Mixed Lefschetz Theorem and Hodge-Riemann Bilinear Relations

Geometry and Combinatorics

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Hodge Decomposition

Theorem: Let X be a smooth compact Kähler manifold. Then $H^d(X, \mathbb{C})$ decomposes as:

$$H^d(X, \mathbb{C}) = \bigoplus_{p+q=d} H^{p,q} \; ; \quad H^{q,p} = \overline{H^{p,q}},$$

where $H^{p,q}$ may be described as the set of cohomology classes admitting a representative of bidegree (p, q).

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Corollary: The odd Betti numbers of X are even.

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Remark: The Hodge decomposition is compatible with the algebra structure:

$$H^{p,q} \cup H^{p',q'} \subset H^{p+p',q+q'}$$

Hard Lefschetz Theorem

Theorem (Lefschetz, Hodge): Let X be a k-dimensional smooth compact Kähler manifold and $\omega \in H^{1,1}(X) \cap H^2(X, \mathbb{R})$ a Kähler class. Let L_ω denote multiplication by ω. Then for all $\ell \leq k$ the map

$$L_\omega^\ell : H^{k-\ell}(X) \to H^{k+\ell}(X)$$

is an isomorphism.

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Corollaries:

- $h^2 \geq 1$, where $\quad h^\ell := \dim H^\ell_{\mathbb{C}}(X, \mathbb{C})$.

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- $h^0 \leq h^2 \leq h^4 \leq \cdots$
- $h^1 \leq h^3 \leq h^5 \leq \cdots$

Lefschetz Decomposition

We define the *primitive cohomology*

$$H_0^{k-\ell}(X) := \{\alpha \in H^{k-\ell}(X) \,:\, L_\omega^{\ell+1}\alpha = 0\}$$

Then

$$\begin{aligned} H^{k-\ell}(X) &= H_0^{k-\ell}(X) \oplus L_\omega \cdot H^{k-\ell-2}(X) \\ &= H_0^{k-\ell}(X) \oplus L_\omega \cdot H_0^{k-\ell-2}(X) \\ &\quad \oplus L_\omega^2 \cdot H_0^{k-\ell-4}(X) + \cdots \end{aligned}$$

Hodge-Riemann Bilinear Relations

Define a real bilinear form Q on $H^*(X, \mathbb{C})$ by

$$Q(\alpha, \beta) = (-1)^{\frac{d(d-1)}{2}} \int_X \alpha \cup \beta \,,$$

where $\deg(\alpha) = d$ and the integral is assumed to be zero if $\deg(\alpha \cup \beta) \neq 2k$, where $k = \dim_{\mathbb{C}}(X)$.

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Then the Hodge decomposition is Q-orthogonal and:

$$i^{p-q}\; Q(\alpha, L_\omega^\ell\, \bar{\alpha}) \geq 0$$

for any

$$\alpha \in H^{p,q}(X) \cap H_0^{p+q}(X, \mathbb{C}) \,; \quad p + q = k - \ell\,.$$

Moreover, equality holds if and only if $\alpha = 0$.

Example: $\dim_{\mathbb{C}}(X) = 2$

Hodge Diamond

$$
\begin{array}{ccccc}
 & & H^{0,0} & & \\
 & H^{1,0} & \downarrow L_\omega & H^{0,1} & \\
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Example continued: Hodge Inequality

- The sign of the intersection form $\langle\ ,\ \rangle$ on $H^{1,1}_{\mathbb{R}}(X)$ is

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If $\langle\alpha,\alpha\rangle \geq 0$, then $\forall\beta$: $\langle\alpha,\beta\rangle^2 \geq \langle\alpha,\alpha\rangle\langle\beta,\beta\rangle$

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- Dually, if C_1, C_2 are curves in X then:

$$\langle C_1, C_2\rangle^2 \geq \langle C_1, C_1\rangle\langle C_2, C_2\rangle \qquad \text{(Hodge Inequality)}$$

Mixed Hard Lefschetz Theorem

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$$L_{\omega_1} \cdots L_{\omega_\ell} : H^{k-\ell}(X) \to H^{k+\ell}(X)$$

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Theorem: Let $\ell \leq k$ and $\omega_1, \ldots, \omega_{\ell+1}$ be Kähler classes. Then

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where

$$\tilde{H}_0^{k-\ell}(X) := H^{k-l}(X) \cap \ker(L_{\omega_1} \cdots L_{\omega_\ell} L_{\omega_{\ell+1}})$$

Mixed Hodge-Riemann Bilinear Relations

Theorem: Let $\ell \leq k$ and $\omega_1, \ldots, \omega_{\ell+1}$ be Kähler classes. Then

$$i^{p-q}\, Q(\alpha, L_{\omega_1} \cdots L_{\omega_\ell}\, \bar{\alpha}) \geq 0$$

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$$\alpha \in H^{p,q}(X) \cap \tilde{H}_0^{k-\ell}(X)\,; \quad p + q = k - \ell\,.$$

Moreover, equality holds if and only if $\alpha = 0$.

Simple corollary

Suppose ω_1, ω_2 are Kähler classes. Then MHL says that

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If a middle cohomology class is primitive for one Kähler class it must have a primitive component relative to any other Kähler class.

Example: $\dim X = 2$

For $\dim X = 2$, MHL is the statement that:

$$L_{\omega_1} L_{\omega_2} : H^0(X) \xrightarrow{\cong} H^4(X)\,.$$

This is equivalent to $(*)$ and follows from HRR since Q is positive in $\ker L_{\omega_1} \cap H^{1,1}_{\mathbb{R}}$ and negative in $\mathrm{Im} L_{\omega_2} \cap H^{1,1}_{\mathbb{R}}$.

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Similarly, MHR reduces to showing that:

$$Q(1, L_{\omega_1} L_{\omega_2} \cdot 1) = \int_X \omega_1 \cup \omega_2 > 0$$

which again follows from $(*)$.

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Suppose now that $\dim X = 4$, then given a Kähler class ω:

$$H^{2,2} = H_0^{2,2} \oplus L_\omega H_0^{1,1} \oplus L_\omega^2 H_0^{0,0}$$

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Sign no longer implies existence of primitive component.

In fact, I don't know of an elementary argument to show that

$$\ker(L_{\omega_1}) \cap \mathrm{Im}(L_{\omega_2}) \cap H^{2,2}(X) = \{0\}$$

A little history

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- [Dinh and Nguyên, 2006] Proof of mixed theorems in the compact Kähler case.

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Billera and Lee (1981) proved, combinatorially, that these conditions were sufficient.

Stanley (1980) proved that they were necessary using the cohomology of toric varieties.

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The remaining McMullen conditions reflect the algebra structure of $H^*(X_P)$. So, this proves one direction of the conjecture for simple integral polytopes.

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For integral simplicial polytopes the associated toric variety is an orbifold and all the arguments go through.

A general simplicial polytope may be deformed to a rational one without changing the combinatorial type.

For non-simplicial polytopes the *h*-numbers have been generalized by Stanley (1987). For integral polytopes they are intersection cohomology Betti numbers.

In the non-simplicial case, deformation techniques do not necessarily work since certain combinatorial types are not realized by rational polytopes.

Cohomology algebras of polytopes

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- Combinatorial constructions of cohomology algebras associated with polytopes which reduce to H^* or IH^* when appropriate.
- They satisfy the Lefschetz Theorems and Hodge-Riemann bilinear relations relative to appropriate intersection form.
- They have an even grading so we may define a Hodge structure concentrated in bidegree (p, p).

Alexandrov-Fenchel Inequality

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Alexandrov-Fenchel Inequality (1936):

$$MV(P_1, P_2, \ldots, P_k)^2 \geq MV(P_1, P_1, \ldots, P_k) \cdot MV(P_2, P_2, \ldots, P_k).$$

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The AF inequality is related to the isoperimetric and Bruns-Minkowski inequalities.

Bernstein's Theorem

Bernstein's Theorem: Let P_i be integral and

$$f_i(t) := \sum_{\alpha \in P_i \cup \mathbb{Z}^k} u^i_\alpha \, t^\alpha$$

be Laurent polynomials supported in P_i. Then, for generic coefficients,

$$MV(P_1, \ldots, P_k) = \#\{t \in (\mathbb{C}^*)^k : f_1(t) = \cdots = f_k(t) = 0\}.$$

Algebro-geometric proof of Alexandrov-Fenchel

Khovanskii (1978), Teissier (1978): Consider a suitable (toric) compactification X of the surface in the torus

$$\{t \in (\mathbb{C}^*)^k : f_3(t) = \cdots = f_k(t) = 0\}.$$

Then AF is equivalent to the Hodge inequality for the curves

$$C_i = \mathrm{closure}_X\{f_i = 0\}, \quad i = 1, 2$$

provided $MV(P_1, P_2, P_3, \ldots, P_k) = \langle C_1, C_2 \rangle$ (i.e. no solutions at infinity).

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Gromov (1990): Interpret $MV(\,*\,,\,*\,, P_3, \ldots, P_k)$ as a mixed form:

$$Q(\,*\,, L_{\omega_3} \cdots L_{\omega_k}\,*\,)$$

in $H^{1,1}(Y)$ for a suitable toric variety Y.

Polarized Hodge-Lefschetz Modules

Let V_* be a $\mathbb{Z}$-graded finite-dimensional real vector space, k a positive integer, and Q a non-degenerate real bilinear form of parity $(-1)^k$. Let $\mathfrak{a} \subset \mathfrak{o}_{-2}(V, Q)$ be an abelian subspace and $N_0 \in \mathfrak{a}$. Then $(V_*, Q, \mathfrak{a}, N_0)$ is said to be a *polarized Hodge-Lefschetz module of weight k* if the following are satisfied:

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- There is a bigrading

$$V_{\mathbb{C}} = \bigoplus_{0 \le p,q \le k} V^{p,q}; \quad V^{q,p} = \overline{V^{p,q}}, \qquad (**)$$

such that

$$(V_\ell)_{\mathbb{C}} = \bigoplus_{p+q=\ell+k} V^{p,q}.$$

Hence, the bigrading $(**)$ restricts to a Hodge structure of weight $k + \ell$ on V_ℓ.

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We let $\mathcal{K}$ denote the connected cone containing N_0 of elements in $\mathfrak{a}$ for which the Lefschetz and polarization properties are preserved. (*Polarizing cone*)

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$\mathfrak{a}$ is a vector space of maps conewise linear in the normal fan of P. The polarization cone consists of strictly convex maps.

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$$V_\ell = \bigoplus_{p+q=k+\ell} (I^{p,q}(W, F_0))_{\mathbb{R}}$$

Example 3 continued

Theorem (Griffiths, Schmid, Kaplan, C.): If $N_0 \in \mathcal{K}$, then

$$(V_*, Q, \mathfrak{a}, N_0)$$

is a polarized Hodge-Lefschetz module. Moreover, every polarized Hodge-Lefschetz module arises in this way.

Main Result

Theorem : The mixed versions of the Hard Lefschetz Theorem, the Lefschetz Decomposition, and the Hodge-Riemann Bilinear Relations hold for Polarized Hodge-Lefschetz modules.

Descent Lemma

Suppose $(V_*, Q, \mathfrak{a}, N_0)$ is a polarized Hodge-Lefschetz module of weight k and let $T \in \mathcal{K}$. Set:

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Theorem [CKS, 1987]: $(\tilde{V}_*, \tilde{Q}, \tilde{\mathfrak{a}}, \tilde{N}_0)$ is a polarized Hodge-Lefschetz module of weight $k - 1$ with bigrading

$$\tilde{V}^{p,q} = T \cdot V^{p+1,q+1}.$$

Example: $k = 2$

$$
\begin{array}{llllll}
V_2 & & & V^{2,2} & & \\
V_1 & & V^{2,1} & \downarrow T & V^{1,2} & \\
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$$\mathrm{Im}\, T \cap \ker N_0 \cap V_0 = \{0\}$$

Mixed Hard Lefschetz Theorem

Theorem: Let $(V_*, Q, \mathfrak{a}, N_0)$ be a polarized Hodge-Lefschetz module of weight k, $\ell \leq k$, and $T_1 \ldots T_\ell \in \mathcal{K}$. Then the map

$$T_1 \cdots T_\ell \colon V_\ell \to V_{-\ell}$$

is an isomorphism.

Proof of the Mixed Hard Lefschetz Theorem

It suffices to show that for $T_1, \ldots, T_\ell \in \mathcal{K}$, the map:

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Consider $\tilde{V} = T_2 \cdots T_\ell \cdot V$, then

$$u \in T_2 \cdots T_\ell \cdot V_\ell = \tilde{V}_1$$

Now, by the Descent Lemma T_1 satisfies the Lefschetz property in $\tilde{V}$, therefore:

$$\ker T_1 \cap \tilde{V}_1 = \{0\}.$$

Therefore $u = 0$, and the result follows inductively.