_p H^{k-p,p}(\mathcal{X}_u) \arrow[d] \arrow[drr] & & \\
U \arrow[rr, "\Phi", "\text{Period mapping}"'] & & D
\end{tikzcd}$$

$$u \longmapsto \Phi(u)$$

$$T_u U \xrightarrow{\Phi_{*u}} T_{\Phi(u)} D$$

Variations of Hodge structure

- Abstract version of the period mapping for families of smooth projective varieties.

Variations of Hodge structure

- Abstract version of the period mapping for families of smooth projective varieties.
- A *variation of Hodge structure (VHS)* is, essentially, a submanifold of D...

Variations of Hodge structure

- Abstract version of the period mapping for families of smooth projective varieties.
- A *variation of Hodge structure (VHS)* is, essentially, a submanifold of D...
- that meets a certain condition called *transversality*:

$$\mathrm{Im}(d\Phi_{F^*}) \subset (T_h D)_{F^*}.$$

Variations of Hodge structure

- Abstract version of the period mapping for families of smooth projective varieties.
- A *variation of Hodge structure (VHS)* is, essentially, a submanifold of D...
- that meets a certain condition called *transversality*:

$$\mathrm{Im}(d\Phi_{F^*}) \subset (T_h D)_{F^*}.$$

- Infinitesimal level: the subspaces $E \subset T_{H_0} D$ that are tangent to VHS are called *infinitesimal variations* of Hodge structure, *IVHS*.

$$\{IVHS\} \leftrightarrow \{\text{abelian subspaces of } \mathfrak{g}_{\mathbb{C}}^{-1,1}\}.$$

Mixed Hodge structures

- *Mixed Hodge structure (MHS)*: (W_*, F^*) with
  - W_* increasing filtration of H
  - F^* decreasing filtration of $H \otimes \mathbb{C}$

  such that F^* induces HS of weight j on $Gr_j^{W_*}$.

Mixed Hodge structures

- *Mixed Hodge structure (MHS)*: (W_*, F^*) with
  - W_* increasing filtration of H
  - F^* decreasing filtration of $H \otimes \mathbb{C}$

  such that F^* induces HS of weight j on $Gr_j^{W_*}$.
- *Polarized mixed Hodge structure (PMHS)* of weight k on H:

  such that

Mixed Hodge structures

- *Mixed Hodge structure (MHS)*: (W_*, F^*) with
  - W_* increasing filtration of H
  - F^* decreasing filtration of $H \otimes \mathbb{C}$

  such that F^* induces HS of weight j on $Gr_j^{W_*}$.
- *Polarized mixed Hodge structure (PMHS)* of weight k on H:
  - MHS (W_*, F^*)

  such that

Mixed Hodge structures

- *Mixed Hodge structure (MHS)*: (W_*, F^*) with
  - W_* increasing filtration of H
  - F^* decreasing filtration of $H \otimes \mathbb{C}$

  such that F^* induces HS of weight j on $Gr_j^{W_*}$.
- *Polarized mixed Hodge structure (PMHS)* of weight k on H:
  - MHS (W_*, F^*)
  - $N \in F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}$

  such that

Mixed Hodge structures

- *Mixed Hodge structure (MHS)*: (W_*, F^*) with
  - W_* increasing filtration of H
  - F^* decreasing filtration of $H \otimes \mathbb{C}$

  such that F^* induces HS of weight j on $Gr_j^{W_*}$.
- *Polarized mixed Hodge structure (PMHS)* of weight k on H:
  - MHS (W_*, F^*)
  - $N \in F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}$
  - Bilinear form Q

  such that

Mixed Hodge structures

- *Mixed Hodge structure (MHS)*: (W_*, F^*) with
  - W_* increasing filtration of H
  - F^* decreasing filtration of $H \otimes \mathbb{C}$

  such that F^* induces HS of weight j on $Gr_j^{W_*}$.
- *Polarized mixed Hodge structure (PMHS)* of weight k on H:
  - MHS (W_*, F^*)
  - $N \in F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}$
  - Bilinear form Q

  such that
  - $N^{k+1} = 0$

Mixed Hodge structures

- *Mixed Hodge structure (MHS)*: (W_*, F^*) with
  - W_* increasing filtration of H
  - F^* decreasing filtration of $H \otimes \mathbb{C}$

  such that F^* induces HS of weight j on $Gr_j^{W_*}$.
- *Polarized mixed Hodge structure (PMHS)* of weight k on H:
  - MHS (W_*, F^*)
  - $N \in F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}$
  - Bilinear form Q

  such that
  - $N^{k+1} = 0$
  - $W_* = (W(N)[-k])_*$

Mixed Hodge structures

- *Mixed Hodge structure (MHS)*: (W_*, F^*) with
  - W_* increasing filtration of H
  - F^* decreasing filtration of $H \otimes \mathbb{C}$

  such that F^* induces HS of weight j on $Gr_j^{W_*}$.
- *Polarized mixed Hodge structure (PMHS)* of weight k on H:
  - MHS (W_*, F^*)
  - $N \in F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}$
  - Bilinear form Q

  such that
  - $N^{k+1} = 0$
  - $W_* = (W(N)[-k])_*$
  - $Q(F^a, F^{k-a+1}) = 0$

Mixed Hodge structures

- *Mixed Hodge structure (MHS)*: (W_*, F^*) with
  - W_* increasing filtration of H
  - F^* decreasing filtration of $H \otimes \mathbb{C}$

  such that F^* induces HS of weight j on $Gr_j^{W_*}$.
- *Polarized mixed Hodge structure (PMHS)* of weight k on H:
  - MHS (W_*, F^*)
  - $N \in F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}$
  - Bilinear form Q

  such that
  - $N^{k+1} = 0$
  - $W_* = (W(N)[-k])_*$
  - $Q(F^a, F^{k-a+1}) = 0$
  - the HS of weight $k+l$ induced by F^* on $\ker(N^{l+1} : \mathrm{Gr}^{W_*}_{k+l} \to \mathrm{Gr}^{W_*}_{k-l-2})$ is polarized by $Q(\cdot, N^l \cdot)$.

Nilpotent orbits

- *Nilpotent orbit*: horizontal map $\theta : \mathbb{C}^r \to \check{D}$ of the form

$$\theta(z) := \exp(\sum z_j N_j) \cdot F^*$$

for $F^* \in \check{D}$ and $\{N_1, \ldots N_r\} \subset F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}$ commuting subset such that $\theta(z) \in D$ for all large $\mathrm{Im}(z)$.

Nilpotent orbits

- *Nilpotent orbit*: horizontal map $\theta : \mathbb{C}^r \to \check{D}$ of the form

$$\theta(z) := \exp(\sum z_j N_j) \cdot F^*$$

for $F^* \in \check{D}$ and $\{N_1, \ldots N_r\} \subset F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}$ commuting subset such that $\theta(z) \in D$ for all large $\mathrm{Im}(z)$.

- *Schmid's nilpotent orbit Theorem*: associated to every PVHS over $(\Delta^*)^r \times \Delta^m$ there is a nilpotent orbit $\{N_1, \ldots, N_r; F^*\}$.

Nilpotent orbits

- *Nilpotent orbit*: horizontal map $\theta : \mathbb{C}^r \to \check{D}$ of the form

$$\theta(z) := \exp(\sum z_j N_j) \cdot F^*$$

  for $F^* \in \check{D}$ and $\{N_1, \ldots N_r\} \subset F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}$ commuting subset such that $\theta(z) \in D$ for all large $\mathrm{Im}(z)$.
- *Schmid's nilpotent orbit Theorem*: associated to every PVHS over $(\Delta^*)^r \times \Delta^m$ there is a nilpotent orbit $\{N_1, \ldots, N_r; F^*\}$.
- The nilpotent operators N_j are the logarithms of the monodromy operators.

Nilpotent orbits

- If $\{N_1, \ldots, N_r; F^*\}$ is a nilpotent orbit, then

Nilpotent orbits

- If $\{N_1, \ldots, N_r; F^*\}$ is a nilpotent orbit, then
  1. $N_j^{k+1} = 0$ where k is the weight of the PHS in D.

Nilpotent orbits

- If $\{N_1, \ldots, N_r; F^*\}$ is a nilpotent orbit, then
  1. $N_j^{k+1} = 0$ where k is the weight of the PHS in D.
  2. Every $N \in C(N_1, \ldots, N_r)$ defines the same weight filtration W_*^C.

Nilpotent orbits

- If $\{N_1, \ldots, N_r; F^*\}$ is a nilpotent orbit, then
  1. $N_j^{k+1} = 0$ where k is the weight of the PHS in D.
  2. Every $N \in C(N_1, \ldots, N_r)$ defines the same weight filtration W_*^C.
  3. $((W^C[-k])_*, F^*)$ is a PMHS, polarized by every $N \in C(N_1, \ldots, N_r)$.

Nilpotent orbits

- If $\{N_1, \ldots, N_r; F^*\}$ is a nilpotent orbit, then
  1. $N_j^{k+1} = 0$ where k is the weight of the PHS in D.
  2. Every $N \in C(N_1, \ldots, N_r)$ defines the same weight filtration W_*^C.
  3. $((W^C[-k])_*, F^*)$ is a PMHS, polarized by every $N \in C(N_1, \ldots, N_r)$.
- Conversely, if $F^* \in \check{D}$, and $\{N_1, \ldots, N_r\}$ are commuting nilpotent elements of $F^{-1}\mathfrak{g}_{\mathbb{C}} \cap \mathfrak{g}$ that satisfy the conditions 1, 2 and 3 for some $N \in C(N_1, \ldots, N_r)$, then $\{N_1, \ldots, N_r; F^*\}$ is a nilpotent orbit.

Canonical bigradings

- MHS $\rightsquigarrow$(canonical) bigrading $I^{*,*}$ of $H \otimes \mathbb{C}$ such that

$$I^{q,p} = \overline{I^{p,q}} \mod \oplus_{a<p,b<q} I^{a,b}.$$

Canonical bigradings

- MHS $\rightsquigarrow$(canonical) bigrading $I^{*,*}$ of $H \otimes \mathbb{C}$ such that

$$I^{q,p} = \overline{I^{p,q}} \mod \oplus_{a<p,b<q} I^{a,b}.$$

- Bigrading $I^{*,*}\mathfrak{g}_{\mathbb{C}}$ of $(W_*\mathfrak{g}_{\mathbb{C}}, F^*\mathfrak{g}_{\mathbb{C}})$.

Canonical bigradings

- MHS $\rightsquigarrow$(canonical) bigrading $I^{*,*}$ of $H \otimes \mathbb{C}$ such that

$$I^{q,p} = \overline{I^{p,q}} \mod \oplus_{a<p,b<q} I^{a,b}.$$

- Bigrading $I^{*,*}\mathfrak{g}_{\mathbb{C}}$ of $(W_*\mathfrak{g}_{\mathbb{C}}, F^*\mathfrak{g}_{\mathbb{C}})$.
- $\mathfrak{p}_a := \oplus_b I^{a,b}\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_- := \oplus_{a<0}\mathfrak{p}_a$.

Canonical bigradings

- MHS $\rightsquigarrow$(canonical) bigrading $I^{*,*}$ of $H \otimes \mathbb{C}$ such that

$$I^{q,p} = \overline{I^{p,q}} \mod \oplus_{a<p,b<q} I^{a,b}.$$

- Bigrading $I^{*,*}\mathfrak{g}_{\mathbb{C}}$ of $(W_*\mathfrak{g}_{\mathbb{C}}, F^*\mathfrak{g}_{\mathbb{C}})$.
- $\mathfrak{p}_a := \oplus_b I^{a,b}\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_- := \oplus_{a<0}\mathfrak{p}_a$.
- Then

$$(T\check{D})_{F^*} = \mathfrak{g}_- \quad \text{and} \quad (T_h\check{D})_{F^*} = \mathfrak{p}_{-1}.$$

Asymptotic description of PVHS

- $\Phi : (\Delta^*)^r \times \Delta^m \to D$ a PVHS

Asymptotic description of PVHS

- $\Phi : (\Delta^*)^r \times \Delta^m \to D$ a PVHS
- $$\Phi(z,t) = \exp(\sum z_j N_j) \cdot \underbrace{\Psi(\exp(2\pi i z), t)}_{\Delta^{r+m} \to \check{D},\ \text{holomorphic}}$$

  where

  - $(N_, \ldots, N_r; F^*)$ is a nilpotent orbit, $F^* = \Psi(0,0)$.

Asymptotic description of PVHS

- $\Phi : (\Delta^*)^r \times \Delta^m \to D$ a PVHS
- $$\Phi(z,t) = \exp(\sum z_j N_j) \cdot \exp(\Gamma(\exp(2\pi i z), t)) \cdot F^*,$$

  where
  - $(N_, \ldots, N_r; F^*)$ is a nilpotent orbit, $F^* = \Psi(0,0)$.
  - $\Gamma : \Delta^r \times \Delta^m \to \mathfrak{g}_-$ is holomorphic.

Asymptotic description of PVHS

- $\Phi : (\Delta^*)^r \times \Delta^m \to D$ a PVHS
-

$$\Phi(z, t) = \exp(\sum z_j N_j) \cdot \exp(\Gamma(\exp(2\pi i z), t)) \cdot F^*,$$

  where
  - $(N_, \ldots, N_r; F^*)$ is a nilpotent orbit, $F^* = \Psi(0, 0)$.
  - $\Gamma : \Delta^r \times \Delta^m \to \mathfrak{g}_-$ is holomorphic.
- More compact $\Phi(z, t) = \exp(X(z, t)) \cdot F^*$ for $X : (\Delta^*)^r \times \Delta^m \to \mathfrak{g}_-$.

Asymptotic description of PVHS

- $\Phi : (\Delta^*)^r \times \Delta^m \to D$ a PVHS
- $$\Phi(z,t) = \exp(\sum z_j N_j) \cdot \exp(\Gamma(\exp(2\pi i z), t)) \cdot F^*,$$

  where
  - $(N_, \dots, N_r; F^*)$ is a nilpotent orbit, $F^* = \Psi(0,0)$.
  - $\Gamma : \Delta^r \times \Delta^m \to \mathfrak{g}_-$ is holomorphic.
- More compact $\Phi(z,t) = \exp(X(z,t)) \cdot F^*$ for $X : (\Delta^*)^r \times \Delta^m \to \mathfrak{g}_-$.
- Horizontality $\Leftrightarrow \exp(-X) d \exp(X) = dX_{-1} \in \mathfrak{p}_{-1}$

Asymptotic description of PVHS

- $\Phi : (\Delta^*)^r \times \Delta^m \to D$ a PVHS
- $$\Phi(z,t) = \exp(\sum z_j N_j) \cdot \exp(\Gamma(\exp(2\pi i z), t)) \cdot F^*,$$
  where
  - $(N_, \dots, N_r; F^*)$ is a nilpotent orbit, $F^* = \Psi(0,0)$.
  - $\Gamma : \Delta^r \times \Delta^m \to \mathfrak{g}_-$ is holomorphic.
- More compact $\Phi(z,t) = \exp(X(z,t)) \cdot F^*$ for $X : (\Delta^*)^r \times \Delta^m \to \mathfrak{g}_-$.
- Horizontality $\Leftrightarrow \exp(-X) d \exp(X) = dX_{-1} \in \mathfrak{p}_{-1}$
- In particular, $dX_{-1} \wedge dX_{-1} = 0$ for
  $$X_{-1} = \sum z_j N_j + \Gamma_{-1}.$$

Construction of degenerating PVHS

- Given a nilpotent orbit $(N_, \ldots, N_r; F^*)$

Construction of degenerating PVHS

- Given a nilpotent orbit $(N_, \ldots, N_r; F^*)$
- and $\Gamma_{-1} : \Delta^r \times \Delta^m \to \mathfrak{p}_{-1}$ holomorphic

Construction of degenerating PVHS

- Given a nilpotent orbit $(N_{,} \ldots, N_r; F^*)$
- and $\Gamma_{-1} : \Delta^r \times \Delta^m \to \mathfrak{p}_{-1}$ holomorphic
- such that

$$dX_{-1} \wedge dX_{-1} = 0$$

for

$$X_{-1} = \sum z_j N_j + \Gamma_{-1}.$$

Construction of degenerating PVHS

- Given a nilpotent orbit $(N_{,} \ldots, N_r; F^*)$
- and $\Gamma_{-1} : \Delta^r \times \Delta^m \to \mathfrak{p}_{-1}$ holomorphic
- such that

$$dX_{-1} \wedge dX_{-1} = 0$$

  for

$$X_{-1} = \sum z_j N_j + \Gamma_{-1}.$$

- Then, there exist PVHS that degenerate to the given data.

IVHS of degenerating PVHS

- Φ a PVHS on $\mathcal{W} := (\Delta^*)^r \times \Delta^m$ with nilpotent orbit $(N_, \ldots, N_r; F^*)$.

IVHS of degenerating PVHS

- Φ a PVHS on $\mathcal{W} := (\Delta^*)^r \times \Delta^m$ with nilpotent orbit $(N_, \ldots, N_r; F^*)$.
- For $w_0 \in \mathcal{W}$,

$$d\Phi_{w_0} : (T\mathcal{W})_{w_0} \to (T_h\check{D})_{\Phi(w_0)} \subset \oplus_a \hom(Gr_a^{\Phi(w_0)}, Gr_{a-1}^{\Phi(w_0)})$$

IVHS of degenerating PVHS

- Φ a PVHS on $\mathcal{W} := (\Delta^*)^r \times \Delta^m$ with nilpotent orbit $(N_, \ldots, N_r; F^*)$.
- For $w_0 \in \mathcal{W}$,

$$d\Phi_{w_0} : (T\mathcal{W})_{w_0} \to (T_h\check{D})_{\Phi(w_0)} \subset \oplus_a \hom(Gr_a^{\Phi(w_0)}, Gr_{a-1}^{\Phi(w_0)})$$

- $I^{*,*}$ bigrading of MHS. $J^* := \oplus_b I^{*,b}$ grading of F^*.

IVHS of degenerating PVHS

- Φ a PVHS on $\mathcal{W} := (\Delta^*)^r \times \Delta^m$ with nilpotent orbit $(N_, \ldots, N_r; F^*)$.
- For $w_0 \in \mathcal{W}$,

$$d\Phi_{w_0} : (T\mathcal{W})_{w_0} \to (T_h\check{D})_{\Phi(w_0)} \subset \oplus_a \hom(Gr_a^{\Phi(w_0)}, Gr_{a-1}^{\Phi(w_0)})$$

- $I^{*,*}$ bigrading of MHS. $J^* := \oplus_b I^{*,b}$ grading of F^*.
- $L^* := \exp(X(w_0)) \cdot F^*$ grading of $\Phi(w_0) = \exp(X(w_0)) \cdot F^*$.

IVHS of degenerating PVHS

- Φ a PVHS on $\mathcal{W} := (\Delta^*)^r \times \Delta^m$ with nilpotent orbit $(N_, \ldots, N_r; F^*)$.
- For $w_0 \in \mathcal{W}$,

$$d\Phi_{w_0} : (T\mathcal{W})_{w_0} \to (T_h\check{D})_{\Phi(w_0)} \subset \oplus_a \hom(Gr_a^{\Phi(w_0)}, Gr_{a-1}^{\Phi(w_0)})$$

- $I^{*,*}$ bigrading of MHS. $J^* := \oplus_b I^{*,b}$ grading of F^*.
- $L^* := \exp(X(w_0)) \cdot F^*$ grading of $\Phi(w_0) = \exp(X(w_0)) \cdot F^*$.
- $\hom(Gr_a^{\Phi(w_0)}, Gr_{a-1}^{\Phi(w_0)}) \simeq \hom(L^a, L^{a-1}) \simeq \hom(J^a, J^{a-1})$.

IVHS of degenerating PVHS

- Φ a PVHS on $\mathcal{W} := (\Delta^*)^r \times \Delta^m$ with nilpotent orbit $(N_, \ldots, N_r; F^*)$.
- For $w_0 \in \mathcal{W}$,

$$d\Phi_{w_0} : (T\mathcal{W})_{w_0} \to (T_h\check{D})_{\Phi(w_0)} \subset \oplus_a \hom(Gr_a^{\Phi(w_0)}, Gr_{a-1}^{\Phi(w_0)})$$

- $I^{*,*}$ bigrading of MHS. $J^* := \oplus_b I^{*,b}$ grading of F^*.
- $L^* := \exp(X(w_0)) \cdot F^*$ grading of $\Phi(w_0) = \exp(X(w_0)) \cdot F^*$.
- $\hom(Gr_a^{\Phi(w_0)}, Gr_{a-1}^{\Phi(w_0)}) \simeq \hom(L^a, L^{a-1}) \simeq \hom(J^a, J^{a-1})$.
- Under the isomorphisms, $d\Phi_{w_0} : (T\mathcal{W})_{w_0} \to \oplus_a \hom(J^a, J^{a-1})$

$$d\Phi = dX_{-1}.$$

Limit of IVHS

- $X_{-1} = \sum z_j N_j + \Gamma_{-1}$

Limit of IVHS

- $X_{-1} = \sum z_j N_j + \Gamma_{-1}$
- $\operatorname{Im} d\Phi_{w_0} = \langle N_j + \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial s_j} 2\pi i s_j, \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial t_l} \rangle \subset \mathfrak{p}_{-1}$

Limit of IVHS

- $X_{-1} = \sum z_j N_j + \Gamma_{-1}$
- $\operatorname{Im} d\Phi_{w_0} = \langle N_j + \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial s_j} 2\pi i s_j, \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial t_l} \rangle \subset \mathfrak{p}_{-1}$
- $\lim_{w_0 \to (0,0)} d\Phi_{w_0} = \underbrace{\langle N_j, \frac{\partial \Gamma_{-1}(0,0)}{\partial t_l} \rangle}_{\mathfrak{a}^{\Phi}_{\infty}} \subset \mathfrak{p}_{-1}$

Limit of IVHS

- $X_{-1} = \sum z_j N_j + \Gamma_{-1}$
- $\operatorname{Im} d\Phi_{w_0} = \langle N_j + \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial s_j} 2\pi i s_j, \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial t_l} \rangle \subset \mathfrak{p}_{-1}$
- $\lim_{w_0 \to (0,0)} d\Phi_{w_0} = \underbrace{\langle N_j, \frac{\partial \Gamma_{-1}(0,0)}{\partial t_l} \rangle}_{\mathfrak{a}^{\Phi}_{\infty}} \subset \mathfrak{p}_{-1}$
- $\mathfrak{a}^{\Phi}_{\infty}$ is abelian (limit of abelian subspaces).

Limit of IVHS

- $X_{-1} = \sum z_j N_j + \Gamma_{-1}$
- $\operatorname{Im} d\Phi_{w_0} = \langle N_j + \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial s_j} 2\pi i s_j, \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial t_l} \rangle \subset \mathfrak{p}_{-1}$
- $\lim_{w_0 \to (0,0)} d\Phi_{w_0} = \underbrace{\langle N_j, \frac{\partial \Gamma_{-1}(0,0)}{\partial t_l} \rangle}_{\mathfrak{a}^{\Phi}_{\infty}} \subset \mathfrak{p}_{-1}$
- $\mathfrak{a}^{\Phi}_{\infty}$ is abelian (limit of abelian subspaces).

Definition

An *infinitesimal variation of Hodge structure at infinity (IVI)* is a triple $(\{N_1, \dots, N_r; F^*\}; \mathfrak{a})$ where

Limit of IVHS

- $X_{-1} = \sum z_j N_j + \Gamma_{-1}$
- $\operatorname{Im} d\Phi_{w_0} = \langle N_j + \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial s_j} 2\pi i s_j, \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial t_l} \rangle \subset \mathfrak{p}_{-1}$
- $\lim_{w_0 \to (0,0)} d\Phi_{w_0} = \underbrace{\langle N_j, \frac{\partial \Gamma_{-1}(0,0)}{\partial t_l} \rangle}_{\mathfrak{a}^\Phi_\infty} \subset \mathfrak{p}_{-1}$
- $\mathfrak{a}^\Phi_\infty$ is abelian (limit of abelian subspaces).

Definition

An *infinitesimal variation of Hodge structure at infinity (IVI)* is a triple $(\{N_1, \ldots, N_r; F^*\}; \mathfrak{a})$ where

- $\{N_1, \ldots, N_r; F^*\}$ is a nilpotent orbit

Limit of IVHS

- $X_{-1} = \sum z_j N_j + \Gamma_{-1}$
- $\operatorname{Im} d\Phi_{w_0} = \langle N_j + \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial s_j} 2\pi i s_j, \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial t_l} \rangle \subset \mathfrak{p}_{-1}$
- $\lim_{w_0 \to (0,0)} d\Phi_{w_0} = \underbrace{\langle N_j, \frac{\partial \Gamma_{-1}(0,0)}{\partial t_l} \rangle}_{\mathfrak{a}_\infty^\Phi} \subset \mathfrak{p}_{-1}$
- $\mathfrak{a}_\infty^\Phi$ is abelian (limit of abelian subspaces).

Definition

An *infinitesimal variation of Hodge structure at infinity (IVI)* is a triple $(\{N_1, \ldots, N_r; F^*\}; \mathfrak{a})$ where

- $\{N_1, \ldots, N_r; F^*\}$ is a nilpotent orbit
- $\mathfrak{a} \subset \mathfrak{p}_{-1} \subset \mathfrak{g}_{\mathbb{C}}$ is an abelian subspace

Limit of IVHS

- $X_{-1} = \sum z_j N_j + \Gamma_{-1}$
- $\operatorname{Im} d\Phi_{w_0} = \langle N_j + \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial s_j} 2\pi i s_j, \frac{\partial \Gamma_{-1}(s_0,t_0)}{\partial t_l} \rangle \subset \mathfrak{p}_{-1}$
- $\lim_{w_0 \to (0,0)} d\Phi_{w_0} = \underbrace{\langle N_j, \frac{\partial \Gamma_{-1}(0,0)}{\partial t_l} \rangle}_{\mathfrak{a}^{\Phi}_{\infty}} \subset \mathfrak{p}_{-1}$
- $\mathfrak{a}^{\Phi}_{\infty}$ is abelian (limit of abelian subspaces).

Definition

An *infinitesimal variation of Hodge structure at infinity (IVI)* is a triple $(\{N_1, \ldots, N_r; F^*\}; \mathfrak{a})$ where

- $\{N_1, \ldots, N_r; F^*\}$ is a nilpotent orbit
- $\mathfrak{a} \subset \mathfrak{p}_{-1} \subset \mathfrak{g}_{\mathbb{C}}$ is an abelian subspace
- $\langle N_1, \ldots, N_r \rangle \subset \mathfrak{a}$.

Integrability of IVIs

Theorem

- *The IVHS of every PVHS degenerate to an IVI.*

Integrability of IVIs

Theorem

- *The IVHS of every PVHS degenerate to an IVI.*
- *Every IVI arises as limit of the IVHS of (the germ of) a PVHS.*

Dimensional bounds for IVHS

Theorem (Carlson, Kasparian, Toledo, Mayer)

There are sharp upper bounds for the dimension of IVHS. In the case $k = 2$,

$$\dim \leq \begin{cases} \text{if } h^{2,0} > 1, \begin{cases} \frac{1}{2} h^{2,0}(h^{1,1} - 1) + 1, & \text{if } h^{1,1} \text{ is odd} \\ \frac{1}{2} h^{2,0} h^{1,1}, & \text{if } h^{1,1} \text{ is even} \end{cases} \\ \text{if } h^{2,0} = 1, h^{1,1}. \end{cases}$$

Dimensional bounds for IVHS

Theorem (Carlson, Kasparian, Toledo, Mayer)

There are sharp upper bounds for the dimension of IVHS. In the case $k = 2$,

$$\dim \leq \begin{cases} \text{if } h^{2,0} > 1, \begin{cases} \frac{1}{2} h^{2,0}(h^{1,1} - 1) + 1, & \text{if } h^{1,1} \text{ is odd} \\ \frac{1}{2} h^{2,0} h^{1,1}, & \text{if } h^{1,1} \text{ is even} \end{cases} \\ \text{if } h^{2,0} = 1, h^{1,1}. \end{cases}$$

- Since every IVI can be integrated to a PVHS, the previous bounds hold for IVI.

Dimensional bounds for IVHS

Theorem (Carlson, Kasparian, Toledo, Mayer)

There are sharp upper bounds for the dimension of IVHS. In the case $k = 2$,

$$\dim \leq \begin{cases} \text{if } h^{2,0} > 1, \begin{cases} \frac{1}{2}h^{2,0}(h^{1,1} - 1) + 1, & \text{if } h^{1,1} \text{ is odd} \\ \frac{1}{2}h^{2,0}h^{1,1}, & \text{if } h^{1,1} \text{ is even} \end{cases} \\ \text{if } h^{2,0} = 1, h^{1,1}. \end{cases}$$

- Since every IVI can be integrated to a PVHS, the previous bounds hold for IVI.
- Furthermore, explicit constructions prove that they remain sharp for all IVIs.

Dimensional bounds for IVHS

Theorem (Carlson, Kasparian, Toledo, Mayer)

There are sharp upper bounds for the dimension of IVHS. In the case $k = 2$,

$$\dim \leq \begin{cases} \text{if } h^{2,0} > 1, \begin{cases} \frac{1}{2} h^{2,0}(h^{1,1} - 1) + 1, & \text{if } h^{1,1} \text{ is odd} \\ \frac{1}{2} h^{2,0} h^{1,1}, & \text{if } h^{1,1} \text{ is even} \end{cases} \\ \text{if } h^{2,0} = 1, h^{1,1}. \end{cases}$$

- Since every IVI can be integrated to a PVHS, the previous bounds hold for IVI.
- Furthermore, explicit constructions prove that they remain sharp for all IVIs.
- Same results hold for all weights k.

The case $h^{2,0} = h^{1,1} = 3$

$j^{*,*}$	nilpotent cones	max dim IVI
$j^{2,0} = j^{1,1} = 3$	$\{0\}$	4
$j^{2,1} = j^{1,1} = 1,\ j^{2,0} = 2$	dim 1	4
$j^{2,2} = 1,\ j^{2,0} = 2,\ j^{1,1} = 3$	dim 1, 2 and 3	all cases 3
$j^{2,2} = j^{1,1} = j^{2,1} = j^{2,0} = 1$	dim 1 and 2	all cases 3
$j^{2,2} = 2,\ j^{2,0} = 1,\ j^{1,1} = 3$	dim 1, 2 and 3	all cases 3
$j^{2,2} = j^{1,1} = 3$	dim 1, 2 and 3	all cases 3

Table: MHS, nilpotent cones and IVIs obtained when $k = 2$ and $h^{2,0} = h^{1,1} = 3$

Nilpotent orbits rather than MHS

- $k = 2$, $j^{2,2} = j^{1,1} = 2d$.

Nilpotent orbits rather than MHS

- $k = 2$, $j^{2,2} = j^{1,1} = 2d$.
- Fix real basis where $Q = \begin{pmatrix} & & \mathbb{I}_{2d} \\ & -\mathbb{I}_{2d} & \\ \mathbb{I}_{2d} & & \end{pmatrix}$,

Nilpotent orbits rather than MHS

- $k = 2$, $j^{2,2} = j^{1,1} = 2d$.
- Fix real basis where $Q = \begin{pmatrix} & & \mathbb{I}_{2d} \\ & -\mathbb{I}_{2d} & \\ \mathbb{I}_{2d} & & \end{pmatrix}$,
- $\phi \in \mathfrak{p}_{-1} \Rightarrow \phi = \begin{pmatrix} A & \\ & A^t \end{pmatrix}$ with $A \in \mathbb{C}^{2d \times 2d}$

Nilpotent orbits rather than MHS

- $k = 2$, $j^{2,2} = j^{1,1} = 2d$.
- Fix real basis where $Q = \begin{pmatrix} & & \mathbb{I}_{2d} \\ & -\mathbb{I}_{2d} & \\ \mathbb{I}_{2d} & & \end{pmatrix}$,
- $\phi \in \mathfrak{p}_{-1} \Rightarrow \phi = \begin{pmatrix} A & & \\ & A^t & \end{pmatrix}$ with $A \in \mathbb{C}^{2d \times 2d}$
- $[\phi, \psi] = \begin{pmatrix} & & \\ A^t B - B^t A & & \end{pmatrix}$.

Nilpotent orbits rather than MHS

- $k = 2$, $j^{2,2} = j^{1,1} = 2d$.
- Fix real basis where $Q = \begin{pmatrix} & & \mathbb{I}_{2d} \\ & -\mathbb{I}_{2d} & \\ \mathbb{I}_{2d} & & \end{pmatrix}$,
- $\phi \in \mathfrak{p}_{-1} \Rightarrow \phi = \begin{pmatrix} A & & \\ & & \\ & A^t & \end{pmatrix}$ with $A \in \mathbb{C}^{2d \times 2d}$
- $[\phi, \psi] = \begin{pmatrix} & & \\ & & \\ A^t B - B^t A & & \end{pmatrix}$.
- $N_a := \begin{pmatrix} E_{a,a} & & \\ & & \\ & E_{a,a} & \end{pmatrix}$, $N_0 = N_1 + \cdots + N_{2d}$.

Nilpotent orbits rather than MHS

- $k = 2$, $j^{2,2} = j^{1,1} = 2d$.
- Fix real basis where $Q = \begin{pmatrix} & & \mathbb{I}_{2d} \\ & -\mathbb{I}_{2d} & \\ \mathbb{I}_{2d} & & \end{pmatrix}$,
- $\phi \in \mathfrak{p}_{-1} \Rightarrow \phi = \begin{pmatrix} A & \\ & A^t \end{pmatrix}$ with $A \in \mathbb{C}^{2d \times 2d}$
- $[\phi, \psi] = \begin{pmatrix} & \\ A^t B - B^t A & \end{pmatrix}$.
- $N_a := \begin{pmatrix} E_{a,a} & \\ & E_{a,a} \end{pmatrix}$, $N_0 = N_1 + \cdots + N_{2d}$.
- N_0 polarizes the MHS.

Nilpotent orbits rather than MHS

- Nilpotent orbit $(\{N_1, \ldots, N_{2d}\}, J^{*,*})$. $\mathfrak{a} \subset \mathfrak{p}_{-1}$ abelian and containing the nilpotent elements $\Rightarrow$ "matrix components" are diagonal $\Rightarrow \dim \mathfrak{a} \leq 2d$.

Nilpotent orbits rather than MHS

- Nilpotent orbit $(\{N_1, \ldots, N_{2d}\}, J^{*,*})$. $\mathfrak{a} \subset \mathfrak{p}_{-1}$ abelian and containing the nilpotent elements $\Rightarrow$ "matrix components" are diagonal $\Rightarrow \dim \mathfrak{a} \leq 2d$.
- Nilpotent orbit $(\{N_0\}, J^{*,*})$. If $A := \begin{pmatrix} a\mathbb{I} + i\alpha & \alpha \\ \alpha & a\mathbb{I} - i\alpha \end{pmatrix}$ for $a \in \mathbb{C}$, $\alpha \in \mathbb{C}^{d\times d}$ *symmetric*. $\mathfrak{a}$ is the space of those "matrix components" $\Rightarrow \dim \mathfrak{a} = \frac{1}{2}d(d+1) + 1$

Nilpotent orbits rather than MHS

- Nilpotent orbit $(\{N_1, \ldots, N_{2d}\}, J^{*,*})$. $\mathfrak{a} \subset \mathfrak{p}_{-1}$ abelian and containing the nilpotent elements $\Rightarrow$ "matrix components" are diagonal $\Rightarrow$ $\dim \mathfrak{a} \leq 2d$.
- Nilpotent orbit $(\{N_0\}, J^{*,*})$. If $A := \begin{pmatrix} a\mathbb{I} + i\alpha & \alpha \\ \alpha & a\mathbb{I} - i\alpha \end{pmatrix}$ for $a \in \mathbb{C}$, $\alpha \in \mathbb{C}^{d\times d}$ *symmetric*. $\mathfrak{a}$ is the space of those "matrix components" $\Rightarrow$ $\dim \mathfrak{a} = \frac{1}{2}d(d+1) + 1$
- In the first case *dim* $\leq 2d$ while in the second, $\dim \geq \frac{1}{2}d(d+1) + 1$. If $d \geq 3$ both dimensions don't agree.

MHS of Hodge-Tate type

- Bigrading $J^{a,a}$ for $0 \leq a \leq k$ with $\dim J^{a,a} = n$.

MHS of Hodge-Tate type

- Bigrading $J^{a,a}$ for $0 \leq a \leq k$ with $\dim J^{a,a} = n$.
- $N_0 \in \mathfrak{p}_{-1}$ such that $N_0 : J^{a,a} \to J^{a-1,a-1}$ is isomorphism.

MHS of Hodge-Tate type

- Bigrading $J^{a,a}$ for $0 \leq a \leq k$ with $\dim J^{a,a} = n$.
- $N_0 \in \mathfrak{p}_{-1}$ such that $N_0 : J^{a,a} \to J^{a-1,a-1}$ is isomorphism.
- $(N_0, J^{*,*})$ is a nilpotent orbit

MHS of Hodge-Tate type

- Bigrading $J^{a,a}$ for $0 \leq a \leq k$ with $\dim J^{a,a} = n$.
- $N_0 \in \mathfrak{p}_{-1}$ such that $N_0 : J^{a,a} \to J^{a-1,a-1}$ is isomorphism.
- $(N_0, J^{*,*})$ is a nilpotent orbit
- $(N_0, J^{*,*}, \mathfrak{a})$ is an IVI

MHS of Hodge-Tate type

- Bigrading $J^{a,a}$ for $0 \leq a \leq k$ with $\dim J^{a,a} = n$.
- $N_0 \in \mathfrak{p}_{-1}$ such that $N_0 : J^{a,a} \to J^{a-1,a-1}$ is isomorphism.
- $(N_0, J^{*,*})$ is a nilpotent orbit
- $(N_0, J^{*,*}, \mathfrak{a})$ is an IVI
- Max dim $\mathfrak{a} \Leftrightarrow$ max dim abelian subalgebra of symmetric matrices in $\mathfrak{gl}(n, \mathbb{C})$

MHS of Hodge-Tate type

- Bigrading $J^{a,a}$ for $0 \leq a \leq k$ with $\dim J^{a,a} = n$.
- $N_0 \in \mathfrak{p}_{-1}$ such that $N_0 : J^{a,a} \to J^{a-1,a-1}$ is isomorphism.
- $(N_0, J^{*,*})$ is a nilpotent orbit
- $(N_0, J^{*,*}, \mathfrak{a})$ is an IVI
- Max dim $\mathfrak{a} \Leftrightarrow$ max dim abelian subalgebra of symmetric matrices in $\mathfrak{gl}(n, \mathbb{C})$
- (Carlson–Toledo), $n = 2\alpha + \beta$ with $\beta = 0, 1$

$$\dim \mathfrak{a} \leq \begin{cases} \frac{1}{2}\alpha(\alpha+1) + \beta + 1 \text{ for } n > 1 \\ 1 \text{ for } n = 1. \end{cases}$$

MHS of Hodge-Tate type

- Bigrading $J^{a,a}$ for $0 \leq a \leq k$ with $\dim J^{a,a} = n$.
- $N_0 \in \mathfrak{p}_{-1}$ such that $N_0 : J^{a,a} \to J^{a-1,a-1}$ is isomorphism.
- $(N_0, J^{*,*})$ is a nilpotent orbit
- $(N_0, J^{*,*}, \mathfrak{a})$ is an IVI
- Max $\dim \mathfrak{a} \Leftrightarrow$ max dim abelian subalgebra of symmetric matrices in $\mathfrak{gl}(n, \mathbb{C})$
- (Carlson–Toledo), $n = 2\alpha + \beta$ with $\beta = 0, 1$

$$\dim \mathfrak{a} \leq \begin{cases} \frac{1}{2}\alpha(\alpha+1) + \beta + 1 \text{ for } n > 1 \\ 1 \text{ for } n = 1. \end{cases}$$

- (up to conjugation) only one maximal if n is even and two if n is odd.