Extensions of cocommutative coalgebras and a Hochschild - Kostant - Rosenberg type theorem

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Introduction

Given a commutative algebra A over a field k, it is a well known result that the first Hochschild homology group $HH_1(A)$ is isomorphic to the A-module of Kähler differentials $\Omega^1(A)$, and the pair $(\Omega^1(A), d : A \to \Omega^1(A))$ is therefore a universal object for derivations from A into symmetric A-bimodules M. Dually, $HH^1(A) \cong Der_k(A) = \{f \in Hom_k(A, A) \ / f(ab) = af(b) + bf(a) \ \forall a, b \in A\}$.

This object has been studied for a long while, as its description is connected with important properties of the algebra A. As one of the best-known examples, let us recall the Hochschild - Kostant - Rosenberg theorem [13], and its reciprocal statement [2, 3], which tells us that for a perfect field k the algebra $HH_*(A)$ (for A an essentially finite type commutative algebra) is isomorphic to the exterior algebra $\Lambda_A(\Omega^1(A|k))$ if and only if A is smooth.

We define homologically the concept of a smooth coalgebra and study the properties of this kind of coalgebras.

If C is a topological cocommutative k-coalgebra (the topology may be the discrete one), we define an object Ω^1_C and a coderivation $d : \Omega^1_C \to C$ such (Ω^1_C, d) that is universal for coderivations of cosymmetric C-bicomodules into C. We show that this object is isomorphic to the first cohomology group $Hoch^1(C)$ associated to the coalgebra ([6], [8], [17]).

The behaviour of Ω^1_C with respect to localizations is studied. It turns out that $Hoch^1(C)$ localizes in more general situations than those described in [9]. For the higher cohomology groups, we prove that in certain

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cases there is an isomorphism with the n-th component of the graded-cocommutative coalgebra $\Lambda^*_C(\Omega^1_C)$ obtained from Ω^1_C, which is described in section 6. This fact gives also localization results.

Finally we prove:

Theorem If k is a field, $char(k) = 0$ and C is a cocommutative coalgebra satisfying either one of the following hypothesis:

- C is a smooth algebraic coalgebra and $k.e_i \wedge k.e_i$ is finite dimensional for every group-like $e_i \in C$.
- C is a smooth local topological coalgebra with group-like e provided with a topology verifying the conditions stated in Proposition 5.1, $C = \bigcup_{n\in\mathbb{N}_0} \wedge^n(k.e)$

then $Hoch^*(C)$ is isomorphic to the exterior coalgebra on Ω^1_C.

In fact, this theorem gives an answer to a problem which arises in [11] (Section 5). In this paper, the authors give a proof of Hochschild - Kostant - Rosenberg theorem using G-algebras and expect that a dual version for coalgebras will hold. Their problem is the lack of a definition dualizing regular sequences. Such a definition appears here in the hypothesis of Theorem 3.8. It is worth to notice that they do not mention a key problem: *Hoch* theory does not localize well except under certain conditions; in section 7 we solve this localization problem.

The contents of the paper are as follows:

In section 1 we define what a "smooth coalgebra" is. Our definition is given in terms of extensions. Section 2 is devoted to the definition of local coalgebras in terms of group-like elements and to study its properties. A useful result is proven here: it is a "Nakayama's Lemma" for local coalgebras. Smoothness is also for coalgebras, a local concept.

In section 3 we construct Ω^1_C. When C is (cocommutative) smooth, Ω^1_C turns out to be, as expected, an injective cosymmetric C-comodule.

We study the behaviour of the universal object Ω^1_C with respect to localizations and also prove that Ω^1_C is a free comodule in the smooth local case.

We state at the end of this section an equivalence between the structure of the graded coalgebra associated to C (which is now local) and the existence of a "Koszul resolution".

We describe in section 4 the structure of $gr(C)$. In fact, we prove a structure theorem for cocommutative local smooth coalgebras (propositions 4.4 and 4.5).

We prove in section 5 that Ω^1_C is isomorphic to $Hoch^1(C)$, then the last one commutes with localizations.

Since we are working with topological coalgebras, we prove a result allowing us to calculate the topological version of $Hoch^*$ in terms of "resolutions".

The definition of the exterior coalgebra on Ω^1_C is given in section 6. We also study in detail the example of the coalgebra of distributions supported on a smooth compact manifold X, which suggests that a coalgebra version of a Hochschild - Kostant - Rosenberg type theorem exists.

Finally, in section 7 we give the arguments allowing us to pass from the global case to the local case, and we then prove our main theorem.

We will suppose that the field k is algebraically closed. As we are interested in the cohomology groups $Hoch^*$, this assumption is not restrictive in the sense that $Hoch^*(M, C|k) \otimes \overline{k} = Hoch^*(M \otimes \overline{k}, C \otimes \overline{k}|\overline{k})$ (with this notation, $|k$, resp $|\overline{k}$, means that the tensor products are taken over k, resp. over $\overline{k}$, and $\overline{k}$ denotes the algebraic closure of k). C will be always a topological k-coalgebra, unless the contrary is stated, even if we use the algebraic notation. In particular, usual coalgebras will be considered as topological ones with the discrete topology, and the same will hold for comodules, C^e will denote $C \otimes C^{op}$ and the category of C-bicomodules will be identified with C^e-comodules. All topologies considered will be Hausdorff. Given any C-bicomodule, ρ^- and ρ^+ will respectively denote the left and right structure morphisms. We want to thank Mariano Suárez Álvarez for his helpful (mathematical and formatting) comments. We also thank Jean-Louis Loday for his remarks on the relation between commutative extensions and Harrison cohomology.

1 Smooth coalgebras

Given a commutative k-algebra A, there are several equivalent definitions of what smoothness of A means. One of them says that A is smooth if and only if the second cohomology Harrison group is trivial for every symmetric A-bimodule M, or equivalently, every extension $0 \to M \to B \to A \to 0$ of commutative algebras with M a square zero ideal of B, is split by an algebra morphism. The well known Hochschild - Kostant - Rosenberg theorem says that if A is an essentially finitely generated commutative smooth algebra over a perfect field, then its Hochschild homology groups $HH_n(A)$ are isomorphic to $\Omega^n(A)$, $\forall n \in \mathbb{N}$. In fact, both conditions are equivalent [3, 13].

This section is devoted to the definition and study of basic properties of smooth coalgebras. As the definition makes use of the Hochschild cohomology groups $H^*(M, C)$ defined by Doi for M an arbitrary C-bicomodule, we begin by recalling some results and definitions from [6].

Definition 1.1 *Given a k-coalgebra C, an* **extension** *of C is a coalgebra D such that C is a subcoalgebra of D. The exact sequence*

$$0 \longrightarrow C \longrightarrow D \xrightarrow{p} D/C \longrightarrow 0$$

endows D/C with a structure of D-bicomodule. A structure of (non-counital) coalgebra on D/C is given by $\Delta_{D/C}(\overline{d}) := (p \otimes p)\Delta_D(d)$ $(d \in D)$.

Moreover, if $D = C \wedge C$ (i.e. $\Delta(D) \subseteq C \otimes D + D \otimes C$) we have that $\Delta_{D/C} = 0$ and also $\overline{D} := D/C$ is a C-bicomodule. Denote by $p : D \to \overline{D}$ the canonical projection. In this case, given a k-linear map $\psi : D \to C$ extending id_C, $\rho^+ \circ p = (p \otimes \psi) \circ \Delta$ and analogously for $\rho^- : \overline{D} \to C \otimes \overline{D}$.

If $f : D \to C \otimes C$ is defined by $f := (\psi \otimes \psi)\Delta - \Delta\psi$, then $f(x) = 0$ for $x \in C$, so, there exists $\overline{f} : \overline{D} \to C \otimes C$ such that $\overline{f} \circ p = f$.

Lemma 1.2 *Given an extension D of C and f as above, $\overline{f}$ is a 2-cocycle in the complex $(Hom(\overline{D}, C^{\otimes *}), \delta^*)$.*

Proof: Since p is surjective, it is enough to see that $\delta^2(\overline{f}) \circ p = 0$. This follows immediately from $D = C \wedge C$. ∎

The following result was proved in [6]:

Lemma 1.3 ([6], Lemma 7) *Given k-linear maps $\psi_1, \psi_2 : D \to C$ such that $\psi_i|_C = id_C$ $(i = 1, 2)$, let $\overline{f}_1$ and $\overline{f}_2$ be defined as above. Then there exists $h \in Hom_k(\overline{D}, C)$ such that $\overline{f}_1 - \overline{f}_2 = \delta^1(h)$.*

The above lemmas show that there is an element of $H^2(\overline{D}, C)$ associated to each extension of coalgebras. Doi proved ([6], Theorem 4) that, in the above situation, $[\overline{f}] = 0$ in $H^2(\overline{D}, C)$ if and only if there exists a coalgebra morphism $\psi : D \to C$ such that $\psi|_C = id_C$.

Consequently, given a coalgebra C and a C-bicomodule M, the equivalence classes of extensions of C by M are in 1-1 correspondence with elements of $H^2(M, C)$. As we are interested in cocommutative coalgebras, we will consider only cosymmetric bicomodules and cocommutative extensions. This class of extensions is in 1-1 correspondence with a subgroup of $H^2(M, C)$. Given a 2-cocycle $[f] \in H^2(M, C)$ with M cosymmetric, we say that f is symmetric if $f(m) = \sigma_{12}(f(m))$, where σ_{12} is the transposition of the first and second factors. We remark that if g is an arbitrary 2-cocycle, then $\widehat{g} := \sigma_{12}(g)$ is also a 2-cocycle; assuming $1/2 \in k$, the space of 2-cocycles can be decomposed into the direct sum of subspaces corresponding to the eigenvalues 1 and -1 of σ_{12}. The boundary of a 1-cocycle is always symmetric, so the decomposition of cocycles gives a decomposition of $H^2(M, C) = H^2(M, C)^{sym} \oplus H^2(M, C)^{antisym}$. It is clear that symmetric 2-cocycles correspond to cocommutative extensions and viceversa. Our definition of smoothness of C is as follows:

Definition 1.4 *Given a cocommutative k-coalgebra, we say that C is* **smooth** *if and only if $H^2(M, C)^{sym} = 0$ for every cosymmetric C-bicomodule M.*

Remark: The subspace $H^2(M,C)^{sym}$ is analogous to the second Harrison cohomology group for commutative algebras and symmetric bimodules. A Harrison-type theory might be defined for cocommutative coalgebras, but we are not going to consider other degrees because every property proved in this work depends only on extensions.

Next, we give an alternative description of smoothness:

Proposition 1.5 *The following statements are equivalent:*

1. *C is k-smooth.*
2. *Given an extension of cocommutative k-coalgebras $0 \to D \to E \to M \to 0$ with $E = D \wedge D$ (and so M is a D-cosymmetric bicomodule), and a k-coalgebra morphism $\psi : D \to C$, the following diagram may be completed with a morphism ϕ of k-coalgebras:*

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r] \arrow[d, "\psi"'] & E \arrow[r] \arrow[dl, dashed, "\phi"] & M \arrow[r] & 0 \\ & C & & & \end{tikzcd}$$

Proof: 2. $\Rightarrow$ 1.) Let M be a cosymmetric C-bicomodule. Consider an extension $0 \to C \to E \to M \to 0$ with E cocommutative. M is then a cosymmetric E-bicomodule; the exactness of the sequence is equivalent to $E = C \wedge C$. By hypothesis, the sequence is split by a coalgebra morphism ϕ, and we see that as $H^2(M,C)^{sym}$ classifies commutative extensions, $H^2(M,C)^{sym} = 0$.

1. $\Rightarrow$ 2.) Given an extension $0 \to D \to E \to M \to 0$ where M is a cosymmetric D-bicomodule, E and D are cocommutative and $\psi : D \to C$ a coalgebra morphism, consider the push-out P of C and E over D.

P is a cocommutative coalgebra provided of monomorphisms $C \to P$ and $E \to P$, with coproduct given by:

$$\Delta_P : P \to P \otimes P$$
$$\begin{cases} \Delta_P(c) = \Delta_C(c) \\ \Delta_P(e) = \Delta_E(e) \end{cases}$$

We want to see that $P = C \wedge C$, or equivalently that P/C is a C-bicomodule. In fact, $P/C \cong M$, the C-structure on P/C being induced by the D-structure on M and the morphism ψ.

Now our hypothesis is that C is smooth. As extensions are classified by $H^2(C,P/C)^{sym}$, which is null, it follows that there exists a coalgebra morphism $\sigma : P \to C$ splitting the exact sequence $0 \to C \to P \to M \to 0$. Defining $\phi := \sigma \circ \overline{\psi}$ we obtain the desired morphism splitting $0 \to D \to E \to M \to 0$. ∎

Given a complete topological cocommutative coalgebra C (we consider in particular the discrete case) and a multiplicative subset $S \subset C'$, the localization $C_{[S]}$ is another topological coalgebra provided of a morphism $\pi : C_{[S]} \to C$, universal for coalgebra morphisms $f : D \to C$ from topological cocommutative coalgebras, such that the elements of S define, by the C'-action, invertible endomorphisms in D, or, equivalently $s \circ f$ is invertible in the algebra D' for all $s \in S$.

The construction of $C_{[S]}$ and of localization of comodules $M_{[S]}$ has been carried out in [18] for linear topologies and in [9] for the locally convex case.

As expected, the smoothness property localizes well:

Proposition 1.6 *Let C be a cocommutative smooth k-coalgebra. Then $C_{[S]}$ is smooth for any multiplicatively closed subset S of C'.*

Proof: Let us consider an extension $0 \to D \to E \to M \to 0$ with D and E cocommutative (and hence M cosymmetric), and a coalgebra morphism $\nu : D \to C_{[S]}$. As C is smooth, there exists $u' : E \to C$ such that $u' \circ i = \pi \circ \nu$. Given $s \in S$, $s \circ u' \circ i = s \circ \pi \circ \nu$ is an invertible element in D', as $s \circ \pi$ is invertible in $(C_{[S]})'$, so, there exists $\widetilde{t} \in D'$ such that $i^*(s \circ u').\widetilde{t} = 1$. As i^* is an epimorphism, there exists $t \in E'$ such that $i^*(t) = \widetilde{t}$ and then, for some $m \in M'$, we have $t.(s \circ u') = 1 + m$ and this is invertible because m is nilpotent (in fact $m^2 = 0$). As a consequence, $s \circ u'$ is invertible for all $s \in S$ and so u' factors through $C_{[S]}$. ∎

2 Local Coalgebras

In this section (where C will always denote a cocommutative k-coalgebra) we study local coalgebras, i.e. coalgebras obtained after "localization at maximal ideals".

Note that *coalgebra* means either *algebraic coalgebra* or *topological coalgebra*.

Definition 2.1 *Given C, a* **coideal** *D is a C-comodule provided of an epimorphism $\phi : C \to D$ of (cosymmetric) C-comodules.*

Remark: In this situation, $Ker(\phi)$ is a subcoalgebra of C, and as C-comodule, D is isomorphic to $C/Ker(\phi)$. We remark that this definition is dual to the definition of an ideal in ring theory. There, ideals are subobjects of rings such that their quotients are also rings; here, a coideal is a quotient such that the Kernel of the projection is a (sub)coalgebra. Our definition of coideal differs from Sweedler's; we prefer ours in order to preserve duality.

Definition 2.2 *A coideal D of C is called* **maximal** *if $Ker(\phi)$ is a one dimensional subcoalgebra (necessarily isomorphic to the base field k).*

If $(D, \phi : C \to D)$ is a maximal coideal and $f : k \to Ker(\phi)$ is the corresponding isomorphism of coalgebras, then $e = f(1)$ is a group-like element in C. Conversely, given e define ϕ as the projection $C \to C/k.e$. We shall keep in mind in what follows this correspondence between "points" of the coalgebra, its maximal coideals and group-like elements of C. As an example, let us take $C = k[x]^0$ (see [16] for the definition of $(-)^0$) with k algebraically closed, $k[x]^0 = \oplus_{\lambda \in k} k[s].e^{\lambda s}$. It is easy to see that the only group-like elements of C are the exponentials $e^{\lambda s}$. Then C has 'as many points as' elements of k.

When k is algebraically closed, maximal coideals always exist. This follows because it is enough to find a group-like element in a finite dimensional subcoalgebra $\widetilde{C} \subset C$, and such an element always exists because algebra morphisms $\widetilde{C}^* \to k$ always exist.

Now, given a maximal coideal D in C, $C' - D'$ is clearly a multiplicative subset of C'; we are able to construct $C_{[C'-D']}$, which is a topological coalgebra. We shall denote it by C_D.

We next define the notion of prime coideal of a coalgebra:

Definition 2.3 *A coideal D with kernel K of a coalgebra C is* **prime** *if the restriction map*

$$C^* \cong Com_C(C, C) \to Com_C(K, K)$$

is such that for all $f \in C^*$, $f|_K : K \to K$ is null or an epimorphism.

Notice that the restriction map $f \longmapsto f|_K : K \to K$ is well-defined because f is C colinear and K is a subcoalgebra. We also remark that, with this definition, every maximal coideal is prime.

Definition 2.4 *A coalgebra C (over an algebraically closed field) will be called a* **local coalgebra** *if it has a unique group-like element.*

Remark: given a maximal coideal D in C, the exact sequence $0 \to k \to C \to D \to 0$ is split by a coalgebra morphism $\epsilon : C \to k$, so localization is exact for this sequence; we conclude that $0 \to k \to C_D \to D_D \to 0$ is exact ($k_D = k$ because D' acts by isomorphisms on k), and in particular we see that D_D is a maximal coideal in C_D.

Proposition 2.5 *With notations as above, and given a maximal coideal D of C (corresponding to a subcoalgebra $k.x$ of C), C_D is a local coalgebra.*

Proof: Consider a subcoalgebra K of C_D corresponding to a maximal coideal; denoting by $Im(K)$ (the closure of) the image of the subcoalgebra K by the canonical map $\pi : C_D \to C$, we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & C_D \arrow[r] \arrow[d, "\pi"'] & C_D/K \arrow[r] & 0 \\
0 \arrow[r] & Im(K) \arrow[r] & C & &
\end{tikzcd}$$

$(Im(K))^{\perp}$ is a subset of $D' = (k.x)^{\perp}$, so that $Im(K) = k.x$, because they have the same dimension. As a consequence, if C_D has a maximal coideal, it is unique.

Also, if $s \in S = C' - D' = \{f \in C' \ / \ f(x) \neq 0\}$, then $s.x = (1 \otimes s)\Delta(x) = s(x).x$. So we can define the element $\{\frac{1}{s(x)}.x\}_{s \in S} \in \prod_{s \in S}(k.x)^{(s)}$, where $(k.x)^{(s)} = k.x, \forall s \in S$. Denoting by λ_s the element $\frac{1}{s(x)}.x$, we have, for $t \in S$,

$$t.\lambda_{st} = \frac{1}{(st)(x)} t(x).x = \frac{1}{s(x)t(x)} t(x).x = \lambda_s$$

so, $\{\lambda_s\}_{s \in S} \in (k.x)_{[S]}$. This proves that there is at least one maximal coideal (notice that the argument assuring the existence of maximal coideals was stated only in the algebraic context). ∎

We finish this section with a description of injective comodules over local coalgebras. We will prove that given a finitely cogenerated injective comodule M over a local coalgebra C, it is C-free. In the algebraic category the hypothesis of being finitely cogenerated is not necesary, this fact was already known (see [18]). The methods in [18] differ from ours. We use a dualization of a Nakayama's Lemma, and this allows us to treat the topological case (at least for finitely cogenerated comodules) in the same way as the algebraic case.

We need a previous result, which is the analogue to Nakayama's Lemma for algebras.

Lemma 2.6 *Consider a cocommutative local coalgebra C with group-like x and a C-comodule M, together with an injection $M \hookrightarrow C^{(I)}$ for some set I; suppose that I is finite in the topological case. If the composition:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\rho"] \arrow[rr, bend right] & C \otimes M \arrow[r, "\pi_K \otimes id"] & C/K \otimes M
\end{tikzcd}$$

is a monomorphism for some subcoalgebra $K \subseteq C$, then $M = 0$.

Proof: Let K be a subcoalgebra of C. Then K contains at least an irreducible subcoalgebra of C, but there is only one, so $k.x \subseteq K$.

This is clear in the algebraic context (see [16]), for the topological case it needs a proof:

Consider the inclusion $k.x \hookrightarrow C$; by restriction it induces an algebra map $r : C' \to (k.x)' = k$ with kernel $(k.x)^{\perp}$, which is a maximal ideal of C'. The composition

$$C' \xrightarrow{\pi} K' \longrightarrow K'/\pi((k.x)^{\perp})$$

is clearly a surjection, and $(k.x)^{\perp}$ maps to zero, then it induces a surjection $C'/(k.x)^{\perp} \to K'/\pi((k.x)^{\perp})$. Since $C'/(k.x)^{\perp}$ is a field, it is a monomorphism or it is zero. In this last case on has $k.x = K$, in particular K contains an irreducible subcoalgebra of C. If it is a monomorphism then $K'/\pi((k.x)^{\perp})$ is isomorphic to k and this proves that $\pi((k.x)^{\perp})$ is a maximal ideal in K'. Dualizing the diagram

$$\begin{tikzcd}
C' \arrow[d, "\pi"'] \arrow[dr, "r"] & \\
K' \arrow[r] & k
\end{tikzcd}$$

one obtains the commutative diagram

$$\begin{tikzcd}
C'' & C \arrow[l] \\
K'' \arrow[u] & K \arrow[l] \arrow[u] & k \arrow[l, dashed] \arrow[ul] \arrow[ll, bend left]
\end{tikzcd}$$

The dashed arrow exists because $K = K'' \cap C$, as x is the image of $1 \in k$, the assertion is proved.

If $(\pi_K \otimes id) \circ \rho$ is a monomorphism, then $(\pi_{k.x} \otimes id) \circ \rho$ is also a monomorphism. It is then sufficient to make the proof for $K = k.x$.

First step: suppose that M is a subcomodule of C.

Since C is cocommutative, M is a subcoalgebra and hence, if $M \neq 0$, it contains an irreducible subcoalgebra; but there is only one, so $k.x \subseteq M$.

By hypothesis, the composition $M \to C \otimes M \to C/k.x \otimes M$ is injective, but the element $x \in M$ is in the kernel of this composition; this is a contradiction unless $M = 0$.

Second step: Suppose that there is an injection $M \hookrightarrow C^m$ for some $m \in \mathbb{N}$. Let us take $n = \min\{m \in \mathbb{N} | M$ embeds in $C^m\}$. Suppose $n > 0$.

The case $n = 1$ has already been dealt with in the first step, suppose then $n > 1$. Let $\phi : M \to C^n$ denote an embedding and take $N = M \cap \phi^{-1}(C^{n-1} \oplus 0)$.

If $N = 0$, then the composition $M \xrightarrow{\phi} C^n \xrightarrow{\pi_n} C$ is injective, because if $z \in M \cap Ker(\pi_n \circ \phi)$ then $\pi_n(\phi(z)) = 0$, $z \in N = 0$, and this contradicts the minimality of n, so $N \neq 0$. Now consider the diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[d, hook] & C \otimes N \arrow[r] \arrow[d, hook] & C/k.x \otimes N \arrow[d, hook] \\
M \arrow[r] & C \otimes M \arrow[r] & C/k.x \otimes M
\end{tikzcd}$$

If the composition in the bottom row is a monomorphism, then so is the composition in the top row, because the columns are monomorphisms. Also $\phi|_N : N \to C^{n-1} \oplus 0$ is injective; the inductive hypothesis implies that $N = 0$, and this is a contradiction.

Third step: When the coalgebra C is algebraic, consider T an arbitrary finite dimensional subcomodule of M, if $(\pi \otimes id_M) \circ \rho_M$ is a monomorphism, so is its restriction to T. Since a finite dimensional comodule is finitely cogenerated (consider the structure morphism $T \to C \otimes T \cong C^{dim_k(T)}$), the second step applies to T and we see that $T = 0$. But M is the union of its finite dimensional subcomodules, clearly now, $M = 0$. ∎

Proposition 2.7 *Let C be a local coalgebra and M an injective finitely cogenerated C-comodule. Then M is a free C-comodule.*

Proof: Let us suppose $M \neq 0$ and let n be as in the proof of the above lemma. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & C^n \\
0 \arrow[r] & (k.x)^n \cap M \arrow[r] \arrow[u, hook] & (kx)^n \arrow[u, hook]
\end{tikzcd}$$

1. $M \cap (kx)^n \neq \{0\}$: Identifying $C \otimes M$ to a subspace of $C \otimes C^n \cong (C \otimes C)^n$ and then considering the composition

$$M \to C \otimes M \to C/k.x \otimes M$$

we see that the image of an element $z = (z_1, \ldots, z_n) \in M$ is zero if z is such that the image of $(0, \ldots, z_i, \ldots, 0)$ is null for all i , and this is true if and only if $z_i = \epsilon(z_i).x$ for all i. Then $z = (z_1, \ldots, z_n) = (\epsilon(z_1).x, \ldots, \epsilon(z_n).x)$, so if $M \cap (k.x)^n = 0$ this composition is injective and by Nakayama's Lemma $M = 0$, a contradiction. Then $M \cap (k.x)^n$ is a non-zero subspace of $(k.x)^n$. Let $\{m_1, \ldots, m_s\}$ be a basis of it.

2. Consider the maps $\epsilon^n : C^n \to (k.x)^n$ ($(c_1, \ldots, c_n) \mapsto (\epsilon(c_1).x, \ldots, \epsilon(c_n).x)$ and $\phi : M \to C^s = C \otimes (k.m_1 \oplus ... \oplus k.m_s)$ given by $\phi(m) = m^{-1} \otimes \epsilon^n(m^0)$. Notice that $\epsilon^n(M) \subseteq (k.x)^n \cap M = k.m_1 \oplus ... \oplus k.m_s$. Since ϕ is a morphism of C-comodules, $Ker(\phi)$ is a subcomodule of M. We want to show that it is zero. Looking again at the diagram

$$\begin{tikzcd} Ker(\phi) \arrow[r] \arrow[d, hook] & C \otimes Ker(\phi) \arrow[r] \arrow[d, hook] & C/k.x \otimes Ker(\phi) \arrow[d, hook] \\ M \arrow[r] & C \otimes M \arrow[r] & C/k.x \otimes M \end{tikzcd}$$

we observe that $\phi|_{(k.x)^n \cap M} = id_{(k.x)^n \cap M}$ and hence $Ker(\phi) \cap (k.x)^n = 0$ then, again by Nakayama's Lemma, $Ker(\phi) = 0$.

3. Next we want to show that the cokernel of ϕ is zero. Let us denote K_ϕ the cokernel of ϕ and consider the short exact sequence

$$0 \to M \to C^s \to K_\phi \to 0$$

As M is injective, the sequence is split, and $C^s \cong M \oplus K_\phi$ (identifying K_ϕ with its image by the splitting). But $K_\phi \cap (k.m_1 \oplus \ldots \oplus k.m_s) = 0$ because $(k.m_1 \oplus \ldots k.m_s) = (k.x)^n \cap M$ and $K_\phi \cap M = 0$. This proves that $K_\phi = 0$ because $(k.m_1 \oplus \ldots \oplus k.m_s)$ is the kernel of the composition $C^s \to C \otimes C^s \to C/k.x \otimes C^s$ which induces by restriction the composition $K_\phi \to C \otimes K_\phi \to C/k.x \otimes K_\phi$. ∎

We conclude this section with two lemmas that will be necessary later, but are interesting on their own.

Lemma 2.8 *Let C and D be two cocommutative k-coalgebras with $k = \overline{k}$. Then*

1. *If C and D are local, then $C \otimes D$ is also local.*
2. *If C and D are smooth, then $C \otimes D$ is also smooth.*

Proof: 1. We remark that $C \otimes D$ is the product in the category of cocommutative k-coalgebras, the projections being $p_C = 1 \otimes \epsilon : C \otimes D \to C$ and $p_D = \epsilon \otimes 1 : C \otimes D \to D$. If $\phi : E \to C \otimes D$ is a coalgebra morphism and E is cocommutative, then ϕ determines two coalgebra morphisms $\phi_C = p_C \circ \phi$, $\phi_D = p_D \circ \phi$, and $\phi(x) = \phi_C(x) \otimes \phi_D(x)$ for all $x \in E$. If e is a group-like element of $C \otimes D$ then it corresponds to a coalgebra morphism $k \to C \otimes D$, and as a consequence e must be of the type $e = e_C \otimes e_D$ with e_C (resp. e_D) a group-like element of C (resp. D). But if C and D have unique group-like elements, then the same holds for $C \otimes D$.

2. Let

$$0 \to C \otimes D \to E \to M \to 0$$

be an extension of cocommutative coalgebras with $E = (C \otimes D) \wedge_E (C \otimes D)$. We must produce a coalgebra splitting. Consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r] & E \oplus_{p_D} D \arrow[r] & M_D \arrow[r] & 0 \\ 0 \arrow[r] & C \otimes D \arrow[r] \arrow[u, "p_D"] \arrow[d, "p_C"'] & E \arrow[r] \arrow[u] \arrow[d] & M \arrow[r] \arrow[u] \arrow[d] & 0 \\ 0 \arrow[r] & C \arrow[r] & E \oplus_{p_C} C \arrow[r] & M_C \arrow[r] & 0 \end{tikzcd}$$

where M_C and M_D are the respective Cokernels; which are cosymmetric. One sees easily that the induced maps $M \to M_C$ and $M \to M_D$ are both surjective, so $\Delta_{M_C} = 0 = \Delta_{M_D}$, or, in other words, that $E \oplus_{p_C} C =$

$C\wedge_{E\oplus_{p_C}C}C$ and similarly for $E\oplus_{p_D}D$ (notice that these are cocommutative coalgebras). Now because C and D are smooth, there are coalgebra splittings of both extensions; let us denote them by $s: E\oplus_{p_C}C \to C$ and $t: E\oplus_{p_D}D \to D$. Composing them with the canonical projections $E \to E\oplus_{p_C}C$ and $E \to E\oplus_{p_D}D$ we obtain two coalgebra morphisms $S: E \to C$ and $T: E \to D$. Now the coalgebra morphism $(S\otimes T)\circ\Delta_E : E \to C\otimes D$ is the desired splitting for the extension of $C\otimes D$ by M. ∎

Lemma 2.9 *Let C be a cocommutative smooth coalgebra and K a subcoalgebra of C such that there exists a coalgebra morphism $\psi : C \to K$ with $\psi|_K = id_K$. Then K is smooth. In particular, if C is smooth and $C = K_1 \otimes K_2$, then K_1 and K_2 are smooth.*

Proof: Consider $0 \to D \to E \to M \to 0$ an extension of cocommutative coalgebras with $E = D\wedge_E D$, and $f : D \to K$ a coalgebra morphism. Since K is a subcoalgebra of C, we can consider f as a morphism from D to C (see diagram)

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r] \arrow[d, "f"'] & E \arrow[r] \arrow[d, dashed, "\widetilde{f}"] & M \arrow[r] & 0 \\ & K \arrow[r, hook, shift left] & C \arrow[l, shift left, "\psi"] & & \end{tikzcd}$$

The dashed morphism $\widetilde{f} : E \to C$ such that $\widetilde{f}|_D = f$ exists because C is smooth; if we define $\overline{f} : E \to K$ by $\overline{f} := \psi\circ\widetilde{f}$ we obtain the desired extension of f. ∎

3 Universal object for coderivations

Given a field k and a k-coalgebra (C, Δ, ϵ), there is an object L_C and a coderivation $d : L_C \to C$ such that the pair (L_C, d) is universal for k-coderivations of bicomodules M into C [6, 8]. L_C is the cokernel of $\Delta : C \to C\otimes C$. The sequence

$$0 \longrightarrow C \xrightarrow{\Delta} C\otimes C \longrightarrow L_C \longrightarrow 0$$

is k-split by means of $s : C\otimes C \to C$, $s(c\otimes c') := \epsilon(c)c'$. L_C is then a topological C-bicomodule with the quotient topology of $C\otimes C/Im(\Delta)$.

If C is a cocommutative topological coalgebra, we shall be concerned with complete topological C-bicomodules M which are C-cosymmetric (i.e. $\sigma_{12}\circ\rho^+_M = \rho^-_M$). In this context, a continuous map $f : M \to C$ is called a **coderivation** if

$$\Delta\circ f = (id\otimes f)\circ\rho^-_M + (f\otimes id)\circ\rho^+_M = (1+\sigma_{12})\circ(id\otimes f)\circ\rho^-_M$$

We consider the subcomodule of L_C consisting of the cosymmetric elements and denote it Ω^1_C, i.e.

$$\Omega^1_C := Sym(C\otimes C/Im(\Delta)) = (C\wedge_{C^e} C)/C$$

Proposition 3.1 *Ω^1_C is a universal object for coderivations in the category of cosymmetric bicomodules.*

Proof: Ω^1_C is by definition a cosymmetric C-bicomodule. We must exhibit a coderivation $d : \Omega^1_C \to C$. It is given by

$$d([z]) := (\epsilon\otimes id - id\otimes\epsilon)(z) \qquad (z\in C\otimes C)$$

Observe that d is well defined on L_C because if $z\in Im(\Delta)$ then $(\epsilon\otimes id - id\otimes\epsilon)(z) = 0$.

It is easy to see that d is a coderivation and that (Ω^1_C, d) has the universal property with respect to coderivations from cosymmetric bicomodules into C, using that Ω^1_C is a subobject of L_C. ∎

We shall call Ω^1_C the comodule of Kähler differentials of C.

Given a complete topological cocommutative coalgebra C (we consider in particular the discrete case) and a multiplicative subset $S\subset C'$, we want to establish the relation between the $C_{[S]}$ -bicomodules $\Omega^1_{C_{[S]}}$

and $\left(\Omega^1_C\right)_{[S]}$. In fact, we shall prove that they are isomorphic when C has a topology with the same or less open sets than the topology induced by C'' via the canonical evaluation map (for example when C is reflexive, or when C is the dual of some other space).

We shall begin by proving that a coderivation $D : M \to C$ from a cosymmetric C-bicomodule M, induces a coderivation $D_{[S]} : M_{[S]} \to C_{[S]}$ such that the following diagram is commutative:

$$\begin{tikzcd} M_{[S]} \arrow[r, "D_{[S]}"] \arrow[d, "\pi_M"] & C_{[S]} \arrow[d, "\pi_C"] \\ M \arrow[r, "D"] & C \end{tikzcd}$$

In order to do so, we first define a coderivation $M_{[S]} \to M''_{[S]} \to C''_{[S]}$, analogous to the Leibniz rule "$d(\frac{f}{g}) = \frac{1}{g}df - \frac{f}{g^2}dg$". More precisely:

$$M_{[S]} \to M''_{[S]} \to C''_{[S]}$$
$$\{m_s\}_{s\in S} \mapsto \{c_t\}_{t\in S}$$

where

$$c_t(-) = m_t(- \circ D) - m_{t^2}(-.(t \circ D))$$

The family $\{c_t\}_{t\in S} \in \prod_{t\in S} C''$. In order to see that it is an element of $C''_{[S]}$ we have to verify that $r.c_{tr} = c_t,\ \forall r, t \in S$. This follows from the formula $(h.g) \circ D = h.(g \circ D) + g.(h \circ D) \quad (h,\ g \in C')$, which implies that $r.c_{tr}(f) = c_t(f)\ \ \forall f \in C'$.

Notice that M' is a C'-module by means of the structure map of M, namely $M' \otimes C' \to (C \otimes M)' \xrightarrow{\rho^*} M'$ and $h \circ D$ is an element of M'. The above formula is a straightforward consequence of the definition of coderivation.

We have to show now that $Im(D_{[S]}|_{M_{[S]}}) \subseteq C_{[S]}$. To see that ϕ is not only in C'' but also in C, we must find an element $c_\phi \in C$ such that $\phi(f) = f(c_\phi)$ for all $f \in C'$. In our case,

$$c_t(f) = f(D(m_t)) - f((1 \otimes (t \circ D))\rho(m_{t^2}))$$

as a consequence $c_t = D(m_t) - (1 \otimes (t \circ D))\rho(m_{t^2}) \in C$. The map $\{m_s\}_{s\in S} \mapsto \{c_t\}_{t\in S}$ is continuous because the topology of C is induced by the inclusion $C \to C''$.

The universal coderivation $d : \Omega^1_C \to C$ gives thus a coderivation $d_{[S]} : (\Omega^1_C)_{[S]} \to C_{[S]}$, and by the universal property of $\Omega^1_{C_{[S]}}$ a $C_{[S]}$-bicomodule map $(\Omega^1_C)_{[S]} \to \Omega^1_{C_{[S]}}$.

On the other hand, $\pi_C : C_{[S]} \to C$ induces a C-bicomodule map $\Omega^1_{C_{[S]}} \to \Omega^1_C$ and therefore a $C_{[S]}$-bicomodule map $\widetilde{\pi} : \Omega^1_{C_{[S]}} = (\Omega^1_{C_{[S]}})_{[S]} \to (\Omega^1_C)_{[S]}$. Next we show that these maps are inverses to each other.

Proposition 3.2 *The $C_{[S]}$-bicomodules $\Omega^1_{C_{[S]}}$ and $(\Omega^1_C)_{[S]}$ are isomorphic.*

Proof: With notation as above, let $\phi : (\Omega^1_C)_{[S]} \to \Omega^1_{C_{[S]}}$ and $\psi : \Omega_{C_{[S]}} \to (\Omega_C)_{[S]}$ be defined by: $\phi\left(\{m_t\}_{t\in S}\right) = \overline{(id \otimes d_{[S]}) \circ \rho^-(\{m_t\}_{t\in S})}$ and $\psi\left(\sum \overline{\{x_t\}_{t\in S} \otimes \{y_s\}_{s\in S}}\right) = \left\{\frac{1}{r}.\sum \overline{x_1 \otimes y_1}\right\}_{r\in S}$, both extended by continuity.

In order to see that $\phi \circ \psi = id_{\Omega_{C_{[S]}}}$ it is enough to prove, by the universal property of $\Omega^1_{C_{[S]}}$, that $d_{C_{[S]}} \circ \phi \circ \psi = d_{C_{[S]}}$. Again it is sufficient to see that $\pi_C \circ d_{C_{[S]}} \circ \phi \circ \psi = \pi_C \circ d_{C_{[S]}}$ and this is true due to the following four equalities that we will prove in turn:

$$\pi_C \circ d_{C_{[S]}} = d_C \circ \widetilde{\pi} \tag{1}$$

$$\widetilde{\pi} \circ \phi = \pi_{\Omega_C} \tag{2}$$

$$\pi_{\Omega_C} \circ \psi = \widetilde{\pi} \tag{3}$$
$$d_C \circ \widetilde{\pi} = \pi_C \circ d_{C_{[S]}} \tag{4}$$

(1): $\pi \circ d_{C_{[S]}}(\{m_t\}_{t\in S}) = d(m_1) - (1 \otimes (\epsilon \circ d)) \circ \rho(m_1) = d(m_1)$ because $\epsilon \circ d = 0$.

(2): $\pi_{\Omega_C}(\{\overline{m_t}\}_{t\in S}) = \overline{\pi \otimes \pi(\{m_t\}_{t\in S})}$. Also,

$$= \widetilde{\pi} \circ \phi(\{\overline{m_t}\}_{t\in S}) = \overline{(\pi \otimes \pi) \circ (id \otimes d_{[S]}) \circ \rho_{(\Omega_C)_{[S]}}(\{m_t\}_{t\in S})} =$$

$$= \overline{(\pi \otimes \pi \circ d_{[S]}) \circ \rho_{(\Omega_C)_{[S]}}(\{m_t\}_{t\in S})} = \overline{(id \otimes d) \circ (\pi \otimes \pi_{\Omega_C}) \circ \rho_{(\Omega_C)_{[S]}}(\{m_t\}_{t\in S})}$$

Using that $(\pi \otimes \pi_{\Omega_C}) \circ \rho_{(\Omega_C)_{[S]}}(\{m_t\}_{t\in S}) = \rho_{\Omega_C} \circ \pi_{\Omega_C}(\{m_t\}_{t\in S})$, we have that the last expression equals:

$$\overline{(id \otimes d) \circ \rho_{\Omega_C} \circ \pi_{\Omega_C}(\{m_t\}_{t\in S})} = \overline{(id \otimes d) \circ \rho_{\Omega_C}(m_1)} =$$

$$= \overline{(id \otimes (\epsilon \otimes 1) - (1 \otimes \epsilon)) \circ \rho_{\Omega_C}(m_1)} = \overline{(id \otimes (\epsilon \otimes 1)) \circ \rho_{\Omega_C}(m_1)} - \overline{((id \otimes (1 \otimes \epsilon)) \circ \rho_{\Omega_C}(m_1)}$$

The first one of this last two terms equals $\pi_{\Omega_C}(\{\overline{m_t}\}_{t\in S})$ because $(\epsilon \otimes 1) \circ \rho = id$. So we have to see that the other term equals zero, or equivalently that $(id \otimes (1 \otimes \epsilon)) \circ \rho_{\Omega_C}(m_1) \in Im(\Delta_C)$. If $m = x \otimes y$ this is true because $id \otimes (1 \otimes \epsilon)) \circ \rho(x \otimes y) = \Delta(x)\epsilon(y)$; it is also true when m is a linear combination of elementary tensors, and finally when m belongs to the closure of the algebraic span of elementary tensors, using continuity and the fact that $Im(\Delta)$ is closed.

(3): $\pi_{\Omega_C}\psi\left(\sum \overline{\{x_t\}_{t\in S} \otimes \{y_s\}_{s\in S}}\right) = \pi_{\Omega_C}\left(\left\{\frac{1}{r}.(\sum \overline{x_1 \otimes y_1})\right\}_{r\in S}\right) = \pi_{\Omega_C}\left(\{\sum \overline{x_r \otimes y_1}\}_{r\in S}\right) = \sum \overline{x_1 \otimes y_1} = \widetilde{\pi}\left(\sum \overline{\{x_t\}_{t\in S} \otimes \{y_s\}_{s\in S}}\right)$

(4): $d\left(\widetilde{\pi}\left(\sum \overline{\{x_t\}_{t\in S} \otimes \{y_s\}_{s\in S}}\right)\right) = d\left(\sum \overline{x_1 \otimes y_1}\right) = \sum \epsilon(x_1)y_1 - \epsilon(y_1)x_1$

On the other hand

$$\pi d_{C_{[S]}}\left(\sum \overline{\{x_t\}_{t\in S} \otimes \{y_s\}_{s\in S}}\right) = \pi\left(\sum \overline{\epsilon(\{x_t\}_{t\in S})\{y_s\}_{s\in S} - \{x_t\}_{t\in S}\epsilon(\{y_s\}_{s\in S})}\right) =$$

$$= \pi\left(\sum \epsilon(x_1)\{y_s\}_{s\in S} - \{x_t\}_{t\in S}\epsilon(y_1)\right) = \sum \epsilon(x_1)y_1 - \epsilon(y_1)x_1$$

The proof that the composition in the other sense is the identity uses the same equalities:

$$\psi \circ \phi = id_{(\Omega_C)_{[S]}} \Leftrightarrow \pi_{\Omega_C} \circ \psi \circ \phi = \pi_{\Omega_C}$$

By (3): $\pi_{\Omega_C} \circ \psi = \widetilde{\pi}$ and using (2) $\widetilde{\pi} \circ \phi = \pi_{\Omega_C}$, then $(\pi_{\Omega_C} \circ \psi) \circ \phi = \widetilde{\pi} \circ \phi = \pi_{\Omega_C}$. ∎

Example: Consider an algebraically closed field k of characteristic 0 and $A = k[x]$. It is a Hopf algebra with comultiplication induced by $\Delta(x) = x \otimes 1 + 1 \otimes x$ and antipode $x \mapsto -x$. Although A^* is not a bialgebra, there is an object denoted A^0 which is the biggest subset of A^* such that it is a Hopf algebra with the dual structure of $k[x]$ [16]. So, $A^0 \subseteq k[x]^* = k[|x|]$. The isomorphism $\gamma : k[|x|] \to k[|s|]$

$$\sum a_n x^n \mapsto \sum a_n n! s^n$$

gives the identification $A^0 = k[s, e^{\lambda s}]_{\lambda \in k} = \oplus_{\lambda \in k} k[s]e^{\lambda s} \subset k[|s|]$, where $\Delta(s) = s \otimes 1 + 1 \otimes s$ and the algebra structure is the usual one in $k[|s|]$ viewed as formal power series (the exponentials are forced to be group-like).

Also, the coalgebra $k[x]^0$ can be considered the topological dual of $A = k[x]$ with respect to the following linear topology:

Given $\lambda \in k$ and $n \in \mathbb{N}$, take the k-vector space $V_{n,\lambda} = \langle (x-\lambda).x^m,\ m \geq n \rangle$ and $\{V_{n,\lambda}\}_{(n,\lambda)\in\mathbb{N}\times k}$ as a basis of neighbourhoods of 0. Then the elements of the continuous dual of A are sequences $\{a_n\}_{n\in\mathbb{N}_0}$ in $k[|x|]$ such that there exists $\lambda \in k$ and $n \in \mathbb{N}_0$ with $a_{m+1} = \lambda a_m$, $\forall m \geq n$, i.e. they differ modulo a polynomial from the series representing $\frac{1}{1-\lambda x}$, and these series are exactly those ones giving exponentials when we apply the isomorphism $\gamma : k[|x|] \cong k[|s|]$. As $\overline{k} = k$, $\{(x-\lambda)\}_{\lambda\in k}$ is nothing but the set of irreducible polynomials in $k[x]$, which correspond to semisimple (finite dimensional) representations of $k[x]$, the collection of them allowing the computation of $k[x]^0$. If k is not algebraically closed, one replaces the family of polynomials $\{(x-\lambda)\}_{\lambda\in k}$ by the family of irreducible polynomials of $k[x]$.

Consider $\{V \subset A : \exists\ I \subset V,\ I$ a finite codimensional ideal$\}$, then $A^0 = A'$ (the topological dual with respect to the topology defined by this family of subspaces). Since $Hom_{cont}(V, W')$ is isomorphic to $Hom_{cont}(W, V')$ (by $(f \mapsto f' \circ i_W)$ we have that $\underset{\leftarrow_i}{lim}\ V_i' = \left(\underset{\rightarrow_i}{lim}\ V_i\right)'$, and so $A^0_{[S]} = (A_S)'$. However, the continuous dual is taken with respect to the direct limit topology in A_S. It agrees with the final topology of the canonical map $A \to A_S$, but it needs not be the same topology giving $(A_S)^0$.

Nevertheless, if $A = k[x]$, $S = \{1, x, x^2, x^3, \ldots\}$ and $C = k[x]^0 = k[s, e^{\lambda s}]_{\lambda\in k} = \oplus_{\lambda\in k} k[s]e^{\lambda s}$ then the element $x \in S$ induces the derivation operator in C, which is an isomorphism when restricted to the components corresponding to exponentials $e^{\lambda s}$ with $\lambda \neq 0$. We obtain by direct computation that $C_{[S]} = k[s, e^{\lambda s}]_{\lambda\in k-0} = \oplus_{\lambda\in k-0} k[s]e^{\lambda s}$. The canonical map $C_{[S]} \to C$ is the inclusion $k[s, e^{\lambda s}]_{\lambda\in k-0} \to k[s, e^{\lambda s}]_{\lambda\in k}$. Given another coalgebra D and a map $f : D \to C$ such that $.x$ is an isomorphism in D, polynomials cannot belong to $Im(f)$, and then f factorizes through $C_{[S]}$. On the other hand, $A_S = k[x, x^{-1}]$, and $A^0_S = k[s, e^{\lambda s}]_{\lambda\in k-0}$.

Proposition 3.3 *Given a topological algebra A, $C = A'$ and a multiplicative subset S of $Z(A)$, then:*

1. $C_{[S]} = (A_S)'$.
2. $\Omega^1_C = (\Omega^1(A))'$.
3. $(\Omega^1_C)_{[S]} = (\Omega^1(A)')_{[S]} = (\Omega^1(A)_S)' = (\Omega^1(A_S))' = \Omega^1_{C_{[S]}}$.

Proof:
1. This is proved in [18] for linear topologies and in [9] for the locally convex case.
In order to see 2., consider an arbitrary C-comodule M. Then,

$$Com_C(M, \Omega^1_C) = Coder(M, C) = Coder(M, A') = Der(A, M') = Hom_A(\Omega^1(A), M') = Com_C(M, \Omega^1(A)')$$

Finally, one of the equalities of 3. states that $\Omega^1(A)$ localizes for a topological algebra A. The proof of this fact is similar to the coalgebra case, but no assumptions concerning the topology of A are needed, simply define maps $\frac{adb}{s} \mapsto \frac{a}{s}d(\frac{b}{1})$ and $\frac{a}{s}d(\frac{b}{t}) \mapsto \frac{a}{s}(\frac{1}{t}db - \frac{b}{t^2}dt)$. The other equalities have already been proved. ∎

The following proposition shows taht, similarly to the case of algebras, smoothness of a coalgebra C has consequences on the structure of the comodule of differentials:

Proposition 3.4 *If C is a k-smooth cocommutative coalgebra, then Ω^1_C is an injective C-comodule.*

Proof: Given a diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "f"'] & M \arrow[dl, dashed, "\overline{f}"] \\ & \Omega^1_C & \end{tikzcd}$$

where M and N are C comodules, and $i,\ f$ are morphisms of comodules (i is a monomorphism), we need a morphism $\overline{f}$ extending f.

By the universal property of Ω^1_C, f corresponds to a coderivation $\nabla_f : N \to C$. Consider $C \oplus N$ as cocommutative k-coalgebra, with structure given by: $\Delta((c,n)) = (\Delta_C(c), \rho^-(n), \rho^+(n), 0)$ and similarly for $C \oplus M$ (note that cosymmetry of N implies cocommutativity of $C \oplus N$). Then we have:

$$\begin{tikzcd} 0 \arrow[r] & C \oplus N \arrow[r, "(id_C, i)"] \arrow[d, "id_C \oplus \nabla_f"'] & C \oplus M \arrow[r] \arrow[dl, dashed, "\alpha"] & M/i(N) \arrow[r] & 0 \\ & C & & & \end{tikzcd}$$

The dotted arrow α exists because (id_C, i) is a coalgebra morphism and $C \oplus M = (C \oplus N) \wedge (C \oplus N)$.

The above diagram commutes, so the equality $\alpha = id_C \oplus D_\alpha$, necessarily holds (for some coderivation $D_\alpha : M \to C$, extending ∇_f). This coderivation corresponds to a C-colinear morphism $\overline{f} : M \to \Omega^1_C$, such that $\overline{f} \circ i = f$, because $D_\alpha \circ i = \nabla_f$. ∎

Given a cocommutative coalgebra C and a coideal D, consider the usual exact sequence

$$0 \to K \to C \to D \to 0$$

(K is a subcoalgebra) and let γ be the composition map

$$D \xrightarrow{\rho^+} D \otimes C \xrightarrow{id \otimes \pi} D \otimes D$$

We recall from [16] that $K \wedge K = \{x \in C \ / \ \Delta(x) \in C \otimes K + K \otimes C\} = Ker((\pi_K \otimes \pi_K) \circ \Delta)$.There is an inclusion $K \to K \wedge K$. The comodules of Kähler differentials of both coalgebras are related by the following proposition:

Proposition 3.5 *1. There is an exact sequence of C-comodules*

$$0 \longrightarrow \Omega^1_K \xrightarrow{i} \Omega^1_C \square_C K \xrightarrow{\delta_K} (K \wedge K)/K$$

2. If D is a maximal coideal, so that $K = k.x$, then the monomorphism $\Omega^1_C \square_C k \to (K \wedge K)/K$ is an isomorphism.

Proof:

1. The map i is the composition of the inclusion of $\Omega^1_K = Sym(K \otimes K/\Delta(K)) \subseteq Sym(C \otimes C/\Delta(C)) = \Omega^1_C$ with the structure map giving the isomorphism $\Omega^1_C \cong \Omega^1_C \square_C C$. We notice that if an element z belongs to $\Omega^1_K \subseteq \Omega^1_C$, then $\rho(z) \in \Omega^1_C \otimes K$, and so the image of this composition is included in $\Omega^1_C \square_C K$.

The map δ_K is defined by $\delta_K = p_K \circ (d \otimes \epsilon) : \Omega^1_C \square_C K \to (K \wedge K)/K$ where $d : \Omega^1_C \to C$ is the universal coderivation and $p_K : C \to C/K$ is the canonical projection. The image of $d \otimes \epsilon$ is contained in $K \wedge K$ because the domain is $\Omega^1 \square_C K$ and not $\Omega^1_C \square_C C$.

It is then sufficient to prove that, for each K-comodule T, the following sequence is exact:

$$0 \to Com_K(T, \Omega^1_K) \to Com_K(T, \Omega^1_C \square_C K) \to Com_K(T, (K \wedge K)/K)$$

The first term is $Coder_k(T, K)$, the second one is (by adjunction) isomorphic to $Com_C(h_K(K,T), \Omega^1_C)$ and then to $Coder_k(T, C)$. The third one may be considered as embedded into $Com_C(T, D)$, so we get

$$0 \to Coder_k(T, K) \to Coder_k(T, C) \to Com_C(T, (K \wedge K)/K) \subseteq Com_C(T, D)$$

Observe that $h_K(K,T)$ exists for all T, because it can be identified with T, considered as K-comodule and then as C-comodule. For a definition of h (in the algebraic context) see for example [6]. Also, for any K-comodule T considered as a C-comodule the image of any C-colinear map from T to a C-comodule M is contained in $M \square_C K$, then $Com_K(T, \Omega^1_C \square_C K) = Com_C(T, \Omega^1_C)$. The exactness of last sequence follows by inspection.

2. In case $K = k.e$, Ω^1_K is zero and so δ_K is a monomorphism. In order to see that δ_K is an epimorphism it is possible to define an explicit splitting, but we notice that the coalgebra $k.e$ is trivially smooth, and so the result is a particular case of the following proposition:

Proposition 3.6 *Let K be a smooth subcoalgebra of a cocommutative coalgebra C. Then the sequence*

$$0 \to \Omega^1_K \to \Omega^1_C \square_C K \to (K \wedge K)/K \to 0$$

is split exact.

Proof: We only have to prove that the map $\Omega^1_C \square_C K \to (K \wedge K)/K$ is split surjective, but this is equivalent to the fact that the induced maps

$$(*)\ Com_K(T, \Omega^1_C \square_C K) \to Com_K(T, (K \wedge K)/K)$$

are surjective for every K-comodule T. The domain is isomorphic to $Com_K(h_K(K,T), \Omega^1_C) \cong Coder_k(T, C)$.

First remark: given a coderivation $\delta : T \to C$ one has that $Im(\delta) \subseteq K \wedge K$.
Second remark: the map $\pi \circ \delta : T \to (K \wedge K)/K$ is K-colinear.
Third remark: since K is a smooth coalgebra, the exact sequence, where D is considered as K-bicomodule with null-structure:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, shift right, "i"'] & C \arrow[l, dashed, shift right, "r"'] \arrow[r, shift right, "\pi"'] & D \arrow[l, dashed, shift right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

The sequence splits; C is then isomorphic (as a vector space) to $D \oplus K$ and r is a coalgebra morphism. If $r' : C \to K$ is another coalgebra morphism, then $r' \circ s : D \to K$ is a K-coderivation.

Given $f : T \to (K \wedge K)/K$ a morphism of K-comodules, we construct the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, dashed] \arrow[d, equal] & E \arrow[r, "p_2"] \arrow[d, "p_1"] & T \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & K \arrow[r] & K \wedge K \arrow[r, "\pi"'] & (K \wedge K)/K \arrow[r] & 0 \end{tikzcd}$$

where E is the pull-back of f along π. The diagram may be completed with the dashed arrow because K is the kernel of the projection on the second factor p_2. Also, as K is smooth and E is provided of a coalgebra structure, $E \cong K \oplus T$. Then we obtain a coderivation $\delta_f : T \to K$, and hence the map $(*)$ is an epimorphism. In particular, choosing $T = (K \wedge K)/K$ and $f = id$, we get the splitting of the sequence. ∎

Corollary 3.7 *In the situation of the proposition, if C is smooth then $(K \wedge K)/K$ is a K-injective comodule.*

Proof: As we have a retraction of $\Omega^1_C \square_C K \to (K \wedge K)/K$ it is enough to see that $\Omega^1_C \square_C K$ is K-injective, but C is smooth, then Ω^1_C is C-injective and so $\Omega^1_C \square_C K$ is K-injective.

Remark that the proof above implies that if Ω^1_C is finitely cogenerated as C-comodule then $(K \wedge K)/K$ is finitely cogenerated as K-comodule. Looking for a moment at the dual situation (an ideal I of an algebra A and the quotient algebra A/I) it is clear after the corollary that the object corresponding to $(K \wedge K)/K$ is I/I^2.

If K corresponds to a maximal ideal then K is local, the above map is an isomorphism and $(K \wedge K)/K$ is a free K-comodule ($(K \wedge K)/K \cong K^n$ for some $n \in \mathbb{N}$). This statement is much less clear that in the case of algebras, because we cannot speak of generators or linear combinations, so it becomes necessary to express it in terms of morphisms. We can dualize, obtaining $((K \wedge K)/K)^* \cong (K^n)^* \cong (K^*)^n$.

Let us first take $n = 1$. Denote by $u : (K \wedge K)/K \to K$ the isomorphism and by f the element $u^*(\epsilon)$ (where u^* is the isomorphism $u^* : K^* \to ((K \wedge K)/K)^*$ obtained as the transpose of u), $f \in ((K \wedge K)/K)^* \cong D^*/(D^*)^2$ (with D as above, remark that D^* is an ideal of C^*). Also $((K \wedge K)/K)^*$ is embedded into $(K \wedge K)^*$ and we look at $f \circ \pi$ as an element of $(K \wedge K)^*$, vanishing over K.

Consider the short sequence

$$(*) \quad 0 \longrightarrow K \longrightarrow C \xrightarrow{(f\circ\pi).} C \longrightarrow 0$$

The composition is clearly zero because K is a subcoalgebra and $K = Ker(\pi)$. Consider also

$$(**) \quad 0 \longrightarrow Ker((f\circ\pi).) \longrightarrow C \xrightarrow{(f\circ\pi).} C \longrightarrow 0$$

In case $(**)$ is k-split (note that in the algebraic case this is not an additional assumption and in the topological case the map $\Delta : C \hookrightarrow C^e$ is always split by means of $\epsilon \otimes 1$), by dualization we obtain the commutative diagram:

$$\begin{tikzcd}
0 & K^* \arrow[l] \arrow[d] & C^* \arrow[l] \arrow[d, equal] & C^* \arrow[l, "(f\circ\pi).^*"'] \arrow[d, equal] & 0 \arrow[l] \\
0 & Ker((f\circ\pi).)^* \arrow[l] & C^* \arrow[l] & C^* \arrow[l, "(f\circ\pi).^*"'] & 0 \arrow[l]
\end{tikzcd}$$

The right horizontal map is a monomorphism (because in this case $D^*/(D^*)^2$ is a C^*/D^*-module of rank one, generated by the class of $(f\circ\pi)$). By the same reasons, we know that both sequences are exact. Then we have that $K = Ker((f\circ\pi).)$ and that the map $(f\circ\pi).: C \to C$ is an epimorphism. This gives the exactness of $(*)$. In this case, we have just obtained a "small" injective resolution of K as C-comodule.

Now we consider again an arbitrary n, and we give an analogue of the Koszul complex for algebras. The following theorem will then allow us to construct a resolution of the coalgebra C as C^e-comodule and this leads to the proof of the coalgebra version of Hochschild - Kostant - Rosenberg theorem (Theorem 7.1):

Theorem 3.8 *Given a local cocommutative smooth coalgebra C (hence every subcoalgebra is local), let us consider $f_1, \dots, f_n \in C^* = Com_C(C, C)$ and $K = \cap_{i=1}^n Ker(f_i.)$. If $\bigcup_{n\in\mathbb{N}}(\Lambda^n K) = C$ then the second statement bellow is a consequence of the first one:*

1. *$(K\wedge K)/K$ is a free K-comodule, finitely cogenerated (i.e., $(K\wedge K)/K \cong K^n \cong \oplus_{i=1}^n K.f_i$, where the f_i's index the copies of K) and $\bigoplus_{n\in\mathbb{N}} \Lambda^{n+1}K/\Lambda^n K$ is isomorphic to the K-free cocommutative coalgebra on $(K\wedge K)/K$, that is $\bigoplus_{n\in\mathbb{N}} \Lambda^{n+1}K/\Lambda^n K \cong K \otimes sh(k^n)$, where $sh(k^n)$ is the graded dual coalgebra of the symmetric algebra $S(k^n)$.*

2. *The sequence $0 \to K \to C \to \oplus_{i=1}^n Cf_i \to \oplus_{i<j} Cf_i \wedge f_j \to \dots \to Cf_1 \wedge \dots \wedge f_n \to 0$ is exact.*

This theorem will be proved in section 4.

4 The associated graded coalgebra and a structure theorem

In this section we give a description of the cocommutative coalgebra C in terms of a shuffle coalgebra (Thm. 4.4 and Prop. 4.5), or a cofree cocomutative coalgebra in Sweedler's terminology. It enables us to finish the proof of Theorem 3.8 by means of an inductive argument.

In case $n = 1$ we have already seen that statement 2. is equivalent to the fact that

$$(2') \quad f. : C \to C \text{ is an epimorphism}$$

In this situation 1. clearly implies both.

Next we want to describe $\oplus_i \Lambda^{i+1}K/\Lambda^i K$ when$(K\wedge K)/K$ is isomorphic to K^n.

Lemma 4.1 *Let C be a coalgebra (in the topological situation we also assume that every epimorphic image of C is a finitely cogenerated C-comodule) and $(K \wedge K)/K$ as above, then the following diagram*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & C \arrow[r, "\pi"] & C/K \arrow[r] \arrow[d, dashed, "\overline{\psi}"] & 0 \\
& (K \wedge K)/K \arrow[urr, hook, "\psi"] \arrow[r, equal] & K^n \arrow[r, hook] & C^n &
\end{tikzcd}$$

may be completed commutatively with a monomorphism of C-comodules $\overline{\psi} : C/K \to C^n$.

Proof: The existence of $\overline{\psi}$ is a consequence of the injectivity of C^n as C-comodule. $Ker(\overline{\psi})$ is a C-subcomodule of C/K. Now consider the diagram

$$\begin{tikzcd}
Ker(\overline{\psi}) \arrow[r, "\rho^+"] \arrow[d, hook, "i"] & Ker(\overline{\psi}) \otimes C \arrow[r, "id\otimes\pi"] & Ker(\overline{\psi}) \otimes C/K \arrow[d, "i\otimes id"] \\
C/K \arrow[r, "\rho^+"] & C/K \otimes C \arrow[r, "id\otimes\pi"] & C/K \otimes C/K
\end{tikzcd}$$

We will prove that $(id \otimes \pi) \circ \rho^+$ is a monomorphism. Then, using Lemma 2.6 we will obtain that $Ker(\overline{\psi}) = 0$. Given $x \in Ker(\overline{\psi})$, suppose that $(id \otimes \pi) \circ \rho^+(x) = 0$. Then $i(x) \in Ker\left((id \otimes \pi) \circ \rho^+_{C/K}\right) = K \wedge K/K$. As a consequence $i(x) \in Ker(\overline{\psi} \circ \psi)$, but $\overline{\psi} \circ \psi$ is a monomorphism and then $x = 0$. ∎

The proof of Theorem 3.8 is inductive, then we consider firstly the case $n = 1$. In this particular case the above lemma says that the map $f. : C \to C$ is defined as the composition of $\pi : C \to C/K$ with the injective extension of the inclusion $K \to C$. Clearly $Ker(f.) = Ker(\pi) = K$.

Then the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & K \wedge K \arrow[r] \arrow[d, equal] & (K \wedge K)/K \arrow[r] \arrow[d, equal, "\sim"] & 0 \\
0 \arrow[r] & K \arrow[r] & K \wedge K \arrow[r, "f."] & K \arrow[r] & 0
\end{tikzcd}$$

As a consequence $f. : \Lambda^2 K \to K$ is an epimorphism. We can now prove:

Proposition 4.2 *If $\bigcup_{n\in\mathbb{N}} \Lambda^n K = C$, then the following statements are equivalent:*

1. *The map $f. : C \to C$ defined in the above paragraph is an epimorphism.*
2. *$gr(f.) : \oplus_{n\in\mathbb{N}} \Lambda^{n+1}K/\Lambda^n K \to \oplus_{n\in\mathbb{N}} \Lambda^{n+1}K/\Lambda^n K$ is an isomorphism.*

Proof: (1) $\Rightarrow$ (2): The fact that $f.$ is an epimorphism if and only if $gr(f.)$ is an epimorphism follows by standard filtration arguments. Next we claim that whenever $Ker(f.) = K$, the map (of degree -1) $gr(f.) : \oplus_{n\in\mathbb{N}} \Lambda^{n+1}K/\Lambda^n K \to \oplus_{n\in\mathbb{N}} \Lambda^{n+1}K/\Lambda^n K$ is a monomorphism: given $x \in \Lambda^{n+1}K$, we claim that $f.x \in \Lambda^n K$. This is true because $x \in \Lambda^{n+1}K \Leftrightarrow \pi^{\otimes n+1}\Delta^n x = 0 \Leftrightarrow f^n.x = 0 \Leftrightarrow f.x \in \Lambda^n K$.

Given $\overline{x} \in \Lambda^{n+1}K/\Lambda^n K$, there is a $\overline{z} \in \Lambda^{n+2}K/\Lambda^{n+1}K$ such that $\overline{f.z} = \overline{x}$. if and only if $f.z - x \in \Lambda^n K$ $\Leftrightarrow f^n.(f.z - x) = 0$. But $f.$ is an epimorphism, then there exists $y \in C$ such that $f.y = x$, and it is clear that $y \in \Lambda^{n+2}K$, so we take $z = y$.

Let us now see (2) $\Rightarrow$ (1): The proof is inductive. Given $x \in C$ we can choose n such that $x \in \Lambda^n K$ and $x \notin \Lambda^{n-1}K$. So $\overline{x} \neq 0$ in $\Lambda^n K/\Lambda^{n-1}K$. Since $gr(f.)$ is an isomorphism, there exists $\overline{y} \in \Lambda^{n+1}K/\Lambda^n K$ such that $gr(f.)(\overline{y}) = \overline{x}$. Then $f.y - x = z \in \Lambda^{n-1}K$. By inductive hypothesis, there exists $z' \in \Lambda^n K$ such that $z = f.z'$. The proof finishes using that $\Lambda^0 K = 0$. ∎

Remark: Inductively, the above proposition provides the equivalence between the following statements:

$$(1)\ f_i.: \overline{K_i} := \cap_{i<j} Ker(f_j.) \to \overline{K_i} \text{ is an epimorphism}$$

$$(2)\ gr(f_i.) : \oplus_{l\geq 0}\Lambda^{l+1}\overline{K}_i/\Lambda^l\overline{K}_i \to \oplus_{l\geq 0}\Lambda^{l+1}\overline{K}_i/\Lambda^l\overline{K}_i \text{ is an isomorphism.}$$

In the algebra case the corresponding statement on $\oplus I^n/I^{n+1}$ follows after noticing that $I = \langle f\rangle$ is not a divisor of 0 because $\{0\}$ is a prime ideal.

Given a smooth local coalgebra C with groupe-like element e, Ω^1_C is C-free. We have that $\cup_n \wedge^n k.e = C$ and the rank of Ω^1_C, by Proposition 3.5 is equal to $dim_k((k.e \wedge k.e)/k.e)$. Now we want to prove that if $\phi \in Com_C(C, C) = C^*$, then $\phi = 0$ or ϕ is an epimorphism (i.e. $\{0\}$ is a prime coideal). Note that in the topological case, we implicitely assume that the topology of C is the inductive limit topology of the system $\{\cup_{n\leq m} \wedge^n K\}_{m\in\mathbb{N}}$. An example of local smooth coalgebra verifying this condition is the coalgebra of distributions over the real line supported at the origin. In this case the Dirac measure δ_0 is the group-like element, a basis is given by $\{\delta_0^{(n)}\}_{n\in\mathbb{N}_0}$ and $\wedge^j(\mathbb{C}.\delta_0) = \langle\delta_0, \delta_0', \ldots, \delta_0^{(j)}\rangle$.Lemma 4.3, theorem 4.4 and proposition 4.5 bellow will achive the proof of the 1-dimensional case.

Lemma 4.3 *If C is local and Ω^1_C contains a subcomodule isomorphic to C, then there exists a coderivation $D : C \to C$ and an element $x \in C^*$ such that $x \circ D = \epsilon$.*

Proof: By hypothesis we identify C with a subcomodule of Ω^1_C, define $\widetilde{D} : C \to C$ as the composition of the inclusion $C \to \Omega^1_C$ with the universal coderivation $d : \Omega^1_C \to C$. In general, if N is a nontrivial subcomodule of Ω^1_C then the inclusion $i_N : N \to \Omega^1_C$ is a non zero morphism, and by the universal property $Com_C(N, \Omega^1_C) \cong Coder(N, C)$ the corresponding coderivation $d \circ i_N$ is not zero. We have then proved that $Ker(d : \Omega^1_C \to C)$ does not contain non trivial subcomodules. In particular $k.e$ is a subcomodule of C and we identify it with a subcomodule of Ω^1_C. By the above discussion, $\widetilde{D}(e) \neq 0$, then there exists $x \in C^*$ such that $x(\widetilde{D}(e)) \neq 0$.

Since C^* is a local algebra and $(k.e)^\perp \cong (C/K)^*$ is a maximal ideal, then $x \circ \widetilde{D}$ is a unit in C^*. So take $u \in C^*$ such that $u * (x \circ \widetilde{D}) = \epsilon$ and define $D := \widetilde{D}(u.-)$. The element $x \in C^*$ and the derivation D have the desired properties. Moreover, since $\epsilon \circ D = 0$ for any coderivation D, we may suppose that $x(e) = 0$, replacing x by $x - x(e).\epsilon$ if necessary. ∎

We remark that as D is a coderivation, the quotient $\widetilde{C} := C/Im(D)$ is also a coalgebra.

Theorem 4.4 *Suppose $char(k) = 0$. Let $sh(k.x)$ denote the shuffle coalgebra on the generator x, i.e. the graded dual of the polynomial ring $k[x]$. In the conditions of the above Lemma, $C \cong \widetilde{C} \otimes sh(k.x)$.*

Proof: With the notations of the above proof, given $t \in C^*$ such that $t(e) = 0$, consider the expression

$$E(D, t)(c) = \sum_{n\geq 0} \frac{D^n}{n!}(t^n.c)$$

Since $C = \bigcup_n \wedge^n(k.e)$, given $c \in C$ there exists $n_0 \in \mathbb{N}$ such that $c \in \Lambda^n(k.e)$ for all $n \geq n_0$. Then $t^n.c = 0$ for $n \geq n_0$ and the sum is finite, so that the map $E(D, t) : C \to C$ is well defined.

Let us take $E : C \to C$ defined by $E(c) := E(D, -x)(c) = \sum_{n\geq 0} \frac{(-1)^n D^n(x^n.c)}{n!}$. It is easy but tedious to see that $Im(E) = Ker(x.)$ (one inclusion is obvious, if $c \in Ker(x.)$ then $c = E(c)$, for the other inclusion one must use a commutation formula for $x.$ and D^n).

Next we define a map $\phi : C \to \widetilde{C} \otimes sh(k.x)$ by $\phi(c) = \sum_{n\geq 0} \overline{x^n.c} \otimes x^n$ (it is a finite sum). Its inverse is given by $\psi : \widetilde{C} \otimes sh(k.x) \to C$, $\psi(\sum_{n\geq 0} \overline{c_n} \otimes x^n) = \sum_{n\geq 0} \frac{D^n}{n!} E(c_n)$. In order to see that both are coalgebra maps, it is necessary to proceed by cases, but no difficulty arises. When verifying that both compositions are the identity, one should use the following facts:

1. $\sum_{i=0}^{2k} \frac{(-1)^i}{(2k-i)!i!} = 0$

2. $x. \circ D = D \circ x. + \epsilon$

3. $E(c) - c \in Im(D)$

4. $\phi\left(\frac{D^n E(c)}{n!}\right) = \overline{c} \otimes x^n$

5. $c = \sum_{n \geq 0} \frac{D^n E(x^n.c)}{n!}$

∎

Remarks: 1. Taking into account the above isomorphisms, the map $x.$ acts on $\widetilde{C} \otimes sh(k.x)$ in the following way: it is the identity on the first coordinate and it acts by $x^n \mapsto x^{n-1}$ $(n \geq 1)$ on the second one. As a consequence, it is an epimorphism.
2. The element x of the above result may be chosen as f. In order to do so, it is sufficient to prove that $f \circ D$ is a unit in C^*. But $f \circ D$ is a unit if and only if $f(D(e)) \neq 0$. Now, $f.D(e) = (f \circ D).e + D(f.e)$ as e is group-like. But $f(e) = 0$ then $f.e = 0$, so $f.D(e) = (f \circ D).e = f(D(e))e$. We want to prove then that $f(D(e)) \neq 0$. This statement is true if and only if $D(e) \neq \lambda.e$ for any $\lambda \in k$. But if $D(e) = \lambda.e$, then $\lambda = \epsilon(D(e)) = 0$, and this is impossible because $x(D(e)) = 1$.

Proposition 4.5 *In the above proposition, $\widetilde{C}$ is identified with $k.e$.*

Proof: Since $Im(D) \subset Ker(\epsilon)$, we have an obvious surjective map $\widetilde{C} = C/Im(D) \to C/Ker(\epsilon) \cong k.e$. Also, there is a map form $k.e$ to $\widetilde{C}$ sending e to $\overline{e}$. It is not null because $E(D, -f) : C \to C$ is a coalgebra morphism with image equal to $Ker(f.) = k.e$. We have that $\overline{e} \neq 0$ because $\phi(e) = \overline{e} \otimes 1 + \overline{f.e} \otimes x + \ldots = \overline{e} \otimes 1 + 0$, and ϕ is an isomorphism.

The composition $k.e \to \widetilde{C} \to k.e$ is the identity. Consider the other composition:

$$\overline{c} \to \epsilon(c).e \mapsto \overline{\epsilon(c).e} = \overline{E(c)} = \overline{c}$$

where the equality $\overline{\epsilon(c).e} = \overline{E(c)}$ follows using $E(c) = (\epsilon \otimes 1)\Delta(E(c))$. ∎

The description of C thus obtained can now be used to give a description of $gr(C) = \oplus_{n \geq 0} \Lambda^{n+1} K / \Lambda^n K$. We look at the subcomodule $\Lambda^i(k.e)$ in $\widetilde{C} \otimes sh(k.x)$ via the isomorphism ϕ. For example when $i = 2$, $(k.e) \wedge (k.e) = Ker(\pi \otimes \pi) = k.e \otimes k.1 \oplus k.e \otimes k.x \cong k.e \oplus (k.e \wedge k.e)/k.e$ and in general $\Lambda^{i+1} K = (k.e \otimes k.1) \oplus (k.e \otimes k.x) \oplus ... \oplus (k.e \otimes k.x^i)$. Then $gr(C) = \oplus_{i \geq 0} (k.e \otimes k.x^i) = k.e \otimes (\oplus_{i \geq 0} k.x^i)$.

In a subsequent paper we shall treat the case of arbitrary characteristic. The advantage of the proof given here for $char(k) = 0$ is that it is completely explicit.

We are now able to treat the case $dim_k((K \wedge K)/K) = n$ and finish the proof of Theorem 3.8. Denote by $\{f_1, ..., f_n\}$ a basis of $((K \wedge K)/K)^*$. As we have $K^n \cong (K \wedge K)/K \subseteq C/K$, restriction gives a surjection $K^\perp \cong (C/K)^* \to ((K \wedge K)/K)^*$, so we will consider $\{f_1, \ldots, f_n\}$ as elements of $K^\perp \subset C^*$. In this situation, define $\overline{C}$ as the C-subcoalgebra $\overline{C} := Ker(f_1.)$ and $\overline{K} = Ker(f_1.) \cap K$.

Lemma 4.6 *The coalgebra $\overline{C}$ is smooth.*

Proof: We recall that $Im(E) = Im(E(D_1, -f_1)) = Ker(f_1.) = \overline{C}$ and that $E|_{Ker(f_1.)} = id$, then the smoothness of $\overline{C}$ is a consequence of Lemma 2.9. ∎

Remark: $\overline{C}$ being a subcoalgebra of the local coalgebra C, is also a local coalgebra.

Considering now the coalgebra $\overline{C} = Ker(f_1.)$, then the rank of $\Omega^1_{\overline{C}} = dim_k(\overline{K} \wedge_{\overline{C}} \overline{K}/\overline{K}) \leq n - 1$ and, by an inductive argument, $gr(\overline{C}) = \oplus_{i \geq 0} \Lambda^{i+1} \overline{K} / \Lambda^i \overline{K} \cong k.e \otimes \left(\underset{i \geq 0}{\oplus} k[x_2, ..., x_n]_i \right)$ where $k[x_2, ..., x_n]_i$ is the i^{th}-component of the shuffle coalgebra on $n-2$ generators; in fact $\overline{C} = k.e \otimes sh(k.x_2 \oplus ...k.x_n)$ and $(\overline{K} \wedge \overline{K})/\overline{K} \cong \oplus_{i=2}^{n} k.f_i$.

Next, we shall make use of the following bicomplex:

$$\begin{tikzcd}
 & 0 & 0 & 0 & & 0 & \\
0 \arrow[r] & K \arrow[u] \arrow[r] & C \arrow[u] \arrow[r] & \oplus_{i=2}^{n} C.f_i \arrow[u] \arrow[r] & \cdots \arrow[r] & C.(f_2 \wedge ... \wedge f_n) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[u, "f_1\cdot"] \arrow[r] & C \arrow[u, "-f_1\cdot"] \arrow[r] & \oplus_{i=2}^{n} C.f_i \arrow[u, "f_1\cdot"] \arrow[r] & \cdots \arrow[r] & C.(f_2 \wedge ... \wedge f_n) \arrow[u, "\pm f_1\cdot"] \arrow[r] & 0 \\
0 \arrow[r] & \overline{K} \arrow[u, "j'"] \arrow[r, "j"] & \overline{C} \arrow[u] \arrow[r] & \oplus_{i=2}^{n} \overline{C}.f_i \arrow[u] \arrow[r] & \cdots \arrow[r] & \overline{C}.(f_2 \wedge ... \wedge f_n) \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

By hypothesis, the columns are acyclic (considering the case $n = 1$), and j induces a quasi-isomorphism between the complexes $(0 \to \overline{K} \to 0 \to ...)$ and $(0 \to \overline{C} \to \oplus_{i=2}^{n} \overline{C}.f_i \to ... \to \overline{C}(f_2 \wedge ... \wedge f_n) \to 0)$, which is, in turn, quasi-isomorphic to the total complex of the bicomplex inside the dotted area.

So we obtain that the sequence

$$0 \to \overline{K} \to C \to \oplus_{i=1}^{n} C.f_i \to \oplus_{i<j} C(f_i \wedge f_j) \to \ldots \to C.(f_1 \wedge \ldots \wedge f_n) \to 0$$

is exact, and this proves Theorem 3.8. ∎

5 Ω^*_C and $Hoch^*(C)$

We recall from [6, 8] that given an algebraic k-coalgebra C there exist two cohomology theories associated to it, denoted $Hoch^*(C)$ and $H^*(C)$. The first is the derived functor of $-\square_{C^e} C$, and the other one is the derived functor of $Com_{C^e}(-, C)$. In the topological case, they are defined as the cohomology groups of the complexes corresponding to the canonical resolution of C as C^e-comodule, for example $Hoch^*(C)$ is the cohomology of the complex

$$0 \longrightarrow C \xrightarrow{b_0} C\widetilde{\otimes}C \xrightarrow{b_1} C\widetilde{\otimes}C\widetilde{\otimes}C \longrightarrow \cdots$$

with $b_0 = \Delta - \sigma_{12}\Delta$, $b_1 = \Delta \otimes id - id \otimes \Delta + \sigma_{132}(\Delta \otimes id)$ and in general $b_n = \sum_{i=1^n}(-1)^{i-1}\Delta_i + (-1)^n \sigma_{1,n,n-1,...,3,2}\Delta_1$, where $\Delta_i = id_{C^{\otimes i-1}} \otimes \Delta \otimes id_{C^{\otimes n-i-1}}$ and σ_{132}, $\sigma_{1,n,n-1,...,3,2}$ denote the cyclic permutations (132) and $(1, n, n-1, ..., 3, 2)$ respectively.

However, Taylor ([19], section 4) has shown that in certain cases (for example for nuclear and Fréchet algebras), the topological version of Hochschild homology behaves similarly to algebraic Hochschild homology. The same holds for coalgebras, namely:

Proposition 5.1 *Given a coalgebra C and an injective resolution of C as C^e-comodule $0 \to C \to X_0 \to X_1 \to X_2 \to ...$, if*

1. *C and X_i ($i \in \mathbb{N}_0$) are nuclear Fréchet spaces and M is Fréchet; or*

2. *C and X_i ($i \in \mathbb{N}_0$) are nuclear complete DF spaces and M is a complete DF space,*

the cohomology of the complex $X_*\square_{C^e}M$ is isomorphic to $Hoch^*(M,C)$ (here $\square_{C^e}$ is defined like in the purely algebraic case, using $\widetilde{\otimes}$ instead of $\otimes$).

We shall need the following Lemma, analogous to Proposition 2.7 of [19]:

Lemma 5.2 *Let $\mathfrak{M}$ be a class of (topological) C-bicomodules containing $C\widetilde{\otimes}M\widetilde{\otimes}C$ and the cokernel of $\rho_M : M \to C\widetilde{\otimes}M\widetilde{\otimes}C$ whenever $M \in \mathfrak{M}$. Let K_p be a sequence of covariant functors from $\mathfrak{M}$ into vector spaces such that*

1. *$K_0(M) = Hoch^0(M,C)$ for all $M \in \mathfrak{M}$.*
2. *$K_p(C\widetilde{\otimes}M\widetilde{\otimes}C) = 0$ for all $p > 0$ and $M \in \mathfrak{M}$.*
3. *Each short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ ($M_i \in \mathfrak{M}$) induces maps $\delta_p : K_p(M_3) \to K_{p+1}(M_1)$ such that*
$$.. \to K_p(M_3) \to^{\delta} K_{p+1}(M_1) \to K_{p+1}(M_2) \to K_{p+1}(M_3) \to ...$$
is exact.
Then $K_p(M) \cong Hoch^p(M,C)$ for all $M \in \mathfrak{M}$.

Proof of the proposition: We shall see that the conditions of Lemma 5.2 are fulfilled. Take $\mathfrak{M}$ as the collection of Fréchet C-bicomodules and, with the notations of the above Lemma, $K_p(M) := H_p(X_*\square_{C^e}M)$ where $0 \to C \to X_*$ is a relatively injective $\mathbb{C}$-split resolution of C as C^e-comodule and each X_i is a nuclear Fréchet relative injective C^e-comodule (for example, take the standard resolution).

If $M \in \mathfrak{M}$, as C is Fréchet and nuclear, then $C\widetilde{\otimes}M\widetilde{\otimes}C \in \mathfrak{M}$ as well (see for example [12]). Concerning $Coker(M \to C\widetilde{\otimes}M\widetilde{\otimes}C)$, it is a quotient of a Fréchet space by a closed subspace, then it is Fréchet.

Also if $Y \in \mathfrak{M}$, the following sequence is exact,
$$0 \to C\widetilde{\otimes}Y \to X_0\widetilde{\otimes}Y \to X_1\widetilde{\otimes}Y \to ...$$
since C and the X_i's are nuclear Fréchet spaces, then $X\square_{C^e}(C\widetilde{\otimes}M\widetilde{\otimes}C) \cong X\widetilde{\otimes}M$ so that $K_p(C\widetilde{\otimes}M\widetilde{\otimes}C) = 0$.

It follows by diagram chasing that $Hoch^0(M) = K_0(M)$. Finally, as X_i is a relative injective C-bicomodule, it is a direct sumand of $C^e\widetilde{\otimes}V_i$ for some topological vector space V_i, and so given a $\mathbb{C}$-split exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ the sequence $0 \to X_i\square_{C^e}M_1 \to X_i\square_{C^e}M_2 \to X_i\square_{C^e}M_3 \to 0$ is exact too. The desired long exact sequence is the cohomology long exact sequence associated to the last short exact sequence of complexes.

Next we want to show that Ω^1_C is isomorphic to $Hoch^1(C)$, turning then the latter into an alternative universal object for coderivations when the coalgebra C is not cocommutative.

Proposition 5.3 *If C is a cocommutative coalgebra, then $Hoch^1(C) \cong \Omega^1_C$.*

Proof: We first define $\gamma : \Omega^1_C \to Hoch^1(C)$ by $\gamma([z]) := z - (\Delta\otimes\epsilon)(z)$, where $[z] \in \Omega^1_C$ is the class of some $z \in C\widetilde{\otimes}C$. The map γ is well defined because if $z = \Delta(c)$, then $\Delta(c) - (\Delta\otimes\epsilon)(\Delta(c)) = \Delta(c) - \Delta(c) = 0$. On the other hand, we can see that $Im(\gamma) \subseteq Hoch^1(C) = Ker(b_1)$ as follows:

Let $[z] \in \Omega^1_C$ and $\pi : C\otimes C \to L_C$ the canonical projection, then $0 = \rho^-([z]) - \sigma_{312}\rho^+([z]) = (id\otimes\pi)(\Delta\otimes id)(z) - (id\otimes\pi)\sigma_{312}(id\otimes\Delta)(z)$, so $(\Delta\otimes id)(z) - \sigma_{312}(id\otimes\Delta)(z) \in Im(id\otimes\Delta)$, and $(\Delta\otimes id)(z) - \sigma_{312}(id\otimes\Delta)(z) = (id\otimes\Delta)(id\otimes id\otimes\epsilon)(\Delta\otimes id)(z) - \sigma_{312}(id\otimes\Delta)(z) = (id\otimes\Delta)(\Delta\otimes\epsilon)(z) - (id\otimes\Delta)\sigma_{12}(z)$.

Computing $b_1(z - (\Delta\otimes\epsilon)(z))$ we obtain $(\Delta\otimes id)(z) - (id\otimes\Delta)(z) + \sigma_{132}(\Delta\otimes id)(z) - (\Delta\otimes id)(\Delta\otimes\epsilon)(z)$.

The above equation says, via the permutation $a\otimes b\otimes c \mapsto b\otimes c\otimes a$ that this expression is zero.

The inverse of γ is given by $\gamma' : Hoch^1(C) \to \Omega^1_C$ ($z \mapsto [z]$). Checking that $Im(\gamma') \in \Omega^1_C$, we immediately obtain that they are inverses. ∎

Example: Let $k = \overline{k}$, $char(k) = 0$, $A = k[x]$ and $C = A^0 \cong \oplus_{\lambda\in k}k[s]e^{\lambda s} \subset k[|s|]$.

Identifying $k[x]\otimes k[x]\cong k[x,y]$ in the standard way, then $I=Ker(m:k[x]\otimes k[x]\to k[x])$ is corresponds to $<x-y>$, so $I/I^2=k[x].(x-y)$.

By definition, $\Omega^1_{A^0}=Sym(A^0\otimes A^0/Im(\Delta))\subseteq\frac{k[s,e^{\lambda s}]_{\lambda\in k}\otimes k[t,e^{\lambda t}]_{\lambda\in k}}{Im(\Delta)}$. We show that $\Omega^1_{A^0}\cong k[s,e^{\lambda s}]_{\lambda\in k}$ defining $\widetilde{f}:k[s,t,e^{\lambda s},e^{\mu t}]_{\lambda,\mu\in k}\to k[s,e^{\lambda s}]_{\lambda\in k}$ by

$$\begin{cases}\widetilde{f}(1)=0\\ \widetilde{f}(s^n)=ns^{n-1} & \text{if } n>0\\ \widetilde{f}(t)=-1\\ \widetilde{f}(s^nt)=-s^n & \text{if } n>0\\ \widetilde{f}(s^nt^m)=0 & \text{if } n\geq 0,\ m>1\\ \widetilde{f}(e^{\lambda s})=\lambda e^{\lambda s}\\ \widetilde{f}(e^{\mu .t})=-1\end{cases}$$

$\widetilde{f}$ derives elements depending on s, and so $Im(\widetilde{f})$ contains all polynomials and exponentials in s. Also, as $(s+t)^n=\sum_{k=0}^n\binom{n}{k}s^{n-k}t^k$, $\widetilde{f}((s+t)^n)=0$, so $\widetilde{f}$ vanishes on $Im(\Delta)$ and we have a well defined map $f:\frac{k[s,t,e^{\lambda s},e^{\mu t}]_{\lambda,\mu\in k}}{Im(\Delta)}\to k[s,e^{\lambda s}]_{\lambda\in k}$. Restricting f to the cosymmetric elements $s(s+t)^n$ with $n\geq 0$, we have $f(s(s+t)^n)=f(\left(\sum_{k=0}^n\binom{n}{k}s^{n+1-k}t^k\right)=f(s^{n+1}+ns^nt+o(t^2))=s^n$. Also $f(s.e^{\lambda(s+t)})=e^{\lambda s}$, then polynomials divisible by s and exponentials belong to $Im(f)$ and we obtain the result describing $\Omega^1_{A^0}$.

In this case, $\Omega^1_{A^0}=\left(\Omega^1(A|k)\right)^0$, where on the right side, $(-)^0$ means the A^0-comodule with the universal property:

$$Hom_A(\Omega^1(A|k),X^*)\cong Com_{A^0}(X,(\Omega^1(A|k))^0)\quad (X \text{ a } C\text{-comodule})$$

As a consequence of Proposition 5.1 and Proposition 3.2, $Hoch^1(C)$ always localizes.

The category of topological C-bicomodules is not an abelian category, so we cannot use the cohomological machinery to prove that $Hoch^*(C)$ localizes.

Despite, we are able to construct long exact sequences for $Hoch^*$, but as we shall see, this argument cannot be used because we don't know if localization is exact and we don't even have a general argument to assure that localization of Fréchet (or DF-spaces) are Fréchet (resp. DF-spaces) except in particular situations.

We shall next make explicit the isomorphisms between $Hoch^1(C_{[S]})$ and $Hoch^1(C)_{[S]}$ in order to detect the obstruction to localization for $n>1$.

We first define $j:Hoch^1(C_{[S]})\to Hoch^1(C)_{[S]}$ $(\{x_s\otimes y_t\}_{s,t\in S}\mapsto\{x_s\otimes y_1\}_{s\in S})$ and $\mu:Hoch^1(C)_{[S]}\to Hoch^1(C_{[S]})$ $(\{x_s\otimes y\}_{s\in S}\mapsto\{x_{rs}\otimes y+\Delta(x_{rs^2}s(y))\}_{r,s\in S})$. We will see through the computations, that μ is well defined. The composition $j\circ\mu$ gives:

$$(j\circ\mu)\left(\{x_s\otimes y\}_{s\in S}\right)=j\left(\{x_{rs}\otimes y+\Delta(x_{rs^2})s(y)\}_{r,s\in S}\right)=\{x_r\otimes y+\Delta(x_r)\epsilon(y)\}_{r\in S}$$

Since $x_r\otimes y_r\in Hoch^1(C)$, $x_{r,1}\otimes x_{r,2}\otimes y-x_r\otimes y_1\otimes y_2+x_{r,2}\otimes y\otimes x_{r.1}=0$ so $\Delta(x^r)\epsilon(y)=0$, implying $j\circ\mu=id_{Hoch^1(C)_{[S]}}$.

The composition $\mu\circ j$ gives:

$$(\mu\circ j)\left(\{x_s\otimes y_t\}_{s,t\in S}\right)=\mu\left(\{x_s\otimes y_1\}_{s\in S}\right)=\{x_{rs}\otimes y_1+\Delta(x_{rs^2})s(y_1)\}_{r,s\in S}$$

In order to see that $\{x_s\otimes y_t\}_{s,t\in S}$ coincides with the last expression, we embed $C_{[S]}$ in $C''_{[S]}=((C')_S)'$ and evaluate both expressions on elements of $A_S=C'_S$. Then:

$$\{x_{rs}\otimes y_1+\Delta(x_{rs^2})s(y_1)\}_{r,s\in S}(\frac{a}{t}\otimes\frac{b}{u})=x_{tu}(a)y_1(b)+x_{tu^2}(ab)s(y_1)$$

Since $\frac{a}{t}\otimes\frac{b}{u}-\frac{a}{tu}\otimes\frac{b}{1}+\frac{ab}{tu^2}\otimes\frac{u}{1}=0$ in $HH_1(A_S)$, then:

$\{x_s \otimes y_t\}_{s,t\in S}(\frac{a}{t} \otimes \frac{b}{u}) = \{x_s \otimes y_t\}_{s,t\in S}(\frac{a}{tu} \otimes \frac{b}{1} - \frac{ab}{tu^2} \otimes \frac{u}{1}) = x_{tu}(a)y_1(b) + x_{tu^2}(ab)s(y_1)$ as we wanted to show. In particular $Im(\mu) \subseteq Hoch^1(C_{[S]})$ and then it follows that μ is well defined.

We remark that we have used that the map $C \to C''$ is injective, which is true if and only if C is a Hausdorff space. We have also used that $Hoch^1(C)$ is Hausdorff, but this is true when C is so, because $Hoch^1(C)$ is a kernel in $C\widetilde{\otimes}C$ and so a (closed) subspace of a Hausdorff space. In higher degrees, $Hoch^n(C)$ need not be Hausdorff. This is the obstruction not allowing the use of the long exact sequence argument.

6 The exterior coalgebra on Ω^1_C and higher degrees of $Hoch^*$

In this section we construct the "exterior coalgebra" on Ω^1_C, considering the particular case of C being the dual of a topological algebra A. We analyze in detail the example $C = \mathcal{D}(X)$ (distributions on a compact smooth manifold). Its n-th component of the exterior coalgebra is isomorphic to $Hoch^n(C)$. As a consequence, it localizes in a more general situation than the case studied in [9]. Moreover, the isomorphism between $Hoch^n(C)$ and Ω^n_C in this case suggested us to compare both objects for any $n \in \mathbb{N}$, for arbitrary coalgebras.

Le C be a topological coalgebra and M a topological bicomodule. The construction of the tensor coalgebra T_kM of M over k in a purely algebraic context is carried out in detail in [16]. Let us define T_CM as the space $\oplus_{n\in\mathbb{N}_0} M^{\square_C n}$, where $M^{\square_C 0} = C$, $M^{\square_C 1} = M$ and $M^{\square_C (n+1)} = M\square_C(M^{\square_C n})$. The coproduct in T_CM is obtained as the transpose of the product in the tensor algebra. Note that the coalgebra map is defined like in the "algebraic" T_kM over the elementary tensors, extended by linearity and next by continuity to the completion, and finally restricted to T_CM which is a subobject of the "topological" T_kM.

Let C be cocommutative. If M is a $\mathbb{Z}_2$-graded comodule, then so is T_CM. For M a cosymmetric C-bicomodule, we consider it as graded with $deg(m) = 1\ \forall m \in M$, and let Λ_CM be the **biggest graded-cocommutative subcoalgebra** of T_CM. It is characterized by a similar universal property with respect to graded-cocommutative coalgebras which are (graded)cosymmetric C-bicomodules (with $deg(c) = 0$, $\forall c \in C$).

Remark: If $C = A^0$, then $\Lambda_C\Omega^1_C \cong (\Lambda_A\Omega^1(A))^0 = \Omega^*(A)^0$, as C-comodules.

Let $(\mathcal{C}^*(C), b)$ be the standard complex whose homology is used to compute $Hoch^*(C)$. There is a comultiplication Δ in $\mathcal{C}^*(C)$ obtained dualizing the construction for algebras.

Then $\mathcal{C}^*(C)$ is a differential graded coalgebra and hence we have got a coalgebra structure in cohomology. It turns out to be graded-cocommutative because $\mathcal{C}^*(C)$ is graded-cocommutative.

The existence of a graded-cocommutative coalgebra structure in $Hoch^*(C)$ implies that the map $Hoch^*(C) \to Hoch^1(C) \cong \Omega^1_C$ lifts to a coalgebra map $Hoch^*(C) \to \Lambda_C\Omega^1_C$.

Next we shall study the behaviour of $\Omega^n(A)$ and $\Omega^n_{A'}$ for a commutative topological algebra A. Let M be a symmetric A-bimodule; even in the topological case, it is a fact that the tensor algebra $T^*_A(M)$ localizes, i.e. if $S \subset A$ is a multiplicative set, then $T^*_{A_S}(M_S) = (T^*_A(M))_S$.

This isomorphism induces an isomorphism between the quotients, so $(\Lambda^*_A(M))_S \cong \Lambda^*_{A_S}(M_S)$. In particular, for $M = \Omega^1(A)$ we have $\Omega^1(A)_S = \Omega^1(A_S)$ and so $\Omega^*(A)_S = \Omega^*(A_S)$.

Denoting by C the continuous dual coalgebra A' and by $\Lambda^*_C(M')$ the graded cosymmetric coalgebra, the fact that $\Lambda^n_C(M') = (\Lambda^n_A(M))'$ is deduced by checking that $(\Lambda^n_A(M))'$ satisfies the corresponding universal property:

$$Hom_{gcCoalg}(X, \Lambda^*_A(M)') = Hom_{gcAlg}(\Lambda^*_A(M), X') = Hom_A(M, X') = Com_C(X, M') = Hom_{gcCoalg}(X, \Lambda^*_C(M'))$$

where $gcCoalg$ = graded cocommutative coalgebras and $gcAlg$ = graded commutative algebras. Taking $M = \Omega^1(A)$, we already know that $\Omega^1_C = (\Omega^1(A))'$. Concerning localization, we have that

$$(\Omega^*_C)_{[S]} = (\Omega^*(A))'_{[S]} = (\Omega^*(A)_S)' = (\Omega^*(A_S)) = \Omega^*_{C_{[S]}}$$

The main example of this situation to be considered in this work is the following:

Let X be a compact smooth manifold, $A = C^\infty(X)$ and $C = A' = \mathcal{D}(X)$. Choose an open covering $X = \cup_{i=1}^n U_i$ consisting of sets U_i homeomorphic to open balls and let $\{\phi_i\}_{i=1,...,n}$ be a partition of unity subordinated to the covering $\{U_i\}_{i=1,...,n}$. Consider $S_i = \{f \in C^\infty(X) \ / \ f(x) \neq 0, \ \forall x \in U_i\}$, then A_{Si} is the topological algebra $C^\infty(U_i)$ (see [15]). The canonical morphism $A \to \oplus_{i=1}^n A_{Si}$ is in this case not only a monomorphism but also a section, with left inverse $A_{Si} \to A$ given by $f \mapsto f.\phi_i$. Consider the following exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \bigoplus_{i=1}^{n} A_{Si} \arrow[r] \arrow[d, equal] & \bigoplus_{i<j} (A_{Si})_{S_j} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & A_{S_1....S_n} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \bigoplus_{i=1}^{n} A_{Si} \arrow[r] \arrow[d, equal] & \bigoplus_{i<j} A_{S_i.S_j} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & A_{S_1....S_n} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C^\infty(X) \arrow[r] & \bigoplus_{i=1}^{n} C^\infty(U_i) \arrow[r] & \bigoplus_{i<j} C^\infty(U_i \cap U_j) \arrow[r] & \cdots \arrow[r] & C^\infty(U_1 \cap ... \cap U_n) \arrow[r] & 0
\end{tikzcd}$$

which is split by an A-linear homotopy s. There is an analogous diagram for any A-module M. In particular, taking $M = \Omega^1(A)$, we obtain a (split) exact sequence

$$0 \to \Omega^1(X) \to \bigoplus_{i=1}^{n} \Omega^1(U_i) \to \bigoplus_{i<j} \Omega^1(U_i \cap U_j) \to ...... \to \Omega^1(U_1 \cap ... \cap U_n) \to 0 \quad (*)$$

where $\Omega^1(U_i)$ denotes $\Omega^1(C^\infty(U_i)) = \Omega^1(C^\infty(X)_{Si}) = (\Omega^1(C^\infty(X))_{Si}$. The $C^\infty(U_i)$-module $\Omega^1(U_i)$ is free with basis $\{dx_i^1, ..., dx_i^n\}$ where x_i^j is the j-th coordinate with respect to the local chart associated to U_i.

In this local case, all definitions of Ω^1 (sections of the cotangent bundle, I/I^2, universal object for derivations) are coincident (to prove it, consider the universal property of each one of this objects). Moreover, they all localize, so the exact sequence $(*)$ gives that all different definitions of $\Omega^1(X)$ coincide.

As $\Omega^n(A_S) = \Lambda^n_{A_S}\Omega^1(A_S) = \Lambda^n_{A_S}\Omega^1(A)_S = (\Lambda^n_A \Omega^1(A))_S$, the n-th. component of Ω^* also localizes.

Following Connes' computations [4], a $\mathbb{C}$-split projective resolution $\{E_i\}_{i\in\mathbb{N}}$ of $C^\infty(X)$ as $C^\infty(X \times X)$-module is obtained by taking $E_i =$ pull-back of $\Omega^i(X)$ over the second projection. Tensoring this resolution by $A \otimes_{A^e} -$, the complex calculating Hochschild homology (its topological version) has $\Omega^i(X)$ in degree i and the differential is null, so $HH_n(A) = \Omega^n(A)$. The resolution being $\mathbb{C}$-split, the dual complex of $\mathcal{D}(X \times X)$-injective comodules is also exact and $\mathbb{C}$-split. The complex obtained by cotensoring with C over C^e calculates then $Hoch^*(C)$ and has Ω^i_C in degree i because if P is a finitely generated projective A^e-module then P' is a C^e-injective finitely cogenerated comodule and $(A \otimes_{A^e} P)' = C\square_{C^e} P'$. Then $C\square_{C^e} E'_n = (\Omega^n(A))' = \Omega^n_C$. We conclude that in this case $Hoch^n(C) = \Omega^n_C$.

Also, if M is a $\mathcal{D}(U_i)$-bicomodule (for some i, $1 \le i \le n$) we can compare the cohomology $Hoch^*(\mathcal{D}(U_i), M)$ with $Hoch^*(\mathcal{D}(X), M)_{[Si]}$. In a more general setting:

Proposition 6.1 *Given a continuous map of nuclear Fréchet (resp. DF) coalgebras $f : D \to C$ such that*

- $D\square_C D \cong D$, *and*
- $Hoch^i(D\widetilde{\otimes}D, C) = 0 \quad \forall i > 0$,

we have $Hoch^i(M, C) = Hoch^i(M, D)$ for all $i \in \mathbb{N}_0$ and arbitrary Fréchet (resp. DF) D-bicomodule M.

As a corollary, $Hoch^*(M, \mathcal{D}(U_i)) = Hoch^*(M, \mathcal{D}(X))$ for all $\mathcal{D}(U_i)$-bicomodule M, taking $C = \mathcal{D}(X)$ and $D = \mathcal{D}(U_i)$.

When $D = C_{[S]}$ for a multiplicative subset $S \subset Z(C')$, the first condition of the above proposition is verified because localization is an "idempotent" functor, while the second condition is satisfied whenever one can prove that localization is exact.

Proof: It is a consequence of Lemma 5.2, taking $K_p(M) := Hoch^p(M,C)$ because

$$Hoch^0(M,C) = C\square_{C^e} M = C\square_{C^e} D^e \square_{D^e} M \cong (D\square_C C\square_C D)\square_{D^e} M \cong (D\square_C D)\square_{D^e} M \cong D\square^e_D M = Hoch^0(M,D)$$

Also $Hoch^p(D\widetilde{\otimes}M\widetilde{\otimes}D,C) = 0$ because $D\widetilde{\otimes}D$ is C-coflat. The condition concerning long exact sequences is also satisfied. ∎

7 Proof of the main Theorem

The aim of this section is to calculate $Hoch^n(C)$ for a smooth coalgebra C. Since we make use of simplified resolutions, we consider two kind of situations.

Theorem 7.1 *Suppose $char(k) = 0$. If C is a cocommutative coalgebra satisfying either one of the hypothesis below:*

- *C is a smooth algebraic coalgebra and $ke_i \wedge k.e_i$ is finite dimensional for every group-like $e_i \in C$.*
- *C is a smooth local topological coalgebra provided of a topology verifying Proposition 5.1 (for $M = C$), $C = \cup_{n\in\mathbb{N}_0} \wedge^n (k.e)$ (e being the unique group-like of C) and $k.e \wedge k.e$ is finite dimensional.*

then $Hoch^*(C)$ is isomorphic to the exterior coalgebra on Ω^1_C.

Proof: The following argument allows us, when considering an arbitrary cocommutative smooth coalgebra C, to reduce the problem to the local case. It works for "algebraic" coalgebras, but it does not work for topological coalgebras.

Given then an algebraic cocommutative smooth coalgebra C, it is well known ([16], Theorem 8.0.5, p.163) that $C = \oplus_{i\in I} C_i$, where I indexes the set of irreducible subcoalgebras of C. As C is a coalgebra over an algebraically closed field k, each C_i contains at least a group-like element e_i, and it cannot contain two of them due to the irreducibility of C_i. Then each C_i is local.

Let us denote by Λ a set indexing the finite subsets of I and by $\{I_\alpha\}_{\alpha\in\Lambda}$ the set of all finite subsets of I. Then $C = \underset{\overrightarrow{\Lambda}}{lim} \oplus_{i\in I_\alpha} C_i$, and since $Hoch^*$ commutes with direct limits, assuming the theorem for the local case,

$$Hoch^*(C) = \underset{\overrightarrow{\Lambda}}{lim}\; Hoch^*\left(\oplus_{i\in I_\alpha} C_i\right) \cong \underset{\overrightarrow{\Lambda}}{lim} \bigoplus_{i\in I_\alpha} Hoch^*(C_i) \cong$$

$$\cong \bigoplus_{i\in I} Hoch^*(C_i) \cong \bigoplus_{i\in I} \Lambda^*(\Omega^1_{C_i}) \cong \Lambda^*\left(\oplus_{i\in I}\Omega^1_{C_i}\right) \cong \Lambda^*(\Omega^1_C)$$

where the second equality follows from the Mayer-Vietoris property of $Hoch^*$ (see [10]).

In the local smooth case (for both situations), the existence of a simple resolution makes the computation of $Hoch^n(C)$ easy. We know that $Hoch^n(C) = Cotor^n_{C^e}(C,C)$ and that C^e is a smooth local coalgebra (see Lemma 2.8). Let $\widetilde{e}$ be the unique group-like element in C^e (in fact $\widetilde{e} = e\otimes e = \Delta(e)$, where e is the unique group-like element of C). The map $\Delta : C \to C^e$ is a monomorphism of coalgebras and, since C and C^e are smooth, then $C \cong \Delta(C) = \cap_{i=1}^n Ker(f_i.)$, $\cup_j \Lambda^j C = C^e$, for a certain "regular sequence" $\{f_1, ..., f_n\}$ in the sense that

$$0 \to C \to C^e \to \oplus_{i=1}^n C^e.f_i \to \oplus_{i<j} C^e(f_i\wedge f_j) \to ... \to C^e.(f_1\wedge ... \wedge f_n) \to 0$$

is an exact sequence. Then $Cotor^*_{C^e}(C,C) = H^*(0 \to C \to \oplus_{i=1}^n C.f_i \to \oplus_{i<j} C(f_i\wedge f_j) \to ... \to C.(f_1\wedge ... \wedge f_n) \to 0)$. Using an analogue of the Künneth formula for coalgebras (see [7]) this is the cotensor product of the cohomology of the complexes

$$0 \longrightarrow C \xrightarrow{f_i.} C \longrightarrow 0$$

In other words, $Cotor^*_{C^e}(C,C)$ is a graded coalgebra isomorphic to the exterior coalgebra on $Cotor^1_{C^e}(C,C)$. Since one always has an isomorphism $Cotor^1_{C^e}(C,C) = Hoch^1(C) \cong \Omega^1_C$, the proof is complete. ∎

Conjecture: The reciprocal statement to the Hochschild - Kostant - Rosemberg theorem proved by [3] and [2] suggests that in the coalgebra case the fact of being smooth is equivalent to the above isomorphism.

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