ON THE PICARD GROUP OF INVOLUTIVE ALGEBRAS AND COALGEBRAS

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Abstract

We study how Hilbert bimodules correspond in the algebraic case to hermitian Morita equivalences and consequently we obtain a description of the hermitian Picard group of a commutative involutive algebra A as the semidirect product of the classical hermitian Picard group of A and the automorphisms of A commuting with the involution. We also obtain similar decomposition results on hermitian Picard groups of involutive coalgebras (C, ω_C), which show, at least in the cocommutative case, that this hermitian Picard group differs considerably from the non hermitian one.

Contents

1 Introduction

The Picard group of a k-algebra A (k a field) has been studied in detail from different points of view, principally when A is commutative (e.g. algebraic geometry, fiber bundles). For example, when X is a paracompact space and $A = C(X)$ is the algebra of complex valued continuous functions on X, then an element of the Picard group of A (over $\mathbb{C}$) corresponds to a line bundle over X ([1]). On the other hand, we know from [6] that the elements of the Picard group of a k-algebra A are isomorphism classes of A-bimodules providing a Morita equivalence between A and A itself, called invertible A-bimodules.

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Being interested in the situation described in [1], our first aim was to describe an analogous situation in the algebraic context, over an arbitrary field k. In fact, we found that the data provided by the inner products when $A = C(X)$ and $k = \mathbb{C}$ was not reflected by Morita equivalences but by hermitian Morita equivalences. This fact led us to the definition of algebraic Hilbert bimodules. Once this established, the definiton of hermitian Picard group of a k-algebra A (not necessarily commutative) is natural and is equivalent to the definition given by [8].

We then prove explicitely the following result:

Theorem (3.11): Let A be a commutative involutive k-algebra. Then there exists an split exact sequence of groups

$$1 \longrightarrow hCPic_k(A, \omega_A) \longrightarrow hPic_k(A, \omega_A) \longrightarrow hAut_k(A) \longrightarrow 1$$

where $hAut_k(A) = \{\alpha \in Aut_k(A)$ commuting with the involution of A $\}$ and $hCPic_k(A, \omega_A)$ is the subgroup of $hPic_k(A, \omega_A)$ consisting of (isomorphism classes of) A-symmetric A-bimodules. In other words,
$hPic_k(A, \omega_A) \cong hCPic_k(A, \omega_A) \rtimes hAut_k(A)$.

As examples show, this result provides an important tool to compute explicitely the Picard group of some algebras.

The situation relating the (hermitian) Picard group of a k-coalgebra C with the (hermitian) Morita - Takeuchi equivalence of C itself is similar. Of course, this is clear in the finite dimensional case, as the (hermitian) Picard groups of C and C^* are isomorphic. However, the analogy also holds for infinite dimensional k-coalgebras [10]. We define the hermitian Picard group of an involutive coalgebra, and we prove that in the cocommutative case, the following result holds:

Theorem (4.6): If C is a cocommutative involutive k-coalgebra, then
$hPic_k(C, \omega_C) \cong hCPic_k(C, \omega_C) \rtimes hAut_k(C)$.

This theorem, analogous to theorem 3.11 remarks an important difference between $Pic_k(C, \omega_C)$ and $hPic_k(C, \omega_C)$. Torrecillas and Zhang proved in [10] that if C is cocommutative k-coalgebra, then $Pic_k(C) \cong Aut_k(C)$. Here, in the involutive case, the classes of the C-cosymmetric bicomodules are not trivial, explicitely, $hCPic_k(C, \omega_C) \cong UZ(C^*)^+/UZ(C^*).\overline{UZ(C^*)}$ (Proposition 4.5) so, they depend on the dual algebra C^*. This fact suggests that in many infinite dimensional cases, the difference between $hPic_k(C, \omega_C)$ and $hPic_k(C^*, \omega_{C^*})$ depends on the respective automorphisms groups.

The contents of this work are the following:

In section 2 we recall definitions and properties of hermitian Morita theory from a point of view that will be useful afterwards.

The aim of section 3 is to introduce the notion of Hilbert bimodule, whose name comes from the fact that these modules play the role of Hilbert bimodules in the C^*-algebra context. We also provide an alternative definition of the hermitian Picard group of an involutive algebra. After having obtained some technical results, we prove Theorem 3.11.

We also prove in Proposition 3.14 that the subgroup of the hermitian Picard group of an algebra consisting of elements that are symmetric over the hermitian center is invariant under hermitian Morita equivalences.

Section 4 is devoted to coalgebras. After recalling the definition of hermitian Morita - Takeuchi equivalences from [4], we define $hPic_R(C,\omega_C)$, where C is an R-coalgebra, and describe its relation with $Pic_R(C,\omega_C)$ (Proposition 4.5). Finally we prove Theorem 4.6.

k will denote a commutative unital ring, A and B will denote associative unital k-algebras with involutions ω_A and ω_B, i.e. k-linear antimultiplicative morphisms such that $\omega_A^2 = Id_A$, and the same for B. C and D will be involutive k-coalgebras, with involutions ω_C and ω_D.

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2 Hermitian Morita theory

We first focus our attention on hermitian Morita equivalence. The given definition is not the same as Hahn's one [6], but it is equivalent to it [3]. Our purpose in changing it is to emphasize the role played by the k-linear bijection Θ of Hahn's definition.

We begin by recalling the basic definition of Morita equivalence:

Definition 2.1 *A and B are* **Morita equivalent** *if and only if the categories ${}_A mod$ and ${}_B mod$ are equivalent.*

It is well-known (see [7]) that A and B are Morita equivalent if and only if mod_A and mod_B are equivalent categories, and this fact corresponds to the existence of a progenerator bimodule ${}_A P_B = {}_A P_{End_A(P)}$ provided of bimodule isomorphisms $\mu : P \otimes_B P^{*_A} \to A$ (the trace map) and $\tau : P^{*_A} \otimes_A P \to B$ (defined by $\tau(f \otimes p)p' = f(p')p \ \forall p, p' \in P$ and $f \in P^{*_A}$). From now on, we will consider only Morita equivalence contexts of this type, and we will write $P^* = Q = P^{*_A}$.

Remark: Choosing $\mu : P \otimes_B Q \to A$ and $\tau : Q \otimes_A P \to B$ as above, the following equalities hold:

1. $\mu(p \otimes q)p' = p\tau(q \otimes p')$ and
2. $\tau(q \otimes p)q' = q\mu(p \otimes q) \ \forall p, p' \in P,\ q, q' \in Q$

Now suppose that (A, ω_A) and (B, ω_B) are two k-involutive algebras which are Morita equivalent and, with the notations as above, that we also have a k-linear isomorphism $\Theta : P \to Q$ satisfying

$$\Theta(a.p.b) = \omega(b).\Theta(p).\omega_A(a) \ \forall a \in A,\ b \in B,\ p \in P$$

As in every Morita equivalence, there are k-algebra isomorphisms $End_A(P) \cong B$ and $End_B(Q) \cong A$ given by the compositions

$$End_B(Q) = End_B(P^*) \cong P \otimes_B P^* = P \otimes_B Q \cong^{\mu} A$$

where the first isomorphism is $(p \otimes \phi) \mapsto (\phi' \mapsto \phi\phi'(p))$ and the others are the canonical ones.

$$End_A(P) \cong P^* \otimes_A P = Q \otimes_A P \cong^\tau B$$

Using μ, τ and Θ it is possible to define non-degenerated (in fact non-singular) sesquilinear forms in P and Q as follows:

$$< p, p' >_L := \mu(p \otimes \Theta(p'))$$

$$< q, q' >_L := \tau(q \otimes \Theta^{-1}(q'))$$

As a consequence $End_A(P)$ and $End_B(Q)$ are provided of involutions (both denoted by $(-)^*$) which are adjoint to the above sesquilinear forms, i.e. if $f \in End_A(P)$, f^* is defined as the unique element in $End_A(P)$ satisfying

$$< f(p), p' >_L = < p, f^*(p') >_L \quad \forall p, p' \in P$$

and if $g \in End_B(Q)$, g^* is defined respectively as the unique element in $End_B(Q)$ satisfying

$$< g(q), q' >_L = < q, g^*(q') >_L \quad \forall q, q' \in Q$$

Note: although in the examples of this paper the adjoint morphism corresponding to the sesquilinear forms will always be isomorphism and that in this case the usual term is non-singular or regular, we keep the name non-degenerated because the definition of $A - B$-Hilbert bimodule still makes sense when the inner products are not necessarily surjective emphasizing then the relation with Hilbert bimodules in the context of C^* algebras, where the usual term is non-degenerated, and in many examples the sesquilinear forms are non surjective [1].

The purpose of next proposition is to make clear the role of Θ: (we keep the notations of the above paragraph)

Proposition 2.2 *Let A and B be Morita equivalent, and let let $\Theta : P \to Q$ be a k-isomorphism such that $\Theta(a.p.b) = \omega_B(b)\Theta(p)\omega_A(a)$, $\forall a \in A\ b \in B\ p \in P$. Suppose in addition that μ and τ satisfy the following compatibility conditions:*

1. $\mu(p \otimes \Theta(p')) = \omega_A(\mu(p' \otimes \Theta(p)))$ *and*
2. $\tau(\Theta(p) \otimes p') = \omega_B(\tau(\Theta(p') \otimes p))\ \forall p, p' \in P$

Then, considering $End_A(P)$ and $End_B(Q)$ with involutions defined as above, the isomorphisms $End_A(P) \cong A$ and $End_B(Q) \cong B$ are morphisms in the category of involutive k-algebras, i.e. they commute with the respective involutions.

Proof: we make the computation for $B \cong End_A(P)$, the other is similar.

Identifying $End_A(P)$ with $P^* \otimes_A P$ it is enough to prove the assertion for elements $f \in End_A(P)$ of the form $f = \phi \otimes p$ with $\phi \in P^*$ and $p \in P$, where $(\phi \otimes p)(x) = \phi(x).p = \mu(x \otimes \phi).p$. If $x, y \in P$:

$$\begin{array}{lll} < x, f(y) >_L & = \mu(x \otimes \Theta((\phi \otimes p)(y))) & = \mu(x \otimes \Theta(\mu(y \otimes \phi).p)) \\ & = \mu(x \otimes \Theta(p)\omega_A(\mu(y \otimes \phi))) & = \mu(x \otimes \Theta(p)\mu(\Theta^{-1}(\phi) \otimes \Theta(y))) \\ & = \mu(x \otimes \Theta(p))\mu(\Theta^{-1}(\phi) \otimes \Theta(y)) & = \mu(\mu(x \otimes \Theta(p))\Theta^{-1}(\phi) \otimes \Theta(y)) \\ & = \mu((\Theta(p) \otimes \Theta^{-1}(\phi))(x) \otimes \Theta(y)) & = < (\Theta(p) \otimes \Theta^{-1}(\phi))(x), y >_L \end{array}$$

which proves that $f^* = (\phi \otimes p)^* = \Theta(p) \otimes \Theta^{-1}(\phi) = \tau^{-1}(\omega_A(\tau(\phi \otimes p)).$

Now we recall from [6] the definition of hermitian Morita equivalence.

Definition 2.3 *A and B are* **hermitian Morita equivalent** *, if they are Morita equivalent, i.e. there exist bimodules ${}_AP_B$ and ${}_BQ_A$ such that*

1. *P and Q are k-symmetric (i.e. $\lambda p = p\lambda \ \forall \lambda \in k, p \in P$, idem for Q)*
2. *There exist $\mu : P \otimes_B Q \to A$ and $\tau : Q \otimes_A P \to B$ isomorphisms of A-bimodules and B-bimodules respectively such that:*
   - *(a) $\mu(p \otimes q)p' = p\tau(q \otimes p')$ and*
   - *(b) $\tau(q \otimes p)q' = q\mu(p \otimes q) \ \forall p, p' \in P,\ q, q' \in Q$*

   and in addition
3. *There exists a k-linear bijection $\Theta : P \to Q$ such that*
   - *(a) $\mu(p \otimes \Theta(p')) = \omega_A(\mu(p' \otimes \Theta(p)))$*
   - *(b) $\tau(\Theta(p) \otimes p') = \omega_B(\tau(\Theta(p') \otimes p))$ and*
   - *(c) $\Theta(apb) = \omega_B(b)\Theta(p)\omega_A(a) \ \forall a \in A,\ b \in B,\ p, p' \in P$*

A set of data $(A, B, P, Q, \mu, \tau, \Theta)$ satisfying these conditions will be called an **hermitian Morita equivalence context***.*

Examples:

1. Let (A, ω_A) be an algebra with involution, then (A, ω_A) and $(M_n(A), \omega_{M_n(A)})$, where
$(\omega_{M_n(A)}(m))_{ij} = \omega_A((m)_{ji})$ $(m \in M_n(A))$ are hermitian Morita equivalent. To see this, take $e \in M_n(A)$ the idempotent defined as $(e)_{ij} = \delta_{1i}\delta_{1j}$, $P = eM_n(A)$, $Q = M_n(A)e$, $\mu : P \otimes_{M_n(A)} Q \to eM_n(A)e = A$ and $\tau : Q \otimes_{eM_n(A)e} P \to M_n(A)$ the multiplication, and $\Theta = \omega_{M_n(A)} : P \to Q$.

2. Let A be an algebra with involution and let G be a finite group of k-algebra automorphisms of A commuting with the involution such that $A|A^G$ is a Galois extension. Then A^G and $A \rtimes G$ are hermitian Morita equivalent. The Morita equivalence context is described in [2], using $P = Q = A$. If we take $\Theta = \omega_A : P \to Q$ it turns out that this Morita context is in fact an hermitian Morita context.

3 Algebraic Hilbert bimodules and the hermitian Picard group

This section consists of two parts. In the first one we define the notion of algebraic Hilbert bimodule (or simply Hilbert bimodule). Although our motivation for this definition comes from the C^*-algebra case [1], it turns out that the notion of Hilbert bimodule coincides with the compatible hermitian bimodules of [11].

In the second part we recall from [11] the definition of hermitian Picard group of an involutive k-algebra and, after defining the classical hermitian Picard group, we establish the relation between this group, the hermitian Picard group, and the automorphisms of the involutive algebra A for A commutative.

We begin with the notion of algebraic Hilbert bimodule, which is, as we shall see, closely related to the construction of the hermitian Picard group of an algebra with involution.

Definition 3.1 *A bimodule $M \in {}_A mod_B$ together with two k-bilinear maps $<,>_L$: $M \times M \to A$ and $<,>_R$: $M \times M \to B$ will be called an* **algebraic Hilbert $A-B$-bimodule** *if and only if the maps $<,>_L$ and $<,>_R$ verify the following properties:*

1. *$<,>_L$ is A-linear in the first variable and A-antilinear in the second variable, namely*
   - $< ax, y >_L = a < x, y >_L$ *and*
   - $< x, ay >_L = < x, y >_L \omega_A(a)$ *for all* $a \in A,\ x, y \in M$.
2. *$<,>_R$ is B-linear in the second variable and B-antilinear in the first variable, namely*
   - $< xb, y >_R = \omega_B(b) < x, y >_R$ *and*
   - $< x, yb >_R = < x, y >_R b$ *for all* $b \in B,\ x, y \in M$.
3. $< x, y >_L = \omega_A(< y, x >_L)$ *and* $< x, y >_R = \omega_B(< y, x >_R)$ $\forall x, y \in M$.
4. *The sesquilinear forms $<,>_L$ and $<,>_R$ are non-degenerated.*
5. $< x, y >_L z = x < y, z >_R \quad \forall x, y, z \in M$

In an analogous way to the case when A and B are C^*-algebras, $<,>_L$ and $<,>_R$ will be called respectively left and right inner products.

If in addition we have (with the notation as above) $< xb, y >_L = < x, y\omega_B(b) >_L$ we say that $<,>_L$ is B-balanced and analogously for $<,>_R$ and A.

An interesting consequence of the relations between items 1. 3. and 5. of the previous definition is the following:

Remark: If we consider a Hilbert $A-B$-bimodule M with surjective right inner product, then $<,>_L$ is B-balanced.

Proof: Given $x, y \in M$ and $b \in B$, let $x_b, y_b \in M$ such that $< x_b, y_b >_R = b$. Then

$$\begin{array}{rcll}
< xb, y >_L = & & < x < x_b, y_b >_R, y >_L = & (5) \\
& = & << x, x_b >_L y_b, y >_L = & (1) \\
& = & < x, x_b >_L < y_b, y >_L = & (3) \\
& = & < x, x_b >_L \omega_A(< y, y_b >_L) = & (1) \\
& = & < x, < y, y_b >_L x_b >_L = & (5) \\
& = & < x, y < y_b, x_b >_R >_L = & (3) \\
& = & < x, y\omega_B(b) >_L &
\end{array}$$

The same assertion interchanging left and right inner product and B by A is also true.

A **morphism** $f : (M, <,>_L^M, <,>_R^M) \to (N, <,>_L^N, <,>_R^N)$ of Hilbert bimodules is a bimodule map preserving inner products, i.e. $< f(m), f(m') >_L^N = < m, m' >_L^M$ for all $m, m' \in M$ and similarly for $<,>_R$.

Examples:

1. Let A be a k-algebra with involution ω_A. Then ${}_AM_A = A$ with $< a, a' >_L = a\omega_A(a')$ and $< a, a' >_R = \omega_A(a)a'$ is a Hilbert $A - A$-bimodule.

2. Let $({}_AP_B, {}_BQ_A, \mu, \tau, \Theta)$ be a set of hermitian Morita equivalence data between (A, ω_A) and (B, ω_B), we shall see below that P (resp Q) can be provided of a $<,>_L$ and $<,>_R$, obtaining then an $A - B$ (resp $B - A$) Hilbert bimodule.

3. Let A, B, C be algebras with involution, let $({}_AM_B, <,>_L^M, <,>_R^M)$ and $({}_BN_C, <,>_L^N, <,>_R^N)$ be two Hilbert bimodules with surjective inner products, then $M \otimes_B N$ is a Hilbert bimodule with (surjective) left and right inner products defined by

$$< m \otimes n, m' \otimes n' >_R = < n, < m, m' >_R^M n' >_R^N$$

$$< m \otimes n, m' \otimes n' >_L = < m < n, n' >_L^N, m' >_L^M$$

   The hypothesis of surjectivity is used in order to verify that the inner products are well-defined (explicitely we need the inner products being balanced) and their non degeneracy. For example, for the left inner product, which gives a map $M \otimes_B N \to (M \otimes_B N)^{*A}$ $(m \otimes n \mapsto < -, m \otimes n >_L)$, one identifies $(M \otimes_B N)^{*A} \cong N^{*B} \otimes_B M^{*A}$ whose inverse is explicitely constructed as the tensor product of the inverses of $m \mapsto < -, m >_L^M$ and $n \mapsto < -, n >_L^N$. Notice also that the identification $(M \otimes_B N)^{*A} \cong N^{*B} \otimes_B M^{*A}$ can be done for example when M is finitely generated and projective as B-module, which is our case, because the Morita context induced between A and B by $(M, <,>_L, <,>_R)$ has surjective trace maps, and so the projectivity of M is a consequence of the Morita theorems.

4. Let A and B be algebras with involution, $({}_AM_B, <,>_L, <,>_R)$ a Hilbert bimodule with surjective inner products, then M^{*A} and M^{*B} are Hilbert bimodules. We explicite the inner products in M^{*A}:

   Let $f, g \in M^{*A}$, as $<,>_L: M \times M \to A$ is non-singular, there exist unique elements x_f and x_g in M such that $f = < -, x_f >_L$ and $g = < -, x_g >_L$. The inner products in M^{*A} are defined by:

$$< f, g >_R := < x_f, x_g >_L$$

$$< f, g >_L := < x_f, x_g >_R$$

As a straight consequence of the non-degeneracy of the inner products in a Hilbert bimodule, we have the first parts of the following lemma:

Lemma 3.2 *Let $({}_AM_B, <,>_L, <,>_M)$ be a Hilbert bimodule, $x \in M,\ a \in A$ and $b \in B$.*

1. *If $< y, x >_R = 0\ \forall y \in M$ then $x = 0$.*
2. *If $< x, y >_L = 0\ \forall y \in M$ then $x = 0$.*
3. *Suppose that $<,>_L$ (resp. $<,>_R$) is surjective, then $a.y = 0\ \forall y \in M$ (resp. $y.b = 0\ \forall y \in M$) implies $a = 0$ (resp. $b = 0$).*

Proof: 3. $<,>_L$ being surjective, there exists a finite set $\{x_i, y_i\}_{i\in I} \subseteq M$ such that
$\sum_{i\in I} < x_i, y_i >_L = 1_A$.Then:

$$a = a.1 = \sum_{i\in I} a < x_i, y_i >_L = \sum_{i\in I} < ax_i, y_i >_L = \sum_{i\in I} < 0, y_i >_L = 0$$

The assertion concerning b is analogous.

The existence of an hermitian Morita equivalence $(P, P^{*_A}, \mu, \tau, \Theta)$ between two involutive k-algebras A and B provides P with the structure of a $A-B$-Hilbert bimodule where $<,>_R$ and $<,>_L$ are defined by:

$$< p, p' >_L = \mu(p \otimes \Theta(p'))\ ;\ < p, p' >_R = \tau(\Theta(p) \otimes p')$$

and they are surjective.

Conversely, the existence of an $A-B$-Hilbert bimodule M, with $<,>_R$ and $<,>_L$ surjective, allows us to construct an hermitian Morita equivalence between A and B as follows:

Consider $\Theta : M \to M^{*_A}$ mapping $x \in M$ into $< -, x >_L$.
Θ verifies $\Theta(amb) = \omega_B(b)\Theta(m)\omega_A(a),\ \forall a \in A,\ b \in B,\ m \in M$, one uses the fact that
$< x, ay >_L = < x, y > \omega_A(a)\ \forall x, y \in M,\ a \in A$. If $b \in B$ then $\Theta(xb) = < -, xb >_L$. We want to show that $\Theta(xb) = \omega_B(b)\Theta(x)$.

As the right and left inner products are compatible, $< x, yb >_L z = x < yb, z >_R$ $\forall x, y, z \in M$. As $<,>_R$ is right B-linear the last expression equals $x\omega_B(b) < y, z >_R$, and this is $< x\omega_B(b), y >_L z$.
The A-action on M being faithful (surjectivity of $<,>_L$ plus lemma 3.2), this implies that $< x\omega_B(b), y >_L = < x, yb >_L,\ \forall x, y \in M,\ b \in B$ (as we have already seen after the definition of Hilbert bimodules with a different argument). Then $\Theta(xb)(y) = < y, xb >_L = < y\omega_B(b), x >_L = \Theta(x)(y\omega_B(b)) = (\omega_B(b)\Theta(x))(y)$.

Then define $\mu : M \otimes_B M^{*_A} \to A$ by $\mu(x \otimes f) = < x, \Theta^{-1}(f) >_L$ and $\tau : M^{*_A} \otimes_A M \to B$ by $\tau(f \otimes x) = < \Theta^{-1}(f), x >_R$. It is clear that they are isomorphisms of $A-A$-bimodules (resp. $B-B$-bimodules) and that these constructions are reciprocal.

Remark: We could have begun the construction by taking $\tilde{\Theta} : M \to M^{*_B}$ $(x \mapsto < x, - >_R)$ instead of Θ. This situation shows that, in fact, in an hermitian Morita equivalence $M^{*_A} \cong M^{*_B}$.

Proposition 3.3 *Let A and B be k-algebras with involution. Suppose that for every $b \in B$, $b\omega_B(b) = 0$ implies $b = 0$. Then if $({}_AM_B, <,>_L, <,>_R)$ is any Hilbert bimodule and $m \in M$, $< m, m >_L= 0$ implies $m = 0$.*

Proof: $< m, m >_L= 0 \Rightarrow < m, m >_L x = 0 \ \forall x \in M$, then $0 =< m, m >_L x = m < m, x >_R$ and also

$$0 =< x, m < m, x >_R>_R =< x, m >_R< m, x >_R =< x, m >_R \omega_B(< x, m >_R)$$

then $< x, m >_R= 0$, which holds for every $x \in M$, and so, by Lemma 3.2 $m = 0$.

Remark: 1) If A and B are involutive k-algebras which are hermitian Morita equivalent, and B has the property of the previous proposition, then A doesn't need to share it, take for example $(A, \omega_A) = (\mathbb{C}, id)$ and $(B, \omega_B) = (M_2(\mathbb{C}), (\ )^t)$.

Remark: 2) Given an involutive k-algebra (A, ω_A), one can ask if there exists an involutive k-algebra B hermitian Morita equivalent to A such that B has the property of the previous proposition. The answer is "not necessarily", consider for example $A = k \times k$ with coordinate-wise product and involution $\omega_A(x, y) = (y, x)$. As it is commutative, every k-algebra B Morita equivalent to A contains A in its center.

From now on, A will be a k-algebra with involution denoted by $\omega_A(a) = a^*$ $(a \in A)$. A definition of the hermitian Picard group of an algebra with involution can be found in [8] and in [11], we will give here an alternative (and clearly equivalent) definition, using the notion of Hilbert bimodule.

Definition 3.4 *Let A be a k-algebra with involution ω_A, the* **hermitian Picard group of** *(A, ω_A) is the set of isomorphism classes of invertible Hilbert A-bimodules, i.e. isomorphism classes of triples $[(M, <,>_L, <,>_R)]$ where M is an A-bimodule, $<,>_L$ and $<,>_R$ are respectively left and right inner products with values in A, such that M is invertible as A-bimodule (hence $<,>_L$ and $<,>_R$ are surjective). This set will be denoted by $hPic(A, \omega_A)$.*

Remark: As in the nonhermitian case, $hPic(A, \omega_A)$ is a group with product induced by tensor product over A, unit $[(A, <,>_L, <,>_R)]$ and inverse $[(M, <,>^M_L, <,>^M_R)]^{-1} = [(M^*, <,>^{M^*}_L, <,>^{M^*}_R)]$ (see examples above).

A better understanding of the situation may be obtained, as in the nonhermitian case, by considering the categories $\mathcal{HC}_k$ and $\mathcal{HMC}_k$, where $\mathcal{HC}_k$ has involutive k-algebras as objects and isomorphisms commuting with involutions as morphisms, and $\mathcal{HMC}_k$ has the same objects while the morphisms are hermitian Morita equivalences between objects.

Both $\mathcal{HC}_k$ and $\mathcal{HMC}_k$ are monoidal categories. In the first case, $hAut_k(A)$ is the group of automorphisms of the object A, while in the second one, it is $hPic_k(A, \omega_A)$.

So $hPic_k(-)$ is a functor from the category $\mathcal{HMC}_k$ into groups.

Given involutive k-algebras A and B and a k-algebra isomorphism $f : A \to B$ commuting with ω_A, ω_B we obtain a morphism from A to B in $\mathcal{HMC}_k$, namely:

consider the isomorphism class of the Hilbert $B-A$-bimodule B_f, where if $a \in A,\ x, b \in B$ the bimodule action on B_f is given by $b.x.a := bxf(a)$, the inner products are:

$$< x, y >_R := f^{-1}(\omega_B(x)y) = \omega_A(f^{-1}(x))f^{-1}(y)$$

$$< x, y >_L := x\omega_B(y)$$

($B_f \otimes_A A_{f^{-1}} \cong B$ and $A_{f^{-1}} \otimes_B B_f \cong A$ as Hilbert bimodules). We have then a functor from $\mathcal{HC}_k$ to $\mathcal{HMC}_k$, and so, for any involutive k-algebra A, a map $hAut_k(A) \to hPic_k(A, \omega_A)$.

Taking $A = B$ we can generalize this construction for Hilbert bimodules. Let $\alpha \in hAut_k(A)$ and $(M, <,>_L^M, <,>_R^M)$ a $A-A$-Hilbert bimodule, and denote by $(M_\alpha, <,>_L^{M_\alpha}, <,>_R^{M_\alpha})$ the $A-A$-Hilbert bimodule with the right A-action twisted by α, that is:

$$a.m.a' := am\alpha(a')\ \forall a, a' \in A\ m \in M$$

and with right and left inner products defined by

$$< x, y >_L^{M_\alpha} := < x, y >_L^M$$

$$< x, y >_R^{M_\alpha} := \alpha^{-1}(< x, y >_R^M)$$

Proposition 3.5 *If $M = A$ (with the usual inner products, see first example in Section 3.) and $\alpha, \beta \in hAut_k(A)$, consider the Hilbert A-bimodules $(A_\alpha, <,>_L^{A_\alpha}, <,>_R^{A_\alpha})$, $(A_\beta, <,>_L^{A_\beta}, <,>_R^{A_\beta})$ and $A_\alpha \otimes_A A_\beta$ with the inner products as in the third example of Section 3. Then we have the following isomorphisms of Hilbert bimodules:*

1. *$A_\alpha \otimes_A A_\beta \cong (A_{\alpha\beta} <,>_L^{A_{\alpha\beta}}, <,>_R^{A_{\alpha\beta}})$ by the map $x \otimes y \mapsto x\alpha(y)$.*
2. *$A_\alpha^* \cong A_{\alpha^{-1}}$*

The kernel of the map $hAut_k(A) \to hPic_k(A, \omega_A)$ consists of elements α in $hAut_k(A)$ such that A_α is isomorphic to A as $A-A$-Hilbert bimodule. Denoting by $\phi : A \to A_\alpha$ this isomorphism, if $a \in A$

$$\phi(a) = \phi(a1) = a.\phi(1) = a\phi(1)$$

but also

$$\phi(x) = \phi(1x) = \phi(1).x = \phi(1)\alpha(x)$$

So $\alpha(x) = (\phi(1))^{-1}(x)\phi(1)$ i.e. α is inner. Using the fact that ϕ is a Hilbert morphism, we have

$$1 = < 1, 1 >_L^A = < \phi(1), \phi(1) >_L^{A_\alpha} = \phi(1)\omega_A(\phi(1))$$

Conversely, given $a \in U(A)$ such that $a\omega_A(a) = 1$, then taking $\alpha(x) := axa^{-1} = ax\omega_A(a)$ we obtain an isomorphism of A-bimodules $\phi_a : A \to A_\alpha$ defined by $\phi_a(x) = ax$. The fact that $a\omega_A(a) = 1$ implies that ϕ_a preserves inner products.

We have then proved :

Proposition 3.6 *If A is an involutive k-algebra, then there is an exact sequence*

$$1 \longrightarrow Inn_1(A) \longrightarrow hAut_k(A) \longrightarrow hPic_k(A, \omega_A)$$

where $Inn_1(A)$ is the subgroup of $hAut_k(A)$ consisting of inner automorphisms $x \mapsto axa^{-1}$ such that $a\omega(a) = 1$.

We next define a subgroup of $hPic_k(A, \omega_A)$ for A commutative, recalling that in this case an A-bimodule M is **A-symmetric** if and only if the right and left actions on M agree, i.e. if $am = ma \ \forall m \in M, \ a \in A$.

Definition 3.7 *Let A be a commutative involutive k-algebra, then the* **classical hermitian Picard group of** *A, denoted by $hCPic_k(A, \omega_A)$, is the subset of $hPic_k(A, \omega_A)$ of isomorphism classes of A-symmetric invertible bimodules.*

Remark: As A is A-symmetric and A-symmetry is preserved by tensor product over A, so $hCPic_k(A, \omega_A)$ is in fact a subgroup of $hPic_k(A, \omega_A)$.

Taking into account the exact sequence of Proposition 3.6 we see that if A is commutative, $hAut_k(A)$ is a subgroup of the hermitian Picard group.

If one doesn't take care of the involution of A and considers it simply as a k-algebra, the above discussion is consistent with Theorem 2 of [5]. However, as we shall see in the examples, the existence of an involution strongly changes the elements of the decomposition of the usual Picard group.

The aim of the following lemma is to show that $hPic_k(A, \omega_A)$ is the semidirect product of the classical hermitian Picard group $hCPic_k(A, \omega_A)$ and the automorphisms commuting with the involution, i.e. $hAut_k(A)$.

Lemma 3.8
1. *Given an invertible $A - B$-bimodule M, there is a unique isomorphism $\alpha_M : Z(B) \to Z(A)$ of k-algebras such that*

$$\alpha_M(c)m = mc \ \forall m \in M, \ c \in Z(B)$$

2. *If in addition there exists an antiisomorphism $\Theta : M \to M^{*_A}$ satisfying the conditions 3 (a-c) of Definition 2.3 (in particular M can be provided of a structure of $A - B$-Hilbert bimodule), then α_M commutes with involutions.*

Proof: 1. see [5].

2. Consider the antiisomorphism $\Theta : M \to M^{*_A}$ as above. M is α_M-central, i.e. $mb = \alpha_M(b)m \ \forall b \in Z(B), \ m \in M$. Then if $f \in Hom_A(M, A)$, $b \in Z(B)$ and $m \in M$, as $\alpha_M(b) \in Z(A)$:

$$\begin{aligned} (bf)(m) = & \quad f(mb) & = f(\alpha_M(b)m) \\ = & \ \alpha_M(b)f(m) & = f(m)\alpha_M(b) \\ = & \ (f\alpha_M(b))m & \end{aligned}$$

i.e. $bf = f\alpha_M(b) \ \forall b \in Z(B), \ f \in Hom_A(M, A)$.

Using the antiisomorphism Θ we have for $m \in M$ and $b \in Z(B)$

$$\begin{aligned}\Theta(m)\omega_A(\alpha_M(b)) = &\ \Theta(\alpha_M(b)m) &= \Theta(mb)\\ = &\ \omega_B(b)\Theta(m) &= \Theta(m)\alpha_M(\omega_B(b))\end{aligned}$$

As this holds for any $m \in M$ and the action is faithful, then $\alpha_M(\omega_B(b)) = \omega_A(\alpha_M(b))$

Corollary 3.9 *For A commutative, every invertible A-bimodule M is α-central for some (unique) $\alpha \in Aut_k(A)$. If in addition, M has the structure of a Hilbert $A-A$-bimodule, then $\alpha \in hAut_k(A)$.*

Let us introduce another definition:

Definition 3.10 *Let $(M, <,>_L, <,>_R)$ be a Hilbert $A-A$-bimodule and A commutative. We define $(M^s, <,>^s_L, <,>^s_R)$ (the* **symmetrization** *of M) as $(M_{\alpha^{-1}}, <,>_L^{M_{\alpha^{-1}}}, <,>_R^{M_{\alpha^{-1}}})$ where $\alpha = \alpha_M$ is the automorphism of the above corollary.*

Remark: It is clear that $(M^s, <,>^s_L, <,>^s_R)$ is an A-symmetric Hilbert $A-A$-bimodule, and consequently it defines an element in $hCPic_k(A, \omega_A)$.

Summarizing the above results, we obtain the following:

Theorem 3.11 *Let A be a commutative involutive k-algebra. Then there exists an split exact sequence of groups*

$$1 \longrightarrow hCPic_k(A, \omega_A) \longrightarrow hPic_k(A, \omega_A) \longrightarrow hAut_k(A) \longrightarrow 1$$

The first map is the inclusion and the second is $M \mapsto \alpha_M$. where α_M is the unique automorphism associated to M by Lemma 3.8. The splitting is given by $\alpha \mapsto (A_\alpha, <,>_L^{A_\alpha}, <,>_R^{A_\alpha})$.

This theorem allows us to write the hermitian Picard group of A as the semidirect product $hPic_k(A, \omega_A) \cong hCPic_k(A, \omega_A) \rtimes hAut_k(A)$, the isomorphism given by $M \mapsto (M^s, \alpha_M)$.

Examples: 1) $hPic(C(T^2), \omega) = \mathbb{Z} \rtimes hAut_{\mathbb{C}}(C(T^2))$ ([1]) where $C(T^2)$ is the algebra of continuous complex valued functions in the torus, with involution given by complex conjugation.

2) $hPic_{\mathbb{Q}}(M_n(\mathbb{C}), id) = Gal(\mathbb{C}|\mathbb{Q})$

2') $hPic_{\mathbb{R}}(M_n(\mathbb{C}), \overline{(-)}) = hAut_{\mathbb{Q}}(\mathbb{C}) = Gal(\mathbb{C}|\mathbb{R})$.

3) $hPic_k(k \times k, \overline{(-)}) = hCPic_k(k \times k, \overline{(-)}) \rtimes hAut_k(k \times k) = k^*/(k^*)^2 \rtimes \mathbb{Z}_2$, where $\overline{(x, y)} = (y, x)$.

4) Let $G = \mathbb{Z}_n =< t >$ and let k be a field such that $\frac{1}{|G|} \in k$ and k contains a primitive n^{th}-root ξ of 1 , then

$$k[G] \cong \begin{cases} ke_0 \times (ke_1 \times ke_{n-1}) \times (ke_2 \times ke_{n-2}) \times ... \times (ke_{(n-1)/2} \times ke_{(n+1)/2}) & \text{for } n \text{ odd} \\ ke_0 \times (ke_1 \times ke_{n-1}) \times (ke_2 \times ke_{n-2}) \times ... \times (ke_{n/2}) & \text{for } n \text{ even} \end{cases}$$

where $e_i = \sum_{l=0}^{n-1} (\xi^i t)^l$. Taking $\omega_G(g) = g^{-1}\ \forall g \in G$

$$hPic_k(k[G], \omega_G) \cong \begin{cases} \begin{array}{c} hPic_k(k, id) \times (hPic_k(k \times k, \overline{(-)}))^{\frac{n-1}{2}} \cong \\ \cong k^*/(k^*)^2 \times (k^*/(k^*)^2 \rtimes \mathbb{Z}_2)^{\frac{n-1}{2}} \end{array} & \text{for } n \text{ odd} \\ \\ \begin{array}{c} hPic_k(k, id) \times (hPic_k(k \times k, \overline{(-)}))^{\frac{n-2}{2}} \times hPic_k(k, id) \cong \\ \cong k^*/(k^*)^2 \times (k^*/(k^*)^2 \rtimes \mathbb{Z}_2)^{\frac{n-2}{2}} \times k^*/(k^*)^2 \end{array} & \text{for } n \text{ even} \end{cases}$$

As in the C^*-algebra case, the left and right structures of a Hilbert $A-B$-bimodule are closely related even when A and B are not necessarily commutative, due to the existence of inner products. Next proposition shows explicitly this relation: the right structure is determined by the left one and an automorphism $\alpha \in hAut_k(A)$, and can be regarded as an improvement (for the case we are interested in) of proposition 1.1 [1] for k-algebras with involution instead of C^*-algebras. Notice that in [1], the Hilbert bimodules considered are $A-A$-bimodules, but in fact the proof can be achieved for $A-B$-bimodules with A not necessarily equal to B.

Proposition 3.12 *If* $(M, <,>_L^M, <,>_R^M)$ *and* $(N, <,>_L^N, <,>_R^N)$ *are Hilbert* $A-B$*-bimodules with surjective right inner products and* $\phi : M \to N$ *is an isomorphism of left Hilbert* A*-modules, then, there exists a* k*-algebra automorphism* $\alpha : B \to B$ *commuting with the involution of* B *such that* $\phi : M_\alpha \to N$ *is an isomorphism of Hilbert* $A-B$*-bimodules.*

Proof: α is defined by the equality

$$\alpha(< \phi(x), \phi(y) >_R^N) = < x, y >_R^M \quad \forall x, y \in M$$

A straightforward computation shows that α is well-defined and multiplicative. It also commutes with ω_B, as:

$$\alpha(\omega_B(< \phi(x), \phi(y) >_R)) = \alpha(< \phi(y), \phi(x) >_R) =$$

$$= < y, x >_R = \omega_B(< x, y >_R) = \omega_B(\alpha(< \phi(x), \phi(y) >_R))$$

Considering now $\phi : M_\alpha \to N$, we have to show that it is an isomorphism of $A-B$-Hilbert bimodules.

To see that it is B-linear, as $< M, M >_R = B$, consider $m, x, y \in M$ and let $b = < \phi(x), \phi(y) >_R^N$, then:

$$\begin{array}{llll} \phi(m._\alpha b) & = \phi(m\alpha(< \phi(x), \phi(y) >_R^N)) & = \phi(m < x, y >_R^M) = \\ & = \phi(< m, x >_L^M y) & = < m, x >_L^M \phi(y) = \\ & = < \phi(m), \phi(x) >_L^N \phi(y) & = \phi(m) < \phi(x), \phi(y) >_R^N = \\ & = \phi(m)b \end{array}$$

In order to finish the proof, we must show that ϕ is compatible with the new right product defined in M_α, $<,>_R^{M_\alpha}$. Consider $x, y \in M$, then

$$< x, y >_R^{M_\alpha} = \alpha^{-1}(< x, y >_R^M) = < \phi(x), \phi(y) >_R^N$$

As a corollary we obtain:

Corollary 3.13 *Let $(M, <,>_L^M, <,>_R^M)$ and $(N, <,>_L^N, <,>_R^N)$ be two Hilbert $A-B$-bimodules and let $\phi : M \to N$ be an isomorphism of left Hilbert A-modules. Then ϕ is an isomorphism of Hilbert $A-B$-bimodules if and only if ϕ preserves the right inner products or ϕ is B-linear.*

Proof: It is clear that if ϕ preserves the right inner product then α is the identity and the isomorphism is an isomorphism of $A-B$-bimodules.

It follows by the compatibility of the left and right inner products in N that if ϕ preserves the right action then so it does with the right inner product and it is in fact an isomorphism of $A-B$-bimodules.

The last proposition and corollary show that in fact, the isomorphism class of a Hilbert $A-A$-bimodule depends only on its left Hilbert structure (for example) and on the group $hAut_k(A)$.

Given (A, ω_A), it is always a $Z(A)$-algebra. Denote by $hZ(A) = \{a \in Z(A) \ / \ a = a^*\}$, then $hZ(A)$ is a subalgebra of $Z(A)$ and it is the maximal subalgebra $K \subseteq A$ such that ω_A is K-linear. We define then $hPicent(A, \omega_A) := hPic_{hZ(A)}(A, \omega_A)$.

Remark: If A is commutative and $\omega_A = id_A$ then $hCPic_k(A, \omega_A) = hPicent(A, \omega_A)$. But in general, if $\omega_A \neq Id_A$ $hZ(A) \neq A$ and so the notion of $hZ(A)$-symmetry does not agree with A-symmetry.

Proposition 3.14 *Let (A, ω_A) and (B, ω_B) be two hermitian Morita equivalent k-algebras, then*

$$hPicent(A, \omega_A) \cong hPicent(B, \omega_B)$$

Proof: An element of $hPicent(A, \omega_A)$ is the isomorphism class of an $A-A$-Hilbert bimodule $(M, <,>_L, <,>_R)$ which is $hZ(A)$-symmetric. As (A, ω_A) and (B, ω_B) are hermitian Morita equivalent by means of bimodules P and Q. We consider the Hilbert bimodule structure on P and Q given in the example 2 of Section 3. As $(M, <,>_L, <,>_R)$ is an $A-A$-Hilbert bimodule, then $Q \otimes_A M \otimes_A P$ is a $B-B$-Hilbert bimodule with the structure obtained as the tensor product of the Hilbert bimodules (notice that as P, Q and M are invertible, their inner products are surjective). This procedure gives in fact the isomorphism between $hPic_k(A, \omega_A)$ and $hPic_k(B, \omega_B)$. We will now show that $Q \otimes_A M \otimes_A P$ is $hZ(B)$-symmetric.

The first part of Lemma 3.8 says that there is a unique isomorphism $\alpha_P : Z(B) \to Z(A)$ such that $pb = \alpha(b)p$ for all $b \in Z(B)$, $p \in P$ and the second part of the same lemma guarantees that α sends the elements of $hZ(B)$ into $hZ(A)$.

Given $b \in hZ(B)$, $q \in Q$, $p \in P$ and $m \in M$, $q \otimes m \otimes pb = q \otimes m \otimes \alpha(b)p = q \otimes m\alpha(b) \otimes p$. As M is $hZ(A)$-symmetric, $m\alpha(b) = \alpha(b)m$, so $q \otimes m\alpha(b) \otimes p = q\alpha(b) \otimes m \otimes p$.

But if P is α-central and $Q = P^{*_A}$, then Q is α^{-1}-central, and $q\alpha(b) \otimes m \otimes p = bq \otimes m \otimes p$, so $Q \otimes_A M \otimes_A P$ is a $B-B$-Hilbert bimodule which is $hZ(B)$-symmetric.

Clearly, the construction in the other sense is similar.

4 The hermitian Picard group of an involutive coalgebra

Throughout this section we shall make use of such objects as coalgebras, bicomodules and cotensor products. So we begin this section by recalling from [9] some definitions.

A **coalgebra** over a field k is a k-vector space C provided of a coproduct $\Delta : C \to C \otimes C$ and a counit $\epsilon : C \to k$ which are k-linear and such that Δ is coassociative and $(1 \otimes \epsilon) \circ \Delta = (\epsilon \otimes 1) \circ \Delta = id_C$. C^{op} is the opposite coalgebra of C and it equals C as k-vector spaces, with same counit and coproduct $\Delta^{op} : C^{op} \to C^{op} \otimes C^{op}$ defined as $\Delta^{op} = \sigma_{12} \circ \Delta$.

C is called **cocommutative** if $C = C^{op}$.

A **left** C**-comodule** is a pair (M, ρ^-) where M is a k-vector space and $\rho^- : M \to C \otimes M$ is a k-linear coaction, i.e. $(\Delta \otimes id_M) \circ \rho^- = id_C \otimes \rho^-) \circ \rho^-$. Right C-comodules $(M, \rho^+ : M \to M \otimes C)$ are defined in a similar way. In general, we shall omit ρ^+ and ρ^- in the notation, and say that M is a left or right C-comodule.

C**-bicomodules** are objects (M, ρ^+, ρ^-) such that (M, ρ^+) and (M, ρ^-) are respectively right and left C-comodules, and both coactions are compatible in the following sense: $(id_C \otimes \rho^+) \circ \rho^- = (\rho^- \otimes id_C) \circ \rho^+$. We shall identify C-bicomodules with $C^e = C \otimes C^{op}$ (left) comodules.

Given two left C-comodules M, N, $Com_C(M, N)$ will denote the set of C-colinear maps from M to N.

If M is now a right C comodule, and N is a left C-comodule, the **cotensor product** over C of M and N, denoted by $M \Box_C N$ is the kernel of the map

$$\rho_M^+ \otimes id_N - id_M \otimes \rho_N^- : M \otimes N \to M \otimes C \otimes N$$

The functors $-\Box_C N$ and $M \Box_C -$ are left exact.

From now on, R will be a cocommutative k-coalgebra, and C an R-coalgebra, i.e. a k-coalgebra C together with a coalgebra map $\epsilon_R : C \to R$ making C a cosymmetric R-bicomodule. We shall say that C is an R-involutive coalgebra in case there is a k-linear coalgebra isomorphism $\omega_C : C \to C^{op}$ such that $\omega_C^2 = id_C$ and the following diagram is commutative:

$$\begin{tikzcd} C \arrow[rr, "\omega_C"] \arrow[dr, "\epsilon_R"'] & & C^{op} \arrow[dl, "\epsilon_R"] \\ & R & \end{tikzcd}$$

We recall also from [9] the definition of Morita - Takeuchi context.

Definition 4.1 *A* **Morita - Takeuchi context** *consists of the following set of data:*

- *A $D-C$-bicomodule P and a $C-D$-bicomodule Q (with coactions denoted by $\rho_P^+ : P \to P \otimes C$, etc.).*
- *Bicomodule morphisms $\mu : C \to Q \Box_D P$ and $\tau : D \to P \Box_C Q$ where $P \Box_C Q$ denotes the cotensor product*

- *The compatibility conditions:*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P\Box_C C \arrow[d, "id\Box"] \\ D\Box_D P \arrow[r, "\tau\Box id"] & P\Box_C Q\Box_D P \end{tikzcd} \qquad \begin{tikzcd} Q \arrow[r] \arrow[d] & Q\Box_D D \arrow[d, "id\Box\tau"] \\ C\Box_C Q \arrow[r, "\mu\Box id"] & Q\Box_D P\Box_C Q \end{tikzcd}$$

Remark: The kernels of μ and τ are subcoalgebras of C and D respectively. If both maps are injective, then the context gives an equivalence (see also [9]).

Also we have from [4]:

Definition 4.2 *Two involutive k-coalgebras C and D are* **hermitian Morita - Takeuchi equivalent** *if and only if there exist two bicomodules ${}_DP_C$ and ${}_CQ_D$, bicomodule isomorphisms*
$\mu : C \to Q\Box_D P$ and $\tau : D \to P\Box_C Q$, and a k-isomorphism $\Theta : P \to Q$ satisfying:

1. *(P, Q, μ, τ) is a Morita - Takeuchi context.*
2. *(a) The following diagram is commutative:*

$$\begin{tikzcd} P \arrow[r, "\rho_P^+\circ\rho_P^-"] \arrow[d, "\Theta"] & D\otimes P\otimes C \arrow[d, "\sigma_{13}\circ(\omega_D\otimes\Theta\otimes\omega_C)"] \\ Q \arrow[r, "\rho_Q^+\circ\rho_Q^-"] & C\otimes Q\otimes D \end{tikzcd}$$

We also ask Θ to verify the following compatibility conditions:

(b) $(\Theta^{-1}\otimes\Theta)\circ\mu = \sigma_{12}\circ\mu\circ\omega_C$

(c) $(\Theta\otimes\Theta^{-1})\circ\tau = \sigma_{12}\circ\tau\circ\omega_D$

A pair $(P, \Theta : P \to Q)$ giving an hermitian Morita - Takeuchi equivalence between two k-coalgebras C and D will be called an **hermitian invertible** bicomodule.

Remark: With the notations of [9], the coalgebra D is isomorphic to the coendomorphism coalgebra $e_C(P) = h_C(P, P)$, which is isomorphic to $P\Box_C Q$. The k-linear map Θ allows us to define an involution on $e_C(P)$ via $\sum_i p_i\otimes q_i \mapsto \sum_i \Theta^{-1}(q_i)\otimes\Theta(p_i)$. Condition 2 (c) of definition 4.2 guarantees that the coalgebra isomorphism $D \cong e_C(P)$ commutes with the respective involutions.

The above remark can be regarded as the analog of Proposition 2.2 for coalgebras.

The definition of the Picard group of a coalgebra was given in [10]. We are interested not in arbitrary invertible C-bicomodules, but in those ones that have an additional structure in order to provide an hermitian Morita - Takeuchi equivalence, as these bicomodules are those ones that play the dual role of Hilbert bimodules studied in previous sections.

So, two hermitian invertible C-bicomodules (M, Θ_M), (N, Θ_N) will be called **isomorphic** if there is a C-bicomodule isomorphism $\phi : M \to N$ such that the following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\Theta_M"'] & N \arrow[d, "\Theta_N"] \\ h_C(M, C) & h_C(N, C) \arrow[l, "h_C(\phi, C)"] \end{tikzcd}$$

where $h_C(Y, -)$ is the left adjoint to $Y \Box_C -$, that in fact is functorial in both variables, explicitely $h_C(Y, X) = lim_{\to\mu}(Com_C(X_\mu, Y))^*$ for Y quasifinite and $X = lim_{\to\mu} X_\mu$, each X_μ finite dimensional.

Definition 4.3 *The* **hermitian Picard group** *of an involutive R-coalgebra (C, ω_C) is the set of isomorphism classes of R-cosymmetric hermitian invertible C-bicomodules.*

This set is in fact a group with multiplication induced by the cotensor product. It is clear that if M and N are invertible C-bicomodules then so is $M \Box_C N$, we have only to remark that if M and N have the additional hermitian structure given by $\Theta_M : M \to h_C(M, C)$ and $\Theta_N : N \to h_C(N, C)$, then $M \Box_C N$ has $\Theta_{M \Box_C N} = \Theta_M \otimes \Theta_N|_{M \Box_C N}$.

An easy way to see that this morphism is well defined and has the desired "anti-colinear" properties is observing that condition 2.(a) of the definition involving Θ in an hermitian Morita - Takeuchi equivalence is the same as saying that $\Theta : \overline{M} \to h_C(M, C)$ is $C - D$-bicolinear ($C = D$ in our case) with the comodule structures on $\overline{M}$ obtained from those of M reversing sides by means of involutions. Then we have a well defined C-bicomodule morphism

$$\begin{tikzcd} \overline{M \Box_C N} \arrow[r, "\Theta_{M \Box_C N}"] \arrow[d, equal] & h_C(M \Box_C N, C) \arrow[d, equal] \\ \overline{N} \Box_C \overline{M} \arrow[r, "\Theta_N \Box_C \Theta_M"] & h_C(N, C) \Box_C h_C(M, C) \end{tikzcd}$$

Clearly the identity element is $[C, \omega_C : C \to C = h_C(C, C)]$ and the inverse of an element $[M, \Theta_M : M \to h_C(M, C)]$ is $[h_C(M, C), (\Theta_M)^{-1} : h_C(M, C) \to M = h_C(h_C(M, C), C)]$.

There is an obvious morphism from the hermitian Picard group $hPic_R(C, \omega_C)$ into $Pic_R(C, \omega_C)$, that is simply to forget the hermitian structure, i.e.: a typical element of $hPic_R(C, \omega_C)$ (which can be denoted by $[\Theta : M \to h_C(M, C)]$), is mapped to $[M] \in Pic_R(C, \omega_C)$. In order to compute the kernel of this map in a more comfortable way we shall need the following characterization of the algebra $Com_{C^e}(C, C)$.

Proposition 4.4 *Under the identification $Com_C(C, C) \cong C^* = Hom_k(C, k)$ $(f \mapsto \epsilon \circ f)$, the subalgebra $Com_{C^e}(C, C) \subseteq Com_C(C, C)$ is sent onto $Z(C^*)$ (the center of the dual algebra C^*).*

Proof: An element $\phi \in Com_{C^e}(C,C)$ is a k-linear morphism $\phi : C \to C$ such that

$$\sum_{(c)} c_1 \otimes \phi(c_2) = \sum_{(c)} \phi(c_1) \otimes c_2 = \sum_{(\phi(c))} \phi(c)_1 \otimes \phi(c)_2 \ \forall c \in C$$

then, if $\phi \in Com_{C^e}(C,C)$, for every $c \in C$

$$\begin{array}{rlll} \phi(c) = & (\epsilon \otimes 1)\Delta(\phi(c)) = & \sum_{(c)} \epsilon(\phi(c_1))c_2 = & (\epsilon \circ \phi \otimes 1)\Delta(c) \\ = & (\epsilon \otimes 1)\Delta(\phi(c)) = & \sum_{(c)} c_1\epsilon(\phi(c_2)) = & (1 \otimes \epsilon \circ \phi)\Delta(c) \end{array}$$

For any $f \in C^*$, we have

$$f\left(\sum_{(c)} c_1\epsilon(\phi(c_2))\right) = f\left(\sum_{(c)} \epsilon(\phi(c_1))c_2\right)$$

and so

$$\sum_{(c)} f(c_1)(\epsilon \circ \phi)(c_2)) = \sum_{(c)} (\epsilon \circ \phi)(c_1)f(c_2)$$

Denoting by $*$ the product in C^*, this last equality says that $(\epsilon \circ \phi) * f = f * (\epsilon \circ \phi)$ for all $f \in C^*$, i.e. $\epsilon \circ \phi \in Z(C^*)$.

Suppose conversely that $a \in Z(C^*)$. It identifies with $\phi_a = (1 \otimes a) \circ \Delta$, $\phi_a \in Com_C(C,C)$. We want to see that in fact, ϕ_a is a bicomodule morphism.

As $a * f = f * a \ \forall f \in C^*$, then:

$$\sum_{(x)} a(x_1)f(x_2) = \sum_{(x)} f(x_1)a(x_2) \ \forall f \in C^*, \ x \in C$$

$$\Leftrightarrow f\left(\sum_{(x)} a(x_1)x_2\right) = f\left(\sum_{(x)} x_1 a(x_2)\right) \ \forall f \in C^*, \ x \in C$$

$$\Leftrightarrow \sum_{(x)} a(x_1)x_2 = \sum_{(x)} x_1 a(x_2) \ \forall f \in C^*, \ x \in C$$

and then

$$\sum_{(c)} c_1 a(c_2) \otimes c_3 = \sum_{(c)} c_1 \otimes c_2 a(c_3) \ \forall c \in C$$

But this last equality can be writen as

$$\sum_{(c)} \phi_a(c_1) \otimes c_2 = \sum_{(c)} c_1 \otimes \phi_a(c_2) \ \forall c \in C$$

i.e. $\phi_a \in Com_{C^e}(C,C)$.

Once the above characterization obtained, we have:

Proposition 4.5 *If C is an involutive R-coalgebra, then there is an exact sequence of groups*

$$1 \longrightarrow UZ(C^*)^+/(UZ(C^*).\overline{UZ(C^*)}) \longrightarrow hPic_R(C,\omega_C) \longrightarrow Pic_R(C,\omega_C)$$

where $UZ(C^*)$ is the multiplicative subgroup of units of $Z(C^*)$ which are fixed by ω_C^* and
$UZ(C^*).\overline{UZ(C^*)} = \{x.\overline{x} \ : \ x \in UZ(C^*) \ \textit{and} \ \overline{x} = x \circ \omega_C\}$

Proof:

$$\begin{aligned} Ker\,(hPic_R(C,\omega_C) \to Pic_R(C,\omega_C)) = &\ \{\ [\Theta : M \to h_C(M,C)\ /[M] = [C]\ \} \\ = &\ \{\ [\Theta : C \to h_C(C,C) = C\ \} \end{aligned}$$

Consider $\Theta \circ \omega_C : C \to C$. It is an element of $Com_{C^e}(C,C) \cong Z(C^*)$. Being an isomorphism, $\epsilon \circ \Theta \circ \omega_C \in UZ(C^*)$ and, by the commutativity of the diagram:

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "\omega_C"'] & C\Box_C C \arrow[d, "\sigma_{12}(\Theta\otimes\Theta^{-1})"] \\ C \arrow[r, "\mu"] & C\Box_C C \end{tikzcd}$$

$\epsilon\circ\Theta\circ\omega_C = \epsilon\circ\Theta$ in C^*, because $\sum_{(c)} \overline{c_2}\otimes\overline{c_1} = \sum_{(c)} \Theta^{-1}(c_2)\otimes\Theta(c_1)$ and so $(\epsilon\circ\Theta)^{-1} = \epsilon\circ(\Theta^{-1})$. This implies that $\epsilon\circ(\Theta\omega_C) = (\epsilon\circ(\omega\Theta^{-1}))^{-1} = (\epsilon\circ(\Theta^{-1}))^{-1} = \epsilon\circ\Theta$, i.e. $\epsilon\circ\Theta\circ\omega_C$ is fixed by the involution ω_C^*.

As the unit of $hPic_R(C,\omega_C)$ is $([C],,\omega_C)$, a morphism $\Theta : C \to h_C(C,C) = C$ is equivalent to $\omega_C : C \to C = h_C(C,C)$ if and only if there exists a C-bicomodule isomorphism $\phi : C \to C$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[d, "\Theta"'] & C \arrow[d, "\omega_C"] \\ h_C(C,C) & h_C(C,C) \arrow[l, "h_C(\phi,C)"'] \end{tikzcd}$$

Remark that as ϕ is also an isomorphism then $\epsilon\circ\phi \in UZ(C^*)$. We recall that $h_C(C,C) = lim_{\to\mu} Com_C(C_\mu, C)^*$ where $\{C_\mu\}$ is a directed system of k-finite dimensional subcomodules of C such that $lim_{\to\mu} C_\mu = C$. As $\phi : C \to C$ is an isomorphism, then $lim_{\to\mu} C_\mu = lim_{\to\mu}\phi(C_\mu)$, and in order to obtain $h_C(C,C)$ we may use either $\{C_\mu\}$ or $\{\phi(C_\mu)\}$. This is useful to compute $h_C(\phi,C)$, because taking limits in the following commutative diagrams and identifying $h_C(C,C)$ with C:

$$\begin{tikzcd} Com_C(C_\mu,C)^* \arrow[r, "Com_C(\phi|_{C_\mu},C)^*"] & Com_C(\phi(C_\mu),C)^* \\ C_\mu^{**} \arrow[u, "(\epsilon_*)^*"] \arrow[r, "(\phi|_{C_\mu})^{**}"] & (\phi(C_\mu))^{**} \arrow[u, "(\epsilon_*)^*"'] \\ C_\mu \arrow[u, equal] \arrow[r, "\phi|_{C_\mu}"] & \phi(C_\mu) \arrow[u, equal] \end{tikzcd}$$

we can conclude that $h_C(\phi,C) : C \to C$ is nothing but ϕ. Then the condition $[\Theta : C \to C] = [\omega_C : C \to C]$ becomes

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[d, "\Theta"'] & C \arrow[d, "\omega_C"] \\ C & C \arrow[l, "\phi"'] \end{tikzcd}$$

i.e. $\Theta = \phi \circ \omega_C \circ \phi$. This is the same as $\Theta \circ \omega_C = \phi \circ (\omega_C \circ \phi \circ \omega_C)$. By identifying $Com_{C^e}(C,C)$ with $Z(C^*)$, the last equality reads $\epsilon \circ \Theta \circ \omega_C = (\epsilon \circ \phi) * (\overline{\epsilon \circ \phi})$, so the proof is complete.

The decomposition of $hPic_R(C,\omega_C)$ as a semidirect product of the central hermitian Picard group and $hAut_R(C)$ will result as a corollary of the existence of the following diagram:

Theorem 4.6 *Let C be an R-coalgebra, then:*

1. *The following diagram may be completed and commutes*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & Ker(\Psi|_K) \arrow[r, equal] \arrow[d] & Ker(\Psi) \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & K \arrow[r, hook] \arrow[d, "\Psi|_K"'] & hPic_R(C,\omega_C) \arrow[r, dashed, "\widetilde{\alpha}"] \arrow[d, "\Psi"] & hAut_R(Z(C)) \arrow[d] \\
1 \arrow[r] & Picent(C) \arrow[r] & Pic_R(C) \arrow[r, "\alpha"] & Aut_R(Z(C))
\end{tikzcd}$$

where Ψ is the map described in Proposition 4.5, the lowest line is the exact sequence of Theorem 2.10 of[10], and the map $hCPic_R(C,\omega_C) \to hPic_R(C,\omega_C)$ is the inclusion. We recall that in [10], the group $Picent(C)$ is defined as the subgroup of $Pic_R(C)$ consisting of $Z(C)$-cosymmetric invertible C-bicomodules. $K = Ker(\widetilde{\alpha})$.

2. *If C is cocommutative, then the second line splits and so $hPic_R(C,\omega_C) \cong K \rtimes hAut_R(C)$. Also, in this case $K \cong Ker(\Psi|_K) = Ker(\Psi) \cong UZ(C^*)^+/UZ(C^*).\overline{UZ(C^*)}$.*

Before proving the theorem, we need the following lemmas:

Lemma 4.7 *If M is a $C-D$-invertible bicomodule f-cocentral ($f: Z(C) \to Z(D)$ an R-coalgebra isomorphism), then $h_C(M,C)$ is a $D-C$-bicomodule f^{-1}-cocentral.*

Proof: Let M be an f-cocentral $C-D$ invertible bicomodule (i.e. for all $m \in M$ $\sum_{(m)} m_0 \otimes f(\pi(m_{-1})) = \sum_{(m)} m_0 \otimes \pi(m_1)$, where $\rho^-(m) = \sum_{(m)} m_{-1} \otimes m_0 \in C \otimes M$, $\rho^+(m) = \sum_{(m)} m_0 \otimes m_1 \in M \otimes D$ and π denotes both projections on the respective cocenters).

We recall from [9] that the C-right structure of $h_C(M,C)$ is given by comultiplicationin C:

$$h_C(M,C) \to h_C(M, C \otimes C) \cong h_C(M,C) \otimes C$$

and the D-left structure of $h_C(M,C)$ is given by the D-right structure of M, as having a map $h_C(M,C) \to D \otimes h_C(M,C)$ is equivalent to a map $C \to D \otimes h_C(M,C) \otimes M$, and we obtain the last one by means of the identity of $h_C(M,C)$ (and so we have $C \to h_C(M,C) \otimes M$) and the right structure of M.

The fact that $h_C(M,C)$ is f^{-1}-cocentral is equivalent to the commutativity of the following diagram:

$$
\begin{tikzcd}
C \arrow[r, equal] \arrow[dd, "\widetilde{id}_{M^\square}"] & C \arrow[r, equal] \arrow[dd, "\widetilde{id}_{M^\square}"] & C \arrow[r, equal] \arrow[dd, "\widetilde{id}_{M^\square}"] & C \arrow[r, equal] \arrow[d, "(\pi\otimes 1)\Delta"] & C \arrow[d, "(1\otimes\pi)\Delta"] \\
 & & & Z(C)\otimes C \arrow[r, "\sigma_{12}"] \arrow[d, "1\otimes\widetilde{id}_{M^\square}"] & C\otimes Z(C) \arrow[dd, "\widetilde{id}_{M^\square}\otimes 1"] \\
M^\square\otimes M \arrow[r, equal] \arrow[d, "\rho^-\otimes 1"] & M^\square\otimes M \arrow[r, equal] \arrow[d, "1\otimes\rho^+"] & M^\square\otimes M \arrow[r, equal] \arrow[d, "1\otimes\rho^-"] & Z(C)\otimes M^\square\otimes M \arrow[d, "\sigma_{12}"] & \\
Z(D)\otimes M^\square\otimes M \arrow[r, "\sigma_{312}"] & M^\square\otimes M\otimes Z(D) \arrow[r, "\sigma_{23}(1\otimes f)"] & M^\square\otimes Z(C)\otimes M \arrow[r, equal] & M^\square\otimes Z(C)\otimes M \arrow[r, "\sigma_{23}"] & M^\square\otimes M\otimes Z(C)
\end{tikzcd}
$$

where $M^\square$ is a shorthand notation for $h_C(M,C)$. The left and right vertical rectangles commute respectively due to the definitions of the $Z(D)$-left and $Z(C)$-right comodule structures of $h_C(M,C)$. The middle ones commute because of the f-cocentrality of M (left) and right $Z(C)$-colinearity of $\widetilde{id}_{h_C(M,N)}$ (right). So if M is f-cocentral, then $h_C(M,C)$ is f^{-1}-cocentral.

Lemma 4.8 *The image of $\Psi(hPic_R(C,\omega_C))$ under the morphism α in the diagram of Theorem 4.6 is contained in $hAut_R(Z(C))$.*

Proof: We know from [10] that given an invertible C-bicomodule M, it determines an R-automorphism $f : Z(C) \to Z(C)$ such that M is f-cocentral. We want to prove that if M comes from an hermitian Morita - Takeuchi context then f commutes with the involution of C.

For this, consider $\Theta : M \to h_C(M,C)$ the k-isomorphism of the hermitian context. By Lemma 4.7, as in the case of algebras, if M is f-cocentral, then $h_C(M,C)$ is f^{-1}-cocentral.

Denoting by ρ^- and ρ^+ the left and right C-comodule structural morphisms of $h_C(M,C)$, we have, if $m \in M$:

1. $\rho^-(\Theta(m)) = \sum_{(\Theta(m))} \Theta(m)_{-1} \otimes \Theta(m)_0 = \sum_{(\Theta(m))} f^{-1}(\Theta(m)_1) \otimes \Theta(m)_0$.
   Also, $\rho^-(\Theta(m)) = \sum_{(m)} \omega(m_1) \otimes \Theta(m_0)$

2. On the other hand, $\rho^+(\Theta(m)) = \sum_{(\Theta(m))} \Theta(m)_0 \otimes \Theta(m)_1$.
   And $\rho^+(\Theta(m)) = \sum_{(m)} \Theta(m_0) \otimes \omega(m_{-1}) = \sum_{(m)} (\Theta \otimes \omega)(m_0 \otimes m_{-1}) =$
   $= (\Theta \otimes \omega) \sum_{(m)} (m_0 \otimes f(m_1)) = \sum_{(m)} \Theta(m_0) \otimes \omega f(m_1))$.

Then, $\sum_{(m)} \omega(m_1) \otimes \Theta(m_0) = \sum_{(m)} (f^{-1} \otimes 1)\sigma_{12}^2(f \otimes 1)(\omega(m_1) \otimes \Theta(m_0)) =$
(by 1.) $= \sum_{(\Theta(m))} (f^{-1} \otimes 1)\sigma_{12}^2(f \otimes 1) f^{-1}(\Theta(m)_1 \otimes \Theta(m)_0) = \sum_{(\Theta(m))} (f^{-1} \otimes 1)\sigma_{12}(\Theta(m)_0 \otimes \Theta(m)_1) =$
(by 2.) $= \sum_{(m)} (f^{-1} \otimes 1)\sigma_{12}(\Theta(m_0) \otimes \omega f(m_1)) = \sum_{(m)} f^{-1}\omega f(m_1) \otimes \Theta(m_0)$.

Applying $\omega \otimes \Theta^{-1}$, we obtain $\sum_{(m)} \omega f^{-1} \omega f(m_1) \otimes m_0 = \sum_{(m)} m_1 \otimes m_0$, so $\omega f^{-1} \omega f = id$, i.e. $\omega f = f\omega$.

So, the lemma allows us to complete the diagram of the theorem by the dashed arrow.

Proof of the Theorem:

1) Once the maps defined, it is clear that the middle line is exact and, by Proposition 4.4 we have that $Ker(\Psi) \cong UZ(C^*)^+/UZ(C^*)\overline{UZ(C^*)}$.

2) If C is a cocommutative R-coalgebra then $Z(C) = C$ and, by [10], $Picent(C)$ is trivial, so, in this case, $K \cong UZ(C^*)^+/UZ(C^*)\overline{UZ(C^*)}$.

Then we have an exact sequence

$$1 \longrightarrow UZ(C^*)^+/UZ(C^*)\overline{UZ(C^*)} \longrightarrow hPic_R(C,\omega_C) \xrightarrow{\widetilde{\alpha}} hAut_R(C)$$

where $\widetilde{\alpha} = \alpha|_{hPic_R(C,\omega_C)}$.

The splitting of $\widetilde{\alpha}$ is obtained assigning to each morphism β commuting with the involution of C, the element $([{}_\beta C_1], \Theta_\beta)$ in $hPic_R(C,\omega_C)$ where ${}_\beta C_1$ is the C-bicomodule which has the same underlying set and C-right structure of C and C-left structure twisted by β and, $\Theta_\beta : C \to h_C(C,C) \cong C$ is the k-isomorphism defined by $\Theta_\beta(x) = \beta(\omega_C(x))$ $(x \in C)$, verifying $\sigma_{13} \circ (\omega \otimes \Theta_\beta \otimes \omega) \circ \Delta_C = \Delta_C \circ \Theta_\beta$.

It is known that in the finite dimensional case, $Pic_k(C,\omega_C)$ is isomorphic to $Pic_k(C^*,\omega_{C^*})$, and the same holds for the hermitian Picard groups.

We conclude with an infinite dimensional example:

Example: Let k be a field and C the cocommutative coalgebra $k[x]$ with comultiplication $x^n \mapsto \sum_{i=0}^n x^i \otimes x^{n-i}$ and involution $x^n \mapsto (-1)^n x^n$. C^* is the dual algebra of power series $k[[x]]$.

By Theorem 4.6 $hPic_k(C,\omega_C) \cong UZ(C^*)^+/UZ(C^*)\overline{UZ(C^*)} \rtimes hAut_k(C)$.

Also, $hPic_k(C^*,\omega_{C^*}) \cong UZ(C^*)^+/UZ(C^*)\overline{UZ(C^*)} \rtimes hAut_k(C^*)$ because in this particular case, as $k[[x]]$ is a principal ideal domain, $CPic_k(C^*,\omega_{C^*})$ is trivial.

So the difference between the hermitian Picard group of C and of its dual algebra comes from the respective hermitian automorphism groups. For the coalgebra, it is isomorphic to k^*, while for the dual algebra it is bigger. For example, we can send the formal series $\sum_{i\geq 0} a_i x^i$ to $\frac{1}{2}(\sum_{i\geq 0} a_i.(x^i + x^{3i}))$.

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