On slope filtrations

Yves André, CNRS, ENS Ulm Paris

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Abstract: One encounters many "slope filtrations" (indexed by rational numbers) in algebraic geometry, asymptotic analysis (of linear differential equations), ramification theory and p-adic theories.

We outline a unified treatment of their common features, and survey how new ties between various mathematical domains have been woven *via* deep correspondences between different slope filtrations.

I. Four basic examples of slope filtrations.

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I.1. *General setting.* We deal with *descending* filtrations of objects M (of some category $\mathcal{C}$) by subobjects $F^{\geq\lambda} M$ indexed by $\lambda \in \mathbb{Q}$.

The filtration is supposed to be left-continuous and locally constant: it comes from a finite *flag*

$$0 \subset F^{\geq\lambda_1} M \subset \ldots \subset F^{\geq\lambda_r} M = M$$

where the $\lambda_1 > \ldots > \lambda_r$ are the *breaks* of the filtration. It is assumed that one can form the graded pieces $gr^\lambda M$ (in the category $\mathcal{C}$).

It is also assumed that objects of $\mathcal{C}$ have a well-defined *rank* in $\mathbb{N}$ (typically they are linear objects with extra structure, and the rank/dimension refers to the underlying linear structure).

This allows to attach to any object M its Newton polygon:

deg M

0 rk M

The "principle" is that, in the presence of slope filtrations, *one can "unscrew" objects M according to their Newton polygons, functorially in M*.

Although the breaks λ_i are not necessarily integral, it happens in all "natural examples" that the vertices of the Newton polygons always have *integral* coordinates (deg $M \in \mathbb{Z}$).

I.2. *In Asymptotic Analysis:*

the Turrittin-Levelt filtration. A fundamental fact of asymptotic analysis is the ubiquity of Gevrey series of (precise) rational order $s \in \mathbb{Q}$: power series $\sum a_n x^n$ such that $\sum \frac{a_n}{n!^s} x^n$ has finite radius of convergence: namely,

any infinite power series which occurs as a solution (or in the asymptotic expansion of a solution) of a (linear or non-linear) analytic differential equation is Gevrey of some precise order $s \in \mathbb{Q}$.

In the linear case, for $s > 0$, Ramis showed that this reflects (up to an inversion $s = 1/\lambda$) the so-called Turrittin-Levelt filtration of the corresponding differential module M ("localized" at the singularity).

$\partial = x\frac{d}{dx}$, acting on $(K, v) = (\mathbb{C}((x)), ord_0)$.

$\mathcal{C} = \{K\langle\partial\rangle$-modules of finite length$\}$ (*differential modules*)

Any $M \in \mathcal{C}$ is of the form $K\langle\partial\rangle / K\langle\partial\rangle.P$ for some $P = \sum_0^n b_i \, \partial^i$, $b_n = 1$. P factors according to its Newton polygon:

$$(i, -v(b_{n-i}))$$

$NP(M) := NP(P)$ is independent of P: comes from the Turrittin-Levelt slope filtration of M. $\deg M$ is the so-called *irregularity* of M.

Note: This theory is now being generalized to higher dim. (Sabbah, A., Mochizuki).

I.3. *In Number Theory:*

• *Ramification theory:* **the Hasse-Arf filtration.**

(K, v): complete discretely valued field with perfect residue field, $G_K = Gal(K^{sep}/K)$. By analysing the "norm" of $g - id$ acting on finite extensions L/K, ramification theory provides a decreasing sequence of normal subgroups

$$G_K^{(\lambda)} \triangleleft G_K,\ \lambda \in \mathbb{Q}_+$$

$\to$ slope filtration of objects of

$\mathcal{C} = \{F$-linear "finite" representations of $G_K\}$ (*finite Galois representations*)

(F: auxiliary field of car. 0).

$K^{sep} = \{\text{Puiseux series } \sum a_i x^{i/n}\}$
$|$
$K = \mathbb{C}((x))$

K^{sep}
$|$
$K^{tame} = \{\text{Puiseux series}\} \quad (p \nmid n)$
$|$
$K = \bar{\mathbb{F}}_p((x))$

$K^{sep} = \bar{\mathbb{Q}}_p$
$|$
K^{tame}
$|$
$K = \mathbb{Q}_p$

Hasse-Arf: $NP(M)$ has integral vertices (deg M, the so-called *Swan conductor* of M, is an integer).

Note: Recent generalization to the case of non-perfect residue field, using non-archimedean rigid geometry (Abbes-Saito, Xiao).

• *Difference modules:* **the Dieudonné-Manin filtration.**

(K, v): complete discretely valued field of car. 0, ϕ isometry of K of infinite order.

$\mathcal{C} := \{\phi$-modules of finite K-dim.$\}$

$\Phi : M \otimes_{K,\phi} K \cong M$

Any $M \in \mathcal{C}$ is of the form $K\langle\phi\rangle / K\langle\phi\rangle.P$ for some $P = \sum_0^n a_i \phi^i,\ a_n = 1, a_0 \neq 0$. P factors according to its Newton polygon:

$$(i, -v(a_{n-i}))$$

$NP(M) := NP(P)$ is independent of P: comes from a slope filtration of M (which splits if ϕ is invertible).

• *F-isocrystals:* v: p-adic, $\phi =$ power of Frobenius $\to$ (descending) **Dieudonné-Manin filtration**.

• *q-difference equations:*

$(K, v) = (\mathbb{C}((x)), ord_0)$, $\phi(x) = qx$.

(in the context of q-calculus: $n \mapsto n_q = 1 + q + \ldots + q^{n-1}$)

The slope filtration also exists on the (non-complete) field $\mathbb{C}(\{x\})$ of germs of meromorphic functions at 0: the **Adams-Sauloy filtration**, related to the Bézivin's q-analogues of Gevrey series if $|q| \neq 1$...).

I.4. *In Algebraic Geometry:*

the Harder-Narasimhan filtration.

X: smooth projective curve $/\mathbb{C}$

$M \neq 0$: vector bundle on X.

deg $M = \deg\det M$, $\mu(M) = \frac{\deg M}{\mathrm{rk\,M}}$ (slope)

M *semistable* iff for any subbundle $N \neq 0$, $\mu(N) \leq \mu(M)$ (Mumford)

Any M has a (unique) Harder-Narasimhan flag

$$0 \subset M_1 = F^{\geq\lambda_1} M \subset \ldots \subset M_r = M = F^{\geq\lambda_r} M$$

where the $\lambda_1 > \ldots > \lambda_r$, and $gr^{\lambda_i}M$ is semistable of slope $\lambda_i = \mu(gr^{\lambda_i}M)$.

Note: Fundamental in moduli theory. Many generalizations (parabolic bundles, Higgs bundles, higher dimension, derived categories...).

II. A unified context.

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II.1. Additive category $\mathcal{C}$, Short exact sequences (S.E.S):

$$0 \to N \xrightarrow{f} M \xrightarrow{g} P \to 0,$$

$$f = \ker g,\ g = \operatorname{coker} f;$$

N strict subobject, P strict subquotient

$\mathcal{C}$ *quasi-abelian* if

- any morphism f has a kernel and a cokernel

- strict quotients (strict subobjects) preserved by pull-back (push-out). (Schneiders,...)

Ex: Vector bundles on a smooth curve.

Rank function: rk : $\mathcal{C} \to \mathbb{N}$: additive on S.E.S, $\operatorname{rk} M = 0 \Leftrightarrow M = 0$

II.2. Slope functions and slope filtrations.

Slope function: $\mu : \mathcal{C} \setminus 0 \to \mathbb{Q}$,

- $N \to M$ mono+epi $\Rightarrow \mu(N) \leq \mu(M)$

- deg := rk $\times\ \mu$ additive on S.E.S.

Note: if $\mathcal{C}$ abelian, a slope function is just an additive function deg divided out by rk.

M *semistable* iff for any [strict] subobject $N \neq 0$, $\mu(N) \leq \mu(M)$

Slope filtration: separated, exhaustive, left-continuous, descending filtration of objects M by *strict* subobjects $F^{\geq\lambda}M$, $\lambda \in \mathbb{Q}$,

($\rightarrow$ flag: $0 \subset F^{\geq\lambda_1} M \subset \ldots \subset F^{\geq\lambda_r} M = M$,

$\lambda_1 > \ldots > \lambda_r$), satisfying:

- $F^{\geq\lambda}M$ is *functorial* in M,

- $gr^\lambda gr^\lambda M = gr^\lambda M$,

- $\mu(M) := \frac{\sum \lambda \,.\, \mathrm{rk}\, gr^\lambda M}{\mathrm{rk}\, M}$ is a slope function.

Then:

• the associated flag is the unique one with $gr^\lambda M$ semistable of slope λ.

• bijection: $\{$ slope filtrations $\} \leftrightarrow \{$ slope functions$\}$.

Now, $(\mathcal{C}, \otimes)$ F-linear.

Two types of slope filtrations, according to the behaviour w.r.t. $\otimes$:

1) breaks($M_1 \otimes M_2$) $\leq$ max(breaks (M_1), breaks (M_2)),

M semistable slope $\lambda \Rightarrow M^\vee$ too .

Such filtrations split: $M = grM$ (ex: Turrittin-Levelt, Hasse-Arf)

2) M_i semistable slope $\lambda_i, i = 1, 2, \Rightarrow M_1 \otimes M_2$ semistable slope $\lambda_1 + \lambda_2$ and $M_i^\vee$ semistable slope $-\lambda_i$.

Then $\deg M = \deg \det M$ (ex: Dieudonné-Manin, Harder-Narasimhan).

III. Variation of Newton polygons in families.

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III. 1. Families of ϕ-modules.

Family of ϕ-modules parametrized by an algebraic variety S of $char.p$ (F-isocystal $\mathcal{M}/S$)

(Dieudonné-Manin's) $NP(\mathcal{M}_s)$ *lower* semi-continuous (Grothendieck); finitely many possible "degenerations".

Note: - variant for a family of q-difference equations (q fixed),

- similar result for families of linear meromorphic differential equations, in the absence of confluence

- $q \to 1$: q-difference equations $\to$ differential equations

$$\frac{f(qx) - f(x)}{q(x-1)} \to \frac{df}{dx} \quad .$$

III. 2. Families of vector bundles.

$\mathcal{M}/X \times S$, flat family of vector bundles on the smooth projective curve X, parametrized by S

(Harder-Narasimhan's) $NP(\mathcal{M}_s)$ *upper* semi-continuous (Shatz); infinitely many possible "degenerations".

IV. Further slope filtrations and correspondences

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IV.1. *p-adic Galois representations of $\bar{\mathbb{F}}_p((x))$ and p-adic differential equations.*

Analogy: Galois rep's/wildness $\leftrightarrow$ differential equations/irregularity

becomes much more precise in the p-adic setting.

Fontaine-Tsuzuki's functor

$D : Rep_{\mathrm{fin}}\, G_{\bar{\mathbb{F}}_p((x))} \hookrightarrow \mathcal{C} = \{$$p$-adic differential equations over a thin annulus of outer radius 1, which admit a Frobenius structure$\}$

The Christol-Mebkhout slope filtration on $\mathcal{C}$

• $Im\ D$: semisimple objects (my proof uses a general structure theorem for slope filtrations $\rightarrow$ structure of $\mathcal{C}$ $\rightarrow$ structure of objects of $\mathcal{C}$).

• D : sends Hasse-Arf filtration to Christol-Mebkhout filtration (Tsuzuki)

Note. Has just been generalized to local field of char. p with imperfect residue field by Xiao.

IV.2. *p-adic Galois representations of p-adic fields and filtered ϕ-modules.* Fontaine's functor

D : $Rep_{crys}\, G_{\mathbb{Q}_p} \hookrightarrow \mathcal{C} = \{\phi$-modules$/\mathbb{Q}_p$, + fitration of the underlying $\mathbb{Q}_p -$ space$\}$

slope fn $\mu(D) := \frac{\text{break}(\det D)) - v(\phi_{|\det D})}{\text{rk}\, M}$,

whence a slope filtration.

• $Im\ D$: semistable objects of slope 0 (Fontaine-Colmez).

Note: using the infinitesimal generator of the cyclotomic quotient of $G_{\mathbb{Q}_p}$, one builds a link between p-adic Galois rep's and p-adic differential equations over a thin annulus with Frobenius structure (Fontaine, ..., Berger).

(p-adic) diff. eq. $\triangleleft$ (p-adic) Galois rep's $\triangleright$ (p-adic) linear algebra.

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