Toeplitz operators and asymptotic equivariant index

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Introduction

We describe generalized Szegö projectors and Toeplitz operators, which generalize pseudo-differential operators on arbitrary contact manifolds. An important case arises from CR analysis. In this contact context, the analogue of Fourier integral transformations is easy to visualize.

For general elliptic systems of Toeplitz operators there is no K-theoretical or cohomological formula relating the index of such a system to its symbol (because the Toeplitz space is only defined via its "symbol" up to a finite dimensional space, so the index is not defined at all - although there is one for elliptic matrices of Toeplitz operators). However in presence of a compact group action, there is a good notion of "asymptotic trace" (generalizing the Wodzicki residual trace) and "asymptotic index", an avatar of M.F. Atiyah's index theory for relatively elliptic equivariant pseudodifferential operators

Toeplitz operators

In this section we recall the mechanism of generalized Szegö projectors and Toeplitz operators (references:[10, 7, 9]).

As in [10, 7, 9], we call symplectic cone a smooth (paracompact) manifold which is a principal $\mathbb{R}_+^\times$ bundle, equipped with a symplectic form ω homogeneous of degree 1. The Liouville form is its horizontal primitive $\lambda = \rho \lrcorner \omega$ ($\omega = d\lambda$), where ρ denotes the radial (Euler) vector field, infinitesimal generator of homotheties. The basis $X = \Sigma/\mathbb{R}_+^\times$ is an oriented contact manifold; its contact form λ_X (any pull of λ by a smooth section) is defined up to a smooth positive factor, and Σ is canonically identified with the set of positive multiples of λ_X in T^*X.

Microlocal model

We first describe the microlocal model for generalized Szegö projectors given in [3]. Let $(x, y) = (x_1, \ldots, x_p, y_1, \ldots, y_q)$ denote the variable in $\mathbb{R}^{p+q}$. We consider the system of pseudodifferential operators $D = (D_j)$ with

$$D_j = \partial_{y_j} + |D_x| y_j \ (j = 1, \ldots, q)$$

The D_j commute; the complex involutive variety char D is defined by the complex equations $\eta_j - i|\xi|y_j = 0$; it is $\gg 0$, in the sense of [22, 23]. Its real part is the symplectic manifold $\Sigma : \{\eta_j = y_j = 0\}$.

The kernel of D in L^2 is the range of the Hermite operator H (in the sense of [3]) defined by its partial Fourier transform:

$$f \in L^2(\mathbb{R}^p) \mapsto Hf \quad \text{with} \quad \mathcal{F}_x Hf(\xi, y) = (\pi^{-1}|\xi|)^{\frac{q}{4}} e^{-\frac{1}{2}|\xi|y^2} \hat{f}(\xi)$$

The orthogonal projector on $\ker D$ is $S = HH^*$:

$$f \mapsto (2\pi)^{-p} \int_{\mathbb{R}^{2p+q}} e^{i(<x-x',\xi>+i\frac{1}{2}(y^2+y'^2)} (\pi^{-1}|\xi|)^{\frac{q}{2}} f(x',y') dx' dy' d\xi$$

As H, it is a Fourier integral operator, whose complex canonical relation is $\gg 0$, with real part the graph of Id_Σ (Fourier integral operators are described in [21], Fourier integral operators with complex canonical relation are described in [22, 23]).

Generalized Szegö projectors

Let M be a compact manifold, and $\Sigma \subset T^\bullet M$ a symplectic subcone ($T^\bullet M$ denotes T^*M deprived of its zero section). A generalized Szegö projector associated to Σ (or Σ-Szegö projector) is a self adjoint[1] elliptic Fourier integral projector S of degree 0 ($S = S^* = S^2$), whose complex canonical relation $\mathcal{C}$ is $\gg 0$, with real part the diagonal $\operatorname{diag} \Sigma$ (elliptic means that the principal symbol of S does not vanish on Σ).

From [10, 7, 9], we recall:

[1]the requirement that S be self adjoint is convenient but not essential

1) A Σ-Szegö projector S always exists. It is microlocally isomorphic (mod. some elliptic FIO transformation) to the model above.
We will denote $\mathbb{H} \subset C^{-\infty}(M)$ its range. Modulo C^{∞}, it defines a sheaf $\mu\mathbb{H}$ on Σ - a subsheaf supported by Σ of the sheaf of microfunctions on $T^{\bullet}M$.

2) Toeplitz operators defined by S are the operators on $\mathbb{H}$ of the form $u \in \mathbb{H} \mapsto T_P(u) = SPS(u)$ with P a pseudodifferential operator on M. More generally, if P is any FIO whose canonical relation is complex positive, with real part containing $\operatorname{diag}\Sigma$, then SPS is a Toeplitz operator.
Modulo operators of degree $-\infty$ (smoothing operators), Toeplitz operators form a sheaf $\mathcal{A}_\Sigma$ of algebras on Σ, acting on $\mu\mathbb{H}$; $(\mathcal{A}_\Sigma, \mu\mathbb{H})$ is locally isomorphic to the sheaf of pseudodifferential operators in p real variables ($2p = \dim\Sigma$), acting on the sheaf of microfunctions. The principal symbol (principal part) of T_P is $\sigma(P)_{|\Sigma}$.

3) If S, S' are two Σ-Szegö projectors with range $\mathbb{H}, \mathbb{H}'$, S' induces a quasi isomorphism $\mathbb{H} \to \mathbb{H}'$ (the restriction of SS' to $\mathbb{H}$ is a positive (≥ 0) elliptic Toeplitz operator).

More generally, if $\Sigma \subset T^\bullet M, \Sigma' \subset T^\bullet M'$ are two symplectic cones and $f : \Sigma \to \Sigma'$ a homogeneous symplectic isomorphism, there always exists a Fourier integral operator F from M to M', inducing an "elliptic" Fredholm map $\mathbb{H} \to \mathbb{H}'$, e.g. there exists a complex canonical relation $\mathcal{C} \gg 0$ with real part the graph of f, and we may take $F = S' \circ F'$ where F' is any elliptic FIO with canonical relation $\mathcal{C}$ (such elliptic FIO exist, they were called "adapted" in [10, 7]). Thus the pair $\mathcal{A}_\Sigma, \mu\mathbb{H}$ consisting of the sheaf micro Toeplitz operators (i.e. smoothing operators), acting on $\mu\mathbb{H}$ is well defined, up to (non unique) isomorphism: it only depends on the symplectic cone Σ, not on the embedding.

Holomorphic case

An first example of Toeplitz structure is $\Sigma = T^\bullet M$ (M a compact manifold), $S = \mathrm{Id}$: the Toeplitz algebra is the algebra of pseudodifferential operators acting on the sheaf of microfunctions on M.

In general, as noted above, the basis $X = \Sigma/\mathbb{R}_+^\times$ of Σ is a contact manifold, and Σ can be canonically embedded in $T^\bullet X$ as the set of positive multiples of the contact form. An important particular case is the holomorphic case: X is the smooth, strictly pseudoconvex boundary of a Stein complex manifold; the contact form of X is the form induced by $\mathrm{Im}\,\partial\phi$ where ϕ is any defining function ($\phi = 0, d\phi \neq 0$ on X, $\phi < 0$ inside - e.g. if X is the unit sphere bounding the unit ball of $\mathbb{C}^n$, with defining function $\bar{z} \cdot z - 1$, the contact form is $\mathrm{Im}\,\bar{z} \cdot dz_{|X}$). Then the Szegö projector S is the orthogonal projector on the space of boundary values of holomorphic functions in $L^2(X)$ (the fact that it is Fourier integral operator as above was proved in [4]).

The pseudodifferential algebra is a special case of holomorphic Toeplitz algebra: if M is a manifold, it has a real analytic compact manifold; if M^c is a complexification of M, small tubular neighborhoods of M in M^c (for some hermitian metric) are Stein manifold with strictly complex boundary $X \sim S^*M$, and the pseudodifferential algebra of M acting on microfunctions is isomorphic to the Toeplitz algebra of X acting on $\mathbb{H}$. In fact there exists an adapted Fourier integral operator from M to X which defines an isomorphism from $C^{-\infty}(M)$ to $\mathbb{H}(X)$[2] and interchanges pseudodifferential operators on M and Toeplitz operators on X.

[2] e.g. $e^{i\epsilon A}$ with $A = \sqrt{-\Delta}$ for some real analytic Riemannian metric on M, cf [5].

Fourier integral operators

The analogue of Fourier integral transformations is the following: let X, X' be two contact manifolds, S, S' generalized Szegö projectors, anf $f : X \to X'$ a contact isomorphism. The pushforward map $u \mapsto f_* u = u \circ f^{-1}$ does not send $\mathbb{H}$ to $\mathbb{H}'$: we correct it as for Toeplitz operators $T_f(u) = S'(f_* u)$; this behaves as an elliptic Fourier operator attached to the contact map f. All other analogues of F.I.O attached to f are (mod smoothing operators) of the form $u \mapsto A' T_f u$, A' a Toeplitz operator on X'.

The Atiyah-Weinstein problem can be described as follows: If X is a compact contact manifold, and S, S' to Szegö projectors defined by two embeddable CR structures giving the same contact structure, then the restriction of S' to $\mathbb{H}$ is a Fredholm operator $\mathbb{H} \to \mathbb{H}'$ (SS' induces an elliptic Toeplitz operator on $\mathbb{H}$). In this case the spaces $\mathbb{H}, \mathbb{H}'$ and the index are well defined. The Atiyah-Weinstein conjecture computes the index in terms of topological data of the situation (topology of the holomorphic fillings of which X is the boundary).

Equivariant Toeplitz algebra

Let G be a compact Lie group, dg its Haar measure ($\int dg = 1$), $\mathfrak{g}$ its Lie algebra.
Let Σ be a G- symplectic cone (with compact basis), ω its (invariant) symplectic form, λ the Liouville form ($\omega = d\lambda$). As mentioned above, the basis $X = \Sigma/\mathbb{R}_+^\times$ is a G-compact oriented contact manifold; replacing it by its G-mean, we may choose an invariant form λ_X defining the contact structure, and Σ is canonically identified with the set of positive multiples of λ_X in T^*X.

As was shown in [10, 7], the statements of §1 allow a compact group action: if M is a compact G-manifold and Σ is embedded as an invariant symplectic subcone of $T^\bullet M$, there exists an G-invariant generalized Szegö projector associated to Σ [3]; if S' is another one, it induces an equivariant Fredholm map $\mathbb{H} \to \mathbb{H}'$, and more generally if u is an equivariant isomorphism $\Sigma \subset T^\bullet M \to \Sigma' \subset T^\bullet M'$, there exists an equivariant adapted FIO F inducing an equivariant elliptic Toeplitz FIO $\mathbb{H} \to \mathbb{H}'$.

[3] e.g. the Szegö projector of an invariant embeddable CR structure is invariant.

If S is an equivariant generalized Szegö projector, G acts on $\mathbb{H}$ and on the Toeplitz algebra, so as on their microlocalization $\mu\mathbb{H}, \mathcal{A}_{\boldsymbol{\Sigma}}$. The infinitesimal generators of G (vector fields image of elements $\xi \in \mathfrak{g}$) define Toeplitz operators T_ξ of degree 1 on $\mathbb{H}$ and $\mathbb{H}_E$. The elements of G act as unitary Fourier integral operators - or "Toeplitz-FIO's".

The Toeplitz space $\mathbb{H}_E$ (and its Sobolev counterparts) splits according to the irreducible representations of G: $\mathbb{H} = \widehat{\bigoplus} \mathbb{H}_\alpha$.

Equivariant trace

The G-trace and G-index (relative index in [1]) were introduced by M.F. Atiyah in [1] for equivariant pseudo-differential operators on a G-manifold. The G-trace of P is a distribution on G, describing $\operatorname{tr}(g \circ P)$. Here we adapt this to Toeplitz operators.

Below we will use the following extension: an equivariant Toeplitz bundle is the range of an equivariant Toeplitz projector P of degree 0 on some $\mathbb{H}^N$. The symbol of $\mathbb{E}$ is the range of the principal symbol of P; it is an equivariant vector bundle on X; any equivariant vector bundle on X is the symbol of an equivariant Toeplitz bundle. We will denote by $\mathbb{E}^{(s)}$ its space of Sobolev H^s sections.

If $\mathbb{E}, \mathbb{F}$ are two equivariant Toeplitz bundles, there is an obvious notion of Toeplitz (matrix) operator $P : \mathbb{E} \to \mathbb{F}$, and of its principal symbol $\sigma_d(P)$ if it is of degree d, which is a homogeneous vector-bundle homomorphism $E \to F$ on Σ. P is elliptic if its symbol is invertible; then it is a Fredholm operator $\mathbb{E}^{(s)} \to \mathbb{F}^{(s-d)}$ and has an index which does not depend on s.

Definition

We denote $\operatorname{char}\mathfrak{g}$ (characteristic set of $\mathfrak{g}$) the closed subcone of Σ where all symbols of infinitesimal operators $T_\xi, \xi \in \mathfrak{g}$ vanish.

$\operatorname{char}\mathfrak{g}$ contains the fixed point set Σ^G, whose basis is the fixed point set X^G (because G is compact). The base Z of $\operatorname{char}\mathfrak{g}$ is the set of points of X where all Lie generators $L_\xi, \xi \in \mathfrak{g}$ are orthogonal to λ_X. Note that Σ^G is always a smooth symplectic cone and its base X^G a smooth contact manifold; $\operatorname{char}\mathfrak{g}$ and Z may be singular.

Let $\mathbb{E}$ be an equivariant Toeplitz bundle. If $P : \mathbb{E} \to \mathbb{E}$ is a Toeplitz operator of trace class ($\deg P < -n$), the trace function $\operatorname{Tr}_P^G(g) = \operatorname{tr}(g \circ P)$ is well defined; it is a continuous function on G. It is smooth if P is of degree $-\infty$ ($P \sim 0$). If P is equivariant, its Fourier coefficient for the representation α is $\frac{1}{d_\alpha}\operatorname{tr} P_{|\mathbb{H}_\alpha}$ (d_α the dimension of α).

The following result is an immediate adaptation of the similar result of [1] for pseudo-differential operators.

Proposition Let $P : \mathbb{E} \to \mathbb{E}$ be a Toeplitz operator, with $P \sim 0$ near char $\mathfrak{g}$. Then $\mathrm{Tr}_P^G(g) = \mathrm{tr}\, g \circ P$ is well defined as a distribution on G. If P equivariant, $\mathrm{tr}\, P_{|\mathbb{H}_\alpha}$ is well defined (finite), and we have, in distribution sense:

$$\mathrm{Tr}_P^G = \sum \frac{1}{d_\alpha} \, \mathrm{tr}\, P_{|\mathbb{H}_\alpha} \, \chi_\alpha \tag{1}$$

where α runs over the set of irreducible representation of G, with dimension d_α and character χ_α.

We have seen above that this is true if P is of trace class. Let D be a bi-invariant elliptic operator of order $m > 0$ on G, e.g. the Casimir of a faithful representation (with $m = 2$); its image D_X on X defines an invariant Toeplitz operator $\mathbb{E} \to \mathbb{F}$, with characteristic set char $\mathfrak{g}$.

If $P \sim 0$ near Σ, we can divide it repeatedly by D_X (mod. smoothing operators) and get for any N:

$$P = D_X^N Q + R \quad \text{with R} \sim 0$$

The degree of Q is $\deg P - mN$, so it is of trace class if N is large enough. We set $\mathrm{Tr}_P^G = D^N \mathrm{Tr}_Q^G + \mathrm{Tr}_R^G$: this is well defined as a distribution; the fact that it does not depend on the choice of D, N, Q, R is immediate.

Formula 1 for equivariant operators, obviously follows. Note that the series converges in distribution sense, i.e. the coefficients have at most polynomial growth (with respect to the eigenvalues of D).

More generally if we have let an equivariant Toeplitz complex of finite length:

$$(\mathbb{E}, d): \quad \cdots \to \mathbb{E}_j \xrightarrow{d} \mathbb{E}_{i+1} \to \dots$$

i.e. $\mathbb{E}$ is a finite sequence $\mathbb{E}_k$ of equivariant Toeplitz bundles, $d = (d_k : \mathbb{E}_k \to \mathbb{E}_{k+1})$ a sequence of Toeplitz operators such that $d^2 = 0$. Then for a Toeplitz operator $P : \mathbb{E} \to \mathbb{E}, P \sim 0$ near char $\mathfrak{g}$, its equivariant supertrace $\mathrm{Tr}_P^G = \sum (-1)^k \mathrm{Tr}_{P_k}^G$ is well defined; it vanishes if P is a supercommutator.

Equivariant index

Let $\mathbb{E}_0, \mathbb{E}_1$ be two equivariant Toeplitz bundles. We will say that an equivariant Toeplitz operator $P : \mathbb{E}_0 \longrightarrow \mathbb{E}_1$ is G-elliptic (relatively elliptic in [1]) if it is elliptic on $\operatorname{char} \mathfrak{g}$, i.e. the principal symbol $\sigma(P)$, which is a homogeneous equivariant vector bundle homomorphism $E_0 \longrightarrow E_1$, is invertible on $\operatorname{char} \mathfrak{g}$. Then there exists an equivariant $Q : \mathbb{F} \longrightarrow \mathbb{E}$ such that $QP \sim 1_{\mathbb{E}}, PQ \sim 1_{\mathbb{F}}$ near $\operatorname{char} \mathfrak{g}$. The G-index $\operatorname{Ind} I_P^G$ is then defined as the distribution $\operatorname{Tr}^G_{1-QP} - \operatorname{Tr}^G_{1-PQ}$.

More generally, an equivariant complex $(\mathbb{E}, d)$ as above is G-elliptic if the principal symbol $\sigma(d)$ is exact on $\operatorname{char} \mathfrak{g}$. Then there exists an equivariant Toeplitz operator $s = (s_k : \mathbb{E}_k \longrightarrow \mathbb{E}_{k-1})$ such that $1 - [d, s] \sim 0$ near $\operatorname{char} \mathfrak{g}$ ($[d, s] = ds + sd$). The index (Euler characteristic) is the super trace $I^G_{(\mathbb{E},d)} = \operatorname{supertr}(1 - [d, s]) = \sum (-1)^j \operatorname{Tr}^G_{(1-[d,s])_j}$.

bigskip If P is G-elliptic, for any irreducible representation α, the restriction $P_\alpha : \mathbb{E}_{0,\alpha} \to \mathbb{E}_{1,\alpha}$ is a Fredholm operator with index I_α is finite dimensional (resp. more generally the cohomology H^*_α of $d_{|\mathbb{E}_\alpha}$ is finite dimensional), and we have

$$\mathrm{Ind} I_P^G = \sum \frac{1}{d_\alpha} I_\alpha \; \chi_\alpha \qquad (\text{resp. } \mathrm{Ind}^{\mathrm{G}}_{(\mathbb{E},\mathrm{d})} = \sum \frac{(-1)^{\mathrm{j}}}{\mathrm{d}_\alpha} \dim \mathrm{H}^{\mathrm{j}}_\alpha \; \chi_\alpha) \tag{2}$$

Asymptotic index

The G-index Ind_P^G is obviously invariant under compact perturbation and deformation, so for fixed $\mathbb{E}_j$ it only depends on the homotopy class of the symbol $\sigma(P)$. However it does depend on the choice of Szegö projectors: as mentioned, the Toeplitz bundles $\mathbb{E}_j$ are known in practice only through their symbols E_j, and are only determined up to a space of finite dimension, so as the Toeplitz spaces $\mathbb{H}$. However if $\mathbb{E}, \mathbb{E}'$ are two equivariant Toeplitz bundles with the same symbol, there exists an equivariant elliptic Toeplitz operator $U : \mathbb{E} \to \mathbb{E}'$ with quasi-inverse V (i.e. $VU \sim 1_{\mathbb{E}}, UV \sim 1'_{\mathbb{E}}$). This may be used to transport equivariant Toeplitz operators from $\mathbb{E}$ to $\mathbb{E}'$: $P \mapsto Q = UPV$. Then if $P \sim 0$ on X_0, $Q = UPV$ and VUP have the same G-trace, and since $P \sim VUP$, we have $T_P - T_Q \in C^\infty(G)$. Thus the equivariant G-trace or index are ultimately well defined up to a smooth function on G.

Definition

We define the asymptotic G-trace AsTr_P^G as the singularity of the distribution Tr_P^G (i.e. Tr_P^G mod. $C^\infty(G)$).

If $P \sim 0$, we have $\mathrm{Tr}_P^G \sim 0$, i.e. the sequence of Fourier coefficients is of rapid decrease, $O(c_\alpha)^{-m}$ for all m, where c_α is the eigenvalue of D_G in the representation α.

Definition

If P is elliptic on char $\mathfrak{g}$, the asymptotic G-index AsInd_P^G is defined as the singularity of of Ind_P^G.

It only depends on the homotopy class of the principal symbol $\sigma(A)$, and since it is obviously additive we get :

Theorem

The asymptotic index defines an additive map from $K^G_{X-Z}(X)$ to $Sing(G) = C^{-\infty}/C^{\infty}(G)$($Z \subset X$ denotes the basis of char $\mathfrak{g}$).

$K^G_{X-Z}(X)$ denotes the equivariant K-theory of X with compact support in $X - Z$, i.e. the group of stable classes of triples (E, F, u) where E, F are equivariant G-bundles on X, u an equivariant isomorphism $E \longrightarrow F$ defined near Z, with the usual equivalence relations ($(E, F, a) \sim 0$ if a is stably homotopic near Z to an isomorphism on the whole of X). The asymptotic index is also defined for equivariant Toeplitz complexes, exact near Z.

The sequence of Fourier coefficients $\frac{1}{d_\alpha}\mathrm{tr}\, P_\alpha$ has at most polynomial growth w.r. to the eigenvalues of D_G; if $P \sim 0$ it is of rapid decrease. The Fourier coefficients of the asymptotic index are integers: they are completely determined, except for a finite number of them, by the asymptotic index.

Example : let Σ be a symplectic cone, with free positive elliptic action of $U(1)$, i.e. the Toeplitz generator $A = \frac{1}{i}\partial_\theta$ is elliptic with positive symbol (this is the situation studied in [10]). Then the algebra of invariant Toeplitz operators (mod. C^∞) is a deformation star algebra, setting as "deformation parameter" $\hbar = A^{-1}$. char $\mathfrak{g}$ is empty and the asymptotic trace or index is always defined. The asymptotic trace of any element a is the series $\sum_{-\infty}^{\infty} a_k e^{ki\theta}$, $a_k = \mathrm{tr}\ a_{|\mathbb{H}_k}$, mod smooth functions of θ, i.e. the sequence (a_k) is known mod rapidly decreasing sequences. It is standard knowledge that the sequence (a_k) has an asymptotic expansion:

$$a_k \sim \sum_{k \leq k_0} \alpha_j k^{-j}. \tag{3}$$

In this case the asymptotic trace is as well defined by this asymptotic expansion; it encodes the same thing as the residual trace.

Remark. For a general the circle group action, with generator $A = e^{i\theta}$, all simple representations are powers of the identity representation, denoted T, and all representations occurring as indices can be written as sums.

$$\sum_{k \in \mathbb{Z}} n_k T^k \quad (\text{mod. finite sums}) \tag{4}$$

In fact, using the sphere embedding below, it can be seen that the positive and negative parts of the series have a weak periodcity property: they are of the form

$$\frac{P_{\pm}(T, T^{-1})}{(1 - T^{\pm k})^k}$$

for a suitable polynomial $P_{\pm}$ and some integer k; in other words they represent rational functions whose poles are roots of 1, and whose Taylor series have integral coefficients.

K-theory and embedding

It will be convenient (even though not technically indispensable) to reformulate some constructions above in terms of sheaves of Toeplitz algebras and modules, in particular to follow the index in an embedding (§7).
As above we use the following notation: for distributions, $f \sim g$ means that $f - g$ is C^∞; for operators, $A \sim B$ (or $A = B$ mod. C^∞) means that $A - B$ is of degree $-\infty$, i.e. has a smooth Schwartz kernel. If M is a manifold, $T^\bullet M$ denotes the cotangent bundle deprived of its zero section; it is a symplectic cone with base the cotangent sphere $S^*M = T^\bullet M/\mathbb{R}_+$.

As pointed out above, if Σ is a G symplectic cone, the micro sheaf $\mathcal{A}_\Sigma$ of Toeplitz operators acting on $\mu\mathbb{H}$ are well defined with the action of G, up to (non unique) isomorphism, independently of any embedding $\Sigma \to T^\bullet M$. The asymptotic trace AsTr_P^G resp. index AsInd_P^G are well defined for a section P of $\mathcal{A}_\Sigma$ vanishing (resp. invertible) near $\mathrm{char}\,\mathfrak{g}$.

If M is a G-manifold and $X = S^*M$ ($\Sigma = T^\bullet M$), $\mathcal{A}_\Sigma$ identifies with the sheaf of pseudodifferential operators acting on the sheaf $\mu\mathbb{H}$ of microfunctions on X (note that even in that case the exact index problem does not make sense: a Toeplitz bundle $\mathbb{E}$ on X corresponds to a vector bundle on the cotangent E on X, not necessarily the pullback of a vector bundle on M, so $\mathbb{E}$ is in general at best defined up to a space of finite dimension).

It will be convenient to use the language of $\mathcal{E}$-modules. In the C^∞ category $\mathcal{E}$ is not coherent and general $\mathcal{E}$-module theory is not practical. We will just stick to two useful examples.[4]
If $\mathcal{M}$ is an $\mathcal{A}$-module, resp. a complex of $\mathcal{A}$ modules, it corresponds to a system of pseudodifferential (resp. Toeplitz) operators, whose sheaf of solutions is $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mu\mathbb{H})$. E.g. a locally free complex of $(\mathcal{E}, d)$-modules defines a Toeplitz complex $(\mathbb{E}, D) = \mathrm{Hom}\,(L, \mathbb{H})$.

[4] In proof of the Atiyah-Weinstein conjecture we need to patch together two smooth embedded manifolds near their boundaries: this cannot be done in the real analytic category, where things work slightly better

More generally we will say that a $\mathcal{E}$-module $\mathcal{M}$ is "good" if it is finitely generated, equipped with a filtration $\mathcal{M} = \bigcup \mathcal{M}_k$ (i.e. $\mathcal{E}_p \mathcal{M}_q = \mathcal{M}_{p+q}$, $\bigcap \mathcal{M}_k = 0$) such that the symbol $\operatorname{gr} \mathcal{M}$ has a finite locally free resolution. We denote $\sigma(\mathcal{M}) = \mathcal{M}_0 / \mathcal{M}_{-1}$, which is a sheaf of C^∞ modules on the basis X; since there exist global elliptic sections of $\mathcal{E}$, $\operatorname{gr} \mathcal{M}$ is completely determined by the symbol, so as the resolution.

It is elementary that a resolution of $\sigma(\mathcal{M})$ lifts to a "good resolution" of $\mathcal{M}$, i.e. a good finite locally free resolution of $\mathcal{M}$[5].
It is also standard that two resolutions of of $\sigma(\mathcal{M})$ are homotopic, and $\sigma(\mathcal{M})$ has locally finite locally free resolutions it also has a global one (because we are working in the C^∞ category on a compact manifold or cone with compact support, and dispose of partitions of unity); this lifts to a global good resolution of $\mathcal{M}$.

[5] the converse is not true: if d is a locally free resolution of $\mathcal{M}$ its symbol is not necessarily a resolution of the symbol of $\mathcal{M}$ - if only because filtrations must be defined to define the symbol and can be modified rather arbitrarily.

If $\mathcal{M}$ is "good", it defines a K-theoretical element $[\mathcal{M}] \in K_Y(X)$ ($Y = \operatorname{supp}\sigma(\mathcal{M})$), viz. the K-theoretical element defined by the symbol of any good resolution (this does not not depend on the resolution of $\sigma(M)$ since any two such are homotopic).

This works just as well in presence of a G-action (one must choose invariant filtrations etc.).

The asymptotic trace and index extend in an obvious manner to endomorphisms of good complexes or modules:

- if $\mathcal{M} = \mathcal{A}^N$ is free, $\operatorname{End}_{\mathcal{A}}(\mathcal{M})$ identifies with the ring of $N \times N$ matrices with coefficients in the opposite ring $\mathcal{A}^{op}$, and if $A = (A_{ij}$ vanishes near char $\mathfrak{g}$ we set $\mathrm{AsTr}^G(A) = \sum \mathrm{AsTr}^G(A_{jj})$.
- If $\mathcal{M}$ is isomorphic to the range $P\mathcal{N}$ of a projector P in a free module $\mathcal{N}$ (this does not depend on the choice of $\mathcal{N}$ our if $A \in \operatorname{End}\mathcal{A}(M)$ we set $\mathrm{AsTr}^G(A) = \mathrm{AsTr}^G(PA)$.
- If (L, d) is a locally free complex and A is a $A = (A_k)$ endomorphism, vanishing near char $\mathfrak{g}$, we set $\mathrm{AsTr}^G(A) = \sum(-1)^k \mathrm{AsTr}^G(A_k)$ (the Euler characteristic or super trace; if A, B are endomorphisms of opposite degrees

- If $\mathcal{M}$ is a good $\mathcal{A}$-module, (L, d) a good locally free resolution of $\mathcal{M}$, $A \in \mathrm{End}_{\mathcal{A}}(\mathcal{M})$, we set $\mathrm{AsTr}^G(A) = \mathrm{AsTr}^G(\widetilde{A}$, where $\widetilde{A}$ is any extension of A to (L, d) (such an extension exists, and is unique up to homotopy i.e. up to a supercommutator).
- Finally if $\mathcal{M}$ is a locally free complex with symbol exact on $\mathrm{char}\,\mathfrak{g}$, or a good $\mathcal{A}$-module with support outside of $\mathrm{char}\,\mathfrak{g}$, it defines a K-theoretical element $[\mathcal{M}] \in K_Z^G(X)$, and its asymptotic index (the supertrace of the identity), is the image by the index map of theorem4 of $[\mathcal{M}]$.

Remark. The equivariant trace or index are defined just as well for modules admitting a projective resolution (projective meaning direct summand of some $\mathcal{A}^N$, with a projector not necessarily of degree 0). What does not work for these more general objects is the relation to topological K-theory.

Let Σ be a G-symplectic cone, embedded equivariantly in $T^\bullet M$ with M a compact G-manifold, and S an equivariant Szegö projector. As recalled in §1, the range $\mu\mathbb{H}$ of S is the sheaf of solutions of an ideal $I \subset \mathcal{E}_M$. The corresponding $\mathcal{E}_M$-module $\mathcal{M} = \mathcal{E}_M/I$ is good as one can see on the microlocal model.

We have $\mathrm{End}_{\mathcal{E}}(\mathcal{M}) = [I : I]$, the set of ψDO a such that $Ia \subset I$ acting on the right. The map $a \mapsto \mathrm{Tr}_a^G(\mathrm{Tr}_a^G f(1) = fa(1))$ is an isomorphism from $\mathrm{End}_{\mathcal{E}}(\mathcal{M})$ to the algebra of Toeplitz operators mod. C^∞. $\mathcal{M}$ is a $\mathcal{E}, \mathcal{E}'$ bimodule.
If $\mathcal{P}$ is a (good) $\mathcal{E}'$-module, the transfered module is $\mathcal{M} \otimes_{\mathcal{E}'} \mathcal{P}$, which has the same solution sheaf
$(\mathrm{Hom}\,(\mathcal{M} \otimes \mathcal{P}, \mathbb{H}) = \mathrm{Hom}\,(\mathcal{P}, \mathrm{Hom}\,(\mathcal{M}, \mathbb{H})$ and
$\mathrm{Hom}\,(\mathcal{M}, \mathbb{H}) = \mathbb{H}')$. Thus the transfer preserves traces and indices.

This extends obviously to the case where Σ is embedded equivariantly in another symplectic cone $\Sigma \subset \Sigma'$: the small Toeplitz sheaf $\mu\mathbb{H}$ is realized as $\mathrm{Hom}_{\mathcal{A}_\Sigma}(\mathcal{M}, \mu\mathbb{H}')$, with $\mathcal{M} = \mathcal{E}/I$ and $I \subset \mathcal{E}$ is the annihilator of the Szegö projector S of Σ.

Theorem

Let X', X be two compact contact G-manifolds and $f : X \to X'$ be an equivariant embedding. Then the K-theoretical push-forward (Bott homomorphism) $K^G_{X-Z}(X) \to K^G_{X'-Z'}(X')$ commutes with the asymptotic G index.

Let $F : \mathcal{A}_\Sigma \to \mathcal{A}'_\Sigma$ be an equivariant embedding of the corresponding Toeplitz algebras (above f), and let $\mathcal{M}$ be the $\mathcal{A}'_\Sigma$-module associated with the Szegö projector S_Σ. We have seen that transfer $\mathcal{P} \mapsto \mathcal{M} \otimes \mathcal{P}$ preserves the asymptotic index.

Lemma

The K-theoretical element (with support in Σ) $[\mathcal{M}] \in K^G_\Sigma(T^\bullet M)$ is precisely the Bott element used to define the Bott isomorphism $K^G(X) \to K^G_X(X')$.[6]

[6] if $f : X \to Y$ is a map between manifolds (or suitable spaces), the K-theoretical push-forward is the topological translation of the Grothendieck

Proof: We have already noticed that $\mathcal{M}$ is good; it has, locally (and globally), a good resolution. Its symbol is a locally free resolution of $\sigma(\mathcal{M}) = C^\infty(X)/\sigma(I)$. Let us identify a small equivariant tubular neighborhood of Σ with the normal tangent bundle N of Σ in Σ'; N is a symplectic bundle; the ideal I endows it with a compatible positive complex structure N^c, i.e. the first order jet of elements of $\sigma(I)$ are holomorphic in the fibers of N^c; if a, b are such symbols be have $\{a, b\}_N = 0$; $\frac{1}{i}\{a, \bar{a}\}_N \gg 0$). In such a neighborhood a good symbol resolution is homotopic to the Koszul complex : the Koszul complex is the complex (E, d) with $E_p = \bigwedge^{-p}(N^{c*})$ (0 if $p > 0$), the differential d at a point with complex coordinates z of N is the interior product (contraction) $d\omega = z \lrcorner \omega$. The K-theoretical element $[(E, d)] \in G^G_\Sigma(\Sigma')$ is precisely the Bott element.

E.g. if $\Sigma' = \mathbb{C}^N - \{0\}$, with Liouville form $\operatorname{Im} \bar{z}.dz$ [7], with basis the unit sphere $X = S^{2N-1}$), $\mathbb{H}$ the space of holomorphic functions on the sphere $X' = S^{2N-1}$, $X \subset X$ the diameter $z_1 = \cdots = z_k = 0$, Σ', $\mathbb{H}' =$ the functions independent of $z_1, \ldots, z_k$, I is the ideal spanned by the Toeplitz operators T_{∂_k}. The transfer module $\mathcal{M}$ is $\mathcal{A}/I$ with $I = \sum_0^k z_j \mathcal{A}$, its resolution is the standard Koszul complex.

Remark. It is always possible to embed a compact contact manifold in a canonical contact sphere with linear G-action:

[7] the coordinates z_j are homogeneous of degree $\frac{1}{2}$.

Lemma

Let Σ be a G cone (with compact base), λ a horizontal 1-form, homogeneous of degree 1 ($L_\rho\lambda = \lambda, \rho\lrcorner\lambda = 0$, where ρ is the radial vector field, generating homotheties). Then there exists a homogeneous embedding $x \mapsto Z(x)$ of Σ in a complex representation V^c of G such that $\lambda = Im\,\bar{Z}.dZ$

In this construction, Z must be homogeneous of degree $\frac{1}{2}$ as above. This applies of course is Σ is a symplectic cone, λ its Liouville form (the symplectic form is $\omega = d\lambda$ and $\lambda = \rho\lrcorner\omega$. We first choose a smooth equivariant function $Y = (Y_j)$, homogeneous of degree $\frac{1}{2}$, realizing an equivariant embedding of Σ in $V - \{0\}$, where V is a real unitary G-vector space (this always exists if the basis is compact). Then there exists a smooth function $X = (X_j)$ homogeneous of degree $\frac{1}{2}$ such that $\lambda = 2X.dY$. We can suppose X equivariant, replacing it by its mean $\int g.X(g^{-1}x)\,dg$ if need be. Since Y is of degree $\frac{1}{2}$ we have $2\rho\lrcorner dY = Y$ hence $X.Y = \rho\lrcorner = 0$. Finally we get $\lambda = \mathrm{Im}\,\bar{Z}.dZ \quad \text{with } \mathrm{Z} = \mathrm{X} + \mathrm{iY}$ (the coordinates z_j on V are homogeneous of degree $\frac{1}{2}$ so that the canonical form $\mathrm{Im}\,\bar{Z}.dZ$ is of degree 1)

[1] Atiyah, M.F. *Elliptic operators and compact groups.* Lecture Notes in Mathematics, Vol. 401. Springer-Verlag, Berlin-New York, 1974.

[2] Boutet de Monvel L., Leichtnam E., Xiang Tang, Weinstein A. *Asymptitic equivariant index of Toeplitz operators and relative index of CR structures.* arXiv:0808.1365v1; Duistermaat 65 volume, Progress in Math, Birkhaüser.

[3] Boutet de Monvel, L. *Hypoelliptic operators with double characteristics and related pseudodifferential operators.* Comm. Pure Appl. Math. 27 (1974), 585-639.

[4] Boutet de Monvel, L.; Sjöstrand, J. *Sur la singularité des noyaux de Bergman et de Szegö.* Astérisque 34-35 (1976), 123-164.

[5] Boutet de Monvel, L. *Convergence dans le domaine complexe des séries de fonctions propres.* C.R.A.S. 287 (1978), 855-856.

[6]Boutet de Monvel, L. *On the index of Toeplitz operators of several complex variables.* Inventiones Math. 50 (1979) 249-272.

[7] Boutet de Monvel, L. *Symplectic cones and Toeplitz operators* (actes du congrès en l'honneur de Trèves, Sao Carlos), Contemporary Math., vol. 205 (1997) 15-24.

[8] Boutet de Monvel, L. Logarithmic singularity of Toeplitz projectors *Math. Res. Lett. 12, (2005) 401-412, arXiv:math.CV/0412252v1*

[9] Boutet de Monvel, L. *Vanishing of the logarithmic trace of generalized Szegö projectors.* arXiv:math.AP/0604166 v1, Proceedings of the Conference "Algebraic Analysis of Differential Equations" in honor of Prof. T. Kawai, Springer Verlag, 2007.

[10] Boutet de Monvel, L.; Guillemin, V. *The Spectral Theory of Toeplitz Operators*. Ann. of Math Studies no. 99, Princeton University Press, 1981.

[11] Boutet de Monvel L., Leichtnam E., Xiang Tang, Weinstein A. *Asymptotic equivariant index of Toeplitz operators and relative index of CR structures* (to appear).

[13] Boutet de Monvel, L.; Malgrange, B. *Le théorème de l'indice relatif.* Ann. Sci. ENS., 23 (1990), 151-192.

[14] Boutet de Monvel L., Leichtnam E., Xiang Tang, Weinstein A. *On the index of a quantized contact transformation* (en préparation).

[15] M. Englis:*Weighted Bergman kernels and quantization*. Comm. Math. Phys. 227, pages 211-241, 2002.

[16] M. Englis:*Berezin quantization and reproducing kernels on complex domains*. Trans. Amer. Math. Soc. 349, pages 411-479, 1996.

[17] Epstein, C. *Subelliptic Spin$_c$ Dirac operators, I*. Ann. of Math. (2) 166 (2007), no. 1, 183–214.

[18] Epstein, C. *Subelliptic Spin$_c$ Dirac operators, II*. Ann. of Math.(2), 166 (2007), no.3, 723-777.

[19] Epstein, C. *Subelliptic Spin$_c$ Dirac operators, III, the Atiyah-Weinstein conjecture*. Ann. of Math., 168 (2008), 299-365.

[20] Epstein, C. *Cobordism, relative indices and Stein fillings* . arXiv:0705. 1702v1 [math.AP], to appear in the proceedings of the conference in honor of G. Henkin.

[21] Epstein, C.; Melrose, R. *Contact degree and the index of Fourier integral operators.* Math. Res. Lett. 5 (1998), no. 3, 363-381.

[22] Hörmander, L. Fourier integral operators I. Acta Math. 127 (1971), 79-183.

[23] Melin, A.; Sjöstrand, J. *Fourier Integral operators with complex valued phase functions* Lecture Notes 459 (1975) 120-223.

[24] Melin, A.; Sjöstrand, J. *Fourier integral operators with complex phase functions and parametrix for an interior boundary value problem* Comm. P.D.E. 1:4 (1976) 313-400.

[25] A. Weinstein: *Some questions about the index of quantized contact transformations* RIMS Kokyuroku No. 1014, pages 1-14, 1997.