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CYCLIC HOMOLOGY OF A FREE LOOP SPACE

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0-INTRODUCTION

In this work we generalise some results of [VP-B] and [VP-H] concerning the cyclic homology of a free loop space. The interested on this subject is due to a result of Gromoll and Meyer which asserts that the fact of the sequence of Betti numbers $b_q=\dim_k H^q(X^{S^1},k)$ being unbounded above (for X a simply connected smooth manifold) is equivalent to the existence of infinetely many distinct closed geodesics on X in any Riemannian metric.

We also show a duality result on cyclic cohomology which is a cyclic version of the duality theorem for Hochschild homology established by [VP-H].

The main theorems are:

Theorem 1.1: If k is a field and A is a k-differential graded algebra (k-DGA) such that $A_0=k$, there is an isomorphism of k-vector spaces:

$$HH_{-n}(A) \cong HH^n(\bar{\Omega}C) \quad (n \in \mathbb{N})$$

where C is the dual coalgebra of A and $\bar{\Omega}$ is the cobar construction on C ([A]).

Theorem 1.2: Under the same hypothesis of Th.1.1, $HC^{-n}(\Omega C)$ is isomorphic to $HC^-_n(A)$ as $HC^*(k)$ modules, for all $n \in \mathbb{N}$.

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In section I we describe the bar and cobar constructions in this situation. We prove Theorem 1.1 and Theorem 1.2.

In section II we prove that Th.1.1 is equivalent to Theorem II of [VP-H].

In section III we prove the following result:

Theorem 3.1: Let k be a field, X a simply connected finite CW-complex , let $n_X+1=\sup(i/ H^i(X,k)\neq 0)$, $r_X+1=\inf(i\geq 2/H^i(X,k)\neq 0)$ and suppose either char k $> n_X/r_X$ or X is k-formal ([An]).

If $c_q = \dim H^q(ES^1 \times_{S^1} X^{S^1},k)$ $(q \geq 0)$ is. a bounded above sequence, then the k-algebra $H^*(X,k)$ is generated by a single cohomology class.

The converse may not hold if char k is different from zero.

The equivalence of both statements of Th.3.1 has been obtained by Vigué and Burghelea [VP-B] for char k=0.

A dual (homological) version of Theorem 3.1 may be obtained when the k-algebra generated by the singular chains on the Moore loop space of X is finitely generated as k-vector space.

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I- BAR AND COBAR CONSTRUCTION

Let k be a field, (C,δ) a differential graded k-coalgebra (k-DGC) with coproduct Δ and (A,d) its dual k-DGA, such that $A_0 = C^0 = k$.

We begin by recalling from [M] the definition of the bar and cobar construction on A and C, respectively.

Let $\overline{B}(A) = \coprod_{i\geq 0}(s\overline{A})^{\otimes i}$ be the reduced bar construction on A ([M]). If A is a k-DGA, then $\overline{B}(A)$ is a double complex.

We consider in $\mathrm{Tot}\overline{B}(A)$ the differential:

$b':\overline{B}(A) \rightarrow \overline{B}(A)$ defined as:

$$b'(a_0\otimes\ldots\otimes a_n) = \Sigma_{0\leq i\leq n-1}(-1)^i(a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n) + \Sigma_{0\leq i\leq n}(-1)^i(a_0\otimes\ldots\otimes da_i\otimes\ldots\otimes a_n)$$

Then $A^e\otimes B(A)$ is an $A\otimes A^{op}$-A^e-resolution of A, and the Hochschild homology of A can be calculated as follows:

$$HH_n(A) = H_n(A,A) = H_n(A\otimes_{A^e}(A^e\otimes\overline{B}(A)) = H_n(A\otimes\overline{B}(A))$$

where the differential in the last total complex is induced by the differential of $A\otimes_{A^e}(A^e\otimes\overline{B}(A))$ and is explicitely :

$$b(a_0\otimes\ldots\otimes a_n) = \Sigma_{0\leq i\leq n-1}(-1)^i(a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n) + (-1)^n(a_na_0\otimes\ldots\otimes a_{n-1}) + \Sigma_{0\leq i\leq n}(-1)^i(a_0\otimes\ldots\otimes da_i\otimes\ldots\otimes a_n).$$

Remark: $\overline{B}(A)$ is a k-DGC with coproduct Δ_B:

$$\Delta_B(a_1\otimes\ldots\otimes a_n) = \Sigma_{1\leq i\leq n}(a_1\otimes\ldots\otimes a_i)\otimes(a_{i+1}\otimes\ldots\otimes a_n)$$

Cobar construction on C:

We shall denote by $\Delta: C\rightarrow C\otimes C$ the coproduct of C. and by $\overline{C}$ the cokernel of $\varepsilon: k\rightarrow C$.
The cobar construction on C is $\overline{\Omega}C = \coprod_{i\geq 0}(s^{-1}\overline{C})^{\otimes i}$, with differential:

$$\delta(c_1\otimes\ldots\otimes c_n) = \Sigma_{1\leq i\leq n}(-1)^i(c_1\otimes\ldots\otimes\Delta c_i\otimes\ldots\otimes c_n) + \Sigma_{1\leq i\leq n}(-1)^i(c_1\otimes\ldots\otimes\delta c_i\otimes\ldots\otimes c_n)$$

We recall that $\overline{\Omega}C$ is a k-DGA with coproduct ϕ obtained by yuxtaposition.

Next we consider the functors $\overline{\Omega}$: k-DGC$\rightarrow$ k-DGA and $\overline{B}$: k-DGA $\rightarrow$ k-DGC, then:

Proposition:([M]) $\overline{\Omega}$ is a left adjoint to $\overline{B}$.

Corollary 1: $\overline{B}(A)$ is isomorphic to the dual of $\overline{\Omega}C$ as k-DGC (we shall denote by $(\overline{\Omega}C)^*$ the dual of $\overline{\Omega}C$).

Proof: All the isomorphisms that will be considered are isomorphisms of k-DG modules.

It is clear that $(\overline{B}(A))^i$ is $(\overline{B}(C^*))^i \cong \mathrm{Hom}_k(k,\overline{B}(C^*)) \cong \mathrm{Hom}_k(\overline{\Omega}k,(C^i)^*) \cong \mathrm{Hom}_k(\overline{\Omega}k,\mathrm{Hom}_k(C^i,k)) \cong \mathrm{Hom}_k(C^i,\mathrm{Hom}_k(\overline{\Omega}k,k)) \cong \mathrm{Hom}_k(C^i,\overline{B}k) \cong \mathrm{Hom}((\overline{\Omega}C)^i,k) \cong ((\overline{\Omega}C)^*)^i$.

Corollary 2: If $\Omega C = C\otimes_k\overline{\Omega}C$ and $B(A) = A\otimes\overline{B}(A)$, then $(\Omega C)^*$ is isomorphic to $B(A)$ (as k-DG modules).

While the differential b' in $\bar{B}(A)$ induces b on B(A), the differential δ in $\bar{\Omega}C$ induces δ' on ΩC, defined by:

$\delta'(g\otimes f_1\otimes\ldots\otimes f_n) = (\Delta g\otimes f_1\otimes\ldots\otimes f_n)(1+(-1)^{n+1}T) - g\otimes\delta(f_1\otimes\ldots\otimes f_n)$.

Theorem 1.1: $HH_{-n}(A)$ is isomorphic to $HH^n(\bar{\Omega}C)$, where $\bar{\Omega}C$ is considered as k-DGA and we suppose that $A_0=k$.

Proof: $HH_{-n}(A)$ is, by definition of Hochschild homology, $H_{-n}(BA)=H_{-n}(A\otimes\bar{B}A)$.

As A and $\Omega\bar{B}A$ are homotopically equivalent if $A_0=k$ ([Mo]), this last one equals $H_{-n}(\Omega\bar{B}A\otimes\bar{B}A)=H_{-n}(\Omega\bar{B}A)$.

But $(\Omega\bar{B}A)^*$ is isomorphic to $B(\bar{\Omega}C)$ as k-DG modules, then $H_{-n}(\Omega\bar{B}A)$ is $H^n(B(\bar{\Omega}C))$, which is by definition $HH^n(\bar{\Omega}C)$.

Similarly, for the cyclic case we have the following result:

Theorem 1.2: For all $n\in N$, $HC^{-n}(\bar{\Omega}C)$ is isomorphic to $HC^-_n(A)$ if $A_0=k$.

Proof: $HC^{-n}(\bar{\Omega}C)$ is, by definition of cyclic cohomology of an algebra,

$H_n(Hom_{k[u]}(k[u]\otimes B(\bar{\Omega}C), k[u]))$, where u is an indeterminate of degree 2 and $d(u^i\otimes\alpha) = u^i\otimes b\alpha + u^{i-1}\otimes B\alpha$. By [H-J], this last expression is isomorphic to $H_n(k[v]\otimes Hom_k(B(\bar{\Omega}C),k))$, where $dg(v)=-2$.

Since the dual of $B(\bar{\Omega}C)$ is isomorphic to $\Omega(\bar{B}A)$ by a morphism of mixed complexes, $HC^{-n}(\bar{\Omega}C)$ is then isomorphic to $H_n(k[v]\otimes\Omega(\bar{B}A))=H_n(k[v]\otimes(\bar{B}A\otimes\bar{\Omega}(\bar{B}A)))$. As $A_0=k$, the last one equals $H_n(k[v]\otimes(A\otimes\bar{B}A))$ which is, by definition, $HC^-_n(A)$.

Remark: We obtain another proof of the theorem following Corollary VI.3 of [S], from where we copy the notations:

$$HC^n(\bar{\Omega}C) = H_{-n}(Hom_S(k[u]\otimes B\bar{\Omega}C, k[u])) = H_{-n}(Hom_S(k[u], k[u] \,\square_{k[u]}(k[u]\otimes(B\bar{\Omega}C)^*))) =$$

$$H_{-n}(Hom_S(k[u], k[u]\otimes\Omega(\bar{B}A))) = H_{-n}(Hom_S(k[u], k[u]\otimes BA) = HC^-_{-n}(A)$$

II. SOME EQUIVALENCES

Let A be a k-DGA such that $H_0(A)=k$, $H_1(A)$ and $HH_i(A)$ are finite dimensional k-vector spaces. We still denote by C the dual coalgebra of A.

In this paragraph we prove that Theorem1.1 is equivalent to TheoremII of [VP-H]. We first recall this Theorem:

(*) Theorem II ([VP-H]): If A is in the above conditions, then $Hom_k(HH_*(A),k)$ is isomorphic to $HH^{-*}(\Omega\bar{B}A)$, where $\Omega\bar{B}A$ is considered as a k-DGA.

Theorem 2.1 : (*) implies Theorem 1.1.

Proof: We must show that $HH^*(\bar{\Omega}C)$ is isomorphic to $HH_{-*}(A)$.

$HH^*(\bar{\Omega}C)$ is, by definition, $H^*(B\bar{\Omega}C)=H^*(\bar{\Omega}C\otimes\bar{B}\bar{\Omega}C)$. On the other hand, $HH_{-*}(A)$ is isomorphic to $HH_{-*}(\Omega\bar{B}A))$, as a consequence of (*). But $HH_{-*}(\Omega(\bar{B}A))$ is $H_{-*}(B\bar{\Omega}(\bar{B}A))=H_{-*}(\bar{\Omega}(\bar{B}A)\otimes\bar{B}\bar{\Omega}(\bar{B}A))$, and this equals $H^*(\bar{B}\bar{\Omega}C\otimes\bar{\Omega}C) = H^*(B\bar{\Omega}C)$.

We shall now see the converse:

Theorem 2.2: Theorem 1.1 implies (*).

Proof: We want to show that $HH^*(\Omega\bar{B}A)$ is isomorphic to $Hom_k(HH_*(A),k)$.

$HH^*(\Omega\bar{B}A)$ is isomorphic to $HH_{-*}(\bar{\Omega}C)$. But, under the hypotheses $HH_{-*}(\bar{\Omega}C)$ is isomorphic to its double dual, that is to $Hom_k(H^{-*}(\bar{\Omega}C),k)$ which by Theorem 1.1 is isomorphic to $Hom_k(HH_*(A),k)$.

III. Proof of the main theorem

The aim of this section is to prove the following theorem:

Theorem 3.1: Let k be a field, X a simply connected finite CW-complex, let $n_X+1=\sup(i/H^i(X,k)\neq0)$, $r_X+1=\inf(i\geq2/H^i(X,k)\neq0)$ and suppose either char k $>n_X/r_X$ or X is k-formal ([An]).

If $c_q = \dim H^q(ES^1 \times_{S^1} X^{S^1}, k)$ $(q \geq 0)$ is a bounded above sequence, then the k-algebra $H^*(X,k)$ is generated by a single cohomology class.

The converse may not hold if char k is different from zero.

Proof: According to [B-F], $H^*_{S^1}(X^{S^1},k) = HC^*(C_*(\Omega X),k)$, where $C_*(\Omega X)$ is the k-DGA of singular chains in the Moore loop space of X. Let us first suppose that $H^*(X,k)$ is not generated by a single cohomology class.

Then the sequence $(b_q = \dim HH^q(C_*(\Omega X),k))$ is not bounded above ([VP-H]).

Let us consider Connes-Gysin exact sequence:

$$\ldots \to HH^{n-1}(C_*(\Omega X)) \xrightarrow{B^{n-1}} HC^{n-2}(C_*(\Omega X)) \xrightarrow{S^n} HC^n(C_*(\Omega X)) \xrightarrow{I^n} HH^n(C_*(\Omega X)) \to \ldots$$

from which we obtain:

$$0 \to \operatorname{Ker} I^n \to HC^n(C_*(\Omega X)) \to HH^n(C_*(\Omega X)) \to HC^{n-1}(C_*(\Omega X)) \to \operatorname{Im} B^n \to 0$$

then: $\dim HH^n(C_*(\Omega X)) \leq \dim HC^n(C_*(\Omega X)) + \dim HC^{n-1}(C_*(\Omega X))$.

So, if $(c_q = \dim HC^q(C_*(\Omega X)))$ were bounded, (b_q) would also be bounded.

For the converse statement, we follow the steps of ([VP-H], Th.III). Let $(T(V),d)$ be an Adams-Hilton model for X (i.e.:($(T(V),d)$ is quasiisomorphic to $(C_*(\Omega X),d)$([An]), then $H^*_{S^1}(X^{S^1},k)$ is isomorphic to $HC^*(T(V),d)$.

If (Ω^*,d) denotes the k-DGA obtained by dualizing the bar construction on $(T(V),d)$ Theorem 1.2 shows that $HC^*(\Omega^*,d)$ is isomorphic to $HC^-_{-*}(T(V),d)$.

It is also known (Prop.3.1 of [VP-S]) that in the conditions of Theorem 3.1, there is a commutative DGA A which is quasi-isomorphic to (Ω^*,d) and such that $H^*(A) = H^*(X,k)$.

Moreover, if X is k-formal, $(A,d) = (H^*(X,k),0)$. Then $H^*_{S^1}(X^{S^1},k) \cong HC^*(H^*(X,k),0)$.

As in the rational case, there is a quasi-isomorphism between a CDGA of the type $(\Lambda W,d)$ and (A,d). Then, by Lemma 4.1 of [VP-H], the algebra $H^*(\Lambda W)$ is generated by a single class if and only if $\dim W^{odd} \leq 1$.

We have to distinguish two cases, as, if $H(\Lambda W)$ is generated by a single class, then (i) $H(\Lambda W) = (y)$, where $dg(y)$ is odd or (ii) $H(\Lambda W) = k \oplus k.x \oplus k.x^2 \oplus \ldots \oplus k.x^{r-1}$, where $dg(x)$ is even.

In case (i), it is easy to calculate thatthe sequence (c_q) is bounded above.

Unfortunately, in case (ii) the conclusion does not hold. We shall exhibit an example:

If $(\Lambda W,d)$ is generated by x and y, with $dg(y)=1$ and $dy=x^r$, for some $r \geq 0$, then $H(\Lambda W)$ is isomorphic to the polynomial algebra $k[x]/(x^r)$. As we see in [BACH], the negative cyclic homology of this algebra is:

$$HC^-_{2i+1}(k[x]/(x^r)) = k^{(r-1)} \oplus (k/rk) \oplus \prod_{j>i}\Big(\bigoplus_{0 \leq a < r}(k/(a+jr)k)\Big) \oplus (k/rk))$$

$$HC^-_{2i}(k[x]/(x^r)) = \prod_{j>i}(\operatorname{Ann}(r)) \oplus \Big[\bigoplus_{0 \leq a \leq r-1}\Big(\prod_{j>i+1}(\operatorname{Ann}(a+jr)k)\Big]\Big) \quad (i \geq 0).$$

Of course, if $k \supseteq \mathbb{Q}$, the sequence (c_q) is bounded above, as the the second expression, and the second and third terms of the first expression vanish. But if this is not the case, we see that the dimensions grow with i.

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