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CYCLIC HOMOLOGY OF MONOGENIC ALGEBRAS

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Abstract. We compute the cyclic homology of $A = k[X]/ < f >$ for an arbitrary commutative ring k and a monic polynomial f.

0.Introduction.

Let k be an arbitrary commutative unitary ring and f a monic polynomial in $k[X]$. The cyclic homology of $A = k[X]/ < f >$, for k a characteristic zero field was calculated in [M-N] and [K], and for k an arbitrary ring and $f = X^p - 1$ in [C-G-V]. The first general result in arbitrary characteristic appeared in [Bach], where the cyclic homology of $k[X]/ < f >$, for k a field, were calculated.

In [G-G], the authors replaced the complex $B(A)_{\text{norm}}$ of Loday and Quillen by a simpler mixed complex $\tilde{M}(A)$ for the case $f = X^r$. This leads to the computation of cyclic homology of monogenic extensions. It is possible to define the mixed complex $\tilde{M}(A)$ for k an arbitrary ring and f a monic polynomial, and so it is natural to ask if its homology coincides with the cyclic homology of A in every case. The positive answer was given in [L-L] for A an integral domain. In this paper we give a considerably easier and clearer proof of this fact, which is valid for arbitrary k and A.

The paper is divided in two sections. In the first one we show a strong homotopy k-map which induces a quasi-isomorphism from the simplified complex $\tilde{M}(A)$ to the standard complex $B(A)_{\text{norm}}$. An independent proof of this fact has been given by T. Lambre ([L]). As an application, we compute the cyclic homology of A in a simple way which follows the method used in Theorem 2.6 of [G-G]. In section 2 we give an explicit expression in the case $f = X^r + a$, by means of a decomposition of $\tilde{M}(A)$ as a direct sum of r double complexes.

1.Cyclic homology of $k[X]/ < f >$.

We shall use freely the notion of mixed complex first introduced in [K] and [B-O]. To prove our main theorem we need the properties of strong homotopy k-map between two mixed complexes shown in [K,Proposition 1.3], [J,Lemma 2.1] and [G-G,Proposition 1.3]

Let k be an arbitrary commutative ring with 1, $f = X^r + f_{r-1}X^{r-1} + \cdots + f_0 \in k[X]$ a monic polynomial and $A = k[X]/ < f >$. Let us denote by $\tilde{C}(A) = (C_*(A), b_*, B_*)$ the

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normalized mixed complex associated to A. The *Hochschild, cyclic, periodic* and *negative* homology $HH_*(A)$, $HC_*(A)$, $HC_*^{\text{per}}(A)$ and $HC_*^-(A)$ of A are defined (see [L-Q]) as the respective homologies of $\tilde{C}(A)$. In this section we introduce a mixed complex $\tilde{M}(A) = (M_*(A), d_*, D_*)$ simpler than $\tilde{C}(A)$ and show that there exists a quasi-isomorphism from $\tilde{M}(A)$ to $\tilde{C}(A)$.

Since f is monic, we can carry out the division algorithm and denote by $\bar{P}$ the quotient and by $\ddot{P}$ the remainder, i.e. $P = \bar{P}.f + \ddot{P}$, $dg(\ddot{P}) < dg(f)$. The uniqueness of $\bar{P}$ and $\ddot{P}$ is obvious.

We define $\tilde{M}(A)$ as the mixed complex whose graded module is given by $M_n(A) = A$ $\forall n$ and whose -1 and $+1$ differentials ($d_*: M_*(A) \to M_{*-1}(A)$ and $D_*: M_*(A) \to M_{*+1}(A)$ respectively) are defined by $d_{2m}(P) = f'.P$, $d_{2m+1} = 0$, $D_{2m}(1) = 0$, $D_{2m}(X^a) = -aX^{a-1} - m.\overline{f'X^a}$ if $0 < a < r$ and $D_{2m+1} = 0$. The complex $(M_*(A), d_*)$ was obtained in [Bach] by tensoring A with the A^e-resolution $Rs(A)$ of [Bach] which will be denoted here as $R'(A)$.

We shall use the following concept

1.1.Definition: The *degree* of

$$w = \sum \lambda . X^{\alpha_{i(0)}} \otimes \ldots \otimes X^{\alpha_{i(n+1)}} \in A \otimes \bar{A}^n \otimes A \quad (\alpha_{i(j)} < r \quad \forall i, j),$$

is:

$$dg(w) = \max dg(X^{\alpha_{i(0)}} \otimes \ldots \otimes X^{\alpha_{i(n+1)}}),$$

where $dg(X^{\alpha_0} \otimes \ldots \otimes X^{\alpha_{n+1}}) = \sum_{i=0}^{n+1} \alpha_i$. In a similar way we define the degree of an element of $A \otimes \bar{A}^n$.

We have previously defined in [Bach,§1] morphisms of complexes $g_*: (A \otimes \bar{A}^* \otimes A, b') \to R'(A)$ and $h_*: R'(A) \to (A \otimes \bar{A}^* \otimes A, b')$ such that $g_* \circ h_* = id$, $dg(h_{2m}(1 \otimes 1)) = rm$ and $dg(h_{2m+1}(1 \otimes 1)) = rm + 1$. Moreover, it is clear from the definitions that $dg(h_m \circ g_m(w \otimes 1)) \leq dg(w \otimes 1)$ $\forall w \in A \otimes \bar{A}^n$.

1.2.Proposition: $h_* \circ g_*$ is homotopic to the identity by σ_*, where σ_* is the homotopy of A-modules recursively defined by: $\sigma_0 = 0$ and $\sigma_n = \varepsilon_0 \circ (h_n \circ g_n - id - \sigma_{n-1} \circ b')$, with $\varepsilon_0(a) = 1 \otimes a$ for $a \in A \otimes \bar{A}^n \otimes A$.

Proof. We have:

$$\begin{aligned}
b' \circ \sigma_{n+1} + \sigma_n \circ b' &= b' \circ \varepsilon_0 \circ (h_{n+1} \circ g_{n+1} - id - \sigma_n \circ b') + \sigma_n \circ b' \\
&= h_{n+1} \circ g_{n+1} - id - \sigma_n \circ b' - \varepsilon_0 \circ (h_{n+1} \circ g_{n+1} - id - \sigma_n \circ b') + \sigma_n \circ b' \\
&= h_{n+1} \circ g_{n+1} - id - \varepsilon_0 \circ h_n \circ g_n \circ b' + \varepsilon_0 \circ b' + \varepsilon_0 \circ b' \circ \sigma_n \circ b' \\
&= h_{n+1} \circ g_{n+1} - id - \varepsilon_0 \circ h_n \circ g_n \circ b' + \varepsilon_0 \circ b' + \varepsilon_0 \circ (h_n \circ g_n - id - \sigma_n \circ b') \circ b' \\
&= h_{n+1} \circ g_{n+1} - id
\end{aligned}$$

By tensoring g_*, h_* and σ_* by $A \otimes_{A^e}$, we obtain A-maps

$$\bar{g}_*: C_*(A) \to M_*(A), \quad \bar{h}_*: M_*(A) \to C_*(A), \quad \bar{\sigma}_*: C_*(A) \to C_{*+1}(A),$$

verifying $\bar{g}_* \circ \bar{h}_* = id$ and $\bar{h}_* \circ \bar{g}_*$ is homotopic to the identity by $\bar{\sigma}_*$

1.3.Remark: We have

1) $\bar{g}_{2m}(w) = 0$ if $dg(w) < mr$ and $\bar{g}_{2m+1}(w) = 0$ if $dg(w) < mr + 1$.

2) $\bar{\sigma}_n(1 \otimes P_1 \otimes \ldots \otimes P_n) = \gamma \circ \sigma_n(1 \otimes P_1 \otimes \ldots \otimes P_n \otimes 1)$ (with $\gamma(P_0 \otimes Q \otimes P_{n+2}) = P_{n+2} P_0 \otimes Q$ for $Q \in \bar{A}^{n+1}$ and $P_i \in A$).
3) $D_n = \bar{g}_{n+1} \circ B_n \circ \bar{h}_n$.
4) $dg(\bar{\sigma}_n(w)) \leq dg(w)$ for $w \in A \otimes \bar{A}^{\otimes^n}$

Proof. 1) follows easily from the explicit definition of $\bar{g}_*$ given in [Bach,§1], 2) is trivial, for 3) see [Bach, Proposition 2.1]. Let us see 4). Using the fact that $dg(h_n \circ g_n(w \otimes 1)) \leq dg(w \otimes 1)$ we can see that $dg(\sigma_n(w \otimes 1)) \leq dg(w \otimes 1)$. Since $\bar{\sigma}_n(1 \otimes P_1 \otimes \ldots \otimes P_n) = \gamma \circ \sigma_n(1 \otimes P_1 \otimes \ldots \otimes P_n \otimes 1)$, it is clear that $dg(\bar{\sigma}_n(w)) \leq dg(w)$

Now, we are ready to prove the main result of this paper.

1.4.Theorem: There exists an strong homotopy k-map $(G_*^{(i)})_{i \geq 0}$ from $\tilde{M}(A)$ to $\tilde{C}(A)$ with $G_*^{(0)} = \bar{h}_*$.

Proof. We shall build up the maps $G_j^{(i)}$ by induction on i and j. Let $n \geq 0$. Assume that we have already built $G_*^{(i)}$ $(0 \leq i \leq t)$ and $G_j^{(t+1)}$ $(0 \leq j \leq n-1)$, such that:

i) $G_*^{(0)} = \bar{h}_*$
ii) $G_{j+1}^{(i)} \circ D_j - D_{2i+j} \circ G_j^{(i)} = b_{2i+j+2} \circ G_j^{(i+1)} - G_{j-1}^{(i+1)} \circ d_j$, for $0 \leq i < t$ or $i = ta$ $\quad 0 \leq j \leq n-1$ (where we consider $G_{j-1}^{(i+1)} = d_j = 0$ if $j = 0$)
iii) $dg(G_{2j}^{(i)}(X^a)) \leq a + rj$ and $dg(G_{2j+i}^{(i)}(X^a)) \leq a + 1 + rj$

Let $n = 2m$ and $0 \leq a \leq r-1$. If

$$T = G_{n+1}^{(t)} \circ D_n(X^a) - B_{2t-n} \circ G_n^{(t)}(X^\alpha) + G_{n-1}^{(t+1)} \circ d_n(X^a),$$

then it is clear that $dg(T) \leq a + rm$.From this fact and Remark 1.3 (1) it follows that $\bar{g}_{2t+n+1}(T) = 0$ for $t > 0$. For $t = 0$, as by Remark 1.3 (3) $D_n = \bar{g}_{n+1} \circ B_n \circ \bar{h}_n$, we have:

$$\begin{aligned}
\bar{g}_{n+1}(T) &= \bar{g}_{n+1}(G_{n+1}^{(0)} \circ D_n - B_n \circ G_n^{(0)} + G_{n-1}^{(1)} \circ d_n)(X^a) \\
&= (\bar{g}_{n+1} \circ \bar{h}_{n+1} \circ \bar{g}_{n+1} \circ B_n \circ \bar{h}_n - \bar{g}_{n+1} \circ B_n \circ G_n^{(0)} + \bar{g}_{n+1} \circ G_{n-1}^{(1)} \circ d_n)(X^a) \\
&= \bar{g}_{n+1} \circ G_{n-1}^{(1)} \circ d_n(X^a)
\end{aligned}$$

Since $dg(G_{n-1}^{(1)} \circ d(X_n^a)) \leq mr$, using again Remark 1.3 (1) we conclude that $\bar{g}_{n+1}(T) = 0$. Now let $G_n^{(t+1)}(X^a) = -\bar{\sigma}_{2t+n+1}(T)$. From Proposition 1.3 of [G-G], $b_{2t+n+1}(T) = 0$, so:

$$\begin{aligned}
b_{2t+n+2} \circ G_n^{(t+1)}(X^a) &= -b_{2t+n+2} \circ \bar{\sigma}_{2t+n+2}(T) \\
&= (id - \bar{h}_{2t+n+1} \circ \bar{g}_{2t+n+1} + \bar{\sigma}_{2t+n} \circ b_{2t+n+1})(T) = T.
\end{aligned}$$

Finally, iii) follows immediately because $\bar{\sigma}_*$ preserves degree.
For n odd we can repeat the same proof as in the even case

1.5.Corollary: The cyclic, periodic and negative homology A are the respective homologies of $\tilde{M}(A)$.

Proof. See [K,Proposition 1.3] and [J,Lemma 2.1]

2.Cyclic homology of $k[X]/ < X^r + a >$

In this section we shall compute the cyclic homology of $A = k[X]/ < X^r + a >$ $(a \in k)$ by means of a decomposition of the simplified complex $\tilde{M}(A)$.

This gives as a special case the cyclic homology of $k[X]/ < X^r >$ and $k[X]/ < X^r - 1 >$, computed in [Bach] and [C-G-V] respectively.

Let us now show the promised decomposition. $\tilde{M}(A)$ splits into a direct sum $\tilde{M}(A) = \bigoplus_{i=0}^{r-1} \tilde{M}(A)^{(i)}$, where $\tilde{M}(A)^{(i)} = (M_*^{(i)}(A), d_*^{(i)}, D_*^{(i)})$ is the mixed complex of k modules obtained from $M_n^{(i)}(A) = k$, $d_{2m+1}^{(i)} = 0$, $d_{2m}^{(i)}(1) = ra$, $D_{2m+1}^{(i)} = 0$, $D_{2m}^{(0)}(1) = 0$ and $D_{2m}^{(i)}(1) = i + mr$ if $i > 0$ Hence, the SBI sequence splits into a direct sum $\bigoplus_{i=0}^{r-1} \mathrm{SBI}^{(i)}$ of sequences

$$\mathrm{SBI}^{(i)} : \ldots \to H_n^{(i)}(A) \xrightarrow{I} HC_n^{(i)}(A) \xrightarrow{S} HC_{n-2}^{(i)}(A) \xrightarrow{B} H_{n-1}^{(i)}(A) \xrightarrow{I} HC_{n-1}^{(i)}(A) \xrightarrow{S} \ldots,$$

where $H_*^{(i)}(A) = H_*(\tilde{M}(A)^{(i)})$ and $HC_*^{(i)}(A) = HC_*(\tilde{M}(A)^{(i)})$.

2.1.Lemma: If k is a U.F.D and $ra \neq 0$, then $HC_{2m}^{(i)}(A)$ $(0 < m,\ 0 < i < r)$ is freely generated by the $2m$-cycle $(x_{m,0}, \ldots, x_{m,n})$, where

$$x_{m,h} = (-1)^h \left(\prod_{j=m-h}^{m-1} \frac{ra}{(ra : i + jr)} \right) \left(\prod_{j=0}^{m-h-1} \frac{i + jr}{(ra : i + jr)} \right).$$

In particular $HC_{2m}^{(i)}(A) \cong k$ $((na : i + jr)$ is the greatest common divisor of na and $i + jr)$.

Proof. We shall prove it by induction on m. It is clear that $x_{0,0} = 1$ generates $HC_0^{(i)}(A) = k$. Suppose that $(x_{m,0}, \ldots, x_{m,m})$ is a $2m$-cycle which freely generates $HC_{2m}^{(i)}(A)$. It is immediate that $(x_{m+1,0}, \ldots, x_{m+1,m+1})$ is a $2(m+1)$-cycle without torsion. Let $(y_{m+1,0}, \ldots, y_{m+1,m+1})$ be a $2(m+1)$-cycle. Since $(y_{m+1,1}, \ldots, y_{m+1,m+1}) = S((y_{m+1,0}, \ldots, y_{m+1,m+1}))$ belongs to $HC_{2m}^{(i)}(A)$, there exists a unique $\alpha \in k$, such that $y_{m+1,h} = \alpha . x_{m,h-1}$ $(1 \leq h \leq m+1)$. As $(y_{m+1,0}, \ldots, y_{m+1,m+1})$ is a $2(m+1)$-cycle, we have:

$$ra . y_{m+1,0} = (i + mr) . y_{m+1,1} = \alpha (i + mr) . x_{m,0} = \alpha (i + mr) . \prod_{j=0}^{m-1} \frac{i + jr}{(ra : i + jr)}$$

$$= \alpha (ra : i + mr) . \prod_{j=0}^{m} \frac{i + jr}{(ra : i + jr)}.$$

Then $\frac{ra}{(ra:i+mr)} . y_{m+1,0} = \alpha . (\prod_{j=0}^{m} \frac{i+jr}{(ra:i+jr)})$. If the product in brackets is zero, $y_{m+1,0} = x_{m+1,0} = 0$ and if it is different from zero, there exists $\beta \in k$ such that:

$$y_{m+1,0} = \beta . \prod_{j=0}^{m} \frac{i + jr}{(ra : i + jr)} \quad \text{and} \quad \alpha = \beta . \frac{ra}{(ra : i + mr)}.$$

In both cases $(y_{m+1,0}, \ldots, y_{m+1,m+1})$ is a multiple of $(x_{m+1,0}, \ldots, x_{m+1,m+1})$

2.2.Lemma: If k is a principal ideal domain, char $k = 0$ and $a \in k \setminus \{0\}$, then

$$HC_{2m+1}^{(i)}(A) \cong \bigoplus_{j=0}^{m} \left(\frac{k}{< i + jr, ra >} \right) \qquad (m \geq 0,\ i > 0)$$

Proof. Let $i \neq 0$. Since $H_{2m}^{(i)}(A) = 0$, $H_{2m+1}^{(i)}(A) = k / < ra >$ and $B: HC_{2m}^{(i)}(A) \to H_{2m+1}^{(i)}(A)$ sends $(x_{m+1,0}, \ldots, x_{m+1,m+1})$ to $(i + mr) x_{m,0}$ (Lemma 2.1), from the $\mathrm{SBI}^{(i)}$ sequence we obtain short exact sequences

$$0 \to \frac{k}{< (i + mr) x_{m,0}, ra >} \to HC_{2m+1}^{(i)}(A) \to HC_{2m-1}^{(i)}(A) \to 0 \qquad (m \geq 1) \qquad (*)$$

Now, as $(ra : x_{m,0}) = 1$ and k is a principal domain,

$$\frac{k}{< (i + rm) x_{m,0}, ra >} = \frac{k}{< i + mr, ra >}.$$

Hence, $(*)$ splits by means of $\pi: HC_{2m+1}^{(i)}(A) \to \frac{k}{<i+mr,ra>}$, $\pi(\overline{\overline{x}_1, \ldots, \overline{x}_{m+1}}) = \bar{x}_1$

2.3.Theorem: Let $A = k[X] / < X^r + a >$. We have

1) if $ra = 0$, then

$$HC_{2m}(A) = \bigoplus_{i=0}^{r-1} HC_{2m}^{(i)}(A) = k^{(r+m)} \oplus \left(\bigoplus_{i=0}^{r-1} \left(\bigoplus_{j=0}^{m-1} Ann(i + jr) \right) \right)$$

$$HC_{2m+1}(A) = \bigoplus_{i=0}^{r-1} HC_{2m+1}^{(i)}(A) = k^{(m+1)} \oplus \left(\bigoplus_{i=0}^{r-1} \left(\bigoplus_{j=0}^{m-1} \frac{k}{< i + jr >} \right) \right),$$

2) If there exists a morphism of rings $\varphi: k' \to k$, with k' a principal domain, char $k = 0$ and $a \in Im(\varphi)$, then

$$HC_{2m}(A) = k^{(r)} \oplus \left(\bigoplus_{i=0}^{r-1} \left(\bigoplus_{j=0}^{m-1} Ann(i + jr : ra) \right) \right) \oplus \left(Ann(ra) \right)^{(m)}$$

$$HC_{2m+1}(A) = \bigoplus_{i=1}^{r-1} \left(\bigoplus_{j=0}^{m} \frac{k}{< i + jr, ra >} \right) \oplus \left(\frac{k}{< ra >} \right)^{(m+1)},$$

where $M^{(s)}$ denotes the direct sum of s copies of M.

Proof. 1) It is immediate and 2) follows because using the Künneth formula we can consider the case where k is a principal domain and $a \in k \setminus \{0\}$. Then, the result is clear from the previous lemmas

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