Secant varieties and degenerate subvarieties

“Algebraic Geometry, D-modules, Foliations and their interactions”
21 - 26 July, 2008 - Buenos Aires, Argentina

NOTATION

$X \subset \mathbb{P}^r$ irreducible variety (not necessarily smooth) of dimension n

$S^k(X)$ = *embedded* k-secant variety = closure of $\{P \in < x_0, \ldots, x_k >: x_i \in X\}$

N.B. $S^1(X)$ = chordal variety

formal definition

$AS^k(X)$ = *abstract* k-secant variety in $X^{(k+1)} \times \mathbb{P}^r$ = closure of $\{(x_0, \ldots, x_k, P) : P \in < x_0, \ldots, x_k >\}$

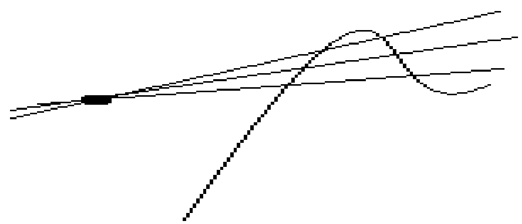
$S^k(X)$ = closure of the image of the secant map

$$AS^k(X) \xrightarrow{\phi_k} \mathbb{P}^r$$

$AS^k(X) =$ *abstract* k-secant variety in $X^{(k+1)} \times \mathbb{P}^r =$
closure of $\{(x_0, \ldots, x_k, P) : P \in < x_0, \ldots, x_k >\}$

dimension of $AS^k(X) = kn + k + n$ (fixed)

dimension of $S^k(X) = kn + k + n - \dim($ general fiber of $\phi_k)$



$AS^k(X) \xrightarrow{\phi_k} \mathbb{P}^r$

X is *k-defective* when ϕ_k has not maximal rank

i.e.

$r \leq kn + k + n$ and ϕ_k is not dominant, or
$r \geq kn + k + n$ and ϕ_k is not generically finite

TERRACINI's LEMMA

The following are equivalent:

- X is *k-defective*

- through $k+1$ general points of X there exists a subvariety $Y \subset X$ of dimension m, whose span has dimension $< mk+m+k-q$ where $q = \max\{0, nk+n+k-r\}$ (a $(k+1)$–filling family of highly degenerate subvarieties)

- the k-th tangential map $\tau_k : X \to X_k$ (i.e. the projection from k general tangent spaces) has positive dimensional general fibers, that is $\dim(X_k) < \dim(X)$

$$\Psi_k$$

EXAMPLES

(1)

X = Veronese surface in $\mathbb{P}^5$
$\dim(S^1(X)) = 4 <$ expected $= 5$
projection from a general tg space $X \to X_1 =$ conic
through two general points of X there is a conic
(a 2-filling family of conics)

(2)

$X = \mathbb{P}^3 \times \mathbb{P}^3 \subset \mathbb{P}^{15}$
$\dim(S^2(X)) = 14 <$ expected $= 15$
projection from two general tg spaces $X \to X_2 = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$
through three general points of X there is a $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$

- CLASSIFICATION OF DEFECTIVE VARIETIES

use the classification of varieties with many degenerate subvarieties

--- , C.Ciliberto.
THE CLASSIFICATION OF DEFECTIVE THREEFOLDS.
Projective Varieties with Unexpected Properties. de Gruyter (2005) 35-49

- k-DEFECTIVITY OF A SPECIFIC VARIETY X

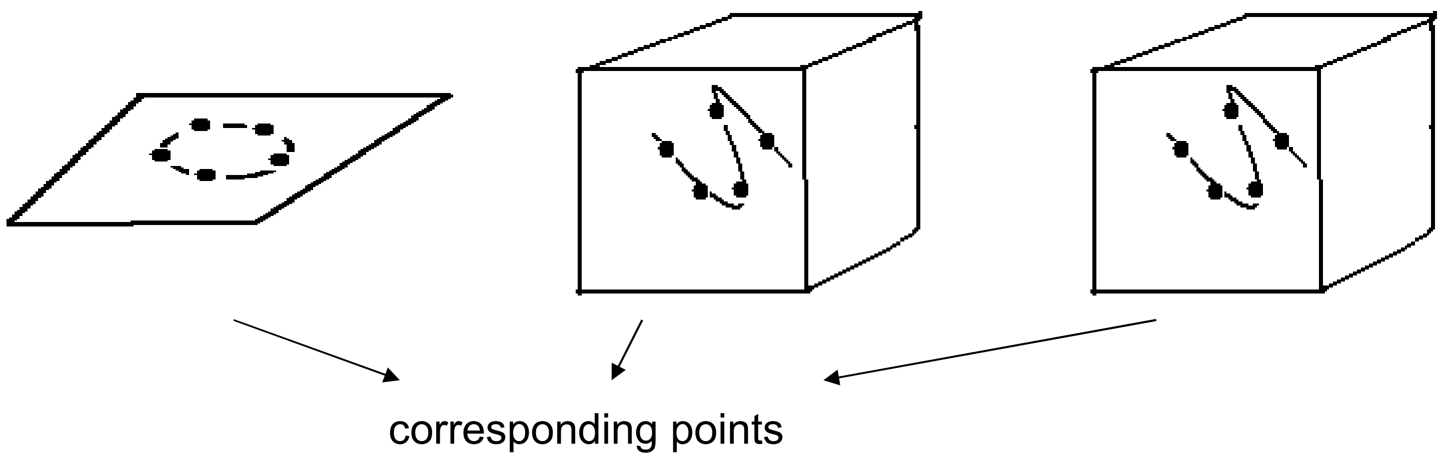
find a (k+1)-filling family of degenerate subvarieties

H.Abo, G. Ottaviani, C. Peterson.
INDUCTION FOR SECANT VARIETIES OF SEGRE VARIETIES.
MathAG/0607191v2

EXAMPLES

(3)

$X = \mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3 \subset \mathbb{P}^{47}$
$k = 4$ (i.e. 5 points)
$\dim(S^4(X)) = 43 <$ expected $= 44$
through five general points of X one can draw
a rational normal curve of $\mathbb{P}^8$



(4)

$X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ any number of times

classical:

$X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is 2-defective

namely through 3 general points of X one can draw a rational normal curve of $\mathbb{P}^4$

No other product of $\mathbb{P}^1$'s is defective

M. Catalisano, T. Geramita, A. Gimigliano.
private communication

METHOD: use induction to reduce to a case in which

taking "general" points and computing one gets the result

No other product of $\mathbb{P}^1$'s is defective

ALTERNATIVE METHOD (proposal): prove the *non-existence* of (k+1)-filling family of degenerate subvarieties

e.g.

prove the non-existence of degenerate curves $C \subset \mathbb{P}^{2k}$ through $k+1$ general points in a product $X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (s times) (s>4)

$\Updownarrow$

prove the non-existence s linear pencils *general enough* on C, whose sum has dimension $\leq 2k$

almost obvious when $s \gg k$

such method is particularly interesting when X is a Segre product

SELDOM EXPLOITED

--- , C.Ciliberto.
THE CLASSIFICATION OF DEFECTIVE THREEFOLDS.
Projective Varieties with Unexpected Properties. de Gruyter (2005) 35-49

	description	k	r	$s^{(k)}$	δ_k	ϵ_k	n_k
(1)a	$X \subset$ cone over 2–uple of 3fold of min. deg in $\mathbb{P}^{k+1}$, vertex = point	≥ 2	$4k+2$	$4k+1$	1	2	1
(1)b	$X \subset$ cone over 2–uple of 3fold of min. deg in $\mathbb{P}^{k+1}$, vertex = line	≥ 2	$4k+3$	$4k+2$	1	2	2
(2)a	$X =$ 2–uple of hypersurface of deg ≥ 3 in $\mathbb{P}^4$	3	14	13	1	2	1
(2)b	$X \subset$ cone over 2–uple of hypersurface of deg ≥ 3 in $\mathbb{P}^4$	3	15	14	1	2	2
(3)a	$X =$ 2–uple of 3fold of deg k in $\mathbb{P}^{k+1}$ whose curve sections have $p_a = 1$	≥ 4	$4k+2$	$4k+1$	1	2	1
(3)b	$X \subset$ cone over 2–uple of 3fold of deg k in $\mathbb{P}^{k+1}$ whose curve sections have $p_a = 1$	≥ 4	$4k+3$	$4k+2$	1	2	2
(4)	$X =$ 2–uple of 3fold of deg k in $\mathbb{P}^{k+1}$ whose curve sections have $p_a = 0$	≥ 4	$4k+3$	$4k+2$	1	2	2
(5)	$X =$ 2–uple of 3fold of deg ≥ 5 in $\mathbb{P}^5$ contained in a quadric	4	19	18	1	2	2
(6)	$X =$ 2–uple of 3fold of deg $k+1$ in $\mathbb{P}^{k+1}$ whose curve sections have $p_a = 2$	≥ 4	$4k+3$	$4k+2$	1	2	2
(7)a	$X \subset$ cone over 2–uple of surface of min. deg k in $\mathbb{P}^{k+1}$, vertex $= \mathbb{P}^k$	≥ 1	$4k+3$	$4k+2$	1	2	2
(7)b	$X \subset$ cone over 2–uple of surface of min. deg k in $\mathbb{P}^{k+1}$, vertex $= \mathbb{P}^{k-1}$	≥ 1	$4k+2$	$4k+1$	1	2	1
(8)	$X \subset$ cone over 2–uple of surface of deg ≥ 3 in $\mathbb{P}^3$, vertex = line	2	11	10	1	2	2
(9)	$X \subset$ cone over 2–uple of surface of deg $k+1$ in $\mathbb{P}^{k+1}$, whose curve sections have $p_a = 1$, vertex $= \mathbb{P}^{k-1}$	≥ 3	$4k+3$	$4k+2$	1	2	2
(10)	$X \subset$ cone over surface not k–weakly defective, vertex $= \mathbb{P}^{k-1}$	≥ 1	$\geq 4k+3$	$4k+2$	1	1	2
(11)	$X \subset$ cone over curve, vertex $= \mathbb{P}^{2k}$	≥ 1	$\geq 4k+3$	$4k+2$	1	2	2
(12)a	$X \subset$ cone over curve, vertex $= \mathbb{P}^{2k-1}$	≥ 1	$4k+2$	$4k+1$	1	2	1
(12)b	$X \subset$ cone over curve, vertex $= \mathbb{P}^{2k-1}$	≥ 1	$\geq 4k+3$	$4k+1$	2	2	1
(13)a	$X =$ 2–uple of 3fold of min. deg in $\mathbb{P}^{k+2}$	≥ 1	$4k+5$	$4k+2$	1	1	1
(13)b	$X =$ projection from point of 2–uple of 3fold of min. deg k in $\mathbb{P}^{k+2}$	≥ 1	$4k+4$	$4k+2$	1	1	1
(13)c	$X =$ projection from line of 2–uple of 3fold of min. deg k in $\mathbb{P}^{k+2}$	≥ 1	$4k+3$	$4k+2$	1	1	1
(14)	$X \subset \mathbb{P}^{4k+3} \cap$ (Segre of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$) and X dominates both $\mathbb{P}^{k+1}$'s	≥ 1	$4k+3$	$4k+2$	1	1	2

--- , C.Ciliberto.
THE CLASSIFICATION OF DEFECTIVE THREEFOLDS.
Projective Varieties with Unexpected Properties. de Gruyter (2005) 35-49

(14)	$X \subset \mathbb{P}^{4k+3} \cap$ (Segre of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$) and X dominates both $\mathbb{P}^{k+1}$'s

a more precise description is given through the paper

PROPOSITION 6.11. *Let X be an irreducible, non–degenerate threefold in the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$, $k \geq 2$, which lies in the 2-uple embedding of $\mathbb{P}^{k+1}$. Suppose the two projections of X to $\mathbb{P}^{k+1}$ span $\mathbb{P}^{k+1}$. Then these projections coincide and are immersions of X in $\mathbb{P}^{k+1}$. Furthermore the dimension s of the linear span of X satisfies $s > 4k+3$ unless:*

(a) *$s = 4k+1$ and the image of X in $\mathbb{P}^{k+1}$ is a threefold Y of minimal degree;*
(b) *$s = 4k+2$ and the image of X in $\mathbb{P}^{k+1}$ is a threefold Y of degree k with curve sections of arithmetic genus 1;*
(c) *$s = 4k+3$ and the image of X in $\mathbb{P}^{k+1}$ is:*
   (c_1) *either a threefold Y of degree $k+1$ with curve sections of arithmetic genus 2 or*
   (c_2) *a threefold Y of degree k which is the projection in $\mathbb{P}^{k+1}$ of a threefold of minimal degree in $\mathbb{P}^{k+2}$. In the latter case Y is described in Lemma 1.6.*

The threefolds in case (a) are $(k-1)$–defective, whereas the threefolds in cases (b), (c) are minimally k–defective.

CLASSIFICATION OF DEFECTIVE PRODUCTS $X \times \mathbb{P}^1$

X = curve : none

--- , C.Ciliberto.
THE GRASSMANNIANS OF SECANT VARIETIES OF CURVES ARE NOT DEFECTIVE.
Indag.Math. 13 (2002) 23-28

X = surface :

THEOREM 5.1. *let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projective surface which is minimally $(1,k)$-defective. Then k is even and X is in the following list:*

(1) *X is contained in a cone with vertex of dimension $\frac{k}{2}-1$ and not smaller over a curve C with $\dim(\langle C \rangle) \geqslant \frac{3}{2}k+1$;*

(2) *$k=4$ and X is the 3-Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$;*

(3) *$X \subset \mathbb{P}^{2k+1}$ is a rational normal scroll $S(a_1,a_2)$ with $a_1 \geqslant \frac{k}{2}$.*

--- , C.Ciliberto.
THE CLASSIFICATION OF (1,K)-DEFECTIVE SURFACES.
Geom. Ded. 111 (2005) 107-123.

CLASSIFICATION OF DEFECTIVE PRODUCTS $X \times \mathbb{P}^1$

X = threefold: k=2 (Coppens, Cools)

Theorem 5.1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate irreducible smooth threefold. If X is $(1,2)$-defective, then X is isomorphic to one of the following varieties:*

i) the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$;

ii) the rational normal scroll $S(1,1,2)$ of minimal degree 4 in $\mathbb{P}^6$;

iii) the rational normal scroll $S(1,1,3)$ of minimal degree 5 in $\mathbb{P}^7$;

iv) the rational normal scroll $S(1,2,2)$ of minimal degree 5 in $\mathbb{P}^7$;

--- , F.Cools.
CLASSIFICATION OF (1,2)-DEFECTIVE THREEFOLDS.
preprint

The previously introduced method (control of families of degenerate subvarieties) is the main tool

REMARK

There are several filling families of degenerate varieties in a defective variety

the span of k+1 tg spaces is tangent along a positive dimensional subvariety	⇐	**defective**	⇒	**every hyperplane tg at k+1 general points is tangent along a positive dimensional subvariety**
↓ Γ_k				↓ Σ_k

the converse is false in general

varieties for which these properties hold are

k - WEAKLY DEFECTIVE

the classification of weakly defective varieties is known up to surfaces

---, C.Ciliberto.
WEAKLY DEFECTIVE VARIETIES
Trans. Am. Math. Soc. 354 (2002) 151-178

k-weakly defective and $r > nk + n + k$

- through $k+1$ general points of X there exists a subvariety $Y \subset X$ of dimension m, whose span has dimension $\leq mk + m + k$ (a $(k+1)-$filling family of highly degenerate subvarieties)

compare with k-defectivity!

again a problem of (k+1)-filling families of degenerate subvarieties

Why weakly defective varieties are interesting?

--- , C.Ciliberto.
ON THE CONCEPT OF k-TH SECANT ORDER OF A VARIETY.
J. London Math. Soc. 73 (2006) 436-454.

COROLLARY 2.7. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective, non–degenerate variety of dimension n. Assume $s^k(X) = n(k+1)+k < r$, i.e. X is not k–defective, and $S^k(X)$ is a proper subvariety of $\mathbb{P}^r$. Then $\mu_k(X) = 1$ unless X is k–weakly defective.*

IDENTIFIABILITY

if not, see e.g.

C.Ciliberto, M. Mella, F. Russo.
VARIETIS WITH ONE APPARENT DOUBLE POINT.
J. Alg. Geom. 13 (2004) 475-512.

PROBLEM: weakly defectivity of Veronese varieties, Segre products etc.

VERONESE → M. Mella
SINGULARITIES OF LINEAR SYSTEMS AND THE WARING PROBLEM. preprint.

IDENTIFIABILITY of Segre products

THEOREM (R. ELMORE, P. HALL, A. NEEMAN)
If $s \gg k$, the Segre product $X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (s times)
is k-identifiable (not k-weakly defective).

(H. KASAHARA, S. KATSUMI)
Extension to higher dimensional products

ALTERNATIVE METHOD (proposal): prove the *non-existence*

of (k+1)-filling family of degenerate subvarieties C

prove the non-existence s linear series *general enough* on C,
whose sum has dimension $\leq 2k$ (for curves)

TASK: find improved (sharp?) bounds for the condition "s>>k"

work in progress (C. BOCCI, --- , F. COOLS)

already the case of $\mathbb{P}^3$'s
is interesting → DNA !

REMARK

There are several filling families of degenerate varieties in a defective variety

the span of k+1 tg spaces is tangent along a positive dimensional subvariety ⇦ **defective** ⇨ **every hypeprplane tg at k+1 general points is tangent along a positive dimensional subvariety**

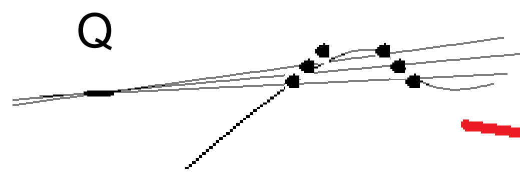
$$\Gamma_k \neq \Psi_k \neq \Sigma_k$$

so we have the families Ψ_k, Γ_k, Σ_k on a k-defective variety X

WHICH RELATIONS AMONG THESE FAMILIES?

WHICH RELATIONS AMONG THESE FAMILIES AND THE ENTRY LOCI?

For $Q \in S^k(X)$ general, the *entry locus* E_Q is the closure of the set $\{x_0 \in X : \text{there are } x_1, \dots, x_k \in X \text{ such that } Q \in < x_0, \dots, x_1 >\}$



ENTRY LOCUS of Q

PROPOSITION $\Psi_k \subseteq \Gamma_k \subseteq \Sigma_k$.
If X is k-defective, then $E_Q \subseteq \Gamma_k$ and Ψ_k is a flat deformation of E_Q.

the span of k+1 tg spaces is tangent along a positive dimensional subvariety

COROLLARY If X is k-defective and $\Psi_k = \Gamma_k$, then $E_Q = \Psi_k = \Gamma_k$.

X is TAMELY DEFECTIVE

TWO GENERAL NON-SENSE

Knowing the entry locus E_Q is usually interesting.

It is much easier to compute Γ_k than E_Q or Ψ_k.

EXAMPLE

Let Y be a homogeneous form (hypersurface) of degree d in $\mathbb{P}^q$

Is it possible to express Y as the determinant of a 2×2 matrix of forms of degree $< d$?

$$Y = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

TRUE (q=2)

↓

E. Carlini, --- , T.Geramita. HYPERSURFACES CONTAINING COMPLETE INTERSECTIONS. preprint

FALSE (q>2)

↓

Noether - Lefschetz

WHAT HAS TO DO WITH SECANTS?

$\mathbb{P}^N$ = space of hogeneous forms of degree d in $\mathbb{P}^q$

$S_a \subset \mathbb{P}^N$, S_a = variety of forms splitting in a product of a form of degree a and a form of degree $d - a$.

$$Y = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad a = \deg(A) = \deg(B)$$

⇕

Y belongs to the secant variety of S_a

If Y is a determinant, which are its determinantal expressions?

⇕

Find the entry locus of Y w.r.t the secant variety of S_a

S_a is wildly defective

$\Gamma_1 =< A, D > \cap < B, C > \cap S_a$ $\qquad E_Y =???$

MAXIMALLY DEFECTIVE VARIETIES

$X \subset \mathbb{P}^r$ $\qquad\qquad$ $S^k(X) \neq \mathbb{P}^r$

how small can r be?

THEOREM (F. ZAK) X smooth, $k = 1$. Then $s^{(k)}(X) \geq \frac{3}{2}n + 1$.

border cases (Severi varieties)

(i) *the Veronese surface* $V_{2,2}$ *in* $\mathbb{P}^5$;
(ii) *the* 4*–dimensional Segre variety* Seg$(2,2)$ *in* $\mathbb{P}^8$
(iii) *the* 8*–dimensional Grassmann variety* $\mathbb{G}(1,5)$ *in* $\mathbb{P}^{14}$
(iv) *the* 16*–dimensional* E_6*–variety in* $\mathbb{P}^{26}$.

REMARK: the following varieties are k-defective

(i) *the* $(k+1)$*–dimensional Veronese variety* $V_{2,k+1}$ *in* $\mathbb{P}^{\frac{k(k+3)}{2}}$;
(ii) *the* $2(k+1)$*–dimensional Segre variety* Seg$(k+1,k+1)$ *in* $\mathbb{P}^{k^2+4k+3}$;
(iii) *the* $4(k+1)$*–dimensional Grassmann variety* $\mathbb{G}(1,2k+3)$ *in* $\mathbb{P}^{\binom{2k+4}{2-1}}$.

~~(iv)~~

EXTENSION OF ZAK'S THEOREM

--- , C.Ciliberto.
ON THE DIMENSION OF SECANT VARIETIES.
preprint (2008).

Theorem 7.4. *Let $X \subset \mathbb{P}^r$ be a non–degenerate variety of dimension n. Assume X is a R_k–variety (or X is k–smooth). Then*

$$\text{either} \quad S^k(X) = \mathbb{P}^r \quad \text{or} \quad s^{(k)}(X) \geq \frac{k+2}{2}n + k.$$

technical condition, which extends smoothness

border cases (k-Severi varieties)

(i) *the $(k+1)$–dimensional Veronese variety $V_{2,k+1}$ in $\mathbb{P}^{\frac{k(k+3)}{2}}$;*
(ii) *the $2(k+1)$–dimensional Segre variety $\mathrm{Seg}(k+1, k+1)$ in $\mathbb{P}^{k^2+4k+3}$;*
(iii) *the $4(k+1)$–dimensional Grassmann variety $\mathbb{G}(1, 2k+3)$ in $\mathbb{P}^{\binom{2k+4}{2-1}}$.*

Theorem 7.4. *Let $X \subset \mathbb{P}^r$ be a non–degenerate variety of dimension n. Assume X is a R_k–variety (or X is k–smooth). Then*

$$\text{either} \quad S^k(X) = \mathbb{P}^r \quad \text{or} \quad s^{(k)}(X) \geq \frac{k+2}{2}n + k.$$

Ingredients for the proof:

1) An extension of Zak's theorem on tangencies

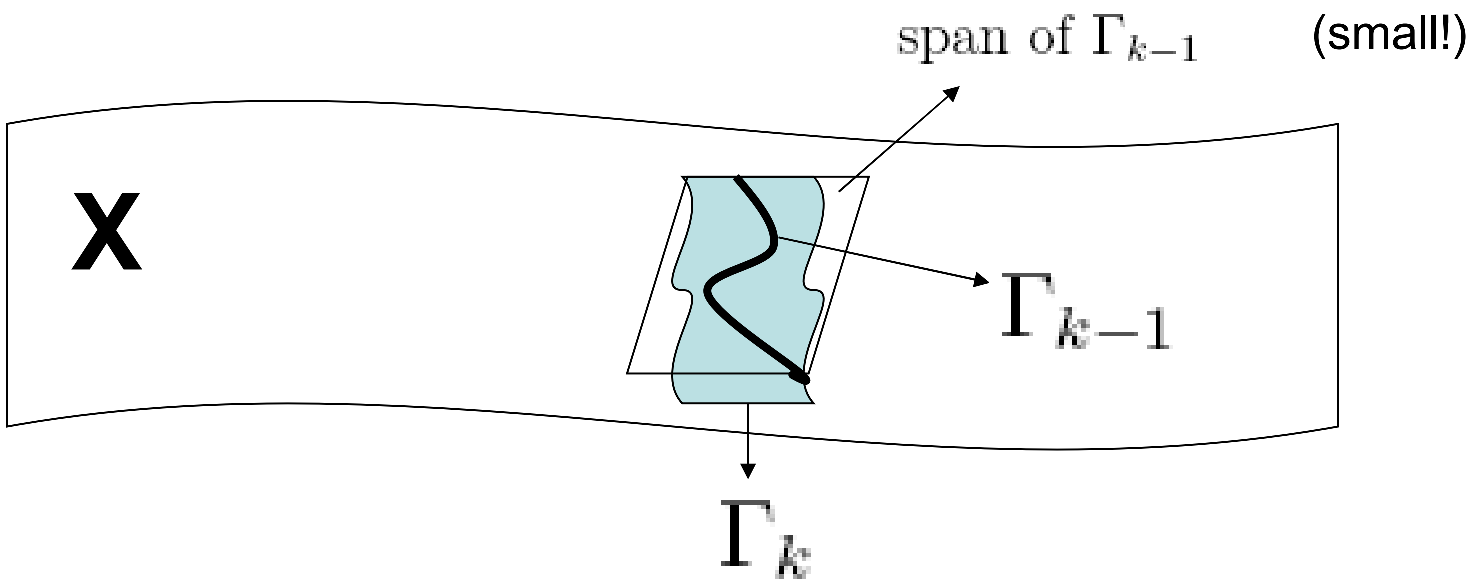
Theorem 6.1. *Let $X \subset \mathbb{P}^r$ be a non–degenerate variety and let $L \neq \mathbb{P}^r$ be a proper linear subspace of $\mathbb{P}^r$ which is J_k–tangent to X along a pure subvariety Y. Then:*

$$\dim(L) \geq \dim(X) + \dim(J^k(Y)) \geq \dim(X) + \dim(S^{k-1}(Y)). \tag{6.1}$$

2) The study of the hierarchical structure of Γ_k and Ψ_k

2) The study of the hierarchical structure of Γ_k and Ψ_k

Γ_{k-1} sits inside Γ_k and since it is
highly degenerate, it forces the defectivity of Γ_k.
Then apply the formula recursively.



Γ_k is forced to be very defective

play induction

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THANK YOU FOR YOUR ATTENTION