Morita - Takeuchi equivalence, cohomology of coalgebras and Azumaya coalgebras

Marco A. Farinati* - Andrea Solotar*,1

Contents

1 Introduction **1**

2 Morita - Takeuchi equivalence **3**
2.1 Morita theory for coalgebras . . . 3
2.2 Hermitian Morita - Takeuchi equivalence . . . 6
2.3 Cohomology theories for coalgebras . . . 11
2.4 Hermitian cohomology theories for involutive coalgebras . . . 13

3 Invariance of cohomology theories **17**
3.1 Morita - Takeuchi invariance of cohomology theories . . . 17
3.2 Morita - Takeuchi invariance of hermitian cohomology theories . . . 20

4 Coseparable and Azumaya coalgebras **21**
4.1 Azumaya coalgebras . . . 21
4.2 Change of base coalgebra . . . 26
4.3 Hermitian Picard group of an involutive coalgebra . . . 28

1 Introduction

Given two k-algebras (k a commutative unital ring) A and B, the study of their representation theory involves the concept of Morita equivalence.

Namely, two k-algebras are Morita equivalent if and only if their categories of modules (for example left modules) are equivalent. It is well-known ([D-I, McC]) that cyclic homology and Hochschild homology are Morita invariant, furthermore, in [F-S] we studied the behaviour of dihedral homology and positive Hochschild homology with respect to hermitian Morita equivalence when both k-algebras A and B are provided by k-linear involutions, and shown the invariance of the above theories in that context.

*Dto. de Matemática Facultad de Cs. Exactas y Naturales. Universidad de Buenos Aires. Ciudad Universitaria Pab I. 1428 - Buenos Aires - Argentina. e-mail: asolotar@dm.uba.ar mfarinat@dm.uba.ar Research partially supported by UBACYT EX230

[1]Research member of CONICET (Argentina)

However, if one works with a bialgebra H, the interest must be focused not only on the category of modules over H but also on the category of comodules over H (and of course on the category of objects which are at the same time modules and comodules over H, these two structures being compatible).

As in the case of algebras, for C and D k-coalgebras there is a theory called Morita - Takeuchi equivalence ([D, T]) which plays the rol of Morita equivalence for algebras, i.e.:

Two k-coalgebras C and D are Morita - Takeuchi equivalent if and only if their categories of comodules (left comodules for example) are equivalent.

If this is the case, the categories of C-bicomodules and D-bicomodules are also equivalent.

In [D] Doi defines two natural cohomology theories of a coalgebra C with coefficients in a bicomodule N related to the derived functors of Com and $Cotor$, where Com denotes the morphisms in a comodule category, in a similar way as Hochschild cohomology and homology are respectively related to Hom and Tor functors. More precisely by means of relative injective resolutions: $H^*(N,C) = Ext^*_{C^e/k}(N,C)$ and $Hoch^*(N,C) = H^*(\mathbf{X}.\square_{C^e}N)$, (where C^e is the enveloping coalgebra of C, $C^e = C \otimes C^{op}$ and $\mathbf{X}.$ is a k-relative C^e-injective resolution of the bicomodule C). Both of them are useful to describe certain properties of C, like coseparability and cocentrality.

The main theorems of this paper show the invariance of this cohomology theory with respect not only to Morita - Takeuchi equivalence, but to k-congruences, which is a weaker condition than Morita - Takeuchi equivalence and will be defined in Section 3, and state that they are also invariant under Azumaya extensions of k. We also show that they are invariant respect to coseparable change of base.

Now we write down these results:

Theorem A: If C and D are two k-coalgebras Morita - Takeuchi equivalent and N is a C-bicomodule, then

$$H^*(N,C) \cong H^*(F(N),D)$$

$$Hoch^*(N,C) \cong Hoch^*(\tilde{F}(N),D)$$

where F is the k-congruence functor from the category of C-bicomodules to the category of D-bicomodules and $\tilde{F}$ is a k congruence associated with F.

Theorem B: If C is a k-coalgebra which is Azumaya over k then $H^n(C,C) = 0$ for $n > 0$ and $H^0(C,C) = k$. The same holds for $Hoch^n(C,C)$.

Theorem C: If R is a cocommutative coseparable k-coalgebra, and C is an R-coalgebra, then the Hochschild homology and cohomology of C, calculated with respect to k or R as base coalgebra are respectively isomorphic.

With the same motivations of [F-S], that is, intending not to left aside when it exists an important property of the coalgebra as having an involution, we define hermitian versions of $Hoch^*$ and H^*, which apply not to the category of C-bicomodules but to its subcategory consisting of C-bicomodules compatible with the involution. So, we are interested also to know when these categories are equivalent for two involutive coalgebras (C, ω_C) and $D, \omega_D)$, i.e. on hermitian Morita - Takeuchi equivalences. Finally, we prove the following result:

Theorem D: If C is an involutive coalgebra, which is Azumaya over k, then its hermitian Picrad group does not depend on the involution.

In this work, k will always be a commutative unital ring, C, D will denote k-coalgebras, R will be a cocommutative k-coalgebra and $\otimes$ will mean $\otimes_k$. For $V_1, ..., V_n$ k-vector spaces, $\tau \in S_n$, $\sigma_\tau : V_1 \otimes ... \otimes V_n \to V_{\tau(1)} \otimes ... \otimes V_{\tau(n)}$ will denote $\sigma_\tau(v_1 \otimes ... \otimes v_n) = v_{\tau(1)} \otimes ... \otimes v_{\tau(n)}$. For a coalgebra C, the enveloping coalgebra $C \otimes C^{op}$ will be denoted by $C^e = C \otimes C^{op}$ and for an R coalgebra, $C^{eR} = C \square_R C^{op}$. If M is a bicomodule, ρ_M^+ will denote the right structure and ρ_M^- the left structure. We consider only k-flat coalgebras.

This paper is organized as follows:

In section 2 we recall the basic definitions and properties concerning Morita - Takeuchi equivalence of coalgebras and both cohomology theories of coalgebras. As an application we compute $H^*(k[x], k[x])$ and $Hoch^*(k[x], k[x])$ using simplified resolutions. We then focus our attention on hermitian structures, considering the relation between the existence of an involution on the coalgebra and hermitian Morita - Takeuchi equivalence, and giving a characterization of the category of compatible bicomodules over a coalgebra (proposition 2.7) which allows us to describe hermitian cohomology theories for coalgebras as cohomological functors, defined on an appropriate category.

In section 3 we prove Theorem A and Theorem D and study some properties of a coalgebra which may be characterized by means of the above cohomology theories. We also show that, under certain hypothesis on the coalgebra C, $H^*(N, C)$ may be alternatively defined as the right derived functor of k-coderivations from N to C. This fact is similar to what happens with Hochschild cohomology of an algebra A with coefficients in an A-bimodule M.

Finally, in section 4 after recalling fron [T-V.O-Z] some definitions and basic facts on coseparable coalgebras and cocentral coalgebras, we prove Theorem B and Theorem C. The last part of this section is devoted to the study of dependence of the hermitian Picard group of an involutive coalgebra on its involution. Although in general the involution plays an important rol, if the coalgebra is Azumaya, it does not affect the hermitian Picard group.

2 Morita - Takeuchi equivalence

2.1 Morita theory for coalgebras

Morita equivalence of algebras allows us to study the representations of a k-algebra A by means of the representations of a k-algebra B, Morita equivalent to A, which is in general, simpler. The typical case of Morita equivalence is given by a k-algebra A and a matrix ring $M_n(A)$, $(n \in \mathbb{N})$ and is particulary useful when considering modules over a crossed product ring obtained from an algebra A and a finite group G acting on A, as $A \rtimes G$ is, under some Galois conditions, Morita equivalent to the k-algebra of invariants A^G.

The corresponding equivalence theory for coalgebras is developped in [T] and [L].

In [T-V.O-Z] the authors give a characterization of Azumaya coalgebras and of the Brauer group of a cocomutative coalgebra R in terms of Morita - Takeuchi equivalence. This Brauer group $Br(R)$ is in general not isomorphic to $Br(R^*)$ if R is infinite dimensional as a k-vector space (the finite dimensional case gives an isomorphism).

We begin by recalling from [T] the definition of Morita - Takeuchi equivalence.

Definition 2.1 *Two k-coalgebras C and D are* **Morita - Takeuchi equivalent** *if and only if their categories of left comodules are equivalent.*

It is not hard to see that definition 2.1 does not change if we replace left-comodules by right-comodules, and that if C and D are Morita - Takeuchi equivalent then their bimodule categories are equivalent. However, the converse does not hold.

Examples:

1. Comatrix coalgebra: Let C be a coalgebra and k_n^* the dual space of the matrix algebra $M_n(k)$, $M_n(k)^*$ being a coalgebra by dualising the multiplication and unit of $M_n(k)$.
   $M^n(C) := C \otimes M_n(k)^*$ is called the comatrix coalgebra of degree n over C, it is a coalgebra Morita - Takeuchi equivalent to C.

2. Let A and B be two finite dimensional k-algebras and let $C = A^*$, $D = B^*$. Then A is Morita equivalent to B if and only if C is Morita - Takeuchi equivalent to D. If ${}_AP_B$ and ${}_BQ_A$ are the invertible bimodules giving the Morita equivalence between A and B, then P^* and Q^* give the Morita - Takeuchi equivalence between C and D.

   Before the following example we need to recall some definitions from [T].

   Definition 2.2 *Let P be a right (left) C-comodule. P is* **quasifinite** *if and only if $Com_C(F,P)$ is finite dimensional for all k-finite dimensional right (left) C-comodules F, where $Com_C(-,-)$ denotes the morphisms in the C-comodule category.*

   The fact of P being quasifinite is equivalent to the existence of a left adjoint to the functor $(-) \otimes P : k - mod \to Com_C$. For example, if V is a finite dimensional k-vector space, then $V \otimes C$ is quasifinite. This left adjoint is denoted by $h_C(P,-)$. Then, for P quasifinite there are canonical isomorphisms

   $$Com_C(Y, W \otimes P) \cong Hom_{k-mod}(h_C(P,Y), W)$$

   for every right C-comodule Y and every k-module W.
   Notation: For each $X \in \mathrm{comod}_C$ let's denote $h_C(X,X)$ by $e_C(X)$ (coalgebra of coendomorphisms).

3. Consider a right comodule P which is a quasifinite injective cogenerator, and $D = e_C(P)$. P is then a $D-C$-bicomodule. C and D are Morita - Takeuchi equivalent by means of the bicomodules P and $Q = h_C(P, C)$.

4. The following example, which partially motivated this work, is closely related to the paper [C-D-R]. Our approach to the subject comes from the point of view of invariants of the action of a group on a coalgebra C.

   Let C be a k-coalgebra, G a finite group (such that the order of G is invertible in k) acting on C by automorphisms. Then the subset of C of coinvariants by the action of G is a subcolagebra C^G of C which, under suitable (Galois) hyphotesis is Morita-Takeuchi equivalent to the crossed-product coalgebra $C \rtimes G$, i.e. the k-module $C \otimes k^{(G)}$ equipped with the coproduct defined by:

$$\Delta_{C \rtimes G} : C \rtimes G \to (C \rtimes G) \otimes (C \rtimes G)$$

$$\Delta_{C \rtimes G}(c \otimes \delta_g) = \sum_{(c),h} c_{(1)} \otimes \delta_h \otimes h^{-1}(c_{(2)}) \otimes \delta_{h^{-1}g}$$

Remark: It is well-known that there are pairs of finite dimensional algebras such that their bimodule categories are equivalent but their left module categories are not (for example $\mathbb{H}$ and $\mathbb{R}$). The second example shows that an analog statement holds for coalgebras.

As Takeuchi has proven in [T], definition 2.1 is the same thing as having a Morita - Takeuchi equivalence context, i.e. a $D-C$-bicomodule P and a $C-D$-bicomodule Q (with coactions denoted by $\rho_P^+ : P \to P \otimes C$, etc.) and bicomodule isomorphisms

$$P \square_C Q \cong D \ ; \ Q \square_D P \cong C$$

where $P \square_C Q$ denotes the cotensor product, i.e. the kernel of the morphism

$$P \otimes Q \xrightarrow{\rho_P^+ \otimes Id_Q - Id_P \otimes \rho_Q^-} P \otimes C \otimes Q$$

As the cohomology theories that will be considered deal with C-bicomodules, we shall need the following lemma:

Lemma 2.3 *Let C and D be Morita - Takeuchi equivalent coalgebras, by means of bicomodules ${}_DP_C$ and ${}_CQ_D$ such that $P \square_C Q \cong D$ and $Q \square_D P \cong C$ as D and C-bicomodules respectively. Let us consider $P' = P \otimes Q$ and $Q' = Q \otimes P$ and denote by C^e the enveloping coalgebra $C \otimes C^{op}$ (respectively D^e). Then:*

1. *P' is naturally a $(C^e - D^e)$-bicomodule and Q' is naturally a $(D^e - C^e)$-bicomodule.*

2. *The functors*

- $Q'\square_{C^e}(-) : C^e - comod \to D^e - comod$
- $(-)\square_{C^e}P' : C^e - comod \to D^e - comod$
- $P\square_C - \square_C Q : C^e - comod \to D^e - comod$

are naturally equivalent, identifying the categories of left C^e-comodules right C^e-comodules and C-bicomodules (idem for D).

Proof: the structure morphisms of P and Q as right or left comodules over C and D induce (respectively) obvious morphisms:

$$P \otimes Q \to D \otimes P \otimes Q \otimes D \cong D^e \otimes P \otimes Q$$

$$P \otimes Q \to P \otimes C \otimes C \otimes Q \cong P \otimes Q \otimes C^e$$

$$Q \otimes P \to C \otimes Q \otimes P \otimes C \cong C^e \otimes Q \otimes P$$

$$Q \otimes P \to Q \otimes D \otimes D \otimes P \cong Q \otimes P \otimes D^e$$

which are clearly coassociative.

In order to prove the equivalence of the three functors, let us define the natural transformation $\phi_M : Q'\square_{C^e}M \to P\square_C M\square_C Q$ ($M \in C^e$-comod) as follows:

Firstly, there is an isomorphism of k-vector spaces $\phi_M : Q \otimes P \otimes M \to P \otimes M \otimes Q$ which interchanges the coordinates.

The C^e-comodule structure of M gives us the commutative diagram:

$$\begin{tikzcd} Q \otimes P \otimes M \arrow[r, "\phi"] \arrow[d, shift left] \arrow[d, shift right] & P \otimes M \otimes Q \arrow[d, shift left] \arrow[d, shift right] \\ Q \otimes P \otimes C^e \otimes M \arrow[r] & P \otimes C \otimes M \otimes C \otimes Q \end{tikzcd}$$

where both horizontal maps are isomorphisms, and they induce an isomorphism between both kernels, i.e. between $(Q \otimes P)\square_{C^e}M$ and $P\square_C M\square_C Q$. The situation is analogous for $(-)\square_{C^e}P'$.

2.2 Hermitian Morita - Takeuchi equivalence

Morita equivalence is a powerful instrument while working with categories of modules, as it allows us to distinguish between algebras by means of its representation categories. However, in many cases, as it is shown in [F-S], it neglects an important tool provided by the algebra itself: involutions. This fact leads [H] and [F-McE] to study hermitian Morita equivalences. In the same sense, while Morita - Takeuchi has proven to be useful in the study of coalgebras and its representation categories (comodule categories), involutions of coalgebras are left aside by it. This fact gave us a motivation to define and study an hermitian Morita - Takeuchi equivalence for coalgebras.

Let C and D be k-coalgebras with involutions ω_C and ω_D, i.e $\omega_C : C \to C^{op}$ is a k-coalgebra isomorphism such that $\omega_C^2 = id_C$ and the same for ω_D. Remembering that a Morita - Takeuchi equivalence between coalgebras C and D is given by two bicomodules ${}_DP_C$ and ${}_CQ_D$ provided of isomorphisms of bicomodules $\mu : C \to Q\Box_D P$ and $\tau : D \to P\Box_C Q$, what we need in order to consider the involutions ω_C and ω_D is to relate the two possible ways of passing from ${}_C comod$ to ${}_D comod$, namely:

As C is involutive, the categories ${}_C comod$ and $comod_C$ are equivalent, because if $M \in {}_C comod$ with left coaction ρ^-, then $M \to M \otimes C$ $(m \mapsto \sigma_{12}((\omega_C \otimes id_M)(\rho^-(m)))$ provides M of the structure of a right C-comodule.

We have therefore two functors from ${}_C comod$ to ${}_D comod$:

$$\begin{tikzcd}
{}_C comod \arrow[r, "P\Box_C -"] \arrow[d, "\phi_{\omega_C}"] & {}_D comod \\
comod_C \arrow[r, "-\Box_C Q"] & comod_D \arrow[u, "\phi^{-1}_{\omega_D}"']
\end{tikzcd}$$

As in the case of algebras, we have:

Proposition 2.4 *The natural isomorphism of these two functors is equivalent to the existence of a k-linear isomorphism $\Theta : P \to Q$ such that the following diagram is commutative:*

$$\begin{tikzcd}
P \arrow[r, "\rho_P^+ \circ \rho_P^-"] \arrow[d, "\Theta"] & D \otimes P \otimes C \arrow[d, "\sigma_{13} \circ (\omega_D \otimes \Theta \otimes \omega_C)"] \\
Q \arrow[r, "\rho_Q^+ \circ \rho_Q^-"] & C \otimes Q \otimes D
\end{tikzcd}$$

Proof: First suppose that there exists a natural transformation $\eta : P\Box_C - \to \phi^{-1}_{\omega_D} \circ (-\Box_C Q) \circ \phi_{\omega_C}$. Take $\Theta = \eta_C : P\Box_C C \to \overline{(\overline{C}^C \Box_C Q)}^D$, where if $M \in {}_C comod$, then $\overline{M}^C \in comod_C$, by means of ϕ_{ω_C}, and similarly for D. Of course $P\Box_C C \cong P$, and $\overline{(\overline{C}^C \Box_C Q)}^D \cong Q$ as k-vector space. The structure of the first one as left D-comodule is given by the left D-structure of P. As we have an isomorphism of $D - C$-comodules, we obtain that applying the D-left coaction on P is the same as applying it on $\overline{(\overline{C}^C \Box_C Q)}^D$, but this is the same as the composition of the involution on D with the right D-coaction on Q. So we have that $(id \otimes \Theta) \circ \rho_P^- = (id \otimes \omega_D) \circ \rho_Q^+ \circ \Theta$.

Similarly, the C-right structure of P is sent by Θ into the C-right structure of $\overline{(\overline{C}^C \Box_C Q)}^D$, which coincides with the C-right action of $\overline{C}^C \Box_C Q$, given by the C-left structure of C and the involution of C.

Suppose conversely, that a k-isomorphism $\Theta : P \to Q$ is given, such that the diagram

$$\begin{tikzcd} P \arrow[r, "\rho_P^+ \circ \rho_P^-"] \arrow[d, "\Theta"] & D \otimes P \otimes C \arrow[d, "\sigma_{13} \circ (\omega_D \otimes \Theta \otimes \omega_C)"] \\ Q \arrow[r, "\rho_Q^+ \circ \rho_Q^-"] & C \otimes Q \otimes D \end{tikzcd}$$

is commutative, and let M be a left C-comodule. In order to define the natural equivalence between the two functors of this proposition, we first consider the k-linear map:

$$\sigma_{12} \circ (\Theta \otimes id_M) : P \otimes M \to \phi_{\omega_C}(M) \otimes Q$$

$$p \otimes m \mapsto m \otimes \Theta(p)$$

The property of Θ with respect to ρ_P^+ and ρ_Q^- says that the following diagram can be completed

$$\begin{tikzcd} P \square_C M \arrow[r, dashed] \arrow[d, hook] & \phi_{\omega_C}(M) \square_C Q \arrow[d, hook] \\ P \otimes M \arrow[r, "\sigma_{12} \circ (\Theta \otimes id_M)"] \arrow[d, shift left] \arrow[d, shift right] & \phi_{\omega_C}(M) \otimes Q \arrow[d, shift left] \arrow[d, shift right] \\ P \otimes C \otimes M \arrow[r, "\sigma_{321} \circ (\Theta \otimes id_{C \otimes M})"] & \phi_{\omega_C}(M) \otimes C \otimes Q \end{tikzcd}$$

i.e. that it is well-defined between the cotensor products. We define then η_M by restriction of $\sigma_{12} \circ (\Theta \otimes id_M)$ and considering $\phi_{\omega_C}(M) \square_C Q$ as a left D-comodule via ω_D, i.e. $\eta_M = \sigma_{12} \circ (\Theta \otimes id_M)|_{P \square_C M} : P \square_C M \to \phi_{\omega_D}^{-1}(\phi_{\omega_C}(M) \square_C Q)$ The property of Θ with respect to ρ_P^- and ρ_Q^+ says that η_M is is D-colinear. It is clear that η_M is an isomorphism for every $M \in {}_C comod$. The naturality of η_N is also easily verified.

Remark: $Q = h_C(P, C)$, and also $Q = h_D(P, D)$.

Definition 2.5 *Two involutive k-coalgebras C and D will be* **hermitian Morita - Takeuchi equivalent** *if and only if there exist two bicomodules ${}_D P_C$ ${}_C Q_D$, bicomodule isomorphisms $\mu : C \to Q \square_D P$ and $\tau : D \to P \square_C Q$, and a k-isomorphism $\Theta : P \to Q$ satisfying:*

1. *(P, Q, μ, τ) is a Morita - Takeuchi context.*
2. *(a) The following diagram is commutative:*

$$\begin{tikzcd} P \arrow[r, "\rho_P^+ \circ \rho_P^-"] \arrow[d, "\Theta"] & D \otimes P \otimes C \arrow[d, "\sigma_{13} \circ (\omega_D \otimes \Theta \otimes \omega_C)"] \\ Q \arrow[r, "\rho_Q^+ \circ \rho_Q^-"] & C \otimes Q \otimes D \end{tikzcd}$$

We also ask Θ to verify the following compatibility conditions:

(b) $(\Theta^{-1} \otimes \Theta) \circ \mu = \sigma_{12} \circ \mu \circ \omega_C$

(c) $(\Theta \otimes \Theta^{-1}) \circ \tau = \sigma_{12} \circ \tau \circ \omega_D$

Let us notice the fact that, as the category of $D - C$-bicomodules (resp. $C - D$-bicomodules) ca be regarded as a subcategory of $D^* - C^*$-bimodules (resp. $C^* - D^*$-bimodules), the existence of an antiisomorphism $\Theta : P \to Q$ provides a sesquicolinear form on P.

Examples: 1. Let C be a k-coalgebra with involution ω_C, and let D be $M^n(C)$ the comatrix coalgebra, i.e. $M^n(C) = C \otimes (M_n(k))^*$. Defining the involution $c \otimes e^{ij} \mapsto \omega_C(c) \otimes e^{ji}$ ($c \in C$, $\{e^{ij}\}_{1 \leq i,j \leq n}$ the canonical basis of $(M_n(k))^*$) it is hermitian Morita - Takeuchi equivalent to C.

2. Let C be a k-coalgebra with involution ω_C, G a group acting on C by morphisms of k-coalgebra such that the action commutes with the involution of G. Then $C \rtimes k^G$ (see [C-D-R]) and C^G (the subcoalgebra of fixed points) are hermitian Morita - Takeuchi equivalent under the Galois conditions described in [C-D-R]. The involution on $C \rtimes k^G$ is defined by $c \otimes \delta_g \mapsto g^{-1}(\omega_C(c)) \otimes \delta_{g^{-1}}$, ($c \in C$ and $g \in G$).

If C and D are Morita - Takeuchi equivalent, then the categories of (for example left) C-comodules and D-comodules are equivalent. This fact implies the equivalence of the bicomodule categories over C and D. However, in the involutive case, the adequate representation category to consider is not the bicomodule category but the category of compatible bicomodules. Namely, we consider C-bicomodules M provided of an "involution" ω_M (this is: a k-isomorphism such that $\omega_M^2 = id_M$) making the following diagram commutative:

$$\begin{tikzcd} M \arrow[r, "\rho_M"] \arrow[d, "\omega_M"'] & C \otimes M \otimes C \arrow[d, "\sigma_{13} \circ (\omega_C \otimes \omega_M \otimes \omega_C)"] \\ M \arrow[r, "\rho_M"] & C \otimes M \otimes C \end{tikzcd}$$

Then we have:

Proposition 2.6 *If C and D are hermitian Morita - Takeuchi equivalent, then the categories of compatible C-bicomodules and compatible D-bicomodules are equivalent.*

Proof: The equivalence is given by restriction of the functors

$$F : P \square_C - \square_C Q :_C bicomod \to_D bicomod$$

$$G : Q \square_D - \square_D P :_D bicomod \to_C bicomod$$

It is clear that both compositions are equivalent to the identity. If M is a compatible C-bicomodule, then $F(M) = P \square_C M \square_C Q$ is a compatible D-bicomodule, by means of the involution $\omega_{F(M)} : F(M) \to F(M)$, $\sum_i p_i \otimes m_i \otimes q_i \mapsto \sum_i \Theta^{-1}(q_i) \otimes \omega_M(m_i) \otimes \Theta(p_i)$. After checking easily that $\omega_{F(M)}$ is well-defined, it is clear that $\omega_{F(M)}$ is an involution.

Remark: If $M = C$, then $F(M) = P\Box_C C\Box_C Q \cong P\Box_C Q \cong D$ as bicomodules. But C is also a compatible bicomodule by means of ω_C, then $P\Box_C Q \cong D$ has an involution. Conditions 2.(b) and (c) of definition 2.5 guarantee that this involution and ω_D agree.

In the last part of this section we give another description of the category of compatible bicomodules over an involutive coalgebra, which is more adequate in case $char(k) = 2$.

Proposition 2.7 *If C is an involutive k-coalgebra, then the category of compatible C-bicomodules is equivalent to the category of $C^e \rtimes k^{\mathbb{Z}_2}$-comodules*

Proof: Let us first give the structure of the coalgebra $C^e \rtimes k^{\mathbb{Z}_2}$:
Denoting by $\{e, t\}$ the elements of $\mathbb{Z}_2$ (e the unit), comultiplication in $k^{\mathbb{Z}_2}$ is given by:

$$\delta_e \mapsto \delta_e \otimes \delta_e + \delta_t \otimes \delta_t$$

$$\delta_t \mapsto \delta_e \otimes \delta_t + \delta_t \otimes \delta_e$$

The idea of comultiplication in $C^e \rtimes k^{\mathbb{Z}_2}$ is that one makes the comultiplication in C^e and in $k^{\mathbb{Z}_2}$ respectively, and then interchanges factors, taking into account that δ_t acts as the involution on the elements of C. That means:

$$\begin{aligned}
\Delta(c_1 \otimes c_2 \otimes \delta_e) = \;& \sigma_{34} \circ \sigma_{45}(\Delta_{C^e}(c_1 \otimes c_2) \otimes \delta_e \otimes \delta_e) \\
& + \sigma_{34} \circ \sigma_{45}(\Delta_{C^e}(c_1 \otimes \omega_C(c_2)) \otimes \delta_t \otimes \delta_t) \\
\Delta(c_1 \otimes c_2 \otimes \delta_t) = \;& \sigma_{34} \circ \sigma_{45}(\Delta_{C^e}(c_1 \otimes c_2) \otimes \delta_e \otimes \delta_t) \\
& + \sigma_{34} \circ \sigma_{45}(\Delta_{C^e}(c_1 \otimes \omega_C(c_2)) \otimes \delta_t \otimes \delta_e)
\end{aligned}$$

Next we define a functor from the category of compatible C^e-comodules into $C^e \rtimes k^{\mathbb{Z}_2}$-comodules by assigning the same object, with structure morphism:

$$M \to M \otimes C^e \rtimes k^{\mathbb{Z}_2}$$

$$m \mapsto \rho_M(m) \otimes \delta_e + \rho_M(\omega_M(m)) \otimes \delta_t$$

Conversely, for a $C^e \rtimes k^{\mathbb{Z}_2}$-comodule N, the structure of compatible C^e-comodule is given as follows: Using the counit of $k^{\mathbb{Z}_2}$ and C^e we have two morphisms of coalgebras $C^e \rtimes k^{\mathbb{Z}_2} \to C^e$ and $C^e \rtimes k^{\mathbb{Z}_2} \to k^{\mathbb{Z}_2}$. The first one gives the structure of C^e-comodule, and the involution ω_N is defined considering that if N is a $k^{\mathbb{Z}_2}$-comodule then it is a $(k^{\mathbb{Z}_2})^* = k[\mathbb{Z}_2]$-module.

The compatibility conditions of these two structures come from the structure morphism of N as $C^e \rtimes k^{\mathbb{Z}_2}$-comodule.

2.3 Cohomology theories for coalgebras

In the sequel we shall assume that either k is a field, or C and k are k-injective. In fact, without these hypothesis, we only have to consider relative Ext and $Cotor$ functors (to the category where the morphisms are k-split)

If C is a k-coalgebra, and M is a left (right) C-comodule we consider the functor $Com_C(M, -)$ from left (right) C-comodules to abelian groups. As the category of C-comodules has enough injectives and as the functor is left exact, when we take the derived functors $Ext^n_C(M, -)$ we obtain a cohomological functor which coincides with the above one in degree 0.

In a similar way, taking the cotensor product functor $-\Box_C M : comod_C \to k - mod$, we may consider the derived functor $Cotor^n_C(-, M)$. Doi defines in [D] two different cohomology theories for a coalgebra C with coefficients in a C-bicomodule N.

Namely, the first is:

$H^*(N, C) = Ext^*_{C^e}(N, C)$, which is calculated by taking an injective resolution $\mathbf{X}.$ of C as C^e-comodule and then calculating the homology groups of $Com_{C^e}(N, \mathbf{X}.)$.

It is important to notice that the category of C^e-bicomodules has enough injectives. In particular, for the C^e-bicomodule C we can always choose the canonical resolution $\mathbf{X_n} = C^{\otimes n+2}$ and make use of the fact that $Com_{C^e}(N, C^{\otimes n+2})$ is isomorphic to $Hom_k(N, C^{\otimes n})$.

This theory is closely related to Hochschild cohomology of an algebra with coefficients in a bimodule M. We observe, for example that:

$$H^0(N, C) = \{f \in N^* \text{ such that } (id \otimes f)\rho^- + (f \otimes id)\rho^+ = 0\}$$

$$H^1(N, C) = Coder(N, C)/InCoder(N, C)$$

where $InCoder(N, C)$ denotes the inner coderivations from N to C, that is k-coderivations $f : N \to C$ given by elements of N as follows:

Consider N as a k-vector space, choose $n \in N$, $n \neq 0$ and, completing to a base define $f(m) = \rho^+(m)_n - \rho^-(m)_n$, where $\rho^+(m)_n$ denotes the element in C corresponding to n, in the expression of $\rho^+(m)$, and the same for $\rho^-(m)$.

The second theory, denoted by $Hoch^*(N, C)$, is defined as the cohomology of the complex $(\mathbf{X}.\Box_{C^e} N, d\Box_{C^e} id_N)$ where $\mathbf{X}.$ is as above.

In particular $Hoch^0(N, C) = \{n \in N \text{ such that } \sigma_{12}\rho^+(n) = \rho^-(n)\}$. If $N = C$, then $Hoch^0(C, C)$ is the cocenter of C.

The relation between Doi's cohomology theories on one side and $Ext^*_{C^e}$ and $Cotor^*_{C^e}$ on the other, is the same as the relation between Hochschild cohomology and homology, and $Ext^*_{A^e}$ and $Tor^{A^e}_*$ functor, that is: Doi's cohomology theories may be expressed in terms of $Ext^*_{C^e}$ and $Cotor^*_{C^e}$ when C and k are k-injective.

Example: Computation of $H^*(TV, TV)$ and $Hoch^*(TV, TV)$ for a k-vector space V.

Lemma 2.8 *The following sequence is an injective resolution of TV as TV^e-comodule.*

$$0 \longrightarrow TV \xrightarrow{\Delta} TV \otimes TV \xrightarrow{d} TV \otimes V \otimes TV \longrightarrow 0$$

where d is defined by:

$$d(1 \otimes 1) = 0$$

$$d(1 \otimes w_1...w_r) = 1 \otimes w_1 \otimes w_2...w_r$$

$$d(v_1...v_k \otimes 1) = -v_1...v_{k-1} \otimes v_k \otimes 1$$

$$d(v_1...v_k \otimes w_1...w_r) = v_1...v_k \otimes w_1 \otimes w_2...w_r - v_1...v_{k-1} \otimes v_k \otimes w_1...w_r$$

with $v_i, w_j \in V$, $1 \leq i \leq k$ and $1 \leq j \leq r$.

Proof: It is clear that $TV \otimes TV$ and $TV \otimes V \otimes TV$ are injective as TV-bicomodules and that $\Delta : TV \to TV \otimes TV$ is a bicomodule map. A straightforward computation shows that $d\Delta = 0$ and that d is a morphism of TV-bicomodules.

To prove the exactness, we define an homotopy contraction:

$$h_0 = Id \otimes \epsilon : TV \otimes TV \to TV$$

and

$$h_1 : TV \otimes V \otimes TV \to TV \otimes TV$$

by the formulae:

$$h_1(1 \otimes x \otimes w_1...w_r) = 1 \otimes xw_1...w_r$$

$$h_1(v_1...v_k \otimes x \otimes w_1...w_r) = \sum_{i=0}^{k} v_1...v_i \otimes v_{i+1}...v_k x w_1...w_r$$

where $v_i, x, w_j \in V$, $1 \leq i \leq k$ and $1 \leq j \leq r$.

It is easy to check that $h_0\Delta = Id_{TV}$, $\Delta h_0 + h_1 d = Id_{TV \otimes TV}$ and $dh_1 = Id_{TV \otimes V \otimes TV}$.

Corollary 2.9 *For $n \neq 0, 1$ and for any TV-bicomodule M*

$$H^n(M, TV) = Hoch^n(M, TV) = 0$$

Proof: It is obvious by taking the injective resolution as in the lemma above.

Corollary 2.10 *With the same notations of Lemma 2.8, if $dim_k(V) \geq 2$ then*

$$H^0(TV, TV) = k$$

$$Hoch^0(TV, TV) = TV^\sigma$$

$$H^1(TV, TV) = Coder(TV)/InCoder(TV)$$

$$Hoch^1(TV, TV) = TV_\sigma$$

where $TV^\sigma = \bigoplus_{n \geq 0} V^{\otimes n \sigma}$ and $V^{\otimes n \sigma}$ are the invariants of $V^{\otimes n}$ under the action of $\mathbb{Z}/\mathbb{Z}_n$ (the generator σ of $\mathbb{Z}/\mathbb{Z}_n$ acting on $V^{\otimes n}$ by $\sigma(v_1...v_n) = v_n v_1...v_{n-1}$).

Proof: In order to prove the first isomorphism, consider $\tilde{d}(f)(1) = 0$, $\tilde{d}(f)(v) = 0$ and $\tilde{d}(f)(v_1...v_n) = f(v_1...v_{n-1})v_n - f(v_2...v_n)v_1$

The following diagram is commutative:

$$\begin{tikzcd}
TV^* \arrow[r, "\tilde{d}"] \arrow[d] & Hom_k(TV, V) \arrow[d] \\
Com_{TV^e}(TV, TV^e) \arrow[r] & Com_{TV^e}(TV, TV \otimes V \otimes TV)
\end{tikzcd}$$

where the vertical maps are isomorphims, given on the left by:

$f \mapsto (1 \otimes f \otimes 1)(\Delta \otimes 1)\Delta$

and on the right by sending $g \in Com_{TV^e}(TV, TV \otimes V \otimes TV)$ to $(\epsilon \otimes 1 \otimes \epsilon)g$.

If $dim_k(V) \geq 2$, for $v_1...v_{n-1} \in V^{\otimes n-1}$ we can choose v_n linearly independent from v_1, then the formula $(\tilde{d})(f)(v_1...v_n) = f(v_1...v_{n-1})v_n - f(v_2...v_{n-1})v_1$ implies that $f|_{V^{\otimes n}} = 0\ \forall n > 0$ when $f \in Ker(\tilde{d})$.

For the second one, take $\overline{d} = (\epsilon \otimes id_V \otimes \epsilon \otimes id_{TV}))(d \otimes id_{TV})\rho_{TV}$. It makes the following diagram commutative:

$$\begin{tikzcd}
TV \arrow[r, "\overline{d}"] \arrow[d] & V \otimes TV = \overline{TV} \arrow[d] \\
TV^e \square_{TV^e} TV \arrow[r, "d \square_{TV^e} Id"] & (TV \otimes V \otimes TV) \square_{TV^e} TV
\end{tikzcd}$$

A straightforward computation shows that $\overline{d}(v_1...v_n) = v_1...v_n - v_n v_1...v_{n-1} = (1-\sigma)(v_1...v_n)$.

Remark: If $dim_k(V) = 1$, we choose a generator x, $V = k.x$ and $TV \cong k[x]$. The same formulae as above also hold, but now $\tilde{d}(f)(x^n) = f(x^{n-1})x - f(x^{n-1})x = 0\ \forall f : k[x] \to k$ and $\overline{d}(x^n) = x^n - x^n = 0$, then $H^0(k[x], k[x]) = k[x]^*$, $H^1(k[x], k[x]) = Hom_k(k[x], k.x) = k[x]^*.x$ and $Hoch^0(k[x], k[x]) = k[x]$, $Hoch^1(k[x], k[x]) = k[x] \otimes k.x = \overline{k[x]}$.

2.4 Hermitian cohomology theories for involutive coalgebras

The duality between finite dimensional algebras and coalgebras suggests that in the case of involutive k-coalgebras, there are cohomology theories, dual in the finite dimensional case to positive Hochschild homology and cohomology, that take into account the fact that an involution exists.

We consider first the case $char(k) \neq 2$. As it happens with algebras, the theory may be generalized to the case of characteristic 2.

It is necessary to define an involution at each level of the complex computing both $Hoch^*(M, C)$ and $H^*(M, C)$ commuting with the differentials in order to express them as the direct sum of two subcomplexes. We recall that the complexes computing these cohomologies and the differentials are explicitely calculated by Y.Doi [D], and they are:

- for $H^*(M,C)$:
$$\delta^n : Hom(M, C^{\otimes n}) \to Hom(M, C^{\otimes n+1})$$
$$\delta^n(f) = (id \otimes f) \circ \rho_M^- - \sum_{i=0}^{n} \Delta_i \circ f + (-1)^{n+1}(f \otimes id)\rho_M^+$$
- for $Hoch^*(M,C)$:
$$d^n : C^{\otimes n} \otimes M \to C^{\otimes n+1} \otimes M$$
$$d^n(c_1 \otimes ... \otimes c_n \otimes m) = c_1 \otimes ... \otimes c_n \otimes \rho_M^-(m) + \sum_{i=0}^{n} c_1 \otimes ... \otimes \Delta(c_i) \otimes ... \otimes c_n \otimes m +$$
$$+ (-1)^{n+1}\sigma(c_1 \otimes ... \otimes c_n \otimes \rho_M^+(m))$$
where σ is the cyclic permutation $(123...n+1)$

Let M be a compatible C-bicomodule. Consider, for $n \in \mathbb{N}_0$:
$$\omega_n : C^{\otimes n} \otimes M \to C^{\otimes n} \otimes M$$
$$\omega_n(c_1 \otimes ... \otimes c_n \otimes m) = (-1)^{\frac{n(n+1)}{2}} \overline{c_n} \otimes \overline{c_{n-1}} \otimes ... \otimes \overline{c_1} \otimes \overline{m}$$
where $\overline{c_i} = \omega_C(c_i)$ and $\overline{m} = \omega_M(m)$ for $c_i \in C$ and $m \in M$.

Lemma 2.11 *ω_n commutes with d^n for all $n \in \mathbb{N}$.*

Proof:
$$d^n(\omega_n(c_1 \otimes ... \otimes c_n \otimes m)) = (-1)^{\frac{n(n+1)}{2}} d^n(\overline{c_n} \otimes ... \otimes \overline{c_1} \otimes \overline{m}) =$$
$$= (-1)^{\frac{n(n+1)}{2}} \Bigg(\overline{c_n} \otimes ... \otimes \overline{c_1} \otimes \rho^-(\overline{m}) + \sum_{i=1}^{n} \overline{c_n} \otimes ... \otimes \Delta(\overline{c_{n-i}}) \otimes ... \overline{c_1} \otimes \overline{m}) +$$
$$+ (-1)^{n+1}\sigma(\overline{c_n} \otimes ... \otimes \overline{c_1} \otimes \rho^+(\overline{m}))\Bigg) =$$
$$= (-1)^{\frac{n(n+1)}{2}} \Bigg(\overline{c_n} \otimes ... \otimes \overline{c_1} \otimes (\omega_C \otimes \omega_M)\sigma_{12}\rho^+(m) + \sum_{i=1}^{n} \overline{c_n} \otimes ... \otimes \Delta(\overline{c_{n-i}}) \otimes ... \overline{c_1} \otimes \overline{m}) +$$
$$+ (-1)^{n+1}\sigma(\overline{c_n} \otimes ... \otimes \overline{c_1} \otimes (\omega_M \otimes \omega_C)\sigma_{12}\rho^-(m))\Bigg)$$

On the other hand:
$$\omega_{n+1}(d^n(c_1 \otimes ... \otimes c_n \otimes m)) = (-1)^{\frac{(n+1)(n+2)}{2}} \omega_{n+1} \Big(c_1 \otimes ... \otimes c_n \otimes \rho^-(m) +$$
$$+ \sum_{i=1}^{n} c_1 \otimes ... \otimes \Delta(c_i) \otimes ... \otimes c_n \otimes m + (-1)^{n+1}\sigma(c_1 \otimes ... \otimes c_n \otimes \rho^+(m))\Bigg) =$$

(denoting $\rho^-(m)$ by $\sum_{(m)} c_{(-1)} \otimes m_{(0)}$ and $\rho^+(m)$ by $\sum_{(m)} m_{(-1)} \otimes c_{(0)}$)

$$= (-1)^{\frac{(n+1)(n+2)}{2}} \left(\sum_{(m)} \overline{c_{(-1)}} \otimes \overline{c_n} \otimes ... \otimes \overline{c_1} \otimes \overline{m_{(0)}} + \sum_{i=1}^{n} \overline{c_n} \otimes ... \otimes (\omega_C \otimes \omega_C)\sigma_{12}\Delta(c_i) \otimes ... \otimes \overline{c_1} \otimes \overline{m} + \right.$$

$$\left. +(-1)^{n+1} (\sum_{(m)} \overline{c_n} \otimes ... \otimes \overline{c_1} \otimes \overline{c_{(0)}} \otimes \overline{m_{(-1)}}) \right)$$

The last term of this expression equals the first one of $d^n(\omega_n(c_1 \otimes ... \otimes c_n \otimes m))$, while the first term equals the third of the above one, and the middle terms are equal.

Let us define an involution on the complex computing $H^*(M, C)$ by $\omega_n(f)(m) = \omega_{C^{\otimes n}}(f(\overline{m}))$, where $f : M \to C^{\otimes n}$ and $\omega_{C^{\otimes n}}(c_1 \otimes ... \otimes c_n) = (-1)^{\frac{n(n+1)}{2}} \overline{c_n} \otimes ... \otimes \overline{c_1}$

Lemma 2.12 *This involution commutes with δ^n.*

Proof: it is similar to the proof of Lemma 2.11

We define then the positive and negative cohomology theories for involutive coalgebras as:

$$H^{*+}(M, C) := H^*((Hom(M, C^{\otimes n})^+, \delta^n))$$

where M is a compatible C-bicomodule and $(Hom(M, C^{\otimes n})^+ = \{f \in Hom(M, C^{\otimes n}) \ / \ \omega_n(f) = f\}$. Similarly

$$Hoch^{*+}(M, C) := H^*((C^{\otimes n} \otimes M)^+, d^n)$$

Example: Consider a vector space V of dimension 1, choose a generator x, and the coalgebra $T(V) \cong k[x]$ with involution defined by $x \mapsto -x$. Then

$$H^{0+}(k[x], k[x]) = (\bigoplus_{n \geq 0} k.x^{2n})^*$$

$$H^{1+}(k[x], k[x]) = (\bigoplus_{n \geq 0} k.x^{2n})^*.x$$

$$Hoch^{0+}(k[x], k[x]) = \bigoplus_{n \geq 0} k.x^{2n}$$

$$Hoch^{1+}(k[x], k[x]) = \bigoplus_{n \geq 0} k.x^{2n} \otimes k.x$$

The rest of this section is devoted to the description of these cohomology theories in such a way that it leads us to their invariance with respect to hermitian Morita - Takeuchi equivalence.

The fundamental fact to remark is that $HH^{*+}(-, C)$ and $Hoch^{*+}(-, C)$ are cohomological functors, from the category of compatible C-bicomodules to the category of k-spaces. As we proved in last section, the category of compatible C-bicomodules is isomorphic to the category of $C^e \rtimes k^{\mathbb{Z}_2}$-comodules. So we consider $H^{*+}(-, C)$ and $Hoch^{*+}(-, C)$ defined on this category.

Proposition 2.13 *If C is an involutive k-coalgebra and M is a $C^e \rtimes k^{\mathbb{Z}_2}$ comodule then*

$$Hoch^{*+}(M,C) = Cotor^*_{C^e \rtimes k\mathbb{Z}_2}(M,C)$$

Proof: We have to show that the complex computing $Hoch^{*+}(M,C)$ is obtained by cotensoring upon $C^e \rtimes k^{\mathbb{Z}_2}$ by M an injective resolution of C in the category of $C^e \rtimes k^{\mathbb{Z}_2}$-comodules. We know that $0 \longrightarrow C \xrightarrow{d} C^{\otimes 2} \xrightarrow{d} C^{\otimes 3} \xrightarrow{d} \ldots$ (see [D]) is a C^e-injective resolution of C. So, in order to have a resolution in our new category, it will be sufficient to prove that $C^{\otimes n+2}$ is $C^e \rtimes k^{\mathbb{Z}_2}$-injective ($n \geq 0$). But $C^{\otimes n+2}$ is an injective C^e-comodule, and if $char(k) \neq 2$, it is also a $k^{\mathbb{Z}_2}$-injective comodule because it is injective as $k[\mathbb{Z}_2]$-module by the semisimplicity of $k[\mathbb{Z}_2]$. As every diagram of $C^e \rtimes k^{\mathbb{Z}_2}$-comodules and $C^e \rtimes k^{\mathbb{Z}_2}$-colinear morphisms

$$\begin{tikzcd}
 & C^{\otimes n+2} & \\
0 \arrow[r] & Y_1 \arrow[u, "\phi"] \arrow[r] & Y_2
\end{tikzcd}$$

can be considered as a diagram in the category of C^e-comodules, by the injectivity of $C^{\otimes n+2}$ in this category, it can be completed by a C^e-colinear morphism $\overline{\phi}$

$$\begin{tikzcd}
 & C^{\otimes n+2} & \\
0 \arrow[r] & Y_1 \arrow[u, "\phi"] \arrow[r] & Y_2 \arrow[ul, "\overline{\phi}"']
\end{tikzcd}$$

By the procedure of "averaging" this morphism, we find a morphism $\tilde{\phi} = \frac{\overline{\phi}+\omega\overline{\phi}\omega}{2}$ which is $C^e \rtimes k^{\mathbb{Z}_2}$-colinear. If we replace $\overline{\phi}$ by $\tilde{\phi}$ in the above diagram, it remains commutative. This completes the proof of the $C^e \rtimes k^{\mathbb{Z}_2}$-injectivity of $C^{\otimes n+2}$.

The only thing to verify now is that $C^{\otimes n+2} \square_{C^e \rtimes k\mathbb{Z}_2} M \cong (C^{\otimes n} \otimes M)^+$.

First look at $C^{\otimes n+2} \square_{C^e \rtimes k\mathbb{Z}_2} M$ as a subobject of $C^{\otimes n+2} \square_{C^e} M \cong (C^{\otimes n} \otimes M)$. Then, considering an element of $C^{\otimes n+2} \square_{C^e \rtimes k\mathbb{Z}_2} M$ as an element of $C^{\otimes n+2} \square_{k\mathbb{Z}_2} M$ we obtain that $C^{\otimes n+2} \square_{C^e \rtimes k\mathbb{Z}_2} M = C^{\otimes n+2} \square_{C^e} M \cap \{z \in C^{\otimes n} \otimes M \ / \ \omega(z) = z\} \cong (C^{\otimes n} \otimes M)^+$.

In a similar way, we may prove that

Proposition 2.14 *Under the same hypothesis of the previous proposition,*

$$H^{*+}(M,C) = Ext^*_{C^e \rtimes k\mathbb{Z}_2}(M,C)$$

3 Invariance of cohomology theories

3.1 Morita - Takeuchi invariance of cohomology theories

We begin this section by showing that the cohomology of coalgebras has sometimes an alternative description in terms of coderivations, as it happens with Hochschild cohomology.

Let S denote the class of all k-split C-bicomodule morphisms (i.e. morphisms which have a k-linear left or right inverse). We say that a C-bicomodule I is S-injective in case whenever a monomorphism $f : M \to M'$ is in S, the induced map $f^* : Com_{C^e}(M', I) \to Com_{C^e}(M, I)$ is surjective.

We say that a short exact sequence

$$0 \longrightarrow M \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is S-exact if it is exact and $f \in S$ (or $g \in S$). Other sequences are S-exact if they are made up from short S-exact sequences in the obvious way.

The category of C-bicomodules does not have, in general, enough S-projectives, however, in some cases (i.e. when C is a semiperfect coalgebra [L2]), there are enough S-projectives. This happens for example if there exists a C^*-module monomorphism $f : C \to C^*$. It is clear that the category of C-bicomodules has enough S-injectives.

Let us consider now, for a C-bicomodule N, $Coder(N, C) = \{f : N \to C \text{ such that } \Delta f = (1 \otimes f)\rho^- + (f \otimes 1)\rho^+\}$

$Coder(-, C)$ is a functor from C-bicomodules to abelian groups, let $\tilde{H}^n(-, C)$ be its n-th derived functor (using the class S).

Proposition 3.1 $\tilde{H}^0(N, C) \cong Coder(N, C)$

Proof: this follows at once since $Coder(-, C)$ is a contravariant functor, which is right exact.

It is useful to notice that the functor $Coder(-, C)$ is naturally equivalent to $Com_{C^e}(-, L_C)$, where L_C is the cokernel of $\Delta : C \to C \otimes C$ see [T].

For a C-bicomodule N, let $\hat{H}(-, N)$ be the derived functor (using S) of the functor $Com_{C^e}(-, N)$. Then, if N is the C-bicomodule C with the usual structure, and N' is another C-bicomodule, $\hat{H}(N', C)$ is the cohomology of C with coeficients in N', that is $H^*(N', C)$ as defined in section 2.

Proposition 3.2 *Let C be the C^e-comodule with the usual structure given by comultiplication and let N be a C-bicomodule. Then, if $n \geq 1$:*

$$\tilde{H}^n(N, C) \cong \hat{H}^{n+1}(N, C)$$

Proof: First observe that

$$0 \longrightarrow C \xrightarrow{\Delta} C \otimes C^{op} \longrightarrow L_C \longrightarrow 0$$

is an S-exact sequence ($s : C \otimes C^{op} \to C$, $s(c_1 \otimes c_2) = \epsilon(c_1)c_2$ is a k-linear splitting of the comultiplication).

The functors $Coder(-, C)$ and $Com_{C^e}(-, L_C)$ are naturally equivalent, hence $\tilde{H}^*(N, C) = \hat{H}^*(N, L_C)$ and for $n \geq 1$ we have exact sequences

$$0 = \hat{H}^n(N, C^e) \to \hat{H}^n(N, L_C) \to \hat{H}^{n+1}(N, C) \to \hat{H}^{n+1}(N, C^e) = 0$$

So $\tilde{H}^n(N, C) \cong \hat{H}^{n+1}(N, C)$.

Next, we focus our attention in Theorem A.

In fact we shall prove the following stronger version of this result, and obtain Theorem A as a corollary.

Theorem 3.3 *Let C and D be k-coalgebras such that the category of C-bicomodules and D-bicomodules are equivalent by means of a a $C^e - D^e$-bicomodule P' and a $D^e - C^e$-bicomodule Q' such that $C\square_{C^e}Q' \cong D \cong P'\square_{C^e}C$ as D^e-comodules, and let N be a C-bicomodule. Then*

$$H^*(N, C) \cong H^*(N\square_{C^e}Q', D)$$

and

$$Hoch^*(N, C) \cong Hoch^*(P'\square_{C^e}N', D)$$

Remark: this situation is analog to what happens with Hochschild homology. If A and B are Morita equivalent k-algebras, and M is an A-bimodule, then $H_*(A, M) \cong H_*(B, F(M))$ (F is the equivalence functor) [McC]. However, if we only know that the bimodule categories are equivalent (and $F(A) = B$), the same result holds for Hochschild cohomology [Sc] and for Hochschild homology [F-S].

Proof of the theorem: Consider an injective resolution $\mathbf{X}.$ of C as a C^e-comodule.

As the functor $-\square_{C^e}Q'$ is an equivalence, then $\mathbf{X}.\square_{C^e}Q'$ is an injective resolution of $C\square_{C^e}Q' \cong D$ as a D^e-comodule.

Choosing this injective resolution of D, we have:

$$H^*(N, C) \cong H^*(Com_{C^e}(N, \mathbf{X}.)) \cong H^*(Com_{D^e}(N\square_{C^e}Q', \mathbf{X}.\square_{C^e}Q')) \cong$$

$$\cong H^*(N\square_{C^e}Q', D)$$

The isomorphism for the functor $Hoch^*(-,-)$ follows by the same arguments.

$$Hoch^*(N, C) \cong H^*(\mathbf{X}.\Box_{C^e} N) \cong H^*(\mathbf{X}.\Box_{C^e} C^e \Box_{C^e} N) \cong$$

$$\cong H^*(\mathbf{X}.\Box_{C^e}(Q'\Box_{D^e} P')\Box_{C^e} N) \cong H^*((\mathbf{X}.\Box_{C^e} Q')\Box_{D^e}(P'\Box_{C^e} N)) \cong$$
$$\cong Hoch^*(P'\Box_{C^e} N, D)$$

Corollary 3.4 *If C and D are k-coalgebras Morita - Takeuchi equivalent by means of a $C-D$-bicomodule Q and a $D-C$-bicomodule P, and N is a C-bicomodule, then:*

$$H^*(N, C) \cong H^*(P\Box_C N \Box_C Q, D)$$

and

$$Hoch^*(N, C) \cong Hoch^*(P\Box_C N \Box_C Q, D)$$

Proof: It follows by taking $P' = P \otimes Q$, $Q' = Q \otimes P$ in the above theorem and using Lemma 2.3.

However, this argument does not provide an explicit quasi-isomorphism between $Hoch^*(M, C)$ and $Hoch^*(Q\Box_C M \Box_C P, D)$. Even if we cannot give it in the general case, due to the lack of a characterization of coalgebras which are Morita - Takeuchi equivalent to C in terms of $M^n(C)$ and "idempotents" as it happens for algebras, we are able to show the explicit quasi-isomorphism for the case $D = M^n(C)$.

If N is a C-bicomodule, let us define maps $\phi_r^N : C^{\otimes r-1} \otimes N \to (M^n(C))^{\otimes r-1} \otimes M^n(N)$ $(r \geq 1)$ by

$$\phi_r^N(c_1 \otimes ... \otimes c_{r-1} \otimes x) = \sum_{i_1,...,i_r} c_1 e_{i_1}^{i_2} \otimes c_2 e_{i_2}^{i_3} \otimes ... \otimes c_{r-1} e_{i_{r-1}}^{i_r} \otimes x e_{i_r}^{i_1}$$

where e_i^j stands for the standard basis of $(M_n(k))^*$, and $\psi_r^N : C^{\otimes r-1} \otimes N \to (M^n(C))^{\otimes r-1} \otimes M^n(N)$ $(r \geq 1)$ by

$$\psi_r^N(c_1 e_{i_1}^{j_1} \otimes c_2 e_{i_2}^{j_2} \otimes ... \otimes c_{r-1} e_{i_{r-1}}^{j_{r-1}} \otimes x e_{i_r}^{j_r}) = \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} ... \delta_{i_r}^{j_r} c_1 \otimes c_2 \otimes ... \otimes c_{r-1} \otimes x$$

Lemma 3.5 *1. $(\psi_r^N)_{r\geq 1}$ and $(\phi_r^N)_{r\geq 1}$ commute with the differentials.*

2. $\psi_r^N \circ \phi_r^N = id_{C^{\otimes r-1} \otimes N}$

3. $\phi_r^N \circ \psi_r^N$ is homotopic to the identity of $M^n(C)^{\otimes r-1} \otimes M^n(N)$

Proof: The proofs of 1. and 2. are straightforward. In order to obtain 3. consider the homotopy in low dimentions given by

$$h^1 : M^n(C)^{\otimes 2} \to M^n(C)$$

$$h^1(xe_{i_0}^{j_0}, ye_{i_1}^{j_1}) = \epsilon(x)e_{i_0}^{j_0}ye_{i_1}^{j_1}\delta_1^{i_0}\delta_1^{j_1} \quad x, y \in C$$

A direct computation shows that $h^1d^0 = id_{M^n(C)}$

As a consequence, we have obtained a natural transformation from $Hoch^*(-, C)$ into $Hoch^*(M^n(-), M^n(C))$ which is an isomorphism in degree 0. As both are cohomological functors, we have that ψ_*^N and ϕ_*^N are maps of complexes inducing the isomorphism between $Hoch^*(N, C)$ and $Hoch^*(M^n(N), M^n(C))$.

3.2 Morita - Takeuchi invariance of hermitian cohomology theories

The theorem we intend to prove in this section is, as in the nonhermitian case, stronger than the hermitian Morita - Takeuchi invariance of H^{*+} and $Hoch^{*+}$. In fact, if C and D are involutive coalgebras such that the categories of compatible bicomodules are equivalent, and the equivalence send C to D (as compatible bicomodules with ω_C and ω_D), then their hermitian cohomology theories agree.

The key step of the proof is the characterization of H^{*+} and $Hoch^{*+}$ as cohomological functors.

The bicomodules appearing in the following theorem are always compatible.

Theorem 3.6 *Let C and D be k-coalgebras such that the category of C-bicomodules and D-bicomodules are equivalent by means of a $D^e - C^e$-bicomodule P' and a $C^e - D^e$-bicomodule Q', such that $C\Box_{C^e}Q'$ and $P'\Box_{C^e}C$ are isomorphic to D as D^e-comodules, then if M is a C^e-comodule:*

$$H^{*+}(M, C) \cong H^{*+}(M\Box_{C^e}Q', D)$$

and

$$Hoch^{*+}(M, C) \cong Hoch^{*+}(P'\Box_{C^e}M, D)$$

Proof: It follows from Proposition 2.13 and 2.14 of section 2.4 by using similar arguments to those of Theorem 3.3.

As a corollary, it is now easy to deduce the invariance of $H^{*+}(-, C)$ and $Hoch^{*+}(-, C)$ under hermitian Morita - Takeuchi equivalences.

Remark: If N is a compatible C-bicomodule, the explicit isomorphism between $Hoch^*(N, C)$ and $Hoch^*(M^n(N), M^n(C))$ obtained in Lemma 3.5 commutes with the respective involutions, inducing then an isomorphism between the hermitian cohomologies.

In the case $char(k) = 2$ we just take the result of proposition 2.13 and 2.14 as a definition of the positive cohomology theories of C. It is important to notice that the result of Theorem 3.6 remains valid.

4 Coseparable and Azumaya coalgebras

4.1 Azumaya coalgebras

We begin this section by recalling some definitions from [T-V.O-Z] related to the idea of Azumaya coalgebras.

From now on, R will be a cocomutative k-coalgebra.

Definition 4.1 *An R-coalgebra (C, ϵ_R) is a k-coalgebra C provided of a morphism of k-coalgebras $\epsilon_R : C \to R$.*

As a consequence, C has a R-bicomodule structure defined by:

$$(\epsilon_R \otimes 1)\Delta : C \to R \otimes C$$

and

$$(1 \otimes \epsilon_R)\Delta : C \to C \otimes R$$

It is well-known that the property of a k-algebra being separable is characterized by means of the derivations of A into A-bimodules (or by the α-derivations of A into A-bimodules, where $\alpha \in Aut(A)$, [R-S]).

Definition 4.2 *An R-coalgebra C is said to be coseparable if there exists a C-comodule map $\pi : C\square_R C \to C$ such that $\pi\Delta = Id_C$.*

Proposition 4.3 *Let C be an R-coalgebra. The following statements are equivalent:*

1. *C is R-coseparable.*
2. *There exists an R-colinear map $\rho : C\square_R C \to R$ such that $\rho\Delta = \epsilon_R$, and we have $\sum \epsilon\rho(c_i \otimes d_{i(1)})d_{i(2))} = \sum c_{i(1)}\epsilon\rho(c_{i(2)} \otimes d_i)$ for any $\sum c_i \otimes d_i \in C\square_R C$.*
3. *C is a right (or left) C^{eR}-injective comodule.*

Proof: see [T-V.O-Z]

In particular, if $R = k$, C is a k-coalgebra coseparable if and only if C is C^e-injective.

In the same way, coderivations of a C-bicomodules N into C give an alternative description to the property of coseparability of a k-coalgebra.

The first part of the following proposition has been already proved by Doi [D]. Here we also relate coseparability with the $Hoch^*(-, C)$ functor.

Proposition 4.4 *The following statements are equivalent:*

1. *C is a coseparable k-coalgebra.*

2. *Every coderivation of a C-bicomodule into C is inner (i.e. $H^1(N,C)=0$, for every bicomodule N).*

3. *$H^n(N,C)=0$, for all C-bicomodule N and $n \geq 1$.*

4. *$Hoch^n(N,C)=0$, for all C-bicomodule N and $n \geq 1$, and so C is C^e-coflat.*

Proof: the equivalence between 1. 2. and 3. is given in [D]. We prove that 4. is equivalent to 1.

C is a coseparable k-coalgebra if and only if C is an injective C^e-comodule ([T-V.O-Z]), then taking

$$0 \longrightarrow C \xrightarrow{id} C \longrightarrow 0$$

as C^e-injective resolution of C as a C^e-comodule we have $Hoch^n(N,C)=0$, for all C-bicomodule N and $n \geq 1$.

Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of C^e-comodules, then we have a long exact sequence

$$0 \to C\Box_{C^e}N' \to C\Box_{C^e}N \to C\Box_{C^e}N'' \to Hoch^1(N',C) \to Hoch^1(N,C) \to ...$$

Since $Hoch^1(X,C)=0$ for all C^e-comodule X we have that $C\Box_{C^e}-$ is right exact, i.e. C is C^e-coflat.

Let us suppose now that $Hoch^n(N,C)=0$, for all bicomodule N and $n \geq 1$. Then using the above long exact sequence, $C\Box_{C^e}-$ is right exact. But this is equivalent to the fact of $Com_{C^e}(C,-)$ being exact, and so C is C^e-injective.

The other element necessary to the definition of Azumaya coalgebra is the cocenter of C, which is defined using the functor $h_{C^eR}(-,C)$, left adjoint to the cotensor product. Note that $h_{C^eR}(-,C)$ always exists because C is always C^e-quasifinite.

Definition 4.5 1. *The* **cocenter** *of the k-coalgebra C is $h_{C^e}(C,C)$.*

2. *Similarly, the* **cocenter** *of C as an R-coalgebra is defined as $h_{C^eR}(C,C)$.*

Remark: $H^0(C,C)$ is isomorphic to $(h_{C^e}(C,C))^*$

Definition 4.6 *Let C be an R-coalgebra. C is* **Azumaya** *over R if and only if C is R-cocentral (i.e. $h_{C^eR}(C,C) \cong R$) and R-coseparable.*

In the following paragraphs, we shall introduce the notion of "strong equivalences" and "weak equivalences" between categories of comodules. The idea of this type of equivalences (in the context of module categories) first appeared in [Sc] and [T2]. Schack used it to give an alternative characterization of the Brauer group of a commutative ring.

Let C and D two coalgebras.

Definition 4.7 *1. A* **weak k-congruence** $T : C^e - comod \to D^e - comod$ *is a k-linear equivalence of categories such that* $T(C) \cong D$.

2. A **strong k-congruence** $T : C^e - comod \to D^e - comod$ *is a k-linear equivalence of categories such that* $T(M \Box_C N) \cong T(M) \Box_D T(N)$*, for all* $M, N \in C^e - comod$.

Remark: the Picard group of a coalgebra C over a cocommutative coalgebra R, $Pic_R(C)$ is defined in [T-V.O-Z] for $R = k$ and generalized in [T-Z] as the group of all isomorphic classes of invertible R-cosymmetric C-bicomodules, where M is an invertible C-bicomodule if and only if there exists a bicomodule N such that $M \Box_C N \cong C$ as C-bicomodules. As strong k-congruences preserve invertibility, then a strong k-congruence T preserving R-cosymmetry induces an isomorphism between the Picard groups $Pic_R(C)$ and $Pic_R(D)$.

Proposition 4.8 *If C and D are Morita - Takeuchi equivalent k-coalgebras, then there is a strong k-congruence* $T : C^e - comod \to D^e - comod$.

Proof: If P and Q denote the bicomodules giving the Morita - Takeuchi equivalence, we take $T = P \Box_C - \Box_C Q : C - bicomod \to D - bicomod$. We have $T(C) = P \Box_C C \Box_C Q \cong P \Box_C Q \cong D$ and if M and N are C-bicomodules $T(M \Box_C N) = P \Box_C M \Box_C N \Box_C Q \cong P \Box_C M \Box_C C \Box_C N \Box_C Q \cong P \Box_C M \Box_C Q \Box_D P \Box_C N \Box_C Q = T(M) \Box_D T(N)$

The following statement is a reformulation of Theorem 3.3.

Theorem 4.9 *If there exists a weak k-congruence T between the categories of C-bicomodules and D-bicomodules given by a pair of invertible bicomodules P' and Q' ($T = -\Box_{C^e} Q'$ and $T^{-1} = -\Box_{D^e} P'$), then:*

$$H^*(M, C) \cong H^*(T(M), D)$$

and

$$Hoch^*(M, C) \cong Hoch^*(\tilde{T}(M), D)$$

for any C-bicomodule M, where $\tilde{T}(M) = P' \Box_{C^e} M$

Proposition 4.10 *Weakly congruent k-coalgebras have isomorphic cocenters, and isomorphic lattices of coideals.*

Proof: Let C and D be weakly congruent k-coalgebras, then

$$Z(C) = h_{C^e}(C, C) = h_{D^e}(T(C), T(C)) = h_{D^e}(D, D) = Z(D)$$

The purpose of the next theorem is to show that the property of being an Azumaya coalgebra over a cocommutative k-coalgebra R can be expressed in terms of strong or weak congruences of categories, giving then an alternative description of the Brauer group of R, $B(R)$ (defined in [T-V.O-Z]).

Theorem 4.11 *Let R be a cocommutative k-coalgebra and C a cosymmetric R-coalgebra. The following statements are equivalent:*

1. *The category of C-bicomodules which are R-cocentral is strongly congruent to the category of cocentral R-bicomodules.*
2. *The category of C-bicomodules which are R-cocentral is weakly congruent to the category of cocentral R-bicomodules.*
3. *C is an Azumaya R-coalgebra.*

Proof: 1)$\Rightarrow$2) is clear.

3)$\Rightarrow$1) Taking $P = Q = C$, we have to prove that $P \square_R Q \cong C^{eR}$ and $Q \square_{C^{eR}} P \cong R$. The first isomorphism is obvious, and the second follows from the following facts:

- C being R-coseparable implies that $C \square_{C^{eR}} C \cong h_{C^{eR}}(C, C)$ (see [T-V.O-Z])
- C being R-cosymmetric implies that $h_{C^{eR}}(C, C) \cong h_{C^e}(C, C) = Z(C)$
- C being R-cocentral means that $Z(C) \cong R$

Now we have to show that this congruence is strong:

Let M and N two R-bicomodules. If $M = R$, since $T(R) = C$ it is clear that $T(M \square_R N) \cong T(M) \square_C T(N)$, and the same holds if $M = R^{(I)}$ for some index I. The general case holds because every R-cosymmetric bicomodule M can be embedded in an exact sequence of the form $0 \to M \to R^{(I)} \to R^{(J)}$ and the above isomorphisms.

2)$\Rightarrow$3) $R = R \square_R R^{op}$, then R is injective in the category of R-cosymmetric bicomodules, then $C = T(R)$ is C^{eR} injective i.e. C is R coseparable. R is cocentral by 4.10. This completes the proof.

Corollary 4.12 *If C is Azumaya over R, then $Pic_R(C) \cong Pic_R(R) = Picent(R)$*

The following part of this section is devoted to the proof of Theorem B.

Remark: Let C be a k-coalgebra, as $Hoch^n(M,C) = Cotor^n_{C^e}(M,C)$, it is obvious that if M is injective as a C^e-comodule, then $Hoch^n(M,C) = 0$ for $n > 0$.

Theorem 4.13 *Let C be a k-coalgebra which is k-Azumaya, then*

$$H^*(M,C) \cong H^*(C\square_{C^e}M, k)$$

$$Hoch^*(M,C) \cong Hoch^*(M\square_{C^e}C, k)$$

in particular $H^*(M,C) = Hoch^*(M,C) = 0$ for $* > 0$

Proof: It follows from Theorem 4.9.

Theorem 4.14 *Let C and R be k-coalgebras, R cocommutative such that C is R-Azumaya , then*

$$Ext^*_{C^{eR}}(M,C) \cong Ext^*_R(C\square_{C^{eR}}M, R)$$

$$Cotor^*_{C^{eR}}(C,M) \cong Cotor^*_R(R, C\square_{C^{eR}}M)$$

Proof: By 4.11 there is a k-congruence T between the category of R comodules and C^{eR}-comodules (given by $-\square_R C$ and $-\square_{C^{eR}}C$).

Let $0 \to C \to X_0 \to X_1 \to ...$ be a injective resolution of C as C^{eR}-comodule. As T, being a k-congruence, is exact and preserves injectivity, then $0 \to T(C) \to T(X_0) \to T(X_1) \to ...$ is an R-injective resolution of $T(C) = R$. Choosing this resolution, we have

$$Ext^*_{C^{eR}}(M,R) = H^*(Com_R(T(M), T(\mathbf{X}_*)) \cong$$

$$\cong H^*(Com_{C^{eR}}(M, \mathbf{X}_*)) = Ext^*_{C^{eR}}(M,C)$$

The proof for the *Cotor* functor is similar:

Let $0 \to C \to X_0 \to X_1 \to ...$ as above,

$$Cotor^*_{C^{eR}}(C,M) = H^*(\mathbf{X}_*\square_{C^{eR}}M) \cong H^*(\mathbf{X}_*\square_{C^{eR}}C^{eR}\square_{C^{eR}}M) \cong$$

$$\cong H^*((\mathbf{X}_*\square_{C^{eR}}C)\square_R(C\square_{C^{eR}}M)) = Cotor^*_R(R, C\square_{C^{eR}}M)$$

Remark: Under the same hypothesis of Theorem 4.14, when $dim_k(R) < \infty$ (then so is C), $H^*(C,C) \cong HH^*(C^*,C^*)$ and $Hoch^*(C,C)^* \cong HH_*(C^*)$. It follows that the cohomology theories for finite dimensional coalgebras are invariant under Azumaya extensions if and only if the Hochschild homology and cohomology are so.

The natural question arising immediately is whether this fact holds in general or not, i.e. if $H^*(C,C) \cong H^*(R,R)$ and $Hoch^*(C,C) \cong Hoch^*(R,R)$ when C is Azumaya over R. This is not a straight consequence of 4.9 because an equivalence between the categories of R-comodules and C^{eR}-comodules does not imply an equivalence between the categories of R-bicomodules and C-bicomodules. It is not clear that the proof of [C-W] for algebras can be dualized, because of the lack of the concept of localization with respect to a subset in the coalgebra context.

4.2 Change of base coalgebra

It is evident that $Hoch^*(M,C)$ and $H^*(M,C)$ depend on the choice of the base coalgebra k. However, in some cases, change of base does not change cohomology. This is the case when we have C a k-coalgebra which is also an R-coalgebra, and R is a cocommutative k-coalgebra coseparable over k.

The property of R being coseparable is described in proposition 4.3. Explicitely, the comultiplication $\Delta_R : R \to R \otimes R$ has a left inverse π, with π a morphism of R-bicomodules.

Remark: If R is k-coseparable, then so is R^e.

The most general definition of $Hoch^*(-,C)$ and $H^*(-,C)$ is taking respectively the relative $Cotor^*_{C^e/k}(-,C)$ and relative $Ext^*_{C^e/k}(-,C)$. So, the computation is achieved by taking a k-relative injective resolution of C as C^e-comodule and then applying the respective functor ($-\square_{C^e} M$ or $Com_{C^e}(M,-)$) and calculating the homology of the resulting complex.

From now on, R will be a cocommutative k-coalgebra, and C an R-cosymmetric coalgebra.

First we recall that a morphism $\phi : N \to M$ of C-bicomodules is **R-allowable** (see [Sc] for the corresponding definition for modules) if and only if there exists a morphism f of R-bicomodules, $f : M \to N$ such that $\phi f \phi = \phi$. ϕ is said to be **R-injective** if it satisfies the usual lifting criterium relative to R-allowable monomorphisms. An **injective R-resolution** of a comodule N is an exact sequence $0 \to N \to I_0 \to I_1 \to ...$ in which the morphisms are R-allowable and every I_n is R-relative injective.

It is clear that in the category of C-bicomodules there are enough R-relative injectives since the obvious functors from C-bicomodules to R-bicomodules have both adjoints.

Lemma 4.15 *If R is k-coseparable, then $\phi : N \to M$ is R-allowable if and only if it is k-allowable.*

Proof: The only thing to prove (the converse assertion being clear) is that if ϕ is k-allowable, then it is R-allowable.

Let then $\phi : N \to M$ be a k-allowable morphism of C-bicomodules and let $f : M \to N$ be a k-morphism such that $\phi f \phi = \phi$. Of course f is not necessarily a morphism of R-bicomodules, so we have to modify it, taking into account the structural morphisms of M and N.

Define $\hat{f} : M \to N$ as the composition:

$$M \xrightarrow{\rho_M^{R^e}} R^e \otimes M \xrightarrow{id\otimes f} R^e \otimes N \xrightarrow{id\otimes\rho_N^{R^e}} R^e \otimes R^e \otimes N \xrightarrow{\epsilon\pi\otimes id} k \otimes N \cong N$$

where $\epsilon : R^e \to k$ and $\pi : R^e \otimes R^e \to R^e$ is obtained by the coseparability of R^e. Then:

$$\begin{aligned}(id \otimes \hat{f})\rho_M^{R^e} &= (id \otimes \epsilon\pi \otimes id)(id \otimes id \otimes \rho_N^{R^e})(id \otimes id \otimes f)(id \otimes \rho_M^{R^e})\rho_M^{R^e} = \\ &= (id \otimes \epsilon\pi \otimes id)(id \otimes id \otimes \rho_N^{R^e} f)(\Delta \otimes id_M)\rho_M^{R^e} = \\ &= (id \otimes \epsilon\pi \otimes id)(\Delta \otimes id_{R^e} \otimes id_N)(id \otimes \rho_N^{R^e} f)\rho_M^{R^e}\end{aligned}$$

On the other hand:

$$\begin{aligned}
\rho_N^{R^e}\hat{f} = \rho_N^{R^e}(\epsilon\pi\otimes id)(id\otimes\rho_N^{R^e})(id\otimes f)\rho_M^{R^e} = \\
= (\epsilon\pi\otimes id\otimes id)(id\otimes id\otimes\rho_N^{R^e})(id\otimes\rho_N^{R^e})(id\otimes f)\rho_M^{R^e} = \\
= (\epsilon\pi\otimes id\otimes id)(id\otimes(\Delta\otimes id))(id\otimes\rho_N^{R^e})(id\otimes f)\rho_M^{R^e} = \\
= (\epsilon\pi\otimes id\otimes id)(id\otimes\Delta\otimes id)(id\otimes\rho_N^{R^e}f)\rho_M^{R^e}
\end{aligned}$$

So, they are equal provided that $(id\otimes\epsilon\pi)(\Delta\otimes id) = \epsilon\pi\otimes id)(id\otimes\Delta)$ but this is exactly the coseparability condition of R^e. We have then proven that $\hat{f}$ is a morphism of R^e-comodules.

Finally,

$$\begin{aligned}
\phi\hat{f}\phi = \phi(\epsilon\pi\otimes id)(id\otimes\rho_N^{R^e})(id\otimes f)\rho_M^{R^e}\phi = \\
= \phi(\epsilon\pi\otimes id)(id\otimes\rho_N^{R^e})(id\otimes f)(id\otimes\phi)\rho_N^{R^e} = \\
= \phi(\epsilon\pi\otimes id)(id\otimes\rho_N^{R^e})(id\otimes f\phi)\rho_N^{R^e} = \\
= (\epsilon\pi\otimes id)(id\otimes id\otimes\phi)(id\otimes\rho_N^{R^e})(id\otimes f\phi)\rho_N^{R^e} = \\
= (\epsilon\pi\otimes id)(id\otimes\rho_M^{R^e})(id\otimes\phi f\phi)\rho_N^{R^e} = \\
= (\epsilon\pi\otimes id)(id\otimes\rho_M^{R^e})(id\otimes\phi)\rho_N^{R^e} = \hat{\phi}
\end{aligned}$$

where the second and fifth equalities use that ϕ is a morphism of bicomodules.

Observe that if ϕ is a morphism of bicomodules, then $\hat{\phi} = \phi$ because:

$$\begin{aligned}
\hat{\phi} = (\epsilon\pi\otimes id_M)(id\otimes\rho_M^{R^e})(id\otimes\phi)\rho_N^{R^e} = \\
= (\epsilon\pi\otimes id_M)(id\otimes\rho_M^{R^e})\rho_N^{R^e}\phi = \\
= (\epsilon\pi\otimes id_M)(\Delta\otimes id)\rho_N^{R^e}\phi = (\epsilon\pi\Delta\otimes id_M)\rho_N^{R^e}\phi = \\
= (\epsilon\otimes id)\rho_N^{R^e}\phi = id_M\phi = \phi
\end{aligned}$$

As $\phi\hat{f}\phi = \phi$, ϕ is R-allowable.

Corollary 4.16 *A C-bicomodule I is R-relative injective if and only if it is k-relative injective.*

We return to the computation of $Hoch^*_{C/R}(M,C)$, where M is an R-cosymmetric C-bicomodule, i.e. an C^{eR}-comodule. Taking the canonical injective resolution of C as C^{eR}-comodule $I_* = C^{\square_R *+2}$, we have that as I_n is R-relative injective, then I_* is k-relative injective. So $Hoch^*_{C/k}(M,C)$ is the cohomology of the complex $(I_n\square_{C^e}M, \delta_n\square_{C^e}id)_{n\in\mathbb{N}_0}$. But this complex is isomorphic to $(C^{\square_R n}\otimes M, \overline{\delta_R})$, that is, the one which computes $Hoch^*_{C/R}(M,C)$. So we have:

Theorem 4.17 *If R is a cocommutative k-coalgebra, C an R-cosymmetric coalgebra and M an R-cosymmetric C-bicomodule, then*

$$Hoch^*_{C/R}(M,C) \cong Hoch^*_{C/k}(M,C)$$

and the same holds for H^*

4.3 Hermitian Picard group of an involutive coalgebra

Roughly speaking, it is natural to define the hermitian Picard group of an involutive R coalgebra as the set of isomorphism clases of compatible C-bicomodules which gives an hermitian Morita - Takeuchi equivalence between C and C itself (the details are given in [F-S 2]). In general, this group denoted by $hPic_R(C, \omega_C)$ (where C is a coalgebra over a cocommutative coalgebra R, and the involution ω_C of C is R-colinear) depends on the involution.

Example: If C is finite dimensional over k, then $hPic_R(C, \omega_C) \cong hPic_{R^*}(C^*, \omega_C^*)$ because if $(P, Q, \mu, \tau, \Theta)$ is the hermitian Morita - Takeuchi context giving the equivalence, then $(P^*, Q^*, \mu^*, \tau^*, \Theta^*)$ provides an hermitian Morita equivalence of the R^*-algebra C^*. Consider then for example $hPic_{\mathbb{R}}(\mathbb{C}^*, \omega_{\mathbb{C}}^*)$, where $\omega_{\mathbb{C}}$ is the complex conjugation, then $hPic_{\mathbb{R}}(\mathbb{C}^*, \omega_{\mathbb{C}}^*) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, while $hPic_{\mathbb{R}}(\mathbb{C}^*, id) \cong \mathbb{Z}_2$.

However, we shall prove using the previous result of this section, that if C is an involutive R-coalgebra which is Azumaya over R, then $hPic_R(C, \omega_C)$ does not depend on the involution ω_C. In order to do it, we recall the notion of R-congruence from definition 4.7. The definition of hermitian congruence is the natural one, i.e. an R-congruence T that sends compatible C-bicomodules into compatible D-bicomodules and such that $T(C) \cong D$ as $C^e \rtimes k^{\mathbb{Z}_2}$-comodules.

Proposition 4.18 *Let C and D be two involutive R-coalgebras. If there exists an hermitian R-congruence between the category of $C^{eR} \rtimes k^{\mathbb{Z}_2}$-comodules and $D^{eR} \rtimes k^{\mathbb{Z}_2}$-comodules, then*

$$hPic_R(C, \omega_C) \cong hPic_R(D, \omega_D)$$

Proof: Consider the pair of invertible compatible bicomodules P' and Q' giving the equivalence. Given an element $[M] \in hPic_R(C, \omega_C)$, then $[P' \Box_{C^{eR}} M] \in hPic_R(D, \omega_D)$, and cotensoring upon D^{eR} by Q' gives the inverse morphism.

We finish this section by stating the above mentioned result on Azumaya coalgebras.

Corollary 4.19 *If (C, ω_C) is an involutive coalgebra such that C is h-Azumaya over R (i.e. C^{eR} is hermitian Morita - Takeuchi equivalent to R) then $hPic_R(C, \omega_C)$ does not depend on the involution.*

Proof: $hPic_R(C, \omega_C) \cong hPic_R(R, id)$

Remark: In general $hPic_R(R, id) \neq Pic_R(R)$

References

[C-D-R] S.Caenepeel - S.Dăscălescu - Ş. Raianu: Cosemisimple Hopf algebras coacting on coalgebras. Comm. in Alg. **24**(5) (1996), 1649-1677.

[C-W] G. Cortiñas - Ch. Weibel: Homology of Azumaya algebras. Proc. AMS, **121(1)** (1994), 53-55.

[D-I] K. Dennis - K. Igusa: Hochschild homology and the second obstruction for pseudo-isotopy. Lect. Notes Math. **966** (1982), 7-58.

[D] Y. Doi: Homological coalgebra. J. Math. Soc. Japan. **33** No. 1 (1981), 31-50

[McC] R. McCarthy: L'équivalence Morita et homologie cyclique. C. R. Acad. Sci. Paris. **307** (1988), 211-215.

[F-S] M.A. Farinati - A. Solotar: Morita equivalence for positive Hochschild homology and dihedral homology. Comm. in Alg. **24**(5) (1996), 1793-1807.

[F-S 2] M.A. Farinati - A. Solotar: On the Picard group of involutive algebras and coalgebras. Preprint Dto. Matemática - FCEyN - Univ. de B. Aires, 1996.

[F-McE] A. Fröhlich - Mc. Evett: Forms over rings with involution, J. of Alg.**12** (1969), 79-104.

[H] A. Hahn: An hermitian Morita theorem for algebras with antistructure. J.of Alg. **93** (1985), 215-235.

[L] B.I. Lin: Morita's theorem for coalgebras. Comm. in Alg. **1**(4) (1975), 311-344.

[L2] B.I. Lin: Semiperfect coalgebras. Journal of Alg. **49** (1977), 357-373.

[R-S] M.J.Redondo - A.L.Solotar: α-derivations. Can. Math. J. **38**(4) (1995), 481-489.

[Sc] S.D.Schack: Bimodules, the Brauer group, Morita equivalence, and cohomology. J. of P. and Ap. Alg. **80** (1992), 315-325.

[T] M. Takeuchi: Morita theorems for categories of comodules. J. Fac. Sci. Univ. Tokyo. **24** (1977), 1483-1528.

[T2] M. Takeuchi: $\sqrt{\text{Morita}}$ theory, formal ring laws and monoidal equivalences of categories of bimodules. J.Math.Soc.Japan **39** N.2 (1987), 301-336.

[T-V.O-Z] B.Torrecillas - F.Van Oysteyen - Y.H.Zhang: The Brauer Group of a Cocommutative Coalgebra. J. Alg. **177** (1995), 536-568.

[T-Z] B.Torrecillas - Y.H.Zhang: The Picard group of coalgebras. Comm. in Alg. **24**(7) (1996) 2235-2247.