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Harmonic Analysis and Fractal Geometry: A dynamic Interplay

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Measure Theory

1.1 Measures in abstract spaces

We start with the definition of a measure in an abstract setting.

Definition 1.1. *Let Ω be an arbitrary set. A* measure *μ on Ω is a function defined on the subsets of Ω, satisfying:*

i. $0 \leq \mu(A) \leq +\infty$, *for all* $A \subset \Omega$.

ii. $\mu(\emptyset) = 0$.

iii. If $A \subseteq B$ *then* $\mu(A) \leq \mu(B)$.

iv. $\mu(\cup_n A_n) \leq \sum_n \mu(A_n)$ *for every collection* $\{A_n\}_{n\in\mathbb{N}}$ *in* Ω .

Property iii. is referred to as the *monotonicity* of the measure and Property iv. as the *σ-subadditivity*. Note: In most books on measure theory the set-function μ in Definition 1.1 is called an *outer measure*. We adopt here a different terminology since it is more appropriate for the study of Caratheodory's construction of measures. This is also the point of view in [**?**], [**?**], [**?**] and others.

Next, we characterize a particular class of sets in Ω related to μ.

Definition 1.2. *If μ is a measure on Ω, then we will say that a set $E \subset \Omega$ is μ-*measurable *if for every set $A \subset \Omega$,*

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E).$$

Sometimes it is more convenient to use the next property that is equivalent to measurability

Proposition 1.3. *A set $E \subset \Omega$ is μ-measurable, if and only if for every $A \subset E$, $B \subset \Omega \setminus E$,*

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

The equivalence is straightforward.

We will usually refer to a *μ-measurable* set simply as a *measurable* set when it is clear from the context which measure we are considering.

Remark 1.4. To verify that a set E is μ-measurable, it is sufficient to verify that

$$\mu(A) \ \geq \ \mu(A \cap E) + \mu(A \setminus E) \quad \forall A \subset \Omega \tag{1.1}$$

or

$$\mu(A \cup B) \ \geq \ \mu(A) \, + \, \mu(B) \quad \forall A \subset E, B \subset \Omega \setminus E \tag{1.2}$$

since the other inequality always holds, because μ is a measure.

Furthermore, (1.1) or (1.2) only has to be checked for sets A and B of finite measure, since for infinite measure the inequality is trivially true.

Before our first theorem, we need two more definitions that are standard in measure theory.

Definition 1.5. *Let $\mathcal{A}$ be a class of subsets of a set Ω. We will say that $\mathcal{A}$ is a σ-algebra if*

a) $\emptyset \in \mathcal{A}$.

b) If $A \in \mathcal{A}$ *then* $A^c \in \mathcal{A}$.

c) If $A_1, A_2, \ldots$ *are in* $\mathcal{A}$ *then* $\cup_k A_k \in \mathcal{A}$.

Definition 1.6. *We say that a set function ν defined on a σ-algebra of sets $\mathcal{A}$ is a* countably additive (or σ-additive) measure on $\mathcal{A}$ *if ν satisfies,*

a) $0 \leq \nu(A) \leq +\infty$, *for all* $A \in \mathcal{A}$.

b) $\nu(\emptyset) = 0$.

c) If $A, B \in \mathcal{A}$ *and* $A \subset B$, *then* $\nu(A) \leq \nu(B)$.

d) If $A_1, A_2, \cdots \in \mathcal{A}$ *are disjoint, then* $\nu(\cup_k A_k) = \sum_k \nu(A_k)$.

Note that c) can be obtained from b) and d). We include it in the definition to stress the monotonicity of the measure ν.

Theorem 1.7. *Let μ be a measure on Ω. The class $\mathcal{M}_\mu$ of μ-measurable sets is a σ-algebra that contains the sets of μ-measure zero. Furthermore, the measure μ is countably additive on $\mathcal{M}_\mu$.*

Proof. We will prove first that any set of measure zero is measurable. Assume that $N \subset \Omega$, and $\mu(N) = 0$. Consider $A \subset N$ and $B \subset \Omega \setminus N$.

$$\mu(A) \ \leq \ \mu(N) = 0, \quad \text{so} \quad \mu(A) = 0.$$

$$\mu(A \cup B) \ \geq \ \mu(B) = \mu(A) + \mu(B).$$

Using (1.2) we see that N is measurable.

Let us now show that $\mathcal{M}_\mu$ is a σ-algebra. Conditions a) and b) in Definition 1.5 are immediate consequence of the definition of $\mathcal{M}_\mu$.

We will prove now part c). Let A be an arbitrary set in Ω, and consider first a sequence $E_1, E_2, \ldots$ of *disjoint* measurable sets. An induction argument together with the measurability of E_k, shows that for all n, and arbitrary A

$$\mu(A) \;\geq\; \sum_{k=1}^{n} \mu(A \cap E_k) \;+\; \mu\left(A \setminus \bigcup_{k=1}^{n} E_k\right).$$

So

$$\mu(A) \;\geq\; \sum_{k=1}^{n} \mu(A \cap E_k) \;+\; \mu\left(A \setminus \bigcup_{k=1}^{\infty} E_k\right) \quad \forall n.$$

This implies that

$$\mu(A) \;\geq\; \sum_{k=1}^{\infty} \mu(A \cap E_k) \;+\; \mu\left(A \setminus \bigcup_{k=1}^{\infty} E_k\right), \tag{1.3}$$

and finally using the σ-subadditivity of μ,

$$\mu(A) \;\geq\; \mu(A \cap \bigcup_{k=1}^{\infty} E_k) \;+\; \mu(A \setminus \bigcup_{k=1}^{\infty} E_k). \tag{1.4}$$

If in equation (1.3) we let, $A = \bigcup_k E_k$, we get

$$\mu(\bigcup_k E_k) \;\geq\; \sum_k \mu(E_k),$$

and therefore the σ-subaditivity of μ.

To complete the proof of the Theorem it remains to show that the union of arbitrary measurable sets is measurable.

For this, let us first show that if E_1 and E_2 are measurable, then $E_1 \setminus E_2$ is as well. For, let $A \subset E_1 \setminus E_2$ and $B \subset (E_1 \setminus E_2)^c$. Then,

$$A \bigcup B = (A \bigcup (B \setminus E_2)) \bigcup (B \bigcap E_2)\,,$$

and since $A \bigcup (B \setminus E_2) \subset E_2^c$, and $B \setminus E_2 \subset E_1^c$, by the measurability of first E_2 and then E_1 we have that

$$\mu(A \bigcup B) \;\geq\; \mu(A \bigcup (B \setminus E_2)) + \mu(B \bigcap E_2) \;\geq\; \mu(A) + \mu(B \setminus E_2) + \mu(B \bigcap E_2) \;=\; \mu(A) + \mu(B).$$

Now, since Ω is measurable, and $E_1 \bigcup E_2 = (E_1^c \setminus E_2)^c$, we conclude that any finite union of measurable sets is measurable.

Let now $E_1, E_2, \ldots$ be arbitrary measurable sets, and let

$$D_n \;=\; E_n \setminus \bigcup_{k=1}^{n-1} E_k.$$

The D_n are now measurable, disjoint and $\bigcup_k D_k = \bigcup_k E_k$, which finishes our proof.

Corollary 1.8. *If μ is a measure on Ω, then μ is a countably additive measure on $\mathcal{M}_\mu$.*

Corollary 1.9. *If $A \subset \Omega$ is any set and $E_1, E_2, ...,$ is a sequence of disjoint measurable sets, then*

$$\mu(A \cap \bigcup_k E_k) = \sum_k \mu(A \cap E_k) \quad \textit{and} \quad \mu(A) = \sum_k \mu(A \cap E_k) + \mu(A \setminus \bigcup_k E_k).$$

Proof. Using that the sets E_k are disjoint, equation (1.3) now yields for all $A \subset \Omega$,

$$\mu(A) \geq \sum_k \mu(A \cap E_k) + \mu(A \setminus \bigcup_k E_k). \tag{1.5}$$

Now, since

$$A = \left(\bigcup_k (A \cap E_k)\right) \cup \left(A \setminus \bigcup_k E_k\right)$$

we get the other inequality in (1.5) using the monotonicity of the measure.

Since $\bigcup_k E_k$ is measurable we have

$$\mu(A) = \mu(A \cap \bigcup_k E_k) + \mu(A \setminus \bigcup_k E_k). \tag{1.6}$$

Now for every $B \subset \Omega$, using (1.5) for $A = B \cap \bigcup_k E_k$, we have

$$\mu(B \cap \bigcup_k E_k) \geq \sum_k \mu(B \cap E_k) \quad ,$$

which yields the desired statements.

Definition 1.10. *Let I be any index set and let $\{\mu_\alpha\}_{\alpha \in I}$ be a family of measures on Ω. We define the* **supremum** *of the family $\{\mu_\alpha\}_{\alpha \in I}$ to be the set function ν on Ω such that*

$$\nu(A) = \sup_{\alpha \in I} \mu_\alpha(A).$$

Proposition 1.11. *The supremum of a family of measures on Ω is a measure.*

We omit the proof of this Proposition, since it is an immediate consequence of the definition of the supremum.

Note 1.12. The infimum of a family of measures on Ω is not necessarily a measure. See Exercise 9.

We will now see some kind of continuity properties of a measure.

Theorem 1.13. *Let μ be a measure on Ω.*

a) If $E_1 \subset E_2 \subset E_3 \subset \dots$ is a sequence of measurable *sets, then*

$$\mu(\bigcup_k E_k) = \lim_{k\to+\infty} \mu(E_k).$$

b) If $E_1 \supset E_2 \supset E_3 \supset \dots$ is a sequence of measurable *sets, and $\mu(E_k) < +\infty$ for some k, then*

$$\mu(\bigcap_k E_k) = \lim_{k\to+\infty} \mu(E_k).$$

Exercise: Extend Theorem 1.13 to the following case: Let A be an arbitrary set in Ω. If $E_1 \subset E_2 \subset E_3 \subset \dots$ is a sequence of measurable sets, then

$$\mu(A \cap \bigcup_k E_k) = \lim_{k\to+\infty} \mu(A \cap E_k).$$

If $E_1 \supset E_2 \supset E_3 \supset \dots$ is a sequence of *measurable* sets and $\mu(A \cap E_k) < +\infty$, for some k, then

$$\mu(A \cap \bigcap_k E_k) = \lim_{k\to+\infty} \mu(A \cap E_k).$$

Proof. of Theorem 1.13.

To prove a), we observe that the equality is trivially true, if for some k $\mu(E_k) = +\infty$. Then we can assume that $\mu(E_k) < +\infty \ \forall\ k$. Define $D_1 = E_1$ and $D_k = E_k \setminus E_{k-1}$ then the sets in the sequence $\{D_k\}$ are measurable, disjoint and $\bigcup_k D_k = \bigcup_k E_k$. So

$$\begin{aligned}
\mu(\bigcup_k E_k) = \mu(\bigcup_k D_k) &= \sum_{k=1}^{\infty} \mu(D_k) \\
&= \mu(E_1) + \sum_{k=2}^{\infty} \mu(E_k \setminus E_{k-1}) \\
&= \mu(E_1) + \sum_{k=2}^{\infty} (\mu(E_k) - \mu(E_{k-1})) \\
&= \lim_{k\to+\infty} \mu(E_k).
\end{aligned}$$

To prove b), we can assume that $\mu(E_1) < +\infty$. We first write E_1 as disjoint union of measurable sets:

$$E_1 = \bigcup_{k=1}^{\infty} (E_k \setminus E_{k+1}) \cup \bigcap_{k=1}^{\infty} E_k \tag{1.7}$$

$$
\begin{aligned}
\mu(E_1) &= \sum_{k=1}^{\infty} \mu(E_k \setminus E_{k+1}) + \mu(\bigcap_{k=1}^{\infty} E_k) \\
&= \mu(E_1) - \lim \mu(E_k) + \mu(\bigcap_{k=1}^{\infty} E_k)
\end{aligned}
$$

which completes the proof of b).

In the case that the sequence of sets is not monotone, we still can have some weaker results, as we will see in the next theorem.

Definition 1.14. *For an arbitrary sequence of sets $\{A_k\}_{k\in\mathbb{N}}$, define*

$$A_* = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k \quad \text{and} \quad A^* = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k .$$

The sets A_* and A^* are called the lower limit and upper limit of the sequence $\{A_k\}$ respectively and we will denote them also by

$$A_* = \underline{\lim}\, A_k \quad \text{and} \quad A^* = \overline{\lim}\, A_k .$$

Note that a point belongs to A_* if and only if there exists $n \in \mathcal{N}$ such that the point belong to **all** *the sets A_k with $k \geq n$, and a point belongs to A^* if and only if it belongs to* **infinitely many** *sets A_k. Obviously, $A_* \subset A^*$. When $A_* = A^*$ we will say that the sequence has a limit. In particular this is true when the sequence is monotone.*

Theorem 1.15. *Let μ be a measure on Ω. If $E_1, E_2, \ldots$ are measurable and A is any set, then*

a) $\mu(A \cap E_*) \leq \underline{\lim}\, \mu(A \cap E_k)$.

b) If $\mu(A \cap \bigcup_{k\geq n} E_k) < +\infty$ *for some n, then* $\overline{\lim}\, \mu(A \cap E_k) \leq \mu(A \cap E^*)$.

In particular we have

a') $\mu(E_*) \leq \underline{\lim}\, \mu(E_k)$.

b') If $\mu(\bigcup_{k\geq n} E_k) < +\infty$ *for some n, then* $\overline{\lim}\, \mu(E_k) \leq \mu(E^*)$.

The proof is left as an exercise. Note that putting together a) and b) in Theorem 1.15, we have that if $\mu(A \cap \bigcup_{k\geq n} E_k) < +\infty$, then

$$\mu(A \cap E_*) \leq \underline{\lim}\, \mu(A \cap E_k) \leq \overline{\lim}\, \mu(A \cap E_k) \leq \mu(A \cap E^*).$$

1.1.1 Regularity of a Measure.

Definition 1.16. *Given a measure μ on Ω, and $\mathcal{C}$ a class of subsets of Ω, we will say that μ is $\mathcal{C}$-*regular *if for every set $A \subset \Omega$, there exists $D \in \mathcal{C}$, such that $A \subset D$ and $\mu(A) = \mu(D)$.*

When the class $\mathcal{C}$ is the σ-algebra $\mathcal{M}_\mu$ of the μ-measurable sets, we will say that the measure μ is regular *without mentioning the σ-algebra. That is, a measure μ on Ω is regular, if given $A \subset \Omega$, there exists a measurable set E, such that $A \subset E$ and $\mu(A) = \mu(E)$.*

Regularity is an important property for a measure. In some situations it is convenient to deal with a smaller class of sets, other than all the subsets of Ω. In the next theorem we will see that if the measure is regular then part a) of Theorem 1.13 can be extended to arbitrary (not necessarily measurables) subsets of Ω. However, part b) can not be extended!!. (Can you find an counterexample?)

Theorem 1.17. *Let μ be a measure on Ω and assume that μ is regular. If $A_1 \subset A_2 \subset \dots$ are arbitrary subsets of Ω, then*

$$\mu(\bigcup_k A_k) = \lim_{k \to +\infty} \mu(A_k).$$

The proof of this theorem is left as an exercise.

1.2 Construction of a Measure. The Method I

Definition 1.18. *Let $\mathcal{C}$ be a collection of subsets of Ω, and τ a set function defined on $\mathcal{C}$ such that*

1. $\emptyset \in \mathcal{C}$.
2. $0 \leq \tau(C) \leq +\infty,\ C \in \mathcal{C}$.
3. $\tau(\emptyset) = 0$.

Then τ is called a pre-measure *with domain $\mathcal{C}$.*

In what follow we will see one of the general methods to construct a measure from a pre-measure.

Let $\mathcal{C}$ be a class of subsets of Ω. A *covering* of a set M from $\mathcal{C}$ is a countable family $\{A_k\}$, of elements from $\mathcal{C}$ such that $M \subset \bigcup_k A_k$.

Given a pre-measure τ, we define a set function μ on each set $M \subset \Omega$ by

$$\mu(M) = \inf \sum_k \tau(C_k) \ . \tag{1.8}$$

where the infimum is taken over all the coverings $\{C_k\}_{k\in\mathcal{N}}$ of M by elements of $\mathcal{C}$. We set $\inf\{\emptyset\} = +\infty$. We will usually write,

$$\mu(M) = \inf_{\substack{M\subset\cup A_k\\ A_k\in\mathcal{C}}} \sum_k \tau(A_k).$$

The *size* of the covering $\{A_k\}_{k\in\mathbb{N}}$ is the value $\sum_k \tau(A_k)$. We will say that the set function μ was constructed using Method I.
We immediately have,

Proposition 1.19. *Every set function μ on Ω constructed using Method I, is a measure.*

Proof. See Exercise 2.

Example 1.20. Lebesgue measure in $\mathcal{R}$ is the measure associated to the pre-measure τ defined on the class $\mathcal{C} = \{[a,b) : a \le b\}$ by $\tau([a,b)) = b - a$.

Method I allows the construction of a measure on sets without any structure. Actually, it is easy to verify that any measure μ on a set Ω can be obtained from a pre-measure using Method I, just taking as a pre-measure $\tau = \mu$ on the class of all subsets of Ω. (Prove it!!)

Now we are interested to know what properties are satisfied by measures constructed using Method I. For example, we could ask whether the measure would be regular, which class of sets are contained in the σ-algebra of measurable sets or if the measure coincides with the pre-measure on the domain of the pre-measure. For general pre-measures there is not too much to say. Method I, for example, does not warranty regularity in general. Clearly these properties will depend on the pre-measure and the class on which it is defined. In what follows we will see some particular but very important cases, where we can answer some of these questions.

Proposition 1.21. *Every σ-additive measure on a σ-algebra $\mathcal{F}$ is a pre-measure with domain $\mathcal{F}$.*

The proof is immediate.

The next Theorem shows that in the case that the pre-measure is a σ-additive measure on a σ-algebra, then the measure obtained using Method I is an extension of the σ-additive measure.

Theorem 1.22. *Let ν be a σ-additive measure defined on a σ-algebra $\mathcal{F}$ of sets in Ω. Then the measure λ constructed by Method I using ν as a pre-measure defined on $\mathcal{F}$ is an extension of ν, i.e.*

$$\lambda(E) = \nu(E) \quad \forall\ E \in \mathcal{F} \quad \textit{and} \quad \mathcal{F} \subset \mathcal{M}_\lambda.$$

Furthermore, λ is a regular measure and satisfies that for every $A \subset \Omega$,

$$\lambda(A) = \inf\{\nu(E) : A \subset E \in \mathcal{F}\}, \tag{1.9}$$

with the infimum being attained.

Proof. To show that λ is an extension of ν, we will prove first that $\lambda(E) = \nu(E)$ for each set $E \in \mathcal{F}$.

Since $E \subset \bigcup_k E_k$ with $E_1 = E$, and $E_k = \emptyset, \forall k > 1$, then we have

$$\lambda(E) \;\leq\; \nu(E) \quad \forall\, E \in \mathcal{F}. \tag{1.10}$$

To prove the other inequality, let $E_k \in \mathcal{F}$ $\forall k$ be a covering of E. Then

$$\nu(E) \;\leq\; \nu(\bigcup_k E_k) \;\leq\; \sum_k \nu(E_k).$$

So

$$\nu(E) \;\leq\; \inf_{\substack{E \subset \cup E_k \\ E_k \in \mathcal{F}}} \sum_k \nu(E_k) = \lambda(E).$$

Next we will prove (1.9).

Consider $A \subset \Omega$ and $E_k \in \mathcal{F}$ $\forall k$ with $A \subset \bigcup_k E_k$. Write

$$E = \bigcup_k E_k.$$

Then we have,

$$\lambda(A) \;\leq\; \lambda(E) = \nu(E) \;\leq\; \sum_k \nu(E_k).$$

This shows that for every covering of A, there exists $E \in \mathcal{F}$ such that $A \subset E$ and $\lambda(A) \;\leq\; \nu(E) \;\leq\; \sum_k \nu(E_k)$. So

$$\lambda(A) = \inf_{\substack{A \subset E \\ E \in \mathcal{F}}} \nu(E).$$

We now want to show that the infimum will actually be attained. This is obvious (taking $E = \Omega$), if $\lambda(A) = +\infty$, . Assume then that $\lambda(A) < +\infty$. For each $n \in \mathbb{N}$ choose $E_n \in \mathcal{F}$ such that

$$\nu(E_n) \;\leq\; \lambda(A) \;+\; \frac{1}{n}.$$

We can choose E_n such that $E_n \supset E_{n+1}$. So, setting $E^* = \bigcap_k E_k$ we have,

$$\nu(E^*) = \lim_{k \to \infty} \nu(E_k) \;\leq\; \lambda(A).$$

On the other side $\lambda(A) \;\leq\; \lambda(E^*) = \nu(E^*)$. The set E^* satisfies that $E^* \in \mathcal{F}, A \subset E^*$, and $\nu(E^*) = \lambda(A)$. This ends the proof of (1.9).

To see that λ is regular notice that $\nu(E^*) = \lambda(E^*) = \lambda(A)$ implies the stronger statement that λ is $\mathcal{F}$-regular and then clearly, regular.

To finish the proof of the Theorem we need to show that $\mathcal{F} \subset \mathcal{M}_\lambda$. Assume that $E \in \mathcal{F}$. Choose $A \subset \Omega$ with $\lambda(A) < +\infty$. Using (1.9) we can find $E^* \in \mathcal{F}$ with $A \subset E^*$ and $\nu(E^*) = \lambda(E^*) = \lambda(A)$. So

$$\lambda(A) = \nu(E^*) = \nu(E^* \cap E) + \nu(E^* \cap E^c) \;\geq\; \lambda(A \cap E) + \lambda(A \cap E^c).$$

The last inequality is a consequence of (1.9).

Recall that every measure μ on Ω has associated to it a σ-additive measure ν on the σ-algebra of the measurable sets. The last Theorem says that we can extend ν by an application of Method I. One can think that we can iterate this procedure to obtain more extensions of the σ-additive measure ν. However, the next Theorem shows that after the first extension we don't get anything new.

Theorem 1.23. *Let μ be a measure on Ω, ν the restriction of μ to the σ-algebra $\mathcal{M}_\mu$, and λ the measure constructed by Method I using the pre-measure ν. Then we have,*

i. λ is a $\mathcal{M}_\mu$-regular measure.(In particular λ is regular).
ii. $\mathcal{M}_\mu \subset \mathcal{M}_\lambda$.
iii. If $A \in \mathcal{M}_\lambda$ and $\lambda(A) < +\infty$ then $A \in \mathcal{M}_\mu$.
iv. $\mu \;\leq\; \lambda$.
v. $\lambda = \mu$ if and only if μ is regular.

Proof. i. and ii. are consequences of Theorem 1.22.

To prove iv., let A be an arbitrary set in Ω. Using (1.9) we can choose $E \in \mathcal{M}_\mu$ such that $A \subset E$ and $\lambda(A) = \mu(E) \geq \mu(A)$.

Let us now prove iii. Assume $A \subset \Omega$ is a λ-measurable set of finite λ-measure. As above we can choose $E \in \mathcal{M}_\mu$ with $A \subset E$ and $\lambda(A) = \mu(E)$. Since A is λ-measurable

$$\lambda(E) = \lambda(E \cap A) + \lambda(E \setminus A) < +\infty.$$

So $\lambda(E \setminus A) = 0$ and then, using iv., $\mu(E \setminus A) = 0$.

This shows that $E \setminus A$ is μ-measurable and since $A = E \setminus (E \setminus A)$ then A is μ-measurable.

To finish the proof of the Theorem, it only remains to prove v. For this, assume μ is regular and consider $A \subset \Omega$ to be an arbitrary set. Using the regularity of μ we choose $E \in \mathcal{M}_\mu$ such that $A \subset E$ and $\mu(A) = \mu(E)$. So

$$\mu(A) = \mu(E) = \lambda(E) \;\geq\; \lambda(A) \;\geq\; \mu(A).$$

The last inequality follows from iv. Then $\mu(A) = \lambda(A)$ for all $A \subset \Omega$.

A consequence of last theorem is that starting with any pre-measure, the application of Method I no more than two times produce a regular measure.

$$\tau \to \mu \to \nu \to \lambda$$

1.3 Measures in Metric Spaces

So far we have studied measures defined on a general set Ω without any structure. In the study of measures in metric or more generally topological spaces it is very fruitful to interrelate the topological structure with the measure-theoretic structure. The topological structure is defined by the open sets. One wants to study measures for which the open sets are measurable sets. There exist a "minimal" σ-algebra (in a sense explained below) containing the open sets. This is called the *Borel* σ-algebra. A measure in a metric space with the property that the associated σ-algebra of the measurable sets contains the Borel σ-algebra is called a *Borel measure*. The study of measures in metric spaces is basically the study of Borel measures.

We are going first to describe the concept of minimal σ-algebra.

Theorem 1.24. *Let I be an arbitrary index set and $\{\mathcal{A}_\alpha\}_{\alpha\in I}$ a family of σ-algebras of subsets of Ω. Then the class of subsets of Ω defined as the intersection*

$$\bigcap_{\alpha\in I} \mathcal{A}_\alpha = \{A \subset \Omega : A \in \mathcal{A}_\alpha, \forall \alpha \in I\},$$

is a σ-algebra of subsets of Ω

The proof of Theorem 1.24 is straightforward.

Definition 1.25. *Given a class $\mathcal{C}$ of subsets of Ω, we define the* minimal σ-algebra *containing $\mathcal{C}$ as the intersection of all σ-algebras of Ω containing $\mathcal{C}$, that is*

$$\bigcap \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra of } \Omega \text{ and } \mathcal{C} \subset \mathcal{A}\}.$$

The minimal σ-algebra containing a class $\mathcal{C}$ is characterized as a σ-algebra $\mathcal{H}$ satisfying:

i. $\mathcal{C} \subset \mathcal{H}$.

ii. If $\mathcal{F}$ is a σ-algebra and $\mathcal{C} \subset \mathcal{F}$ then $\mathcal{H} \subset \mathcal{F}$.

Arbitrary unions of σ-algebras are not necessarily σ-algebras. However, given a collection of σ-algebras of Ω there exist a minimal σ-algebra containing the union of the collection.

Definition 1.26. *Let (X, d) be a metric space. The* Borel σ-algebra $\mathcal{B} = \mathcal{B}(X)$ *is the minimal σ-algebra containing the open sets. The elements of the Borel σ-algebra are called* Borel sets. *A* Borel measure *on X is a measure on X where all the Borel sets are measurable. (i.e. μ is a Borel measure on X if $\mathcal{B}(X) \subset \mathcal{M}_\mu$.)*

Equivalently the Borel σ-algebra is the minimal σ-algebra containing the closed sets.

Definition 1.27. *Two subsets A, B of a metric space (X, d) are said to be* metrically separated *if*

$$d(A, B) > 0, \quad \textit{where} \quad d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Definition 1.28. *A measure μ on a metric space (X, d) is a* metric measure *if it satisfies that for all metrically separated A and B in X,*

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Next we will see a nice characterization of Borel measures.

Theorem 1.29. *Let (X, d) be a metric space and μ a measure on X. Then μ is a metric measure if and only if μ is a Borel measure.*

Proof. Assume that μ is a Borel measure and consider A and B two metrically separated sets. Then we can find an open set G such that

$$A \subseteq G \quad \text{and} \quad B \subseteq X \setminus G.$$

Since G is μ-measurable and A and B are separated by G, we have that $\mu(A \cup B) = \mu(A) + \mu(B)$ which proves one implication. To prove the converse we need the following

Lemma 1.30. *Let $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \ldots$ be arbitrary sets in a metric space X and let μ be a metric measure on X. Let $A = \bigcup_n A_n$. If A_n and $A \setminus A_{n+1}$ are metrically separated for all n, then*

$$\mu(A) = \lim_{n \to +\infty} \mu(A_n).$$

Proof. Since $A_k \subseteq A$ for every k, then

$$\lim_{n \to +\infty} \mu(A_n) \le \mu(A).$$

therefore we need to prove that

$$\mu(A) \le \lim_{n \to +\infty} \mu(A_n). \tag{1.11}$$

This inequality is trivial if the right hand side is infinity. Then we will assume that $\lim_{n \to +\infty} \mu(A_n) < +\infty$. Write

$$A = A_n \cup D_{n+1} \cup D_{n+2} \cup \ldots \quad n \ge 1,$$

where $D_k = A_k \setminus A_{k-1}, k \ge 2$. It is easy to see that D_k and D_s are metrically separated if $|k - s| \ge 2$.

Also

$$\sum_{k=1}^{n} \mu(D_{2k}) = \mu(\bigcup_{k=1}^{n} D_{2k}) \leq \mu(A_{2n}) \leq \lim_{k\to+\infty} \mu(A_k) < +\infty \quad n \geq 1.$$

Then the series $\sum_k \mu(D_{2k})$ is convergent. A similar argument proves that $\sum_k \mu(D_{2k+1})$ is convergent and hence $\sum_k \mu(D_k)$ is convergent.

So, for each n

$$\begin{aligned} \mu(A) &\leq \mu(A_n) + \mu(\bigcup_{k>n} D_k) \\ &\leq \mu(A_n) + \sum_{k=n+1}^{\infty} \mu(D_k). \end{aligned}$$

Hence, since the sum on the right hand side of the last equation above vanishes when n goes to infinity, we obtain $\mu(A) \leq \lim_{n\to+\infty} \mu(A_n)$ as required.

To complete the proof of Theorem 1.29, assume that μ is a metric measure. We will prove that the closed sets are measurable sets, then it will follow that every Borel set is measurable (see Definition 1.26).

Let $F \subset X$ be an arbitrary non-empty closed set. Consider sets $A, B \subset X$ such that $A \subset F$ and $B \subset F^C$.

We want to show that $\mu(A \cup B) \geq \mu(A) + \mu(B)$ which will prove the measurability of F.

For each $n \geq 1$ write $F_n = \{x \in X : d(x, F) < \frac{1}{n}\}$ and $B_n = B \setminus F_n$. Note that for $x \in B_n$ we have $d(x, F) \geq 1/n$.

We have $B_1 \subset B_2 \subset \cdots \subset B_n \subset \ldots$ and $B = \bigcup_n B_n$ since $B \cap F = \emptyset$ and $\bigcap_n F_n = F$.

Furthermore, A and B_n are metrically separated for every n. Hence $\mu(A \cup B_n) = \mu(A) + \mu(B_n) \ \ \forall\, n$.

Now

$$\begin{aligned} \mu(A \cup B) &\geq \lim_{n\to+\infty} \mu(A \cup B_n) = \lim_{n\to+\infty} \{\mu(A) + \mu(B_n)\} \\ &= \mu(A) + \lim_{n\to+\infty} \mu(B_n). \end{aligned}$$

To finish the proof, it remains to show that $\lim_{n\to+\infty} \mu(B_n) = \mu(B)$. Note that since μ is not necessarily regular and neither of the sets B_n needs to be necessarily μ-measurable, this equality could be false.

We will show that the sets B_n and $B \setminus B_{n+1}$ are metrically separated then we can apply Lemma 1.30 to get the equality.

Take $x \in B_n$ and $y \in B \setminus B_{n+1}$. For every $z \in F$ we have

$$\frac{1}{n} \leq d(x, z) \leq d(x, y) + d(y, z).$$

Since $B = (B \cap F_{n+1}) \cup B_{n+1}$ and $y \notin B_{n+1}$ we must have $d(y, F) < \frac{1}{n+1}$. So

$$\frac{1}{n} \leq inf_{z\in F}\{d(x,y)+d(y,z)\},$$

which yields

$$\frac{1}{n} \leq d(x,y)+\frac{1}{n+1} \quad \text{and then} \quad d(B_n, B\setminus B_{n+1}) > 0.$$

Definition 1.31. *Given $A \subset X$ and μ a measure on X, we define for each $E \subset X$*

$$\mu_A(E) = \mu(E\cap A). \tag{1.12}$$

It is easy to show that μ_A is a measure on X. We will call this measure the restriction of μ to A.

Definition 1.32. *An* algebra *of sets is a collection of sets that contains the empty set and is closed under complementation and finite unions.*

We have the following Proposition:

Proposition 1.33. *Let μ be a measure on X, $A \subset X$ an arbitrary set and $\mathcal{R}$ an algebra of sets from X. Then*

1. *If E is μ-measurable, then E is μ_A-measurable*
2. *If A is μ-measurable and $\mu(A) < +\infty$, then we have: If μ is $\mathcal{R}$-regular then μ_A is $\mathcal{R}$-regular.*

Remark 1.34. The converse of (1) is false in general.

Proof. We leave (1) as an exercise for the reader. To prove (2) consider $R \in \mathcal{R}$ such that $A \subset R$ and $\mu(R) = \mu(A) < +\infty$. Since $\mu(R) = \mu(R\cap A)+\mu(R\cap A^c)$ and $\mu(R\cap A) = \mu(A) = \mu(R)$, we have that $\mu(R\setminus A) = 0$.

Let $C \subset X$ be an arbitrary set

$$\begin{aligned}\mu(C\cap R) &= \mu((C\cap R)\cap A)+\mu((C\cap R)\cap A^c)\\ &\leq \mu(C\cap A)+\mu(R\setminus A) = \mu(C\cap A).\end{aligned}$$

So $\mu(C\cap R) = \mu(C\cap A)$.

We now choose $T \in \mathcal{R}$ such that $C\cap R \subset T$ and $\mu(C\cap R) = \mu(T)$. Define $H = T\cup R^c$. We have $C \subset H$, $H \in \mathcal{R}$ and

$$\begin{aligned}\mu_A(H) &\leq \mu(H\cap R) = \mu(T\cap R) \leq \mu(T)\\ &= \mu(C\cap R) = \mu(C\cap A) = \mu_A(C).\end{aligned}$$

We just proved that $\mu_A(H) = \mu_A(C)$ and since $C \subset H$ and $H \in \mathcal{R}$, we completed the proof.

Theorem 1.35. *If μ is a Borel measure and $\mu(X) < +\infty$. Then for each $\varepsilon > 0$, and each Borel set B there exists a closed set F_ε and an open set G_ε such that*

$$F_\varepsilon \subset B \subset G_\varepsilon \quad \textit{and} \quad \mu(G_\varepsilon \setminus F_\varepsilon) < \varepsilon. \tag{1.13}$$

If in addition μ is Borel-regular, then (1.13) *holds for μ-measurable sets.*

Proof. We will see that the class of sets that satisfy (1.13) is a σ-algebra that contains the closed sets.

We say that a set A satisfies **P** if

> $\forall\varepsilon > 0$ there exist F_ε closed and G_ε open such that $F_\varepsilon \subset A \subset G_\varepsilon$ and $\nu(G_\varepsilon \setminus F_\varepsilon) < \varepsilon$.

Let

$$\mathcal{M} = \{A \subset X : A \text{ satisfy } \mathbf{P}\}.$$

Clearly the empty set belongs to $\mathcal{M}$. It is also straightforward to see that $B \in \mathcal{M}$ if and only if $B^c \in \mathcal{M}$ and that $\mathcal{M}$ is closed under finite unions.

We will now prove that the class $\mathcal{M}$ is closed under countable unions. Consider $A_k \in \mathcal{M}, \forall k \geq 1$.

Write $A = \bigcup_{k=1}^{+\infty} A_k$ and $B_n = \bigcup_{k=1}^{n} A_k$. Take $\varepsilon > 0$.

For each $n \geq 1$ we can choose a closed set $F_n \subset B_n$ such that $\mu(B_n \setminus F_n) \leq \frac{\varepsilon}{2^{n+2}}$. The sets F_n can be chosen such that $F_n \subset F_{n+1}\ \ \forall n$.

Write $H = \bigcup_{n=1}^{+\infty} F_n$. Now,

$$\mu(A \setminus H) = \mu((\bigcup_n B_n) \setminus (\bigcup_n F_n)) \ \leq\ \mu(\bigcup_n (B_n \setminus F_n)) \ \leq\ \sum_n \mu(B_n \setminus F_n) \ <\ \frac{\varepsilon}{4}. \tag{1.14}$$

On the other side

$$\mu(H \setminus F_m) = \mu(H) - \mu(F_m) \to_m 0.$$

So, there exist a positive integer m_0 such that

$$\mu(H \setminus F_{m_0}) \ <\ \frac{\varepsilon}{4}. \tag{1.15}$$

Write $F = F_{m_0}$. The set F is closed, $F \subset A$ and combining (1.14) and (1.15) we have that,

$$\mu(A \setminus F) \ \leq\ \mu(A \setminus H) + \mu(H \setminus F) \ <\ \frac{\varepsilon}{2}.$$

Now we choose $G = \cup_k G_k$ where $A_k \subset G_k$, G_k is open and $\mu(G_k \setminus A_k) < \frac{\varepsilon}{2^{k+1}}$. Then, using that $(G \setminus A) \subset \bigcup (G_k \setminus A_k)$, we have

$$\mu(G \setminus A) \;\le\; \mu\left(\bigcup_k (G_k \setminus A_k)\right) \;\le\; \sum_k \mu(G_k \setminus A_k) \;<\; \frac{\varepsilon}{2}.$$

So,

$$\mu(G \setminus F) \;\le\; \mu(G \setminus A) + \mu(A \setminus F) \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

which implies that $A \in \mathcal{M}$. This proves that $\mathcal{M}$ is a σ- algebra.

We now want to see that $\mathcal{M}$ contains the closed sets. If F is closed, then F is a G_δ set, that is $F = \bigcap_n G_n$ with $F \subset G_n$ and G_n open for each n. We can choose G_n such that $G_n \supset G_{n+1}$.

Since $\mu(G_n) \searrow \mu(\bigcap_n G_n) = \mu(F)$, then for n large enough we have

$$\mu(G_n \setminus F) = \mu(G_n) - \nu(F) \;<\; \varepsilon.$$

So, the closed sets are in $\mathcal{M}$ and then every Borel set is in $\mathcal{M}$. This proves the first claim of the Theorem.

Assume now that the Borel measure μ is Borel-regular. Let A be a μ-measurable set and $\varepsilon > 0$. We can find a Borel set B such that $B \supset A$ and $\mu(B) = \mu(A)$. By the first part of this Theorem, there exist an open set G, such that $B \subset G$ and

$$\mu(G \setminus B) \;<\; \varepsilon.$$

So

$$\mu(G \setminus A) = \mu(G \setminus B) + \mu(B \setminus A) \;<\; \varepsilon.$$

We need to find now a closed set aproximating A from inside. Using what we just proved since A^c is μ-measurable, we can find an open set U with $A^c \subset U$ and $\mu(U \setminus A^c) < \varepsilon$.

Write $F = U^c$. So F is closed, $F \subset A$ and since $U \setminus A^c = A \setminus F$, we have

$$\mu(A \setminus F) \;<\; \varepsilon.$$

In the case that $\mu(X) = +\infty$ we have the following version of Theorem 1.35.

Theorem 1.36. *Let μ be a Borel measure on X. If B is any Borel set and $\varepsilon > 0$ then,*

1. *If $\mu(B) < +\infty$, then there exists a closed set $F \subset B$ such that $\mu(B \setminus F) < \varepsilon$.*
2. *If $B \subset \bigcup_n G_n$ with G_n open sets and $\mu(G_n) < +\infty$ $\forall n$, then there exists an open set $G \supset B$ such that $\mu(G \setminus B) < \varepsilon$.*
3. *If in addition μ is Borel-regular then (1) and (2) hold for μ-measurable sets.*

Proof. To prove (1) consider the measure μ_B. This is a Borel measure, and $\mu_B(X) < +\infty$. So choose $F \subset B$ such that $\mu_B(B \setminus F) < \varepsilon$, then

$$\varepsilon > \mu_B(B \setminus F) = \mu((B \setminus F) \cap B) = \mu(B \setminus F),$$

which proves (1).

To prove (2), since $B \subset \bigcup_n G_n$, we can apply (1) to the set $G_n \setminus B$ and find for each n a closed set $F_n \subset G_n \setminus B$ with $\mu((G_n \setminus B) \setminus F_n) < \frac{\varepsilon}{2^n}$.

Define $G = \bigcup_n (G_n \cap F_n^c)$. The set G is open, $B \subset G$ and $\mu(G \setminus B) < \varepsilon$.

To prove (3) assume that μ is Borel-regular and let A be an arbitrary μ-measurable set of finite measure. The measure μ_A is finite and Borel-regular by Proposition 1.33. So we can apply Theorem 1.35 to μ_A to find a closed set F with the required properties. To obtain the approximation property by open sets we use a similar argument than above, replacing the Borel set B by an arbitrary μ-measurable set A.

Starting with an arbitrary pre-measure, there is no guarantee that the measure obtained by Method I is a Borel measure.

Caratheodory proposed another way of constructing measures on metric spaces that solved this problem. This is the Caratheodory construction of a measure and we sometimes call it Method II for constructing a measure.

This construction is done in a metric space, since it needs the notion of diameter of a set. In the next section we will describe Method II for the construction of a measure in a metric space.

1.4 Caratheodory construction of a measure - Method II

Let (X, d) be a metric space. If $E \subset X$ is an arbitrary set, define by $|E|$ the *diameter* of E, that is

$$|E| = \sup\{d(x, y) : x, y \in E\} \quad \text{if } E \text{ is bounded}, \tag{1.16}$$

we set $|E| = +\infty$ if E is not bounded and $|\emptyset| = 0$.

Given $\delta > 0$, we will say that a countable family $\{E_\lambda\}_{\lambda \in J}$ is a δ-covering of E from the class $\mathcal{C}$, if

i) $E_\lambda \in \mathcal{C}, \quad \forall \lambda \in J$

ii) $|E_\lambda| < \delta, \quad \forall \lambda \in J$

iii) $E \subset \bigcup_{\lambda \in J} E_\lambda$.

If τ is a pre-measure on X with domain $\mathcal{C}$, we define as before the *size* of the δ-covering with respect to τ to be

$$\sum_{\lambda \in J} \tau(E_\lambda). \tag{1.17}$$

We assign the value $+\infty$ to the expression (1.17) when $\tau(E_\lambda) = +\infty$ for some $\lambda \in J$, or if $\tau(E_\lambda) < +\infty$ and the series in (1.17) is not convergent. With this convention, we have

$$0 \;\leq\; \sum_{\lambda \in J} \tau(E_\lambda) \;\leq\; +\infty.$$

Now we define, for $\delta > 0$

$$\mu_\delta(E) = \inf\{\sum_{\lambda \in J} \tau(E_\lambda),\ \{E_\lambda\}_{\lambda \in J} \text{ is a } \delta\text{-covering of } E \text{ from } \mathcal{C}\}, \tag{1.18}$$

(we adopt again the convention that $\inf\{\emptyset\} = +\infty$), and

$$\mu(E) = \sup_{\delta > 0} \mu_\delta(E). \tag{1.19}$$

Note that $0 \leq \mu_\delta(E) \leq +\infty$ and if $\delta_2 \leq \delta_1$ then $\mu_{\delta_1}(E) \leq \mu_{\delta_2}(E)$. Then the expression (1.19) is equivalent to

$$\mu(E) = \lim_{\delta \to 0^+} \mu_\delta(E).$$

Proposition 1.37. *μ_δ, $\delta > 0$ and μ are measures on X.*

Proof. Set $\mathcal{C}_\delta = \{C \in \mathcal{C} : |C| < \delta\}$. Since μ_δ was constructed with Method I using the pre-measure τ restricted to the class $\mathcal{C}_\delta$, by Proposition 1.19, μ_δ is a measure for every $\delta > 0$. We can use now Proposition 1.11 to conclude that μ is a measure.

Our next step will be to show that every measure constructed in a metric space (X, d), using Method II, will be a metric measure, and as a consequence, a Borel measure.

Theorem 1.38. *Let (X, d) be a metric space and μ a measure on X constructed using Method II. Then μ is a metric measure.*

Proof. Let A, B two metrically separated sets. Since μ is a measure, $\mu(A \cup B) \leq \mu(A) + \mu(B)$. We need to prove that

$$\mu(A \cup B) \;\geq\; \mu(A) + \mu(B).$$

Write $a = d(A, B)$, and choose $0 < \delta < a$. Let $\Upsilon = \{U_k\}_{k \in J}$ be a δ- covering of $A \cup B$ and split this covering in two classes such that $\Upsilon = \Upsilon_A \cup \Upsilon_B$, where $\Upsilon_A = \{U \in \Upsilon : U \cap A \neq \emptyset\}$ and $\Upsilon_B = \Upsilon \setminus \Upsilon_A$. Then clearly Υ_A is a δ-covering of A. To see that Υ_B is a δ-covering of B, it is enough to see that if $U \in \Upsilon$ and $U \cap B \neq \emptyset$ then $U \cap A = \emptyset$, but this is a consequence of the choice of δ.

Therefore for every δ-covering of $A \cup B$ there exist δ-coverings of A and B such that

$$\sum_{k \in J} \tau(U_k) = \sum_{U \in \Upsilon_A} \tau(U) + \sum_{U \in \Upsilon_B} \tau(U).$$

We conclude that

$$\mu_\delta(A \cup B) \ \geq \ \mu_\delta(A) + \mu_\delta(B) \quad \forall \ \delta < a.$$

Then, taking $\lim_{\delta \to 0^+}$ in both sides of the inequality,

$$\mu(A \cup B) \ \geq \ \mu(A) + \mu(B),$$

which completes the proof.

Corollary 1.39. *Let (X, d) be a metric space. Then every measure constructed using Method II is a Borel measure.*

Proof. If μ is constructed using method II, then μ is metric. By Theorem 1.29, μ is a Borel measure.

Remark 1.40. It can be seen that μ_δ is not necessarily a Borel measure (see Exercise). (**See exercise I.1 from Mattila.** [**?**])

Now we will study the regularity of the measures constructed using Method I and Method II.

Theorem 1.41. *Let (X, d) be a metric space and let τ be a pre-measure defined on a class of sets $\mathcal{C}$ with $X \in \mathcal{C}$. If μ is a measure constructed from τ by Method I or Method II, then μ is $\mathcal{C}_{\sigma\delta}$-regular.*

Note 1.42. We denote here by $\mathcal{C}_{\sigma\delta}$ the class of sets that are countable unions of countable intersections of elements from $\mathcal{C}$.

Proof. Assume first that μ is constructed using Method I.

Let A be any subset of X. If $\mu(A) = +\infty$, we have $A \subset X$ and $\mu(A) = \mu(X)$.

Assume now that $\mu(A) < +\infty$. For each $n \geq 1$, there is a covering $\{U_k^n\}_{k \in N}$ such that

$$\sum_k \tau(U_k^n) \ \leq \ \mu(A) + \frac{1}{n}.$$

Write $C = \bigcap_n \bigcup_k U_k^n$.

Since $A \subset C$, clearly $\mu(A) \ \leq \ \mu(C)$. For the other inequality we observe that for each $n \in \mathbb{N}$,

$$\mu(C) \ \leq \ \inf_{\substack{C \subset \cup C_k \\ C_k \in \mathcal{C} \ \forall k}} \left(\sum_k \tau(C_k) \right) \ \leq \ \sum_k \tau(U_k^n) \ \leq \ \mu(A) + \frac{1}{n},$$

therefore $\mu(A) = \mu(C)$ and $C \in \mathcal{C}_{\sigma\delta}$.

This completes the proof for the case that μ was constructed using Method I. Note that without the hypothesis of X being in $\mathcal{C}$, μ is still $\mathcal{C}_{\sigma\delta}$-regular for sets of finite μ-measure (i.e. for each A such that $\mu(A) < +\infty$, there exists $C \in \mathcal{C}_{\sigma\delta}$ such that $A \subset C$, and $\mu(A) = \mu(C)$.)

Now, if μ was constructed using Method II, for the case that $\mu(A) = +\infty$, we use the same argument as before.

Consider now the case that $\mu(A) < +\infty$. For each $\delta > 0$, $\mu_\delta(A) < +\infty$. The measure μ_δ is constructed using Method I from the pre-measure τ restricted to the subclass of $\mathcal{C}$ of sets whose diameter is less than δ. So, μ_δ is $\mathcal{C}_{\sigma\delta}$-regular for finite measure sets.

For each $n \geq 1$ we choose $C_n \in \mathcal{C}_{\sigma\delta}$ such that $\mu_{\frac{1}{n}}(C_n) = \mu_{\frac{1}{n}}(A)$ and $A \subset C_n$.

Write $C = \cap_n C_n$. $C \in \mathcal{C}_{\sigma\delta}$, $A \subset C$. For $0 < \frac{1}{n} < \delta$ we have

$$\mu_\delta(C) \leq \mu_{\frac{1}{n}}(C) \leq \mu_{\frac{1}{n}}(C_n) = \mu_{\frac{1}{n}}(A) \leq \mu(A).$$

So $\mu_\delta(C) \leq \mu(A)$ for each $\delta > 0$, which implies that $\mu(C) = \mu(A)$.

Corollary 1.43. *If μ was constructed by Method I or by Method II then we have*

If every set in $\mathcal{C}$ is open then μ is G_δ-regular.

If every set in $\mathcal{C}$ is a Borel set, then μ is Borel-regular.

In addition, if μ was constructed by Method II in both cases μ is regular.

Proof. If X is not in $\mathcal{C}$, then we can extend τ by $\tau(X) = +\infty$, without changing μ. Then the first two claims follow from the fact that countable unions of open sets or Borel sets are open or Borel sets respectively.

The last claim is a consequence of Theorem 1.38

Note 1.44. Method I does not guarantee the regularity of μ, even if the pre-measure satisfies the hypothesis of the Corollary 1.43. **Please find a counterexample**.

However, as we will see later, in the case of the Lebesgue measure we get regularity.

1.5 Exercises

1. Give a proof of Theorem 1.17.
2. Give a proof of Proposition 1.19
3. Prove Theorem 1.15 and construct counterexamples to show that equality a) and b) of that theorem doesn't hold.

4. Let μ be a measure on Ω. Prove that the measure λ on Ω obtained by application of Method I using as a pre-measure $\tau = \mu$ with domain in all the subsets of Ω, coincide with μ.

5. Let λ a measure on Ω. Assume that λ is regular and $\lambda(\Omega) < +\infty$. Prove that $E \in M_\lambda$ if and only if $\lambda(\Omega) = \lambda(E) + \lambda(\Omega \setminus E)$.

6. Let ν be a σ-additive measure on a σ-algebra $\mathcal{A}$. We say that $\mathcal{A}$ is complete respect to ν, if $\mathcal{A}$ satisfies that if $N \in \mathcal{A}$ and $\nu(N) = 0$, then $A \in \mathcal{A}$, for all $A \subset N$.
Prove that every σ-additive measure on a σ-algebra $\mathcal{A}$ can be extended to a measure $\overline{\nu}$ on a σ-algebra $\overline{\mathcal{A}}$ such that $\overline{\mathcal{A}}$ is complete with respect to $\overline{\nu}$.
Hint: Consider the class
$$\overline{\mathcal{A}} = \{E \subset \Omega : \exists A, B \in \mathcal{A}, A \subset E \subset B, \text{and } \nu(B \setminus A) = 0\}.$$
Prove that $\overline{\mathcal{A}}$ is a σ-algebra that contains $\mathcal{A}$, and define for $E \in \tilde{\mathcal{A}}$, $\overline{\nu}(E) = \nu(B)$, where $A \subset E \subset B$ and $\nu(B \setminus A) = 0$, $A, B \in \mathcal{A}$.
Note that if μ is a measure on Ω, then $\mathcal{M}_\mu$ is complete respect to ν (the restriction of μ to the σ-algebra $\mathcal{M}_\mu$).

7. Using Exercise 6, show that $\mathcal{M}_\mu = \overline{\mathcal{M}_\mu}$.

8. If μ is a measure on Ω which is not regular, then there exists $A \subset \Omega$ such that
$$\mu(A) \ < \ +\infty \quad \text{and} \quad \mu(A) \ < \ \inf\{\mu(E) : E \supset A, E \in \mathcal{M}_\mu\}.$$

9. a) Construct an example of a family of measures on a set Ω such that the infimum of the family is not a measure.
   b) If $\{\mu_\alpha\}_{\alpha \in I}$ is a family of measures on a set Ω, then there exists a measure μ on Ω such that
      i. $\mu(A) \ \leq \ \inf\limits_{\alpha} \mu_\alpha(A) \quad \forall\, A \subset \Omega.$
      ii. If ν is a measure on Ω such that
$$\nu(A) \ \leq \ \inf_{\alpha} \mu_\alpha(A) \quad \forall\, A \subset \Omega, \quad \text{then} \quad \nu \ \leq \ \mu.$$
   c) Conclude from the previous item and Proposition 1.11 that the set of measures on Ω is a complete lattice with the partial order defined by $\mu \preccurlyeq \nu$ if $\nu(A) \ \leq \ \nu(A) \ \ \forall A \subset \Omega$..
   Let us recall that a *complete lattice* is a parcially ordered set where each subset has an infimum and a supremum.